Homework 3 — solutions

(1) As in the lectures, form a graph G_t by putting an edge e if there was ever a cross (transposition) at e before time t. Then G_t is percolation on G with $p = 1 - e^{-t}$, and each cycle γ of π_t is a subset of some component of G_t.

(a) By the Erdős - Rényi theorem, letting C_1 denote the longest component of G_t, and with $t = \beta = \frac{0.99}{n}$:

$$\mathbb{P}(|\gamma_1| \leq c \log n) \geq \mathbb{P}(|C_1| \leq c \log n) \to 1$$

for some $c < \infty$.

(b) For percolation on $\mathbb{Z}$ there is never an ∞ component, hence same is true for the cycles of π_t.

2 a) $H' := H_{XXZ} - h\, S^z$

since $[S^z, S^z] = 0$ clearly, and $U S^z U^{-1} = S^z$, H' is still partially isotropic and homogeneous.

thus the decomposition $\mathcal{H} = \bigoplus_{M=0}^{L} \mathcal{H}_M$ with $\begin{cases} S^z|_{\mathcal{H}_M} = \frac{1}{2}(L-2M)\mathbb{1} \\ S^\pm \mathcal{H}_M \subseteq \mathcal{H}_{M\mp 1} \end{cases}$

is also preserved by H'.

in fact, if $|\Psi_M\rangle \in \mathcal{H}_M$ is an H_{XXZ}-eigenvector with eigenvalue E_M, then

$$H'|\Psi_M\rangle = H_{XXZ}|\Psi_M\rangle - h\, S^z|\Psi_M\rangle = \left(E_M - \frac{h}{2}(L-2M)\right)|\Psi_M\rangle$$

has eigenvalue $E'_M = E_M - \frac{h}{2}(L-2M) = E_M + h(M - L/2)$.

in particular $E'_0 = E_0 - hL/2 = -\frac{JL\Delta}{4} - \frac{hL}{2}$

$$E'_1 = E_1 + h(1 - L/2) = E'_0 + \varepsilon'_1, \quad \varepsilon'_1 := \underset{\overset{\parallel}{J(\Delta - \cos p)}}{\varepsilon_1} + h$$

[and more generally $E'_M = E'_0 + \underbrace{\sum_{m=1}^{M} \varepsilon_1(p_m)}_{\text{magnon contrib. to energy if } h=0} + h M = E'_0 + \sum_{m=1}^{M} \varepsilon'_1(p_m)$
if we use Bethe ansatz]

thus it's clear that we can proceed just as in the lectures to find the eigenvectors, with corresponding eigenvalues modified as above. the story about momentum, $\mathcal{H} = \bigoplus_{p \in \frac{2\pi}{L}\mathbb{Z}_L} \mathcal{H}_p$ etc, is unchanged too.

b) likewise: everything is as for $h=0$, up to the energy shift found above.

(the typo in v2 of the lecture notes is on the bottom of p 12: it should be $\sum_k S^\pm_k S^\mp_{k+1} |l_1, l_2\rangle = |l_1 \pm 1, l_2\rangle + |l_1, l_2 \pm 1\rangle$: both excitations hop in the same direction.)

3 a) We follow the hint. Note that $f(u) := \Lambda_M(a(u), b(u), c(u); \vec{z})$ is of the form $f(u) = g(u)/h(u)$ with

$$g(u) = a^L \prod_{m=1}^{M} (b - 2\Delta b z_m + a z_m) + (-1)^M b^L \prod_{m=1}^{M} (b - 2\Delta a + a z_m)$$
$$h(u) = \prod_{m=1}^{M} (a - b z_m)$$

where, for brevity we omit the argument u of a, b, c, and we use $\Delta = \frac{a^2+b^2-c^2}{2ab}$ to eliminate c^2 in favour of a, b and Δ.

Next notice that g, h are polynomial in a, b, c, whence also entire in u. f has a pole in u whenever $h(u) = 0$, i.e. for u_* such that $a(u_*) = b(u_*) z_m$ for some $1 \le m \le M$. Assuming the z_m are all different, these are simple poles.

