The existence phase transition for two Poisson random fractal models.

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Abstract

In this paper we study the existence phase transition of the random fractal ball model and the random fractal box model. We show that both of these are in the empty phase at the critical point of this phase transition.

1 Introduction

In order to better explain the rest of the paper, we shall start by a rather informal description of the general setup (see for example [1] for details). Let M be the set of bounded subsets of $\mathbb{R}^d$ with non-empty interior, and let $\mathcal{M}$ be some (suitable) σ-algebra on M. We consider a measure μ on $(M, \mathcal{M})$ which is scale invariant in the following sense. If $A \in \mathcal{M}$ is such that $\mu(A) < \infty$, then $\mu(A_s) = \mu(A)$ where $0 < s < \infty$ and

$$A_s := \{K : K/s \in A\}.$$

We will also assume that μ is translation invariant in that $\mu(x + A) = \mu(A)$ for every $A \in \mathcal{M}$. Here of course, $x + A = \{L \subset \mathbb{R}^d : L = x + K \text{ for some } K \in A\}$. Using $\lambda\mu$ where $0 < \lambda < \infty$ as the intensity measure, one can define a Poisson process $\Phi_\lambda(\mu)$ on M. Thus constructed, $\Phi_\lambda(\mu)$ is a scale and translation invariant random collection of bounded sets of $\mathbb{R}^d$. This setup contains many interesting examples such as the Brownian loop soup introduced in [4], and the scale invariant Poisson Boolean model studied for instance in [2] (see also the references therein). Throughout, this latter model will be referred to simply as the *fractal ball model*, and we shall give an exact definition of it in Section 2. In this fractal ball model, the measure μ above is supported on the set of *open* balls of $\mathbb{R}^d$. Of course, one could also consider a process of closed balls, or indeed a mix of open and closed balls. As we will see, the results of this paper are also valid for these cases, see further the remark after the statement of Theorem 1.1.

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Throughout this paper, we will let

$$\mathcal{C}(\Phi_\lambda(\mu)) := \mathbb{R}^d \setminus \bigcup_{K \in \Phi_\lambda(\mu)} K, \tag{1.1}$$ eqn:defCC

and we will usually write $\mathcal{C}(\lambda)$ or simply $\mathcal{C}$. Thus, with μ as above, $\mathcal{C}$ is a scale invariant random fractal and we will be concerned by various properties of $\mathcal{C}(\lambda)$ as λ varies. It is useful to observe that by using a standard coupling, $\mathcal{C}(\lambda)$ is decreasing in λ.

Random fractal models exhibits several phase transitions (see for instance [3]). However, the perhaps two most natural are the *existence* and the *connectivity* phase transitions as we now explain. Define

$$\lambda_e := \inf\{\lambda > 0 : \mathbb{P}(\mathcal{C}(\lambda) = \emptyset) = 1\}.$$

Therefore, for $\lambda > \lambda_e$, $\mathcal{C}(\lambda)$ is almost surely empty, and we say that it is in the empty phase. If instead $\lambda < \lambda_e$, then $\mathbb{P}(\mathcal{C}(\lambda) \neq \emptyset) = 1$. We say that λ_e is the critical point of the existence phase transition. Analogously, we can define

$$\lambda_c := \sup\{\lambda > 0 : \mathbb{P}(\mathcal{C}(\lambda) \text{ contains connected components larger than one point}) = 1\}.$$

Thus, for $\lambda > \lambda_c$, $\mathcal{C}(\lambda)$ is almost surely totally disconnected, while for $\lambda < \lambda_c$, $\mathcal{C}(\lambda)$ will contain connected components.

Of course, whenever such phase transitions occur, it is natural and interesting to ask what happens at the critical points. In [1] it was proven in full generality that

$$\mathbb{P}(\mathcal{C}(\lambda_c) \text{ contains connected components larger than one point}) = 1,$$

so that *at* λ_c the fractal is in the connected phase. Thus, this phase transition is very well understood.

The existence phase transition is much less understood. Hitherto, the only exact results appear to be in dimension 1. Indeed, in [5], exact conditions for when random intervals cover a line were established. However, there has been some progress (see [2]) on the case of the fractal ball model in $d \geq 2$, see Section 2 for a precise statement of these results.

