Introduction to inverse and ill-posed problems
Lecture 1

- The information about the course is here:

  https://www.chalmers.se/sv/institutioner/math/forskning/forskarutbildning/forskarutbildning-matematik/forskarutbildningskurser-matematik/Sidor/Introduction-to-the-theory,-numerical-methods-and-applications-of-ill-posed-problems.aspx

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- Course coordinator, registration for PhD students: Larisa Beilina

  larisa@chalmers.se, room 2089

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- Registration for the course for Master's students:

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- Course literature: L. Beilina, M. Klibanov, *Approximate global convergence and adaptivity for coefficient inverse problems*. Book, available at

  https://www.springer.com/gp/book/9781441978042

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- Projects together with examples of Matlab and C++ programs are available for download at

  www.waves24.com/download

Lectures will be at the Department of Mathematical Sciences, Chalmers Tvärgata 3.

Day	Time	Place	Event
5 November - 12 December	13:15-15:00	MVL14	Lectures
Tuesday	13:15-15:00	MVL14	Lecture
Thursday	13:15-15:00	MVL14	Lecture
16 - 25 June	13.15-15:00	MVL14	Lecture
Tuesday	13:15-15:00	MVL14	Lecture
Thursday	13:15-15:00	MVL14	Lecture
February	13:00-18:00	MVL14	Exam
June	13:00-18:00	MVL14	Exam

26.11, 28.11, 03.12 and lectures in June will be given by the guest lecturer Michel Cristofol, Aix-Marseille University, France.

- Examination at this course consist in the oral presentation of the computer project which can be done in groups by 2-4 persons.
- Programs can be written in Matlab or C++/PETSc.
- Description of projects together with examples of Matlab and C++/PETSc codes are available for download here:

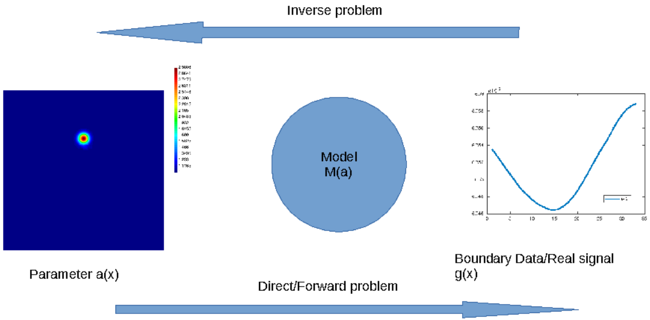
  `www.waves24.com/download`

- To pass this course you should do any one of three computer projects.
- You can work in groups by 2-4 persons.
- Sent final report for every computer assignment with description of your work together with Matlab or C++/PETSc programs to my e-mail. Report should have description of used techniques, tables and figures confirming your investigations. Analysis of obtained results is necessary to present in section “Numerical examples” and summarize results in the section “Conclusion”.
- Matlab programs for solution of least squares problem and C++/PETSc programs for solution of Poisson’s equation on a unite square are available for download from the link

  https://github.com/springer-math/Numerical_Linear_Algebra_Theory_and_Applications

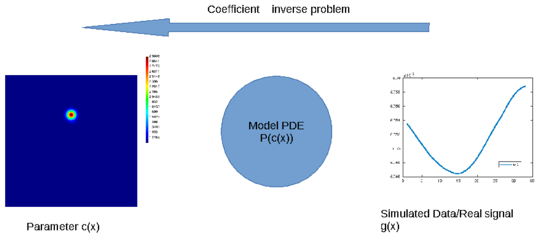
The course plan

- Physical formulations leading to ill- and well-posed problems
- Methods of regularization of inverse problems (Morozov's discrepancy, balancing principle, iterative regularization)
- Parabolic operators, the Dirichlet to Neumann methodology, the Carleman inequalities approach.
- Numerical methods of solution of inverse and ill-posed problems: Lagrangian approach and adaptive optimization, a posteriori error estimation, methods of analytical reconstruction and layer-stripping algorithms, least-squares algorithms and classification algorithms, solution of MRI problems
- Machine learning classification algorithms and neural networks (perceptron algorithm and least squares for classification)



- Inverse and ill-posed problems arise in many real-world applications including medical microwave, optical and ultrasound imaging, MRT, MRI, oil prospecting and shape reconstruction, nondestructive testing of materials and detection of explosives, seeing through the walls and constructing of new materials.

