Optimization of shape in continuum percolation

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Abstract

We consider a version of the Boolean (or Poisson blob) continuum percolation model where, at each point of a Poisson point process in the Euclidean plane with intensity λ, a copy of a given compact convex set A with fixed rotation is placed. To each A we associate a critical value $\lambda_c(A)$ which is the infimum of intensities λ for which the occupied component contains an unbounded connected component. It is shown that $\min\{\lambda_c(A) : A$ convex of area $a\}$ is attained if A is any triangle of area a and $\max\{\lambda_c(A) : A$ convex of area $a\}$ is attained for some centrally symmetric convex set A of area a.

It turns out that the key result, which is also of independent interest, is a strong version of the difference-body inequality for convex sets in the plane. In the plane, the difference-body inequality states that for any compact convex set A, $4\mu(A) \leq \mu(A \oplus \check{A}) \leq 6\mu(A)$ with equality to the left iff A is centrally symmetric and with equality to the right iff A is a triangle. Here μ denotes area and $A \oplus \check{A}$ is the difference-body of A. We strengthen this to the following result: For any compact convex set A there exist a centrally symmetric convex set C and a triangle T such that $\mu(C) = \mu(T) = \mu(A)$ and $C \oplus \check{C} \subseteq A \oplus \check{A} \subseteq T \oplus \check{T}$ with equality to the left iff A is centrally symmetric and to the right iff A is a triangle.

1 Introduction

We begin by introducing the necessary basics on continuum percolation and start by pointing out the book of Meester and Roy [1] as general reference. A *continuum percolation model* is classically defined by a pair (X, F), where X is some random point process and F is a distribution function for some random variable with support in $[0, \infty)$. This is to be interpreted in the following way: Given X, let $\{x_j\}_{j=1}^{\infty}$ be the points of X, i.e. the set of points $u \in \mathbf{R}^2$ such that $X\{u\} = 1$. To each x_j we associate a ball S_j centered at the origin and the S_j's are given independent random radii distributed according to F. (There are some mathematical details hidden here and we refer to [1] for those.) The *occupied component* is defined as the set $S = \cup_{j=1}^{\infty}(\{x_j\} \oplus S_j)$. The fundamental question in continuum percolation theory is whether S contains some *unbounded connected component.*

The most well-known and well-studied continuum percolation model is the *Poisson Boolean model* (or the Poisson blob model) where X is a (spatially homogeneous)

Poisson process with some intensity λ. Here we are only going to consider the simplest possible Poisson Boolean model, namely the model $(X, 1)$, i.e. the model where $S_j = B(0,1)$, the unit ball, for all j. For this model it is known that there exists a critical intensity λ_c such that if $\lambda < \lambda_c$ then S will a.s. contain no unbounded connected component and if $\lambda > \lambda_c$ then S will a.s. contain an unbounded connected component. (Such a component is also a.s. unique, see [1, Theorem 4.6].) Also, by [1, Theorem 3.10], we have

$$0.174 < \lambda_c < 0.843.$$

The question we shall ask ourselves is: If we replace the unit ball $B(0,1)$ by some other convex set A *of the same area*, will this change the probability for an unbounded connected component? More specifically:

(Q) What choice of A maximizes (or minimizes) the probability for an unbounded connected component?

As we shall see it turns out that the key to answering this question is a strengthened version of the so called difference-body inequality for compact convex sets in the Euclidean plane $\mathbf{R}^2$. To make this more precise we need a few definitions.

DEFINITION 1.1 *For two sets A and B in $C(\mathcal{K}(\mathbf{R}^2))$, the family of compact convex sets in the plane, the* **Minkowski sum** *is defined as*

$$A \oplus B = \{a + b : a \in A, b \in B\}.$$

The set rA, $r \in \mathbf{R}$, is given by $rA = \{ra : a \in A\}$. The **reflected set** *of A is the set $\check{A} = -1A$.*

The **difference-body** *of a set $A \in C(\mathcal{K}(\mathbf{R}^2))$ is defined as the set $A \oplus \check{A}$.*

The classical **difference-body inequality**, first established by Rogers and Shephard 1957 (see [3]), states that for any compact convex set A in the plane

$$4\mu(A) \le \mu(A \oplus \check{A}) \le 6\mu(A),$$

with equality to the left iff A is centrally symmetric (i.e. if $\check{A} = A \oplus \{x\}$ for some $x \in \mathbf{R}^2$) and equality to the right iff A is a triangle. Here μ denotes area, i.e. Lebesgue measure on $\mathcal{B}(\mathbf{R}^2)$. (Indeed the difference-body inequality gives a corresponding statement for arbitrary dimension, not only for the $\mathbf{R}^2$-case. See a later remark for the general formulation.) A proof is found in [4, Section 7.3].

Our key result, which is also of independent interest, is the following strengthening of the above.

THEOREM 1.2 *For any $A \in C(\mathcal{K}(\mathbf{R}^2))$ there exist a centrally symmetric convex set C and a triangle T such that $\mu(C) = \mu(T) = \mu(A)$ and*

$$C \oplus \check{C} \subseteq A \oplus \check{A} \subseteq T \oplus \check{T},$$

with equality to the left iff A is centrally symmetric and to the right iff A is a triangle.