In analogy with how the fractal ball model is defined, we can also define the *fractal box model* (again see Section 2) for which the measure μ is supported on boxes of the form $(a, b)^d$ for $a < b$. In this case, $\Phi_\lambda(\mu)$ is then a random scale invariant collection of boxes in $\mathbb{R}^d$. Whenever we need to distinguish between the ball and the box model, we shall write $\mathcal{C}^{ball}$ and $\mathcal{C}^{box}$ etc.

Let v_d be the volume of the unit ball in $\mathbb{R}^d$. The main result of this paper is the following.

thm:main **Theorem 1.1.** *For any $d \geq 1$, we have that $\lambda_e^{ball} = d/v_d$ while $\lambda_e^{box} = d$. Furthermore,*

$$\mathbb{P}(\mathcal{C}^{box}(\lambda_e^{box}) = \emptyset) = \mathbb{P}(\mathcal{C}^{ball}(\lambda_e^{ball}) = \emptyset) = 1.$$

Remarks: The fact that $\lambda_e^{ball} = d/v_d$ is easily deduced from results in [2], while we determine λ_e^{box} by a straightforward second moment argument. Thus, the main contribution of this paper is to determine what happens *at* the critical point of these phase transitions.

If we choose to consider closed balls (boxes) in place of open, then of course we would have that $\mathcal{C}_{closed} \subset \mathcal{C}_{open}$ (using obvious notation). However, when determining λ_e, one sees that the argument does not depend on whether we use open or closed sets so that $\lambda_e(\mathcal{C}_{closed}) = \lambda_e(\mathcal{C}_{open})$. It then follows trivially that Theorem 1.1 holds also for the case of closed balls (boxes).

The rest of the paper is organized as follows. In Section 2 we give precise definitions of our models and also provide some further background. In Section 3, we will prove Theorem 1.1.

2 Models

:defmodels

We start by defining the fractal ball model, although we will later reuse much of the notation for the box model.

Let ν be a locally finite measure on $(0,1]$, and let $\mu = dx \times \nu$ (where dx denotes d-dimensional Lebesgue measure) denote the resulting product measure on $\mathbb{R}^d \times (0,1]$. Then, we let $\Phi_\lambda(\mu)$ be a Poisson process on $\mathbb{R}^d \times (0,1]$ using $\lambda\mu$ as the intensity measure. This definition might seem to clash with $\Phi_\lambda(\mu)$ defined in the introduction (which was a Poisson process on sets). However, this is easily resolved by associating the point $(x,r) \in \mathbb{R}^d \times (0,1]$ with the open ball $B(x,r)$ centered at x and with radius r. Thus, we might write (1.1) as

$$\mathcal{C}(\Phi_\lambda(\mu)) := \mathbb{R}^d \setminus \bigcup_{(x,r)\in\Phi_\lambda(\mu)} B(x,r).$$

Let $A := \{(x,r) \in \mathbb{R}^d \times [\epsilon,1] : o \in B(x,r)\}$, and let $A_{\epsilon^{-1}} = \{(x,r) \in \mathbb{R}^d \times [\epsilon^2,\epsilon] : o \in B(x,r)\}$ (where o denotes the origin). We observe that if $\nu(dr) = r^{-d-1}dr$, then we have that (with $I(\cdot)$ being an indicator function)

$$\mu(A) = \int_\epsilon^1 \int_{\mathbb{R}^d} I(|x| \leq r) dx \nu(dr) = v_d \int_\epsilon^1 r^d r^{-d-1} dr = -v_d \log \epsilon,$$

and an analogous calculation shows that also $\mu(A_{\epsilon^{-1}}) = -v_d \log \epsilon$. With some extra work, one can show that this choice of ν corresponds to a scale invariant model. It is certainly possible to consider other choices of ν, but in this paper we shall focus on the scale invariant case. However, we want to mention the following result from [2] which deals with other choices of ν.

thm:BE **Theorem 2.1** (From [2]). *For the fractal ball model, if $\mathbb{P}(\mathcal{C} = \emptyset) = 1$ then*

$$\int_0^1 u^{d-1} \exp\left(\lambda v_d \int_u^1 r^{d-1}(r-u)\nu(dr)\right) du = \infty, \quad (2.1)$$ eqn:nec

while if

$$\limsup_{u\to 0} u^d \exp\left(\lambda v_d \int_u^1 (r-u)^d \nu(dr)\right) du = \infty, \quad (2.2)$$ eqn:suff

then $\mathbb{P}(\mathcal{C} = \emptyset) = 1$.