Introduction: Inverse and ill-posed problems



- These applications are modelled by acoustic, elastic or electromagnetic wave eq. which include different physical parameters (wave speed *c* - acoustic equation, elasticity parameters λ and μ - elastic equations, dielectric permittivity ε, magnetic permeability μ, conductivity σ - Maxwell's eq.).
- A coefficient inverse problem for a given PDE aims at estimating a spatially distributed coefficient of the model PDE using measurements taken on the boundary of the domain of interest.

Acoustic CIPs

- Acoustic CIPs for acoustic wave equation

$$\frac{1}{c^2(x)} u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (1)$$

$$u(x, 0) = 0, u_t(x, 0) = \delta(x - x_0). \quad (2)$$

- $u(x, t)$ acoustic pressure - we measure it on the boundary
- $c(x)$ speed of sound – want to determine by measured $u(x, t)$
- Applications: medical imaging, electromagnetic, acoustics, geological profiling, new materials

Acoustic CIPs: example

We model the process of electric wave field propagation via a single hyperbolic PDE, which is the same as an acoustic wave equation (1)-(2). The forward problem is the following Cauchy problem

$$\varepsilon_r(x)u_{tt} = \Delta u, \text{ in } \mathbb{R}^3 \times (0,\infty), \quad (3)$$

$$u(x,0) = 0, \; u_t(x,0) = \delta(x - x_0). \quad (4)$$

Here $\varepsilon_r(x)$ is the spatially distributed dielectric constant (relative dielectric permittivity),

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0}, \; \sqrt{\varepsilon_r(x)} = n(x) = \frac{c_0}{c(x)} \geq 1, \quad (5)$$

where ε_0 is the dielectric permittivity of the vacuum (which we assume to be the same as the one in the air), $\varepsilon(x)$ is the dielectric permittivity of the medium of interest, $n(x)$ is the refractive index of the medium of interest, $c(x)$ is the speed of the propagation of the EM field in this medium, and c_0 is the speed of light in the vacuum, which we assume to be the same as one in the air.

Acoustic CIPs: example

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. We assume that the coefficient $\varepsilon_r(x)$ of equation (3) satisfies conditions

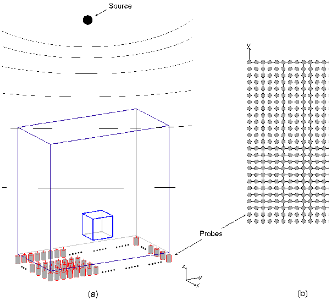
$$\varepsilon_r(x) \in [1, d], \ \varepsilon_r(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{6}$$

$$\varepsilon_r(x) \in C^2\left(\mathbb{R}^3\right). \tag{7}$$

Coefficient Inverse Problem *Suppose that the coefficient $\varepsilon_r(x)$ satisfies (6) and (7). Assume that the function $\varepsilon_r(x)$ is unknown in the domain Ω. Determine the function $\varepsilon_r(x)$ for $x \in \Omega$, assuming that the following function $g(x,t)$ is known for a single source position $x_0 \notin \overline{\Omega}$*

$$u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \tag{8}$$

Reconstruction of dielectrics from experimental data



a) The rectangular prism depicts our computational domain Ω. Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. The signal was measured with the time interval 20 picoseconds with total time 12.3 nanoseconds. b) Schematic diagram of locations of detectors on the bottom side of the prism Ω. The distance between neighboring detectors was 10 mm.

L.Beilina, M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.

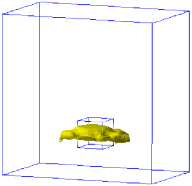
Stage 1. Approximately globally convergent numerical method provides a good approximation for the exact solution ε_{glob}.
Stage 2. Adaptive Finite Element Method refines it via minimization of the corresponding Tikhonov functional with $\varepsilon_0 = \varepsilon_{glob}$:

$$J(u,\varepsilon) = \frac{1}{2}\int_{\Gamma}\int_{0}^{T}(u-\tilde{u})^2 z_{\delta}(t)dsdt + \frac{1}{2}\gamma\int_{\Omega}(\varepsilon-\varepsilon_0)^2 \ dx. \qquad (9)$$

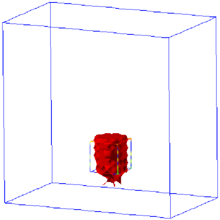
where $\tilde{u}$ is the observed wave field in the model PDE (for example, acoustic wave equation), u satisfies this model PDE and thus depends on ε.