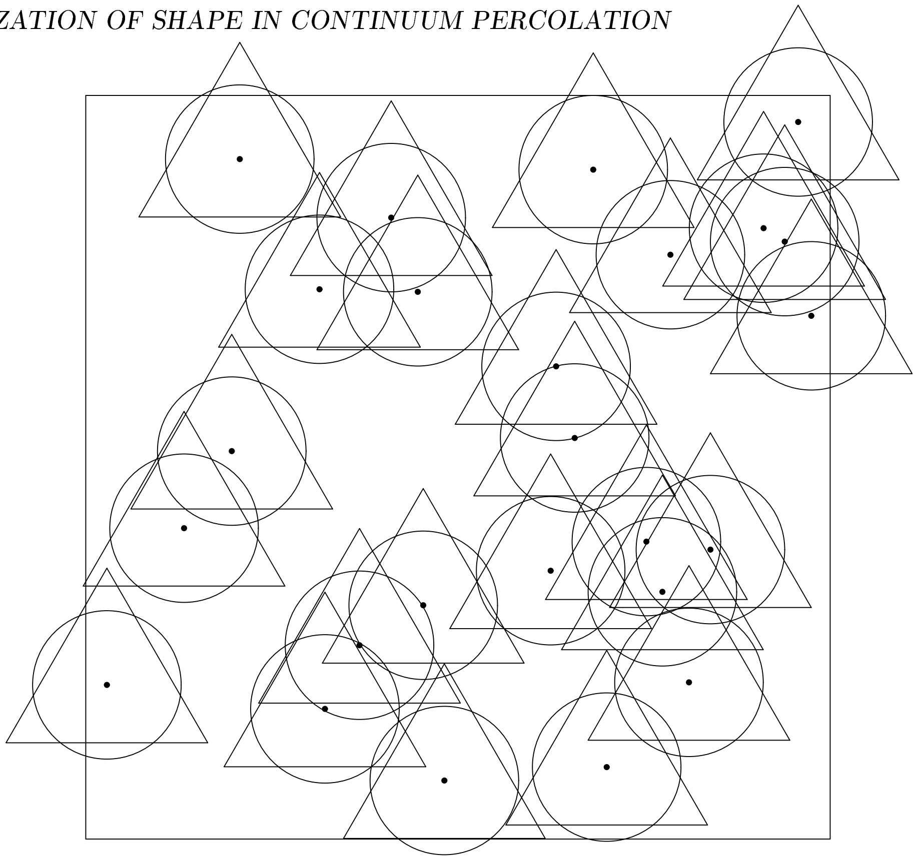

Figure 1: A typical outcome of the models $(X, B(0,1))$ and (X, T_e) with $\lambda = 1/4$. Note the higher connectivity for the triangles; as soon as two circles intersect , the corresponding two triangles also intersect, but the converse often fails.

We note at once that the last part of the theorem will follow automatically from the inclusion part combined with the "ordinary" difference-body inequality.

Clearly Theorem 1.2 is stronger than the ordinary difference-body inequality on $\mathbf{R}^2$. Therefore one could refer to it as a "strong difference-body inequality".

Remark. The ordinary difference-body inequality is, as noted above, not concerned with $\mathbf{R}^2$ only. In general it states for a compact convex set $A \subseteq \mathbf{R}^n$, $n \geq 1$, that

$$2^n \mu_n(A) \leq \mu_n(A \oplus \check{A}) \leq \binom{2n}{n} \mu_n(A),$$

(where μ_n is n-dimensional volume) with equality to the left iff A is centrally symmetric and to the right iff A is a simplex. It is natural to ask if Theorem 1.2 could be generalized in a corresponding way. However, as reading through Section 2 will reveal, trying to use the present methods to give more than a partial proof of such a generalization seems intractable.

Let us now turn back to question (Q) above. To answer that we start by making an extension of the model $(X, 1)$ to a class of models (X, A) where A can be any compact convex set with area π. The occupied component is then defined as $S_A = \cup_{j=1}^{\infty}(\{x_j\} \oplus A)$. (Thus $(X, 1)$ is now the model $(X, B(0,1))$.) Figure 1 illustrates the models $(X, B(0,1))$ and (X, T_e), where T_e is the equilateral triangle with area π, for $\lambda = 1/4$. (Note that T_e satisfies Theorem 1.2 with $A = B(0,1)$.)

Define $\mathcal{F}_A$ to be as in Theorem 1.3, i.e. the set of locally finite counting measures $\gamma = \sum_{i=1}^{\infty} \delta_{y_i}$ on $\mathbf{R}^2$ such that $\cup_{i=1}^{\infty}(\{y_i\} \oplus A)$ contains an unbounded connected component. Our question now is: What choice of A maximizes (minimizes) $P_\lambda(X \in \mathcal{F}_A)$? However with Theorem 1.2 at our disposal this question is simple to answer: The probability in question is maximized for some triangle and minimized for some centrally symmetric set. This follows from simply combining Theorem 1.2 with the fact that $(\{x_i\} \oplus A) \cap (\{x_j\} \oplus A) \neq \emptyset$ iff $x_j - x_i \in A \oplus \check{A}$. This implies that for any A there is a centrally symmetric convex set C and a triangle T both with area π such that C does worse than A and T does better than A. (Note that we automatically also get that this holds uniformly over all distributions of X.)

To each A we can now associate a critical intensity $\lambda_c(A)$ with the obvious interpretation. An immediate consequence of what we just observed is that for any A with $\mu(A) = \pi$ there exist a centrally symmetric convex set C and a triangle T with $\mu(C) = \mu(T) = \pi$ and

$$\lambda_c(T) \leq \lambda_c(A) \leq \lambda_c(C).$$

So far we have assumed that X is a Poisson process, but, as noted in brackets a few lines above, nothing in the argument proving the optimality of some triangle and some centrally symmetric set used any particular feature of X, i.e. this optimality holds in complete generality as far as X is concerned. However for general X the form of a triangle may be crucial, but e.g. in the Poisson process case the probability $P(X \in \mathcal{F}_T)$ will be the same regardless of what triangle T we consider. What we need to require from X in order for this to hold is that X is *invariant under all affine area-preserving transformations, f, of the plane*, i.e. that $P(X \in \mathcal{F}) = P(f(X) \in \mathcal{F})$ for all measurable sets $\mathcal{F}$ in the space of locally finite counting measures on $\mathbf{R}^2$ and all affine f such that $\mu(B) = \mu(f(B))$ for all Borel sets $B \subseteq \mathbf{R}^2$. Clearly a Poisson process has this property, but also any convex combination of Poisson processes with different intensities. The reason that this suffices is that if T and T' are any two triangles with the same area, then there exists some affine area-preserving transformation taking T to T' and since the same transformation takes X to some X' having the same probabilistic behavior as X, no probabilities have changed. In particular $P(X \in \mathcal{F}_T) = P(X' \in \mathcal{F}_{T'}) = P(X \in \mathcal{F}_{T'})$.

Now, summing up what we have done thus far we have proved the following.

THEOREM 1.3 *Let the models (X, A) and the set $\mathcal{F}_A$ be defined as above. Then for any $A \in C(\mathcal{K}(\mathbf{R}^2))$ there exist a centrally symmetric set C and a triangle T such that $\mu(C) = \mu(T) = \mu(A)$ and*

$$P(X \in \mathcal{F}_C) \leq P(X \in \mathcal{F}_A) \leq P(X \in \mathcal{F}_T)$$

uniformly over all distributions of X. Moreover, if X is invariant under affine area-preserving transformations then $P(X \in \mathcal{F}_T) = P(X \in \mathcal{F}_{T'})$ for any two triangles T and T' of the same area and so in particular any triangle of the same area as A will do for the above inequality.

Section 2 contains the proof of Theorem 1.2.

Let us conclude this section with a few remarks.

- Note that it is not only the case that triangles and centrally symmetric sets are optimal in the class of convex sets of some given area for the existence of an unbounded connected component, they are optimal for all increasing connectivity properties. For example if we define $S_A^M = \cup_{j:x_j \in [-M,M]^2}(\{x_j\} \oplus A)$, $A \in C(\mathcal{K}(\mathbf{R}^2))$, and let N_A^M be the number of connected components of S_A^M, then for any A there exist C and T with the same area as A such that $N_T^M \leq_d N_A^M \leq_d N_C^M$.