Remark: Taking $\nu(dr) = r^{-d-1}dr$, one concludes from (2.1) and (2.2) that $\lambda_e^{ball} = d/v_d$. However, simple calculations reveal that (2.2) is not satisfied for $\lambda = d/v_d$, and so we cannot conclude whether $\mathbb{P}(\mathcal{C}(\lambda_e^{ball}) = \emptyset) = 1$. As pointed out in [2], it follows from Theorem 2.1 that if $\nu(dr) = r^{-d-1}(1+2|\log(r)|^{-1})$ and $\lambda = d/v_d$, then $\mathbb{P}(\mathcal{C} = \emptyset) = 1$ while if $\nu(dr) = r^{-d-1}(1-2|\log(r)|^{-1})$ and $\lambda = d/v_d$, then $\mathbb{P}(\mathcal{C} = \emptyset) = 0$. Thus, although their results do not cover the critical case, it comes logarithmically close. Of course, Theorem 1.1 improves on Theorem 2.1 in that we here determine the critical case.

We now turn to the fractal box model. Here, we again use the measures ν and μ as above, but to any $(x, r) \in \mathbb{R}\times(0, 1]$, we associate the open box $X(x, r) := x+(-r/2, r/2)^d$. We then write

$$\mathcal{C}(\Phi_\lambda(\mu)) := \mathbb{R}^d \setminus \bigcup_{(x,r)\in\Phi_\lambda(\mu)} X(x, r).$$

Letting $A = \{(x, r) \in \mathbb{R} \times [\epsilon, 1] : X(x, r) \cap [0, 1]^d \neq \emptyset\}$ we have that

$$\mu(A) = \int_\epsilon^1 \int_{\mathbb{R}^d} I(x \in (-r/2, 1 + r/2)^d)dx\mu(dr) = \int_\epsilon^1 (1 + r)^d r^{-d-1}dr.$$

Similarly, if $A_{\epsilon^{-1}} = \{(x, r) \in \mathbb{R} \times [\epsilon^2, \epsilon] : X(x, r) \cap [0, \epsilon]^d \neq \emptyset\}$ then

$$\mu(A_{\epsilon^{-1}}) = \int_{\epsilon^2}^{\epsilon} \int_{\mathbb{R}^d} I(x \in (-r/2, \epsilon + r/2)^d)dx\mu(dr) = \int_{\epsilon^2}^{\epsilon} (\epsilon + r)^d r^{-d-1}dr = \mu(A),$$

so that also this model is scale invariant.

Whenever convenient, we will write $K \in \Phi$ to mean either a ball or a box, depending on the context.

3 Proofs

sec:proofs

We start this section by introducing some useful notation. First, let

$$\bar{\mathcal{X}}_n := \left\{ x + [0, 1/n]^d : x \in \left(\frac{1}{n}\mathbb{Z}^d\right) \cap [0, 1 - 1/n]^d \right\}.$$

If $\bar{X} \in \bar{\mathcal{X}}_n$, we shall refer to $\bar{X}$ as a level n box. Note that the members $\bar{X}$ of $\bar{\mathcal{X}}_n$ are deterministic, closed boxes. These should not be confused with the open boxes $X(x, r)$ that belong to the Poisson process Φ_λ^{box}.

The interpretation of the following definitions differ depending on whether we are considering the ball model or the box model. However, we believe that this should not lead to any confusion. For these models, we let

$$\Phi_n := \{(x, r) \in \Phi_\lambda(\mu) : 1/n \leq r \leq 1\},$$

and define

$$\mathcal{C}_n := \mathbb{R}^d \setminus \bigcup_{K\in\Phi_n} K$$

(where K is then either a ball or a box). Thus, $\mathcal{C}_n \downarrow \mathcal{C}$. For $m > n$, let

$$\mathcal{C}_m^n := \mathbb{R}^d \setminus \bigcup_{K \in \Phi_m \setminus \Phi_n} K,$$

so that $\mathcal{C}_m^n \cap \mathcal{C}_n = \mathcal{C}_m$, and $\mathcal{C}_m^n, \mathcal{C}_n$ are independent. For any integer n, let

$$M_n := \{\bar{X} \in \bar{\mathcal{X}}_n : \nexists K \in \Phi_n \,:\, \bar{X} \subset K\}.$$