To see that this apparent pole of f is actually removable we compute

$$\operatorname{Res}_{u=u_*} f \underset{\substack{\uparrow \\ \text{for simple} \\ \text{poles}}}{=} \lim_{u \to u_*} \Big((u-u_*) f(u) \Big) \underset{\substack{\uparrow \\ \text{L'Hôpital}}}{=} \frac{g(u_*)}{h'(u_*)}$$

which is ok since $h'(u_*) = \prod_{\substack{n=1 \\ \neq m}}^{M} \big(a(u_*) - b(u_*) z_n\big) \times \big(a'(u_*) - b'(u_*) z_m\big) \neq 0$ if u_* is a simple zero of h.

Thus we calculate

$$g(u_*) \underset{\text{a bit of algebra}}{=} (1 - 2\Delta z_m + z_m^2)\, b(u_*)^{L+M} \times \prod_{\substack{n=1 \\ \neq m}}^{M} (1 - 2\Delta z_n + z_m z_n) \times \Bigg(z_m^L - \prod_{\substack{n=1 \\ \neq m}}^{M} \underbrace{\Big(- \frac{1 - 2\Delta z_m + z_m z_n}{1 - 2\Delta z_n + z_m z_n} \Big)}_{= S(z_m, z_n)} \Bigg) \leftarrow = 0 \text{ iff } m\text{-th BAE is satisfied}$$

really: $S(p(z_m), p(z_n))$ with $p(z)$ inverse to $(z_m = e^{ip_m})$

where we use that $a(u_*) = b(u_*) z_m$ to end up with an expression that only involves $b(u_*)$, Δ and the z's.

b) Direct computation, e.g.

$$\lambda_m - \lambda_n \pm i = \frac{i}{2} \left(\frac{e^{ip_m/2} + e^{-ip_m/2}}{e^{ip_m/2} - e^{-ip_m/2}} - \frac{e^{ip_n/2} + e^{-ip_n/2}}{e^{ip_n/2} - e^{-ip_n/2}} \pm 2 \right)$$

$$= \frac{i}{2} \frac{1}{(e^{ip_m}-1)(e^{ip_n}-1)} \Big(\underbrace{(e^{ip_m}+1)(e^{ip_n}-1)}_{= \cancel{e^{ip_m} e^{ip_n}} - e^{ip_m} + e^{ip_n} \cancel{-1}} - \underbrace{(e^{ip_n}+1)(e^{ip_m}-1)}_{= \cancel{e^{ip_m} e^{ip_n}} + e^{ip_m} - e^{ip_n} \cancel{-1}} \pm 2 (e^{ip_m}-1)(e^{ip_n}-1) \Big)$$

$$= \frac{i}{\cancel{2}} \frac{1}{(e^{ip_m}-1)(e^{ip_n}-1)} \cdot \pm \cancel{2} \Big(\underbrace{(e^{ip_m}-1)(e^{ip_n}-1)}_{= e^{ip_m} e^{ip_n} - e^{ip_m} - e^{ip_n} + 1} \pm e^{ip_n} \mp e^{ip_m} \Big)$$

$$= \frac{i}{(e^{ip_m}-1)(e^{ip_n}-1)} \cdot \begin{cases} +(1 - 2e^{ip_m} + e^{ip_m} e^{ip_n}) \\ -(1 - 2e^{ip_n} + e^{ip_m} e^{ip_n}) \end{cases} \text{ whose ratio is } S(p_m, p_n).$$

c) Recall that $p = p_1 + p_2 \in \frac{2\pi}{L}\mathbb{Z}_L$ is real, so $\mathrm{Im}(p_1) = -\mathrm{Im}(p_2)$.
If $\mathrm{Im}(p_1) = -\mathrm{Im}(p_2) \neq 0$ then the LHS of the BAE (namely: $e^{ip_m L}$) grows or decays exponentially as $L \to \infty$.
The resulting pole or zero, as $L \to \infty$, on the LHS of the BAE should be matched by one on the RHS.
In terms of rapidities it's clear that the latter happens precisely as $|\mathrm{Im}(\lambda_1 - \lambda_2)| \to 1$ asymptotically. Taking into account finite-size corrections this means that (after relabeling $\lambda_1 \leftrightarrow \lambda_2$ if necessary)