- Even if X is invariant under affine area-preserving transformations we cannot say that $P(X \in \mathcal{F}_C)$ is the same for all centrally symmetric convex sets of the same area as we did for triangles; there is not always an affine transformation taking one centrally symmetric convex set to another. Consider e.g. a circle and a square.

- What we have proved so far implies e.g. that $\lambda_c(T) \leq \lambda_c(B(0,1))$ for any triangle T with area π, in particular if T is T_e, the equilateral triangle with that area π. Since $T \oplus \check{T}$ then actually turns out to contain $B(0, \pi^{1/2}3^{1/4}) \supseteq B(0, 2.3)$ it is easily shown that $\lambda_c(T)$ is strictly smaller than $\lambda_c(B(0,1))$. However in general it is not the case that if A is not a triangle, then there exist a triangle T and an $\epsilon > 0$ such that $T \oplus \check{T} \supseteq (1+\epsilon)A \oplus \check{A}$; consider e.g. the case when A is a square. Therefore we cannot say without further ado that for general non-triangles A there exists a triangle T with the same area having a critical intensity which is strictly lower.

- In a model like (X, A) it would be natural to also add random rotations of the A's of the occupied component and one can then ask if the same kind of optimality of triangles still holds. However such a question does not make sense in this case as any fixed area for the A's allows for arbitrarily long "sticks" and such can produce arbitrarily low critical intensities. Perhaps interesting results could be achieved for the model with random rotations with further or other restrictions on the A's.

- Neither the problem addressed in this section nor anything similar to it has, to the very best of my knowledge, been studied before. However, the paper by Meester, Roy and Sarkar [2] may perhaps be thought of as somewhat "similar in spirit". In that paper it is shown for the model (X_F, F), where X_F is a Poisson process with exactly the critical intensity $\lambda_c(F)$ for unbounded connected components, that the fraction of volume covered by the occupied component is *not* independent of F. In particular it is shown that for some intensity $\lambda > \lambda_c(1) (= \lambda_c(B(0,1))$ one can find a two-point distribution F such that if X is a Poisson process with intensity λ, then (X, F) covers the same fraction of volume as $(X, 1)$ but is still subcritical.

Acknowledgment. I am grateful to Marianne Månsson for discussions and for bringing the difference-body inequality to my knowledge.

2 Proof of the strong difference-body inequality

As in the previous section we say that the affine transformation $f : \mathbf{R}^2 \to \mathbf{R}^2$, writing $f(x) = a + Mx$ for some $a \in \mathbf{R}^2$ and some 2×2-matrix M, is *area-preserving* if for all $B \in \mathcal{B}(\mathbf{R}^2)$ we have $\mu(f(B)) = \mu(B)$. As is well known this could equivalently have been defined by requiring that $|\mathbf{det}M| = 1$. The class of affine area-preserving transformations is generated by the following four types of transformations (writing $M = [m_{ij}]_{1 \le i,j \le 2}$):

- *Translations*, i.e. when $M = I_2$,
- *rotations*, i.e. when $a = 0$ and M is orthogonal,
- *tilts*, i.e. when $a = 0$, $m_{11} = m_{22} = 1$ and either $m_{12} = 0$ or $m_{21} = 0$,
- *anisotropic dilations*, i.e. when $a = m_{12} = m_{21} = 0$ and $m_{11} = m_{22}^{-1}$.

Using these types of transformations we shall shape the set A under consideration so as to always make the situation as convenient as possible. When doing so we will not explicitly state the particular sequence of these transformations we have used; it is easily reconstructed by the reader him- or herself.

Let us now start proving the theorem. We begin with the easy part which is the left inclusion: The set $A \oplus \check{A}$ is itself a centrally symmetric set and $\mu(A \oplus \check{A}) = \alpha\mu(A)$ for some $\alpha \in [4,6]$. Now set $C = \alpha^{-1/2}(A \oplus \check{A})$. Then $\mu(C) = \alpha^{-1}\mu(A \oplus \check{A}) = \mu(A)$ and $C \oplus \check{C} = 2\alpha^{-1/2}(A \oplus \check{A}) \subseteq A \oplus \check{A}$ as desired.

The right inclusion will take considerably more work. First observe that in order to prove the existence of a triangle T with the same area as A such that $A \oplus \check{A} \subseteq T \oplus \check{T}$ it suffices to show that one can find some affine transformation f and some triangle T' with the same area as fA such that $fA \oplus \check{fA} \subseteq T' \oplus \check{T}'$. Having done so we simply put $T = f^{-1}T'$ observing that since T is a triangle and since the difference-body operation commutes with linear transformations and disregards translations we have found the T we were looking for.

For the following argument the reader is urged to look at Figure 2 where all important steps are illustrated.

By the compactness of A there exists a triangle $L \subseteq A$ such that $\mu(L) = \sup\{\mu(T) : T \text{ triangle}, T \subseteq A\}$. By making a suitable affine transformation we may assume that L is the triangle $(0,0)(0,1)(1,0)$. Then by the maximality of L we have that $A \subseteq U = (-1,1)(1,1)(1,-1)$. Now let $H \subseteq U$ be the hexagon $(-x,1)(y,1)(1,y)(1,-z)(z,-z)(-x,x)$ where the numbers $x,y,z \in [0,1]$ are chosen so that the line segments $(-x,x)(-x,1)$, $(y,1)(1,y)$ and $(1,-z)(z,-z)$ are parallel to the sides of L and tangent to A. Thus $A \subseteq H$ so that also $A \oplus \check{A} \subseteq H \oplus \check{H}$ and if we can prove the existence of a triangle T with $\mu(T) \le \mu(A)$ such that $H \oplus \check{H} \subseteq T \oplus \check{T}$, then we are done. Unfortunateley such a T does not exist for all A, but at least it does exist for a fairly large class of A's, namely for those A for which $\mu(L) \ge \frac{1}{2}\mu(A)$. To see this assume without loss of generality that $x \ge z \ge y$. (If this is not the case then we can just do another affine transformation to set things right.)