Thus, M_n is the set of level n boxes which are not covered by a single set in the Poisson process Φ_n. Then, let

$$m_n := \{\bar{X} \in \bar{\mathcal{X}}_n : \nexists K \in \Phi_n \,:\, \bar{X} \cap K \neq \emptyset\},$$

which is the set of level n boxes untouched by the Poisson process Φ_n. We see that if $\bar{X} \in m_n$, then in fact $\bar{X} \subset \mathcal{C}_n$. Obviously, $|m_n| \leq |M_n|$ since an untouched box cannot be covered.

The following proposition is a part of Theorem 1.1.

prop:crit **Proposition 3.1.** *For the box model we have that $\lambda_e \geq d$.*

Proof. We start by noting that if $m_n \neq \emptyset$ for infinitely many $n \geq 1$, then $\mathcal{C}_n \cap [0,1]^d \neq \emptyset$ for every $n \geq 1$. Since $\mathcal{C}_n \supset \mathcal{C}_{n+1}$ for every n, and the sets $\mathcal{C}_n \cap [0,1]^d$ are compact, we must then have that

$$\mathcal{C} \cap [0,1]^d = \bigcap_{n=1}^{\infty} \mathcal{C}_n \cap [0,1]^d \neq \emptyset.$$

We will prove that for $\lambda < d$, there exists $c = c(\lambda) > 0$ such that

$$\mathbb{P}(m_n > 0) \geq c, \tag{3.1}$$ eqn:mnpos

for every $n \geq 1$. Then, we can conclude that

$$\mathbb{P}(m_n > 0 \text{ infinitely often}) \geq \limsup_{n \to \infty} \mathbb{P}(m_n > 0) \geq c,$$

by the reverse Fatou's lemma.

We shall proceed by proving (3.1) using a second moment argument. To that end, observe that by translation invariance, for any $\bar{X} \in \bar{\mathcal{X}}_n$,

$$\begin{aligned}
\mathbb{P}(\bar{X} \in m_n) &= \mathbb{P}([0,1/n]^d \in m_n) \qquad (3.2)\\
&= \exp\left(-\lambda \mu(\{(x,r) \in \mathbb{R}^d \times [1/n,1] : \ [0,1/n]^d \cap X(x,r) \neq \emptyset\})\right)\\
&= \exp\left(-\lambda \int_{1/n}^{1} \int_{\mathbb{R}^d} I(x \in (-r/2, r/2 + 1/n)^d) dx \nu(dr)\right)\\
&= \exp\left(-\lambda \int_{1/n}^{1} (r + 1/n)^d r^{-d-1} dr\right)\\
&= \exp\left(-\lambda \int_{1/n}^{1} r^{-d-1} \sum_{k=0}^{d} \binom{d}{k} r^k n^{k-d} dr\right)\\
&= \exp\left(-\lambda \sum_{k=0}^{d} \binom{d}{k} n^{k-d} \int_{1/n}^{1} r^{k-d-1} dr\right)\\
&= \exp\left(-\lambda \log n - \lambda \sum_{k=0}^{d-1} \binom{d}{k} n^{k-d} \left(\frac{n^{d-k} - 1}{d-k}\right)\right).
\end{aligned}$$

Since

$$0 \le \sum_{k=0}^{d-1} \binom{d}{k} n^{k-d} \left(\frac{n^{d-k} - 1}{d-k} \right) \le \sum_{k=0}^{d} \binom{d}{k} = 2^d, \tag{3.3}$$

eqn:ref2

we conclude that

$$e^{-\lambda 2^d} n^{-\lambda} \le \mathbb{P}(\bar{X} \in m_n) \le n^{-\lambda}. \tag{3.4}$$

eqn:Pmn

Therefore,

$$\mathbb{E}[m_n] = n^d \mathbb{P}([0,1/n]^d \in m_n) \ge e^{-\lambda 2^d} n^{d-\lambda}. \tag{3.5}$$