The two-stage numerical procedure

Stage 1. Approximately globally convergent numerical method provides a good approximation for the exact solution.
Stage 2. Adaptive Finite Element Method refines it.



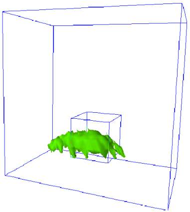
a) $\varepsilon_r^{(5,2)} = 3.9, n^{(5,2)} = 1.97$



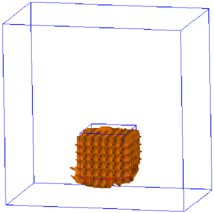
b) $\varepsilon_{r,h} \approx 4.2, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 2.05$

a) A sample of the reconstruction result of the dielectric cube No. 1 (4 cm side) via the first stage. b) Result after applying the adaptive stage (2-nd stage). The side of the cube is 4 cm=1.33 wavelength.

Results of the two-stage procedure, cube nr.2 (big)



a) $\varepsilon_r(5,5) = 3.19, n^{(5,5)} = 1.79$



b) $\varepsilon_{r,h} \approx 3.0, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.73$

a) Reconstruction of the dielectric cube No. 2 (6 cm side) via the first stage. b) The final reconstruction result after applying the adaptive stage (2-nd stage). The side 6 cm=2 wavelength.

L.Beilina, M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.

Elastic CIPs

Let $v(x,t) = (v_1, v_2, v_3)(x,t)$ be vector of displacement. We consider the Cauchy problem for the elastodynamics equations in the isotropic case in the entire space $\mathbb{R}^3$,

$$\begin{aligned} \rho(x)\frac{\partial^2 v}{\partial t^2} - \nabla \cdot \tau = \delta(x_3 - z_0)f(t), \\ \tau = C\epsilon, \\ v(x,0) = 0, v_t(x,0) = 0, \quad x \in \mathbb{R}^3, t \in (0,T), \end{aligned} \tag{10}$$

where $v(x,t)$ is the total displacement generated by the incident plane wave $f(t)$ propagating along the x_3-axis which is incident at the plane $x_3 = z_0$, $\rho(x)$ is the density of the material, τ is the stress tensor, C is a cyclic symmetric tensor and ϵ is the strain tensor which have components

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad i,j = 1,2,3.$$

The strain tensor ϵ is coupled with the stress tensor τ by the Hooke's law

$$\tau_{i,j} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\epsilon_{kl} \tag{11}$$

Elastic CIPs

C is a cyclic symmetric tensor

$$C_{ijkl} = C_{klij} = C_{jkli}.$$

When C_{ijkl} does not depends on $\mathbf{x}$ then material of the domain which we consider is said to be homogeneous. If the tensor C_{ijkl} does not depends on the choice of the coordinate system, then the material of the domain under interest is said to be isotropic. Otherwise, the material is anisotropic.

In the isotropic case the cyclic symmetric tensor C can be written as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}),$$

where δ_{ij} is the Kronecker delta, in which case the equation (11) takes the form

$$\tau_{i,j} = \lambda\delta_{ij}\sum_{k=1}^{3}\epsilon_{kk} + 2\mu\epsilon_{ij}, \tag{12}$$

where λ and μ are Lame coefficients.

Lame coefficients λ and μ are given by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{13}$$

Here, E is the modulus of elasticity, or Young modulus, and ν is the Poisson's ratio of the elastic material. Following relations should be satisfied

$$\lambda > 0, \mu > 0 \iff E > 0, 0 < \nu < 1/2. \tag{14}$$

To write the equation (10) only in terms of v we eliminate the strain tensor ϵ from (10) using (11). Then in the isotropic case equation in (10) writes

$$
\begin{aligned}
&\rho(x)\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial x_1}\left((\lambda + 2\mu)\frac{\partial v_1}{\partial x_1} + \lambda\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_3}{\partial x_3}\right) \\
&\quad - \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)\right) = 0, \\
&\rho(x)\frac{\partial^2 v_2}{\partial t^2} - \frac{\partial}{\partial x_2}\left((\lambda + 2\mu)\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_1}{\partial x_1} + \lambda\frac{\partial v_3}{\partial x_3}\right) \\
&\quad - \frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right)\right) = 0, \\
&\rho(x)\frac{\partial^2 v_3}{\partial t^2} - \frac{\partial}{\partial x_3}\left((\lambda + 2\mu)\frac{\partial v_3}{\partial x_3} + \lambda\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_1}{\partial x_1}\right) \\
&\quad - \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right)\right) - \frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)\right) = \delta(x_3 - z_0)f(t),
\end{aligned}
\tag{15}
$$