We have that $H \oplus \check{H}$ is the centrally symmetric hexagon specified by the three corners $(-x-1, x-y)$, $(-x-1, z+1)$ and $(y-z, z+1)$. For convenience we tilt the plane so that two of the corners of $H \oplus \check{H}$ end up on the x-axis, i.e. we

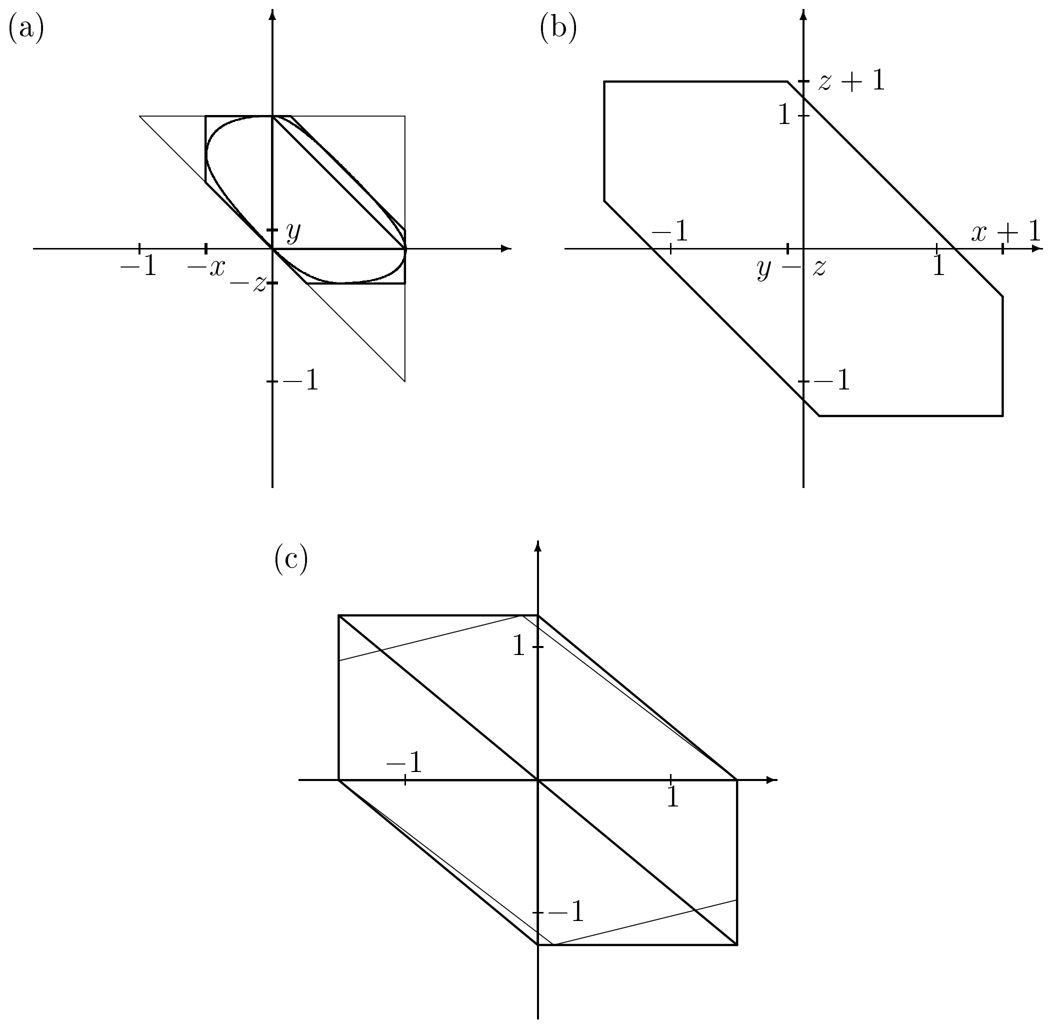

Figure 2: (a) The sets $L \subseteq A \subseteq H \subseteq U$. (b) The set $H \oplus \check{H}$. (c) The sets $H' \oplus \check{H}'$ and $T \oplus \check{T}$, the latter with its six copies of T inscribed.

transform $H \oplus \check{H}$ into the centrally symmetric hexagon $H' \oplus \check{H}'$ specified by the corners $(-x-1, 0)$, $(-x-1, 1+y+z-x)$ and $(y-z, z+1-(z-y)(x-y)/(x+1))$.

Now clearly $H' \oplus \check{H}'$ is contained in $T \oplus \check{T}$ for the triangle $T = (0,0)(0, z+1-(z-y)(x-y)/(x+1))(x+1, 0)$ and we have that

$$\mu(T) = \frac{1}{2}(x+1)(z+1-\frac{(z-y)(x-y)}{x+1}) = \frac{1}{2}(1+x+z+y(x+z)-y^2).$$

However by the convexity of A, $\mu(A) \geq \frac{1}{2}(1+x+y+z)$ so for the inequality $\mu(T) \leq \mu(A)$ to be satisfied we need that $y(x+z) - y^2 \leq y$ which holds as soon as $x + z \leq 1$, in particular if $\mu(L) \geq \frac{1}{2}\mu(A)$ as claimed.

To take care of the cases that were not covered by the above argument we shall, instead of making use of a largest triangle inside A, make use of a largest quadrilateral Q inside A, i.e. we let Q be a quadrilateral contained in A such that $\mu(Q) = \sup\{\mu(B) : B \text{ quadrilateral}, B \subseteq A\}$. The steps in the argument that now

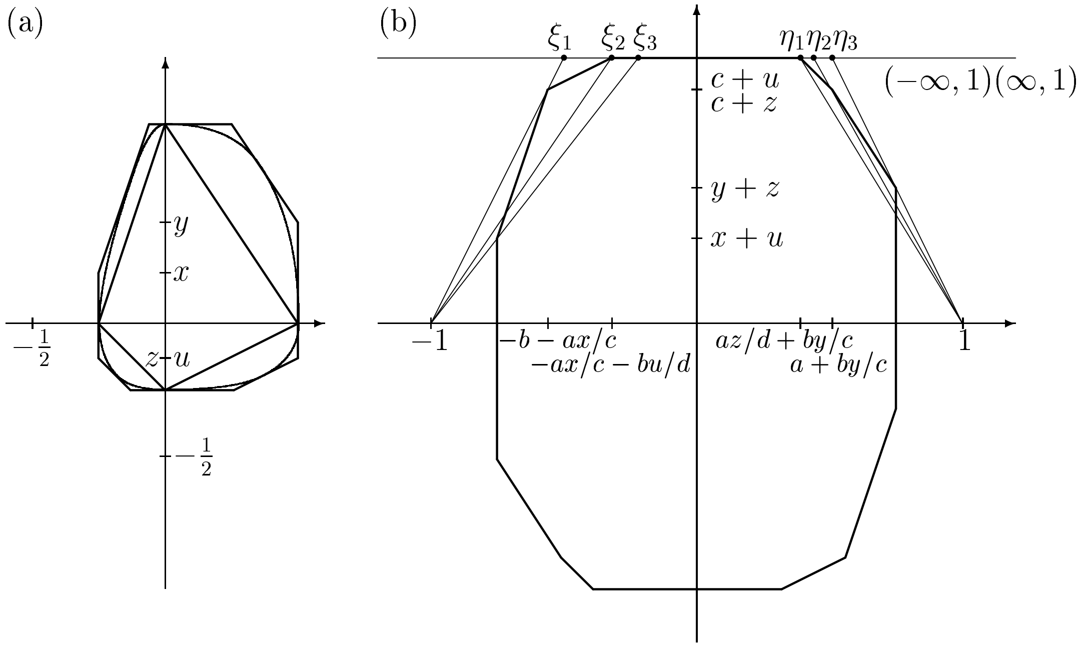

Figure 3: (a) The sets $Q \subseteq A \subseteq O$. (b) The set $O \oplus \check{O}$ with the tangent lines and their intersections with $(-\infty,1)(\infty,1)$ marked with bullets. The first coordinates of these are $\xi_1 = (1-b-ax/c)/(c+u)-1$, $\xi_2 = -ax/c-du/b$, $\xi_3 = (1-a-b)/(x+u)-1$, $\eta_1 = by/c+az/d$, $\eta_2 = 1-(1-a-by/c)/(c+z)$ and $\eta_3 = 1-(1-a-b)/(y+z)$.

follow are illustrated in Figure 3 and again the reader is urged to take a look at that to facilitate reading.