eqn:fmomen

For $\bar{X}_1, \bar{X}_2 \in \bar{\mathcal{X}}_n$ let $R_i^n := \{(x,r) \in \mathbb{R}^d \times [1/n, 1] : \bar{X}_i \cap X(x,r) \neq \emptyset\}$ for $i = 1,2$. We have that $\mu(R_1^n \cup R_2^n) = 2\mu(R_1^n) - \mu(R_1^n \cap R_2^n)$. First, we observe that

$$\begin{aligned} \mu(R_1^n) &= \int_{1/n}^{1} \int_{\mathbb{R}^d} I(X(x,r) \cap \bar{X}_1 \neq \emptyset) dx \nu(dr) \\ &= \int_{1/n}^{1} (r + 1/n)^d r^{-d-1} dr \ge \int_{1/n}^{1} r^{-1} dr = \log n. \end{aligned} \tag{3.6}$$

Next, let $k = (k_1, \ldots, k_d)$ be such that $\bar{X}_2 = \bar{X}_1 + k/n$, and define $k_{\max} := \max\{|k_1|, \ldots, |k_d|\}$. We get that for $k_{\max} \ge 2$,

$$\begin{aligned} \mu(R_1^n \cap R_2^n) &= \int_{(k_{\max}-1)/n}^{1} \int_{\mathbb{R}^d} I(X(x,r) \cap \bar{X}_1 \neq \emptyset, X(x,r) \cap \bar{X}_2 \neq \emptyset) dx \nu(dr) \\ &\le \int_{(k_{\max}-1)/n}^{1} \int_{\mathbb{R}^d} I(X(x,r) \cap \bar{X}_1 \neq \emptyset) dx r^{-d-1} dr = \int_{(k_{\max}-1)/n}^{1} (r + 1/n)^d r^{-d-1} dr \\ &\le -\log((k_{\max} - 1)/n) + 2^d, \end{aligned} \tag{3.7}$$

where the last inequality follows by using the calculations in (3.2) combined with the upper bound of (3.3). Therefore, if $\bar{X}_1 \neq \bar{X}_2$ and $k_{\max} \ge 2$, we have that by using (3.6) and (3.7),

$$\begin{aligned} \mathbb{P}(\bar{X}_1, \bar{X}_2 \in m_n) &= \exp\left(-\lambda \mu(R_1^n \cup R_2^n)\right) = \exp(-2\lambda \mu(R_1^n) + \lambda \mu(R_1^n \cap R_2^n)) \\ &\le e^{-2\lambda \log n + \lambda 2^d - \lambda \log((k_{\max}-1)/n)} = n^{-2\lambda} e^{\lambda 2^d} ((k_{\max} - 1)/n)^{-\lambda} = e^{\lambda 2^d} (n(k_{\max} - 1))^{-\lambda}. \end{aligned} \tag{3.8}$$

If however $k_{\max} \le 1$, then we simply use that

$$\mathbb{P}(\bar{X}_1, \bar{X}_2 \in m_n) \le \mathbb{P}(\bar{X}_1 \in m_n) = \mathbb{P}([0,1/n]^d \in m_n).$$

Thus, by (3.4) and (3.8),

$$\begin{aligned} \mathbb{E}[m_n^2] &= \sum_{\bar{X}_1 \in \bar{\mathcal{X}}_n} \sum_{\bar{X}_2 \in \bar{\mathcal{X}}_n} \mathbb{P}(\bar{X}_1, \bar{X}_2 \in m_n) \\ &\le n^d \left(3^d \mathbb{P}([0,1/n]^d \in m_n) + \sum_{k_{\max}=2}^{n} 2d k_{\max}^{d-1} e^{\lambda 2^d} (n(k_{\max} - 1))^{-\lambda} \right) \\ &\le 3^d n^{d-\lambda} + 2^d d e^{\lambda 2^d} n^{d-\lambda} \sum_{k_{\max}=2}^{n} (k_{\max} - 1)^{d-1-\lambda}. \end{aligned}$$

Here, the first inequality uses that there are n^d possible choices of $\bar{X}_1$, and given the choice of $\bar{X}_1$ there are at most 3^d choices of $\bar{X}_2$ that are either the same as, or immediate neighbours to, $\bar{X}_1$. The remaining boxes $\bar{X}_2$ have $k_{max} \geq 2$.