Elastic CIPs

Assume

$$\lambda(x) \in [1, d_1], d_1 > 0, \ \lambda(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, \ \lambda(x) \in C^2\left(\mathbb{R}^3\right), \quad (16)$$

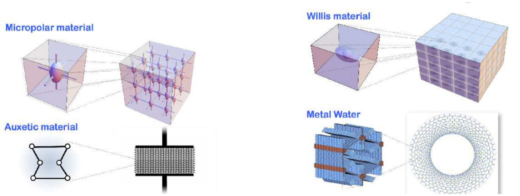
$$\mu(x) \in [1, d_2], d_2 > 0, \ \mu(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, \mu(x) \in C^2\left(\mathbb{R}^3\right), \quad (17)$$

$$\rho(x) \in [1, d_3], d_3 > 0, \ \rho(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, \ \rho(x) \in C^2\left(\mathbb{R}^3\right), \quad (18)$$

Denote $\partial G_T = \partial G \times (0, T)$. We use the following data for our CIPs below:

$$v \mid_{\partial G_T} = f_0(x, t), \quad (19)$$

where the vector function f_0 is known.
Different CIPs can be formulated for (16)-(18), for example:
Inverse Problem (IP1) *Let the function $\rho(x)$ satisfies conditions (18) and this function is unknown in Ω. Determine the function $\rho(x)$ for $x \in \Omega$ assuming that the functions $\lambda(x), \mu(x)$ are known in Ω* and the function *$f_0(x, t)$ in (19) is also known.*
Inverse Problem (IP2) *Let the functions $(\rho, \lambda, \mu)(x)$ satisfies conditions (16), (17), (18) and these functions are unknown in Ω. Determine the functions $(\rho, \lambda, \mu)(x)$ for $x \in \Omega$ assuming that the function $f_0(x, t)$in (19) is known.*



There is a class of materials for which the macroscale properties can be obtained more from such called mechanical *microstructural* design, see Figure for examples of such materials. Practical applications: mechanical cloaking, control and manipulation of waves in fluids and solids, etc. Examples of such materials include nanomaterials such as graphene or carbon nanotubes with extraordinary strength properties. Design of new mechanical metamaterials using computational modeling is one of the applications of elastic CIPs.

Consider a region of space that has no electric or magnetic current sources, but may have materials that absorb electric or magnetic field energy. Then, using MKS units *the MKS system of units is a physical system of units that expresses any given measurement using fundamental units of the metre, kilogram, and/or second (MKS))*, the time-dependent Maxwell's equations are given in differential and integral form by *Faraday's law* :

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{M} \tag{20a}$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{B} \cdot \mathbf{dA} = -\oint_L \mathbf{E} \cdot \mathbf{dL} - \iint_A \mathbf{M} \cdot \mathbf{dA} \tag{20b}$$

Ampere′ s law :

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J} \quad (21a)$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{D} \cdot \mathbf{dA} = \oint_L \mathbf{H} \cdot \mathbf{dL} - \iint_A \mathbf{J} \cdot \mathbf{dA} \quad (21b)$$

Gauss′ law for the electric field :

$$\nabla \cdot \mathbf{D} = 0 \quad (22a)$$

$$\oiint_A \mathbf{D} \cdot \mathbf{dA} = 0 \quad (22b)$$

Gauss′ law for the magnetic field :

$$\nabla \cdot \mathbf{B} = 0 \quad (23a)$$

$$\oiint_A \mathbf{B} \cdot \mathbf{dA} = 0 \quad (23b)$$

In (20) to (23), the following symbols (and their MKS units) are defined:

E	:	electric field (volts/meter)
D	:	electric flux density (coulombs/meter^2)
H	:	magnetic field (amperes/meter)
B	:	magnetic flux density (webers/meter^2)
A	:	arbitrary three-dimensional surface
dA	:	differential normal vector that characterizes surface *A* (meter^2)
L	:	closed contour that bounds surface *A* (volts/meter)
dL	:	differential length vector that characterizes contour *L* (meters)
J	:	electric current density (amperes/meter^2)
M	:	equivalent magnetic current density (volts/meter^2)