We assume that $\mu(A) = 1/2$ and by an affine area-preserving transformation we make sure that $Q = (-a,0)(0,c)(b,0)(0,-d)$ for $a,b,c,d \in [0,1]$ such that $c+d = 1$, $a+b \leq 1$, $a \leq b$, $d \leq c$ and $c/d \leq a/b$. By the maximality of $\mu(Q)$, A is contained in the rectangle $R = (-a,c)(b,c)(b,-d)(-a,-d)$. Let $O \subseteq R$ be the octagon $(-ax/c,c)(by/c,c)(b,y)(b,-u)(bu/d,-d)(-az/d,-d)(-a,-z)(-a,x)$, where x,y,z,u ($x,y \in [0,c]$, $z,u \in [0,d]$) are chosen so that the four line segments $(-a,x)(-ax/c,c)$, $(by/c,c)(b,y)$, $(b,-u)(bu/d,-d)$ and $(-az/d,-d)(-a,-z)$ are parallel to the sides of Q and tangent to A. (If $d = 0$ then $z = u = 0$. We then decide, here and in the sequel, to interpret z/d and u/d as being 1, so that the corresponding line segments vanish.) By the convexity of A we have that $\frac{1}{2} = \mu(A) \geq \frac{1}{2}(a+b+a(x+z)+b(z+y))$, which imposes the extra condition $a(x+z)+b(y+u) \leq 1-a-b$ on x, y, z and u.

Now in analogy with what we did above we will prove (for A's not covered there) that $O \oplus \check{O} \subseteq T \oplus \check{T}$ for some triangle T with $\mu(T) = 1/2$. To be specific we will have $T = (0,0)(t,1)(1,0)$ for some $t \in [0,1]$. Then $T \oplus \check{T}$ is the centrally symmetric hexagon specified by the three corners $(-1,0)$, $(t-1,1)$ and $(t,1)$ while $O \oplus \check{O}$ is the centrally symmetric 12-gon specified by the six corners $(-a-b,x+u)$, $(-b-ax/c,c+u)$, $(-ax/c-bu/d,1)$, $(az/d+by/c,1)$, $(a+by/c,c+z)$ and $(a+b,z+y)$. In order to reach our goal we need to show that if $(r,1)$ is the intersection point of the line $(-\infty,1)(\infty,1)$ with the line through $(1,0)$ tangent to $O \oplus \check{O}$ in the upper right quadrant of the plane and $(l,1)$ is the intersection point of the same line with the line through $(-1,0)$ tangent to $O \oplus \check{O}$ in the upper left quadrant, then $r-l \leq 1$. Since each of the two tangent lines in question can lean onto any of the three corners of the quadrant corresponding to it, we have that the pair r and l may be any of the nine possible pairs of intersection points each corresponding to a pair of corners. Thus showing that $r-l \leq 1$ boils down to checking that the following nine expressions are all non-negative:

(i) $\frac{1-a-b}{x+u} + \frac{1-a-b}{z+y} - 1$,

(ii) $\frac{1-a-b}{x+u} + \frac{1-a-by/c}{c+z} - 1$,

(iii) $\frac{1-a-b}{x+u} - by/c - az/d$,

(iv) $\frac{1-b-ax/c}{c+u} + \frac{1-a-b}{z+y} - 1$,

(v) $\frac{1-b-ax/c}{c+u} + \frac{1-a-by/c}{c+z} - 1$,

(vi) $\frac{1-b-ax/c}{c+u} - by/c - az/d$,

(vii) $\frac{1-a-b}{z+y} - ax/c - bu/d$,

(viii) $\frac{1-a-by/c}{c+z} - ax/c - bu/d$,

(ix) $1 - ax/c - bu/d - by/c - az/d$,

under the conditions

- $c/d \leq a/b$,
- $0 \leq d \leq \frac{1}{2} \leq c \leq 1$
- $c+d=1$,
- $0 \leq a \leq b \leq \frac{1}{2}$,
- $\frac{1}{2} \leq a+b \leq 1$,
- $0 \leq x, y \leq c$,
- $0 \leq z, u \leq d$,
- $a(x+z)+b(y+u) = 1-a-b$.

These conditions were all mentioned above except that the inequality in the last one has been replaced with an equality, which is clearly no less general, and that we demand that $b \leq 1/2$. That we are free to do so follows from the observation that if $b \geq 1/2$, then the triangle $(0,c)(b,0)(0,-d)$ is contained in A and has at least half the area of A, i.e. this represents a situation which falls under what was proved earlier. It is fortunate that we can impose this restriction on b; consider e.g. the case $c=7/8$, $d=1/8$, $a=1/8$, $b=27/32$, $y=u=0$ and $x=z=1/8$. Then (i) begets the value $1/4 + 1/4 - 1 = -1/2$. This shows that neither the present part nor the first part of the proof is enough in itself. Luckily they cover up for each other as we now shall see.

To verify the non-negativity of (i)-(ix) may look as an overwhelming task, but fortunately it is not as bad as it looks. The expressions are non-linear but they are "linear enough" to be handled fairly easily by ad hoc arguments.