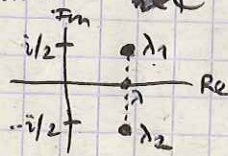
$$\begin{cases} \lambda_1 = \lambda + \frac{i}{2} + \dots \\ \lambda_2 = \lambda - \frac{i}{2} + \dots \end{cases}$$

λ: $\in \mathbb{R}$ to account for $p \in \mathbb{R}$

$\frac{i}{2}$: of the above; check that also $\mathrm{Im}\,\lambda_1 = -\mathrm{Im}\,\lambda_2$

$\dots$: finite-size corrections

picture: $\mathbb{C}$



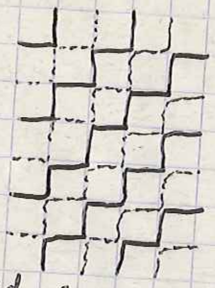
[this is known as a "2-string" of rapidities.]

d) Use that $e^{ip_m} = \frac{\lambda_m + i/2}{\lambda_m - i/2}$:

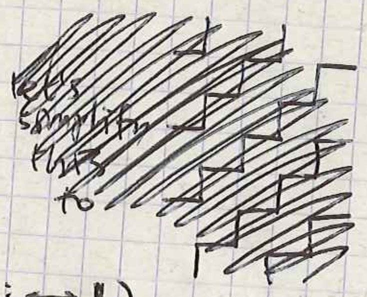
$$\begin{aligned}
\varepsilon_2(p_1, p_2) &= \varepsilon_1(p_1) + \varepsilon_1(p_2) = 2 - \cos p_1 - \cos p_2 \\
&= 2 - \frac{1}{2}\left(e^{ip_1} + \frac{1}{e^{ip_1}} + e^{ip_2} + \frac{1}{e^{ip_2}}\right) \\
&= 2 - \frac{1}{2}\left(\frac{(\lambda + i/2 + \dots) + i/2}{(\lambda + i/2 + \dots) - i/2} + \frac{(\lambda + i/2 + \dots) - i/2}{(\lambda + i/2 + \dots) + i/2}\right. \\
&\qquad \left. + \frac{(\lambda - i/2 + \dots) + i/2}{(\lambda - i/2 + \dots) - i/2} + \frac{(\lambda - i/2 + \dots) - i/2}{(\lambda - i/2 + \dots) + i/2}\right) \\
&= 2 - \frac{1}{2}\left(\frac{\lambda + \cancel{i} + \dots}{\lambda + \dots} + \frac{\lambda + \dots}{\lambda + i + \dots} + \frac{\lambda + \dots}{\lambda - i + \dots} + \frac{\lambda \cancel{- i} + \dots}{\lambda + \dots}\right) \\
&= 2 - \frac{1}{\cancel{2}}\left(\cancel{2}(1 + \dots) + \underbrace{(\lambda + \dots)\frac{(\lambda \cancel{-i} + \dots) + (\lambda \cancel{+i} + \dots)}{(\lambda + i + \dots)(\lambda - i + \dots)}}_{= \cancel{2}(\lambda + \dots)\frac{1}{\lambda^2 + 1 + \dots}}\right) \\
&= 1 - \frac{\lambda^2}{\lambda^2 + 1} + \dots \\
&= \frac{1}{\lambda^2 + 1}
\end{aligned}$$

4) Ground state:

(compatible with periodic boundary conditions for K, L even)



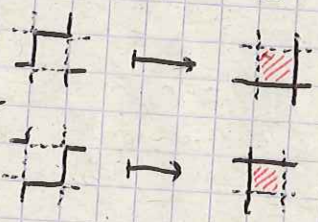
let's simplify this to



with weight $c^N = 1$ (we set $c=1$).

or its spin-reversed ($\cdots \leftrightarrow$ —, & $\vdots \leftrightarrow$ ❙).