In linear, isotropic, nondispersive materials (i.e. materials having field-independent, direction-independent, and frequency-independent electric and magnetic properties), we can relate **D** to **E** and **B** to **H** using simple proportions:

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H} = \mu_r \mu_0 \mathbf{H} \tag{24}$$

where

ε	:	electrical permittivity (farads/meter)
ε_r	:	relative permittivity (dimensionless scalar)
ε_0	:	free-space permittivity (8.854×10^{-12} farads/meter)
μ	:	magnetic permeability (henrys/meter)
μ_r	:	relative permeability (dimensionless scalar)
μ_0	:	free-space permeability ($4\pi \times 10^{-7}$ henrys/meter)

Note that **J** and **M** can act as *independent sources* of E- and H-field energy, $\mathbf{J}_{source}$ and $\mathbf{M}_{source}$.

We also allow for materials with isotropic, nondispersive electric and magnetic losses that attenuate E- and H-fields via conversion to heat energy. This yields

$$\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}; \quad \mathbf{M} = \mathbf{M}_{source} + \sigma^* \mathbf{H} \tag{25}$$

where
σ : electric conductivity (siemens/meter)
σ^* : equivalent magnetic loss (ohms/meter)

Finally, we substitute (24) and (25) into (20a) and (21a). This yields Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} - \frac{1}{\mu} \left(\mathbf{M}_{source} + \sigma^* \mathbf{H} \right) \tag{26}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \left(\mathbf{J}_{source} + \sigma \mathbf{E} \right) \tag{27}$$

We now write out the vector components of the curl operators of (27) and (27) in Cartesian coordinates. This yields the following system of six coupled scalar equations:

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - (M_{source_x} + \sigma^* H_x)\right] \quad (28a)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \left(M_{source_y} + \sigma^* H_y\right)\right] \quad (28b)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - (M_{source_z} + \sigma^* H_z)\right] \quad (28c)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - (J_{source_x} + \sigma E_x)\right] \quad (29a)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \left(J_{source_y} + \sigma E_y\right)\right] \quad (29b)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - (J_{source_z} + \sigma E_z)\right] \quad (29c)$$

Fourier transform

Now we use definition of the Fourier transform for the function $f(x)$. If $f(x)$ is an integrable function in **R** then its Fourier transform is the function $\hat{f}(\xi)$ on **R** such that

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} \exp^{-i\xi x} f(x)\, dx \tag{30}$$

To recover function $f(x)$ from $\hat{f}(\xi)$ is used Fourier inverse transform which is given by the formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp^{i\xi x} \hat{f}(\xi)\, d\xi \tag{31}$$

Fourier transform for Maxwell's system

Now we use definition of the Fourier transform for the functions $E(x,t)$ and $H(x,t)$. If $E(x,t)$ and $H(x,t)$ are integrable functions in $\mathbf{R}^3 \times (-\infty, +\infty)$ then we define their Fourier transforms as the functions $E(x,\omega)$ and $H(x,\omega)$ on $\mathbf{R}^3$ such that

$$\begin{aligned} E(x,\omega) &= \int_{-\infty}^{+\infty} E(x,t) \exp^{-i\omega t} \, dt, \\ H(x,\omega) &= \int_{-\infty}^{+\infty} H(x,t) \exp^{-i\omega t} \, dt \end{aligned} \tag{32}$$

Now, we apply (32) to the system (7)-(27). We multiply system (7)-(27) with $\exp^{-i\omega t}$ and integrate it in time to get

$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t} dt = -\frac{1}{\mu} \int_{-\infty}^{+\infty} \nabla \times \mathbf{E} \exp^{-i\omega t} dt - \frac{1}{\mu}(\mathbf{M}_{source} + \sigma^* \int_{-\infty}^{+\infty} \mathbf{H} \exp^{-i\omega t} dt)$$

$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t} dt = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \nabla \times \mathbf{H} \exp^{-i\omega t} dt - \frac{1}{\varepsilon}(\mathbf{J}_{source} + \sigma \int_{-\infty}^{+\infty} \mathbf{E} \exp^{-i\omega t} dt) \quad (33)$$

In this system we consider $\mathbf{M}_{source} = 0, \sigma^* = 0, \mathbf{J}_{source} = 0$ in accordance with applications in electrical prospecting such that the above system is reduced to the system

$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t} dt = -\frac{1}{\mu} \int_{-\infty}^{+\infty} \nabla \times \mathbf{E} \exp^{-i\omega t} dt$$