First we note that with all other parameters fixed, the expressions are linear in a and b. Changing b keeping all parameters but a fixed and keeping the conditions satisfied, a will change linearly along with it. By this we can conclude that each expression is minimized either when $a = b$ or when $b = 1/2$ (or when $a + b = 1/2$ or $a + b = 1$ which are trivial cases). The cases $a = b$ can be handled in a unified way: If $a = b$, set $a' = a$, $b' = b$, $d' = z' = u' = 0$, $c' = 1$, $x' = x + u$ and $y' = y + z$. Since $a = b$ all conditions are satisfied for the primed quantities. These however correspond to an O' for which it is readily seen that $O \oplus \check{O} \subseteq O' \oplus \check{O}'$, a fact which follows from the facts that $O' \oplus \check{O}'$ is an octagon for which four corners lie on the lines $(1, -\infty)(1, \infty)$ and $(-1, -\infty)(-1, \infty)$ and coincide with four corners of $O \oplus \check{O}$ and that the slopes of the line segments connecting these to the corners on $(-\infty, 1)(\infty, 1)$ and $(-\infty, -1)(\infty, -1)$ are $1/a$ to the left and $1/b$ to the right compared to c/a and d/b to the left and c/b and d/a ($d/a \leq c/b$) to the right for $O \oplus \check{O}$. See Figure 4 for an example. Thus, for $a = b$ (i)-(ix) reduce to four expressions, (i), (iii), (vii) and (ix) (remember that z/d and u/d are interpreted as 1 here) in only the three parameters a, x and y which immediately reduce to two from the condition $a(x + y) = 1 - 2a$. Moreover, since $a = b$, the parameters x and y are interchangeable and so non-negativity of (iii) is equivalent to non-negativity of (vii). Now (ix) becomes $1 - 2a - a(x + y) = 0$, (i) becomes $(1 - 2a)/x + (1 - 2a)/y - 1$ which is minimal if $x = y = (1 - 2a)/(2a)$ and then begets the value $4a - 1 \geq 0$ (as $a = b$ implies $1/4 \leq a \leq 1/2$) and finally (vii) is non-negative iff $1 - 2a - ay - axy$ is non-negative which holds true as $1 - 2a - ay - axy \geq 1 - 2a - a(x + y) = 0$, finishing the proof for $a = b$.

Note that the previous argument works even if $a < b$ as soon as $z \leq u$ since then $a'(x'+z')+b'(y'+u') = a(x+u)+b(y+z) \leq a(x+z)+b(y+u) = 1-a-b = 1-a'-b'$. This allows to *assume from now on* that $z \geq u$.

Next we observe that we may assume that $z = d$: Consider the set O'' corresponding to double-primed quantities in the same way as O correspond to its parameters. Let the double-primed parameters coincide with the cooresponding unprimed ones with the exception that $d'' = z$ and $c'' = 1 - z$. Since $z \geq u$ this gives us a set of parameters satisfying the conditions. Then since $O'' \oplus \check{O}''$ has the same corners on $(1, -\infty)(1, \infty)$ and $(-1, -\infty)$ and $(-1, \infty)$ as $O \oplus \check{O}$ and $c'' \geq c$ it follows from copying the arguments used above, to take care of the cases $a = b$, that $O \oplus \check{O} \subseteq O'' \oplus \check{O}''$.

That we may assume that $z = d$ is good news not only in that we get rid of another parameter, but also in that two of the three corners of $O \oplus \check{O}$ coincide, making e.g. (ii), (v) and (viii) redundant.

From now on we thus work under the assumptions that $b = 1/2$ and $z = d$. Doing this we keep in mind that the last of these two assumptions impose the new restriction $da \leq 1/2 - a$. From now on we also drop the restriction $d/c \leq a/b$; it will no longer be needed. Of course we can also get rid of c by simply writing $c = 1 - d$,

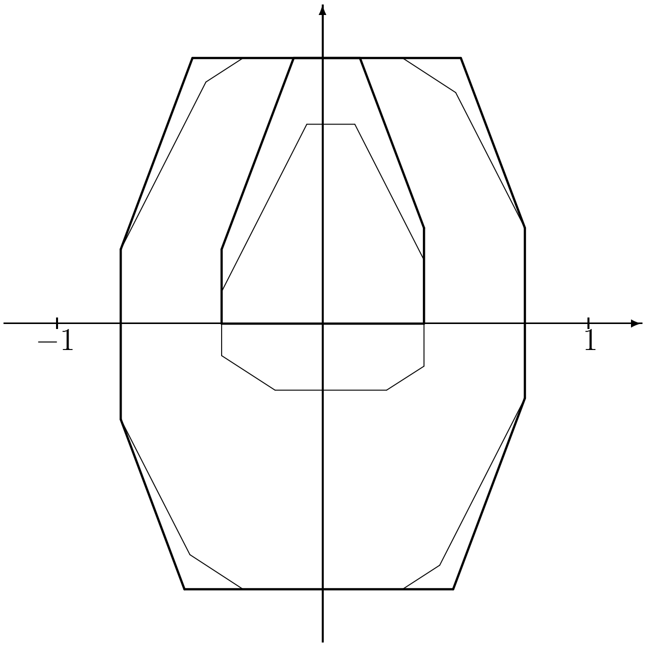

Figure 4: The sets O, O', $O \oplus \check{O}$ and $O' \oplus \check{O}'$ for $a = b = 3/8$, $d = 1/4$, $c = 3/4$, $x = z = 1/8$, $z = 1/8$ and $u = 1/6$. The primed sets are drawn with thick lines and the unprimed sets are drawn with thin lines.

but despite the simplifications we have made thus far, we are still left with six non-linear optimization problems in five variables. Going through this means hard work which does not make exciting reading. Therefore the reader who wishes to simply take our word for that (i)-(ix) are indeed non-negative and does not want the basic ideas obscured by tedious calculations, may skip the rest of the proof and stop reading here.

With $b = 1/2$ and $z = d$, proving that (i) is at least 0 is equivalent to proving the non-negativity of

$$(\frac{1}{2} - a)(x + y + u + d) - (x + u)(y + d)$$

under the given conditions. By inspection of (i) and since we have the condition $ax + (y + u)/2 = 1/2 - a - ad$, we find that when keeping all parameters but x and u fixed, the worst case is when x is maximized at the expense of u. Thus we must take care of two possible scenarios separately:

(1) $u = 0$, $ax + y/2 = 1/2 - a - ad$:

Here we want to show that

$$(\frac{1}{2} - a)(x + y + d) - x(y + d)$$

is non-negative. For fixed x and a this expression is linear in y and d and by considering the original expression for (i) we see that to get the worst case we should maximize d at the expense of y as we must see to that $y/2 + ad = 1/2 - a - ax$. Doing so gives either $y = 0$ and $d = (1/2 - a - ax)/a$ or $d = 1/2$ and $y = 1 - 3a - 2ax$. If the first alternative occurs then we get on insertion

$$\frac{1}{a}((\frac{1}{2} - a)^2 - x(\frac{1}{2} - a - ax)) \geq \frac{1}{a}(1 - \frac{1}{4a})(\frac{1}{2} - a)^2 \geq 0,$$

where the first inequality follows on minimizing for x and the second follows on observing that $y = 0$ implies $a \geq 4$. For the second alternative, note that $d = 1/2$ implies that $x, y \leq 1/2$ and so it follows that in order to get a minimal value, x should be maximized at the expense of y. This either means that $y = 0$ and $x = (1 - 3a)/(2a)$ and our expression then begets the value $a - 1/4 \geq 0$, or $x = 1/2$ and $y = 1 - 4a$ in which case our expression becomes $(1/2 - 2a)^2 \geq 0$.