We see that if $\lambda < d$, then there exists a $C = C(\lambda) > 0$ such that $\mathbb{E}[m_n^2] \leq Cn^{2(d-\lambda)}$. Using (3.5) we conclude that

$$\mathbb{P}(m_n > 0) \geq \frac{\mathbb{E}[m_n]^2}{\mathbb{E}[m_n^2]} \geq \frac{\left(e^{-\lambda 2^d} n^{d-\lambda}\right)^2}{Cn^{2(d-\lambda)}} \geq c,$$

as desired. □

Our next lemma gives a useful consequence of $\mathcal{C}(\lambda)$ surviving, but first we need some more notation. Let $D_n = D_n(\mathcal{C}_n)$ be a minimal collection of boxes in $\bar{\mathcal{X}}_n$ such that

$$\mathcal{C}_n \cap [0,1]^d \subset \bigcup_{\bar{X} \in D_n} \bar{X}.$$

Note that D_n is not necessarily unique, as a point $x \in \mathcal{C}_n$ sitting on the boundary between two boxes $\bar{X}_1$ and $\bar{X}_2$ can be covered by either one of them. If there is more than one way of choosing such a set D_n, we pick one according to some predetermined rule. Let $L_n = |\{\bar{X} \in \bar{\mathcal{X}}_n : \bar{X} \in D_n\}|$.

ma:Dninfty **Lemma 3.2.** *Let $\mathcal{C}(\lambda)$ be either $\mathcal{C}^{ball}$ or $\mathcal{C}^{box}$. For any $\lambda > 0$ we have that*

$$\mathbb{P}(\{\mathcal{C}(\lambda) \cap [0,1]^d \neq \emptyset\} \setminus \{\lim_{n\to\infty} L_{2^n} = \infty\}) = 0.$$

Remarks: The reason for proving Lemma 3.2 along a subsequence $(2^n)_{n\geq 1}$, is that this will avoid unnecessary technical details. It is also all that we need in order to prove Theorem 1.1.

Observe that if $\lim_{n\to\infty} L_{2^n} = \infty$, then $\mathcal{C}_n \cap [0,1]^d \neq \emptyset$ for every $n \geq 1$. As above, it follows that also $\mathcal{C} \cap [0,1]^d \neq \emptyset$.

Proof. Let

$$E_{2^n} := \bigcup_{\bar{X} \in D_{2^n}} \bar{X},$$

and observe that by definition of D_{2^n}, we have that

$$(\mathcal{C}_{2^n} \cap [0,1]^d) \setminus E_{2^n} = \emptyset.$$

We have that for some $\alpha = \alpha(\lambda) > 0$,

$$\mathbb{P}(\mathcal{C}_2 \cap [0,1]^d = \emptyset) = \alpha.$$

By using the FKG inequality for Poisson processes together with the scaling invariance of the models, we conclude that

$$\begin{aligned}
&\mathbb{P}(\mathcal{C}_{2^{n+1}} \cap [0,1]^d = \emptyset | D_{2^n}) \\
&\quad \geq \mathbb{P}(\mathcal{C}_{2^{n+1}}^{2^n} \cap E_{2^n} = \emptyset | D_{2^n}) \geq \prod_{\bar{X} \in D_{2^n}} \mathbb{P}(\mathcal{C}_{2^{n+1}}^{2^n} \cap \bar{X} = \emptyset) = \alpha^{L_{2^n}} > 0.
\end{aligned}$$

Therefore, if there exists $L < \infty$ such that $L_{2^n} \leq L$ for infinitely many n, we can use Lévy's Borel-Cantelli lemma, to conclude that almost surely $\mathcal{C} \cap [0,1]^d = \emptyset$. □

We can now prove our main result.

Proof of Theorem 1.1. The fact that $\lambda_e^{ball} = d/v_d$ is an immediate consequence of Theorem 2.1 as explained in Section 2. Furthermore, Proposition 3.1 shows that $\lambda_e^{box} \geq d$. Therefore, it remains to prove that $\lambda_e^{box} \leq d$ and that both the ball and the box models are in the empty phase at their respective critical points.