$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t} dt = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \nabla \times \mathbf{H} \exp^{-i\omega t} dt - \frac{1}{\varepsilon}\sigma \int_{-\infty}^{+\infty} \mathbf{E} \exp^{-i\omega t} dt \quad (34)$$

Next, we integrate by parts in time integrals $\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t} dt$ and $\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t} dt$ to obtain

$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t} dt = \exp^{-i\omega t} \mathbf{H}|_{-\infty}^{+\infty} + i\omega \int_{-\infty}^{+\infty} \mathbf{H} \exp^{-i\omega t} dt = i\omega \mathbf{H}(x,\omega)$$
$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t} dt = \exp^{-i\omega t} \mathbf{E}|_{-\infty}^{+\infty} + i\omega \int_{-\infty}^{+\infty} \mathbf{E} \exp^{-i\omega t} dt = i\omega \mathbf{E}(x,\omega) \tag{35}$$

and substitute them into (34) to obtain

$$
\begin{aligned}
i\omega\mu\,\mathbf{H}(x,\omega) &= -\nabla\times\mathbf{E}(x,\omega)\\
i\omega\varepsilon\,\mathbf{E}(x,\omega) &= \nabla\times\mathbf{H}(x,\omega) - \sigma\mathbf{E}(x,\omega)
\end{aligned}
\tag{36}
$$

The above system can be rewritten as

$$
\begin{aligned}
\nabla\times\mathbf{E}(x,\omega) &= -i\omega\mu\,\mathbf{H}(x,\omega)\\
\nabla\times\mathbf{H}(x,\omega) &= (i\omega\varepsilon+\sigma)\mathbf{E}(x,\omega)
\end{aligned}
\tag{37}
$$

According to our applications we assume that $\mu = const.$, $\varepsilon = const. > 0$. We introduce new variable $\sigma_\omega := i\omega\varepsilon + \sigma$ to obtain

$$
\begin{aligned}
\nabla\times\mathbf{E}(x,\omega) &= -i\omega\mu\,\mathbf{H}(x,\omega)\\
\nabla\times\mathbf{H}(x,\omega) &= \sigma_\omega\mathbf{E}(x,\omega)
\end{aligned}
\tag{38}
$$

Taking operator of $\nabla\times$ from the first equation in system (38) we have

$$\nabla \times \nabla \times \mathbf{E}(x, \omega) = -i\omega\mu \, \nabla \times \mathbf{H}(x, \omega) \tag{39}$$

Substituting the second equation of the system (38) in the right hand side of (39) we obtain

$$\nabla \times \nabla \times \mathbf{E}(x, \omega) = -i\omega\mu \, \sigma_\omega \mathbf{E}(x, \omega) \tag{40}$$

Statement of the inverse problem in the case of the Fourier transform

Let the function $\sigma_\omega(x) \in C^1(\mathbf{R}^3), x \in \mathbf{R}^3$ (we assume that we approximate $\sigma_\omega(x)$ with the piecewise-linear functions using FEM). Let $\Omega \subset \mathbf{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Determine the coefficient $\sigma_\omega(x) \in \Omega$ assuming that the following function $g(x, \omega)$ is known

$$E(x, \omega)|_{\partial\Omega} = g(x, \omega) \quad \forall (x, \omega) \in \partial\Omega \times (-\infty, +\infty) \tag{41}$$

Examples: CIPs for Maxwell's equation

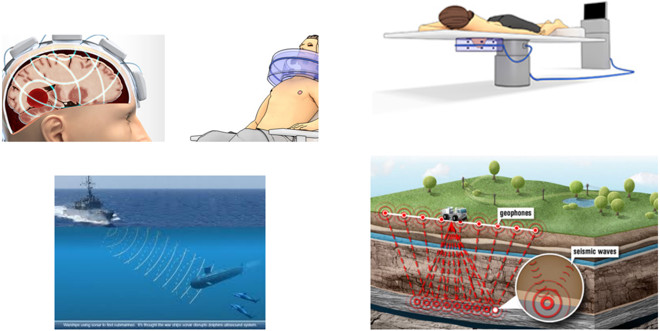


Figure: Biomedical Imaging at the Department of Electrical Engineering at CTH, Chalmers. Top, left: setup of Stroke Finder; microwave hyperthermia in cancer treatment; Top, right: breast cancer detection using microwave tomography. Bottom, left: acoustic imaging, bottom, right: subsurface imaging.