(2) $x = 1 - d$, $(y + u)/2 = 1/2 - 2a$:

The expression under consideration now takes the form

$$(\frac{1}{2} - a)(1 + y + u) - (u + 1 - d)(y + d).$$

Substituting $y = 1 - 4a - u$ we get

$$(\frac{1}{2} - a)(2 - 4a) - (u + 1 - d)(1 + d - 4a - u).$$

Minimizing for u reveals a minimum at $u = d - 2a$ and substituting, the expression becomes

$$(\frac{1}{2} - a)(2 - 4a) - (1 - 2a)^2 = 0.$$

Treating (iii) is equivalent to treating

$$(\frac{1}{2} - a)(1 - d) - (x + u)(a(1 - d) + \frac{1}{2}y).$$

Again, as for (i), the worst case is when x is maximized at the expense of u giving the same two scenarios as above:

(1) We want to prove that

$$(\frac{1}{2} - a)(1 - d) - x(\frac{1}{2} - 2ad - ax)$$

is non-negative (having substituted $y/2 = 1/2 - a - ad - ax$). Being linear in a the expression is minimized when a is taken to one of the two endpoints of the interval of possible values. The interval in question is given by $0 \leq y =$

$1-2a(1+d+x) \leq 1-d$, i.e. $d/(2(1+d+x)) \leq a \leq 1/(2(1+d+x))$. Inserting $a = 1/(2(1+d+x))$ gives us

$$\frac{1}{2(1+d+x)}[(1-d)(d+x) - x(1-d)] = \frac{d(1-d)}{2(1+d+x)}$$

which is non-negative and inserting $a = d/(2(1+d+x))$ yields

$$\frac{1}{2(1+d+x)}[(1-d)(1+x) - x((1-d)(1+d+x) + d(1-d))]$$

$$= \frac{1-d}{2(1+d+x)}(1 - x^2 - 2dx)$$

which is non-negative as $1 - x^2 - 2dx \geq 1 - (1-d)^2 - 2d(1-d) = d^2$.

(2) Substituting $y = 1 - 4a - u$ we want to prove that

$$(\frac{1}{2} - a)(1-d) - (u+1-d)(\frac{1}{2} - a - ad - \frac{1}{2}u)$$

is at least 0. Being linear in a the expression is minimized at one of the two end points of the interval of possible values for a. These are given by the condition $0 \leq y = 1 - 4a - u \leq 1 - d$, i.e. $(d-u)/4 \leq a \leq (1-u)/4$. Trying $a = (d-u)/4$ gives

$$(\frac{1}{2} - \frac{1}{4}d + \frac{1}{4}u)(1-d) - (u+1-d)((\frac{1}{2} - \frac{1}{4}d - \frac{1}{4}d^2 + \frac{1}{4}du - \frac{1}{4}u)$$

$$= \frac{1}{4}[(1-d)(d^2 - du + 2u) - u(2 - d - u - d^2 + du)].$$

Minimizing for u we find the minumum of this at $u = d$, giving the expression the value $2d(1-d) - 2d(1-d) = 0$.

Trying $a = (1-u)/4$ yields

$$\frac{1}{4}(1+u)(1-d) - \frac{1}{4}(u+1-d)(1-u-d+du)$$

$$= \frac{1}{4}[(1-d)(2u + d - du) - u(1-u-d+du)].$$

Again minimizing for u it follows that the minimum is at $u = 0$ which gives the expression the value $d(1-d)/4$ which is non-negative.

Let us now move on to (iv) where we claim that

$$\frac{\frac{1}{2}-\frac{ax}{1-d}}{1-d+u}+\frac{\frac{1}{2}-a}{d+y}-1\geq 0$$

or equivalently that

$$\frac{1}{2}(d+y)(1-d)-ax(d+y)+$$

$$+(\frac{1}{2}-a)(1-d)(1-d+u)-(1-d)(1-d+u)(d+y)$$

is non-negative. This expression is also linear in x and u, but in contrast to (i) and (iii) we cannot in general tell which one to maximize on the others expense, so we have to deal with four cases:

(1) $u=0$, $ax+y/2=1/2-a-ad$:

We need to prove that

$$(d+y)(1-d)-(1-2a-2ad-y)(d+y)+2(\frac{1}{2}-a-d-y)(1-d)^2$$

(where we have substituted $2ax=1-2a-2ad-y$) is at least 0. The expression is linear in a and bounds for a are given by $0\leq ax=1/2-a(d+1)-y/2\leq a(1-d)$, i.e. $(1-y)/4\leq a\leq(1-y)/(2(d+1))$. Trying $a=(1-y)/(2/(d+1))$ yields

$$(d+y)(1-d)+2(\frac{1}{2}-\frac{1-y}{2(d+1)}-d-y)(1-d)^2$$

$$=(d+y)(1-d)-\frac{(1-d)^2}{d+1}(d+y+2d(d+y))$$

$$=(1-d)(d+y)(1-\frac{(1-d)(1+2d)}{1+d})=\frac{2d^2(1-d)(d+y)}{1+d}$$

which is clearly non-negative, and trying $a=(1-y)/4$ corresponds to $x=1-d$ which is a case where (iv) coincides with (i) and is thus already taken care of.

(2) $x=1-d$, $(y+u)/2=1/2-2a$:

As was just mentioned, this falls under the analysis of (i).

(3) $x=0$, $(y+u)/2=1/2-a-ad$:

Here we want to prove that

$$(1-d)[\frac{1}{2}(d+y)+(1-d+u)(\frac{1}{2}-a-d-y)]$$

cannot come below 0. Disregarding the $1-d$-factor and substituting $y = 1-2a-2ad-u$ we want to show the non-negativity of

$$\frac{1}{2}(1+d-2a-2ad-u)+(1-d+u)(a+2ad+u-d-\frac{1}{2}).$$

This expression is linear in a and bound for are given by $(d-u)/(2(d+1)) \leq a \leq (1-u)/(2(d+1))$. Trying the lower one of these yields the expression

$$\frac{1}{2}+(1-d+u)(\frac{1}{2}u-\frac{1}{2}d+\frac{d(d-u)}{2(1+d)}-\frac{1}{2})$$

which is bounded below by

$$\frac{1}{2}-\frac{1}{2}(1-d+u)(1+d-u)=\frac{1}{2}(d-u)^2 \geq 0.$$

With $a=(1-u)/(2(d+1))$ we get the expression

$$\frac{1}{2}d+(1-d+u)(\frac{1}{2}u-d+\frac{d(1-u)}{2(d+1)})$$

which is non-negative iff

$$d(1+d)+(1-d+u)(u-d-2d^2)$$
$$=d(1+d)(d-u)-(1-d+u)(d^2-u)$$

is non-negative. This is clearly the case if $u \geq d^2$ and for $u \leq d^2$ the last expression is bounded below by $d^2(1+d)(1-d)-d^2(1-d+d^2)=d^3(1-2d) \geq 0$.