Obviously, if $\bar{X} \in D_n$, then $\bar{X}$ cannot be covered by a single set in the Poisson process Φ_n. Therefore,

$$L_n = |D_n| \leq |M_n|. \tag{3.9}$$

eqn:DnNn

We proceed by bounding $\mathbb{E}[|M_n|]$ in the case $\mathcal{C} = \mathcal{C}^{box}$. Similar to the proof of Proposition 3.1 we have that for any $\bar{X} \in \bar{\mathcal{X}}_n$,

$$\begin{aligned}
\mathbb{P}(\bar{X} \in M_n) &= \exp\left(-\lambda \int_{1/n}^{1} \int_{\mathbb{R}^d} I(x \in (-r/2 + 1/n, r/2)^d) dx \nu(dr)\right) \\
&= \exp\left(-\lambda \int_{1/n}^{1} r^{-d-1} \sum_{k=0}^{d} \binom{d}{k} r^k (-n)^{k-d} dr\right) \\
&= \exp\left(-\lambda \log n - \lambda \sum_{k=0}^{d-1} \binom{d}{k} (-n)^{k-d} \left(\frac{n^{d-k} - 1}{d-k}\right)\right).
\end{aligned}$$

Furthermore, since

$$\sum_{k=0}^{d-1} \binom{d}{k} (-n)^{k-d} \left(\frac{n^{d-k} - 1}{d-k}\right) \geq -\sum_{k=0}^{d} \binom{d}{k} = -2^d,$$

we conclude that

$$\mathbb{E}[|M_n|] = n^d \mathbb{P}([0, 1/n]^d \in M_n) \leq n^{d-\lambda} e^{\lambda 2^d}. \tag{3.10}$$

eqn:emn

By (3.9),(3.10) together with Lemma 3.2, we have that if $\mathbb{P}(\mathcal{C}^{box} \cap [0,1]^d \neq \emptyset) > 0$, then

$$\lim_{n\to\infty} e^{\lambda 2^d} (2^n)^{d-\lambda} \geq \lim_{n\to\infty} \mathbb{E}[|M_{2^n}|] \geq \lim_{n\to\infty} \mathbb{E}[|D_{2^n}|] = \infty,$$

and so we conclude that we must have $\lambda < d$. This proves that $\lambda_e^{box} \leq d$ and that for $\lambda = d$

$$\mathbb{P}(\mathcal{C}^{box}(\lambda) \cap [0,1]^d \neq \emptyset) = 0.$$

We now turn to the case of $\mathcal{C}^{ball}$. First we observe that for any $K \in \Phi$, $[0,1/n]^d \subset K$ iff the closed ball $\bar{B}(\frac{1}{2n}(1,\ldots,1), \sqrt{d}/(2n)) \subset K$, simply because of the fact that the sets

$K \in \Phi$ are balls. We then get that for some constant $C = C(\lambda) < \infty$, and $n > \sqrt{d}/2$

$$\begin{aligned}
\mathbb{P}(\bar{X} \in M_n) &= \mathbb{P}(\nexists K \in \Phi : \bar{B}(o, \sqrt{d}/(2n)) \subset K) \\
&= \exp\left(-\lambda\mu(\{(x,r) : \bar{B}(o, \sqrt{d}/(2n)) \subset B(x,r)\})\right) \\
&= \exp\left(-\lambda \int_{\sqrt{d}/(2n)}^{1} \int_{\mathbb{R}^d} I(|x| \leq r - \sqrt{d}/(2n)) dx \nu(dr)\right) \\
&= \exp\left(-\lambda \int_{\sqrt{d}/(2n)}^{1} v_d (r - \sqrt{d}/(2n))^d r^{-d-1} dr\right) \\
&= \exp\left(-\lambda v_d \int_{1/n}^{2/\sqrt{d}} (s - 1/n)^d s^{-d-1} ds\right) \leq C n^{-\lambda v_d},
\end{aligned}$$

where the last inequality follows as above. As for the box model, we obtain that for $\lambda_e = d/v_d$

$$\mathbb{P}(\mathcal{C}^{ball}(\lambda_e) \cap [0,1]^d \neq \emptyset) = 0.$$

□

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