Note that the smallest change we can make wrt either ground state is to flip all 'spins' on the four edges surrounding a face of the lattice:

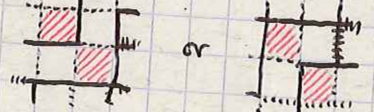


(note: the two local pictures are spin-reversed of each other)
both result in weight change of

$$\left(\frac{a}{c}\right)^2\left(\frac{b}{c}\right)^2 = a^2 b^2$$

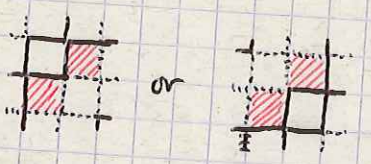
there are N faces for which this can be done.

Next-to-smallest change: flip spins around two faces:
possible results:



again both resulting in the same weight change: $\left(\frac{a}{c}\right)^2\left(\frac{b}{c}\right)^4 = a^2 b^4$

there are N ways to put on the lattice



$\rightsquigarrow \left(\frac{a}{c}\right)^4\left(\frac{b}{c}\right)^2 = a^4 b^2$; # $= N$

or the two faces do not touch;
this clearly gives $(a^2b^2)^2 = a^4 b^4$; $\# = \binom{N}{2} - \overbrace{N \cdot 4}^{\text{correct}}$ for , , , .

(Note that , are not ok with the ice rule.)

summary of results obtained by proceeding in this way:

	weight ($c=1$)	#
ground state	1	1 $\left(\text{really } 2\right.$ by reversing all ⁞ $\leftrightarrow$ ‖ and $\cdots \leftrightarrow$ — $\left.\right)$
	a^2b^2	$N = \#$ faces
	a^2b^4	$N = \#$ faces (for putting, say  ← this one)
	a^4b^2	N likewise
	a^4b^4	$\binom{N}{2} - 4N$ choose any two faces but avoid  see above, not allowed (weight zero)
	a^2b^6	N
	a^6b^2	N
	a^4b^4	$4N = \left(\begin{matrix}\text{include the four}\\ \text{rotated versions}\end{matrix}\right) \times \#$ faces ← note: all have same weight (check!)
	$a^2b^2 \cdot a^2b^4$	irrelevant for us (higher order in a, b)
	$a^2b^2 \cdot a^4b^2$	irrelevant for us
	$(a^2b^2)^3$	irrelevant for us
	a^2b^8	N (irrelevant)
	a^8b^2	N (irrelevant)
	a^4b^6	4N each (but irrelevant) horiz/vert refl versions of this have same weights (check!)
	a^6b^4	
	a^4b^4	$N = \#$ faces

note (and check!): $\circ \leftrightarrow a$, $\times \leftrightarrow b$

call these "disconnected"

⋮ various disconnected possibilities ← all higher order in a, b (check!)

≥ 5 faces	← all higher order in a, b (check!)

⋮

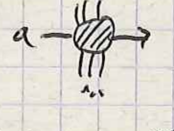
so

$$\frac{1}{2} Z = 1 + N a^2 b^2 + N a^2 b^2 (a^2 + b^2) + \underbrace{\left(\binom{N}{2} - 4N + 4N + N\right)}_{=\binom{N+1}{2} = \frac{1}{2}N(N+1)} a^4 b^4$$

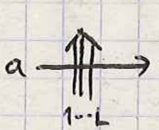
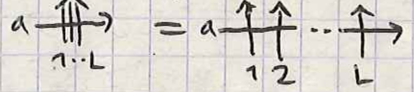
$$+ N a^2 b^2 (a^4 + b^4) + \cdots$$

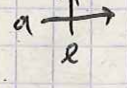
as we wanted to show.

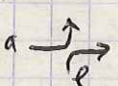
5. Any operator $X_{a\ldots} \in \mathrm{End}(V_a \otimes \ldots)$ is graphically of the form

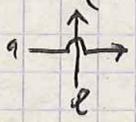
$X_{a\ldots} =$ 

where we fill in the ⊘ in a way that tells us that it's X we're dealing with; e.g.