(4) $u=d$, $ax+y/2=1/2-a-ad-d/2$:

Here non-negativity of (iv) is equivalent to non-negativity of

$$(\frac{1}{2}-a)(1-d)-ax(d+y)-\frac{1}{2}(d+y)(1-d)$$
$$=(\frac{1}{2}-a)(1-d)-(\frac{1}{2}-a-ad-\frac{1}{2}d-\frac{1}{2}y)(d+y)-\frac{1}{2}(d+y)(1-d).$$

Again we use linearity in a and bounds for a follow from the bounds for x giving $(1-d-y)/4 \leq a \leq (1-d-y)/(2(1+d))$. The case $a=(1-d-y)/4$ corresponds to $x=1-d$ which is already taken care of. Putting $a=(1-d-y)/(2(1+d))$ yields the expression

$$\frac{2d+y}{2(1+d)}(1-d)-\frac{1}{2}(d+y)(1-d)$$
$$=\frac{1-d}{2(1+d)}(2d+y-(1+d)(d+y))$$
$$=\frac{d(1-d)}{2(1+d)}(1-d-y) \geq 0.$$

Turning to (vi), the expression

$$\frac{\frac{1}{2}-\frac{ax}{1-d}}{1-d+u}-a-\frac{y}{2(1-d)}$$

is to be shown to be non-negative, a task equivalent to proving non-negativity for

$$\frac{1}{2}(1-d)-ax-a(1-d)(1-d+u)-\frac{1}{2}y(1-d+u).$$

Here it is also hard to tell which of x and u should go before the other, but the expression is also linear in x and y and to get the worst case x should be maximized at the expense of y. This gives us two scenarios:

(1) $y=0$, $ax+u/2=1/2-a-ad$:

We want to show that

$$\begin{aligned}&\frac{1}{2}(1-d)-ax-a(1-d)(1-d+u)\\ &=\frac{1}{2}(1-d)-\frac{1}{2}+a+ad+\frac{1}{2}u-a(1-d)(1-d+u)\\ &=3ad+\frac{1}{2}u-au-\frac{1}{2}d-ad^2+adu\end{aligned}$$

is at least 0. The expression is linear in a and bounds for a are given by $0 \leq ax = 1/2-a(d+1)-u/2 \leq a(1-d)$, i.e. $(1-u)/4 \leq a \leq (1-u)/(2(d+1))$. Trying $a=(1-u)/4$ means to have $x=1-d$ and by inspection of the expression (xi) under treatment we see that this case falls under the analysis of (iii) which is already taken care of. On the other hand, trying $a=(1-u)/(2(d+1))$ means to have $x=0$ and our expression becomes

$$(1-d)(\frac{1}{2}-a+ad-au) \geq (1-d)(\frac{1}{2}-a) \geq 0.$$

(2) $x=1-d$, $(y+u)/2=1/2-2a$:

Since $x=1-d$ we see, as noted above, that this falls under the analysis of (iii).

To prove that (vii) is at least 0 we need to prove that

$$\frac{\frac{1}{2}-a}{d+y}-\frac{ax}{1-d}-\frac{u}{2d}$$

is non-negative, or in other words that

$$(\frac{1}{2}-a)d(1-d)-d(d+y)ax-\frac{1}{2}(d+y)(1-d)u$$

is non-negative. This expression is minimized for x and u when u is maximized at the expense of x and we thus again have two cases:

(1) $x = 0$, $(y+u)/2 = 1/2 - a - ad$:

Our expression now becomes

$$(1-d)[d(\frac{1}{2}-a) - \frac{1}{2}u(d+y)]$$

and disregarding the $1-d$-factor and substituting $y = 1-2a-2ad-u$ it follows that this is non-negative iff

$$d(\frac{1}{2}-a) - \frac{1}{2}u(d+1-2a-2ad-u)$$

is. This expression is linear in a and using the bounds on y we have that $(d-u)/(2(1+d)) \leq a \leq (1-u)/(2(1+d))$. Trying $a = (d-u)/(2(1+d))$ we get

$$\frac{d(1+u)}{2(1+d)} - \frac{1}{2}u = \frac{1}{2(1+d)}(d(1+u) - u(1+d))$$

$$= \frac{d-u}{2(1+d)}$$

which is evidently non-negative. Trying $a = (1-u)/(2(1+d))$ we instead get

$$\frac{d(d+u)}{2(1+d)} - \frac{1}{2}du = \frac{d^2(1-u)}{2(1+d)}$$

which is also non-negative.

(2) $u = d$, $ax + y/2 = 1/2 - a - ad - d/2$: Here we are to prove that

$$(\frac{1}{2}-a)d(1-d) - axd(d+y) - \frac{1}{2}d(d+y)(1-d)$$

is at least 0. However this expression equals

$$d[(1-d)(\frac{1}{2}-a-\frac{1}{2}d-\frac{1}{2}y) - ax(d+y)]$$

$$= d[(1-d)(ax+ad) - ax(1-2a-2ad-2ax)]$$

$$= 2a^2x(1+d+x) + ad^2(1-d-x)$$

which is indeed non-negative.

Finally (ix) remains. Here we want to prove that

$$1 - \frac{ax}{1-d} - \frac{y}{2(1-d)} - a - \frac{u}{2d}$$

is at least 0 which is equivalent to proving non-negativity of

$$2d(1-a)(1-d) - 2adx - dy - u(1-d).$$

Again we see that minimizing for x and u means to maximize u at the expense of x and we must thus treat the same two cases as for (vii):

(1) What we want to prove now is that

$$2d(1-a)(1-d) - dy - u(1-d)$$

$$= (1-d)(2d(1-a) - u) - d(1 - 2a - 2ad - u)$$

is at least 0. Since the expression is linear in u we try $u = 0$, which gives

$$2d(1-a)(1-d) - d(1 - 2a - 2ad) = d(1 - 2d + 4ad)$$

which is non-negative as $d \leq 1/2$, and $u = d$ which gives

$$d(1-d) - d(1 - 2a - 2ad - d) - 2ad(1-d) = 4ad^2)$$

which is also non-negative.

(2) Here we want to show that

$$d(1-d) - 2ad(1-d) - 2adx - d(1 - d - 2a - 2ad - 2ax)$$

$$= d(1-d) - d(1 - d - 2a - 2ad) - 2ad(1-d) = 4ad^2$$

is non-negative, which is clearly true. This finally completes the proof of Theorem 1.2.

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