$T_a =$   motivated by the definition 

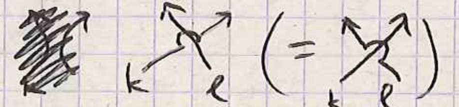
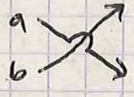
$L_{al} =$   simplest thing we can do: this is our building block

$P_{al} =$  

and we could e.g. draw $\mathrm{id}_{al} = \mathbb{1}_{al}$  

} to suggest what it does with basis vectors (switch from V_a to V_l and reversely, or just pass from V_a to V_a and V_l to V_l, resp.)

(Also note that we would instead draw

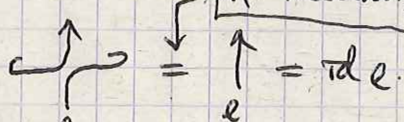
$P_{kl} =$  , $P_{ab} =$ 

$\mathrm{id}_{kl} =$  , $\mathrm{id}_{ab} =$ 

The reason is of course that V_a, V_b are both represented horizontally; V_k, V_l both vertically: these are pairs of horizontal lines.)

a) $\mathrm{tr}_a(X_{a\ldots}) =$  for any $X_a \in \mathrm{End}(V_a \otimes \ldots)$

so

$\mathrm{tr}_a(P_{al}) =$  $= \mathrm{id}_l$.

If necessary, check by acting on basis vectors ; and ‖

Algebraically:

$$P_{al} = \begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix}_{al} = \begin{pmatrix} \begin{pmatrix}1&0\end{pmatrix}_l & \begin{pmatrix}1&0\end{pmatrix}_l \\ \begin{pmatrix}0&1\end{pmatrix}_l & \begin{pmatrix}0&1\end{pmatrix}_l \end{pmatrix}_a = \left(\begin{pmatrix}1&0\end{pmatrix}_a \otimes \begin{pmatrix}1&0\end{pmatrix}\right)_{al} + \left(\begin{pmatrix}0&1\end{pmatrix}_a \otimes \begin{pmatrix}1&0\end{pmatrix}\right)_{al}$$
$$+ \left(\begin{pmatrix}1&0\end{pmatrix} \otimes \begin{pmatrix}0&1\end{pmatrix}\right)_{al} + \left(\begin{pmatrix}0&1\end{pmatrix} \otimes \begin{pmatrix}0&1\end{pmatrix}\right)_{al}$$

$$\mathrm{tr}_a P_{al} = \begin{pmatrix}1&0\end{pmatrix}_l + \begin{pmatrix}0&1\end{pmatrix}_l = \mathrm{id}_l = 1\cdot\begin{pmatrix}1&0\end{pmatrix}_l + 0\cdot\begin{pmatrix}1&0\end{pmatrix}_l + 0\cdot\begin{pmatrix}0&1\end{pmatrix}_l + 1\cdot\begin{pmatrix}0&1\end{pmatrix}_l$$

b) $L_{al}(u_*) = c_* P_{al}$

$\Leftrightarrow$ $b(u_*) = 0$ & $a(u_*) = c(u_*) = c_* = r \sinh\gamma$

$\Updownarrow$ $\quad\quad\quad\quad$ $\nearrow$

$\sinh u_* = 0$ $\quad$ (or $r=0$, but we don't want that)

$u_* \in 2\pi i \mathbb{Z}$ (only solution for generic γ)

[or $u_* \in \pi i + 2\pi i\mathbb{Z}$ & $\gamma \in \pi i \mathbb{Z}$]

c) $H_0 = \log t(u_*)$: first compute

$$t(u_*) = \mathrm{tr}_a \left(L_{aL}(u_*) \cdots L_{a1}(u_*)\right) = c_*^L \,\mathrm{tr}_a (\underbrace{P_{aL} \cdots P_{a1}}_{= P_{a12\cdots L}})$$

the operator that represents the cycle $(a12\cdots L) \in S_{L+1}$

cf.

$$\mathrm{tr}_a (P_{aL} \cdots P_{a1}) = P_{12\cdots L} \quad \text{cf (a)}$$
$$= U$$

so $t(u_*) = c_*^L\, U = c_*^L\, e^{iP}$

and

$$H_0 = \log t(u_*) = L \log(c_*) \mathbb{1} + i\underset{\uparrow}{P} \in \mathrm{End}\,\mathcal{H}$$

is essentially the momentum operator.

d)
$$P_{al} L'_{al}(u_*) = \begin{pmatrix} a'(u_*) & & & \\ & c'(u_*) & b'(u_*) & \\ & b'(u_*) & c'(u_*) & \\ & & & a'(u_*) \end{pmatrix} = r \begin{pmatrix} \cosh\gamma & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \cosh\gamma \end{pmatrix}$$

but $\Delta = \frac{a^2+b^2-c^2}{2ab} = \cosh\gamma$ $\quad$ so $\quad P_{al} L'_{al}(u_*) = r \begin{pmatrix} \Delta & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \Delta \end{pmatrix}$.

$$\vec{S}_a \cdot_\Delta \vec{S}_l = \frac{1}{2}\left(S_a^+ S_l^- + S_a^- S_l^+ + 2\Delta\, S_a^z S_l^z\right) \quad \left[\text{recall } S^+ = \begin{pmatrix} & 1 \\ & \end{pmatrix},\ S^- = \begin{pmatrix} & \\ 1 & \end{pmatrix},\ S^z = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right]$$

!

$$= \frac{1}{4}\begin{pmatrix} \Delta & & & \\ & -\Delta & 2 & \\ & 2 & -\Delta & \\ & & & \Delta \end{pmatrix} = \frac{1}{4}\left(2\begin{pmatrix} \Delta & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \Delta \end{pmatrix} - \Delta\, \mathrm{id}\right) = \frac{1}{2}\begin{pmatrix} \Delta & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \Delta \end{pmatrix} - \frac{\Delta}{4}\mathrm{id}$$

and we see that $P_{al} L'_{al}(u_*) = 2r\left(\vec{S}_a \cdot_\Delta \vec{S}_l + \frac{\Delta}{4}\mathrm{id}\right)$.

e)
$$H_1 = \frac{d}{du}\Big|_{u=u_*} \log t(u) = t(u_*)^{-1}\, t'(u_*)$$
$$= c_*^{-L} U^{-1} \sum_{l\in\mathbb{Z}_L} \mathrm{tr}_a \left(L_{aL}(u_*) \cdots L'_{al}(u_*) \cdots L_{a1}(u_*)\right)$$
$$= c_*^{-1} U^{-1} \sum_{l\in\mathbb{Z}_L} \mathrm{tr}_a \Big(P_{aL} \cdots P_{al+1} \overbrace{P_{al} \underbrace{P_{al} L'_{al}(u_*)}_{=:\check{L}'_{al}(u_*)}}^{=\mathbb{1}} P_{al-1} \cdots P_{a1}\Big)$$

cyclic property of tr_a and operators to the right of $\check{L}'_{al}(u_*)$ don't act on V_l

$$= c_*^{-1} U^{-1} \sum_{l\in\mathbb{Z}_L} \mathrm{tr}_a \Big(\underbrace{P_{al-1} \cdots P_{a1} P_{aL} \cdots P_{al+1} P_{al}}\ \check{L}'_{al}(u_*)\Big)$$

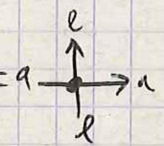
$$= P_{12\cdots l-1\, a\, l+1 \cdots L}$$
$$= P_{1\cdots L}\, P_{al-1} = U \cdot P_{al-1}$$

$$= c_*^{-1} \sum_{l\in\mathbb{Z}_L} \mathrm{tr}_a \left(P_{al-1} \check{L}'_{al}(u_*)\right) = c_*^{-1} \sum_{l\in\mathbb{Z}_L} \mathrm{tr}_a \left(\check{L}'_{l-1,l}\ P_{a,l-1}\right)$$

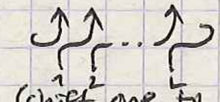
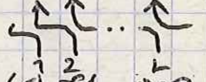
so finally

$$H_1 = \frac{2r}{c_*}\sum_{\ell\in\mathbb{Z}_L}\left(\vec{S}_{\ell-1}\cdot\vec{S}_\ell + \frac{\Delta}{4}\mathrm{id}\right) = \frac{2}{\sinh\gamma}\left(\frac{H_{XXZ}}{-J} + L\frac{\Delta}{4}\right)$$

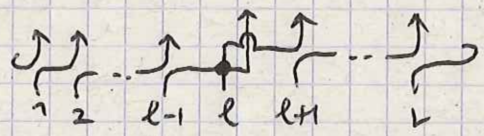
$$= -\frac{2}{J\sinh\gamma}\left(H_{XXZ} + E_0\right) \qquad \text{since } E_0 = -JL\Delta/4$$

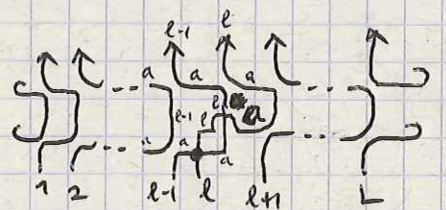
f) draw $P_{a\ell}L'_{a\ell}(u_*) = \check{L}'_{a\ell}(u_*) =$ 

recall that

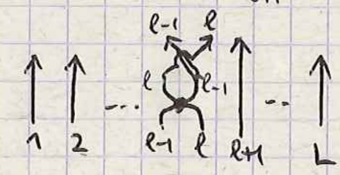
$t(u_*) = c_*^L \times$  (shift one to the right) so $t(u_*)^{-1} = c_*^{-L} \times$  (shift one to the left).

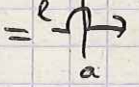
Next:

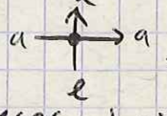
$t'(u_*) = c_*^{L-1} \sum_{\ell\in\mathbb{Z}_L}$  .

so $t(u_*)^{-1}t'(u_*) = \frac{1}{c_*}\sum_{\ell\in\mathbb{Z}_L}$ 

straighten out lines

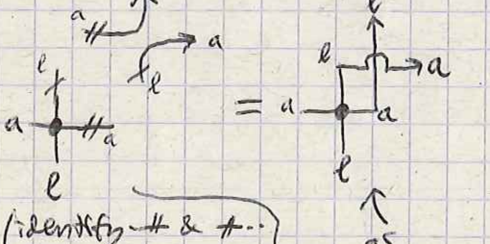
$= \frac{1}{c_*}\sum_{\ell\in\mathbb{Z}_L}$ 

NB. note that the additional $P_{a\ell} =$  later turns into  and is necessary to make sure we get something in $\mathrm{End}(V_1\otimes\cdots\otimes V_L)$

(since , just interpreted as a rotated version of , graphically swaps the two spaces, horizontal & vertical
auxiliary local quantum)

I admit that this was a bit tricky!!

also note that



(identify # & ⫲ and ⫯ & ⫰)

as drawn above

6a) For the graphical version to get the relation for moving D through B see (4.39) in the lecture notes; the corresponding ~~...~~ algebraic expressions are

$$b(u-v)\, D(u)\, B(v) + c(u-v)\, B(u)\, D(v) = B(v)\, D(u)\, a(u-v)$$

where we wrote ~~expressions in~~ the factors in each term such that the factor acting first is on the right, the second one in the middle, and the last-acting factor on the left (even if a, b, c are just functions that can be moved around freely) — just for clarity on how to read it off.

For the commutativity of B's: see (4.36), which is simply

$$a(u-v)\, B(u)\, B(v) = B(v)\, B(u)\, a(u-v).$$

b) Just as in §4.3 of the notes, with arguments of the factors of the form $u-v$ instead of $v-u$; compare (4.40) and (4.41) in the notes.

(The typos ~~...~~ in v2 of the notes, on p45:

$$M_1(u_0; \vec{u}) = -a(u_1)^L \frac{c}{b}(u_1-u_0) \prod_{n=2}^{M} \frac{a}{b}(u_n-u_1)$$ in (4.57)

and on p. (4.63):

$$\left(\frac{b}{a}(u_m)\right)^L = -\frac{c}{b}(u_m-u_0)\frac{b}{c}(u_0-u_m) \prod_{\substack{n=1 \\ \neq m}}^{M} \frac{a}{b}(u_n-u_m)\frac{b}{a}(u_m-u_n)$$