Introduction to inverse and ill-posed problems:

Physical formulations leading to ill- and well-posed problems

Lecture 2

Notations and Definitions

- The theory of Ill-Posed Problems addresses the following fundamental question: *How to obtain a good approximation for the solution of an ill-posed problem in a stable way?*
- A numerical method, which provides a stable and accurate solution of an ill-posed problem, is called the *regularization* method for this problem.
- Foundations of the theory of Ill-Posed Problems were established by three Russian mathematicians: A. N. Tikhonov [T1,TA,T], M.M. Lavrent'ev [L] and V. K. Ivanov [I] in 1960-ies. The first foundational work was published by Tikhonov in 1943 [T].

[T1] A. N. Tikhonov, On the stability of inverse problems, *Doklady of the USSR Academy of Science*, 39, 195-198, 1943

[TA] A. N. Tikhonov and V. Y. Arsenin. *Solutions of Ill-Posed Problems,* Winston and Sons, Washington, DC, 1977.

[T] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov and A.G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, London: Kluwer, London, 1995.

[L] M.M. Lavrentiev, *Some Improperly Posed Problems of Mathematical Physics*, Springer, New York, 1967.

[I] V. K. Ivanov, On ill-posed problems, *Mat. USSR Sb.*, 61, 211–223, 1963.

- Theory of inverse and ill-posed problems is developed further and a lot of new works on this subject are available:

- S. Arridge, Optical tomography in medical imaging, *Inverse Problems*, 15, 841–893, 1999.
- A.B. Bakushinsky and M.Yu. Kokurin, *Iterative Methods for Approximate Solution of Inverse Problems*, Springer, New York, 2004.
- F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, New York, 2006.
- K. Chadan and P. Sabatier, Inverse Problems in Quantum Scattering Theory, Springer, New York, 1989.
- G. Chavent, *Nonlinear Least Squares for Inverse Problems: Theoretical Foundations and Step-by-Step Guide for Applications (Scientific Computation)*, Springer, New York, 2009.
- V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, New York, 2005.
- B. Kaltenbacher, A. Neubauer and O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*, de Gruyter, New York, 2008.
- A. Kirsch, *An Introduction To the Mathematical Theory of Inverse Problems*, Springer, New York, 2011.

Notations and Definitions

Let $u(x)$, $x = (x_1, ..., x_n) \in \Omega$ be a k times continuously differentiable function defined in Ω. Denote

$$D^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

the partial derivative of the order $|\alpha| \leq k$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with integers $\alpha_i \geq 0$. Denote $C^k\left(\overline{\Omega}\right)$ the Banach space of functions $u(x)$ which are continuous in the closure $\overline{\Omega}$ of the domain Ω together with their derivatives $D^{\alpha} u, |\alpha| \leq m$. The norm in this space is defined as

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^{\alpha} u(x)| < \infty.$$

Notations and Definitions

By definition $C^0\left(\overline{\Omega}\right) = C\left(\overline{\Omega}\right)$ is the space of functions continuous in $\overline{\Omega}$ with the norm

$$\|u\|_{C\left(\overline{\Omega}\right)} = \sup_{x\in\Omega} |u(x)|.$$

We also introduce Hölder spaces $C^{k+\alpha}\left(\overline{\Omega}\right)$ for any number $\alpha \in (0,1)$. The norm in this space is defined as

$$\|u\|_{C^{k+\alpha}\left(\overline{\Omega}\right)} := |u|_{k+\alpha} := \|u\|_{C^k\left(\overline{\Omega}\right)} + \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}},$$

provided that the last term is finite. It is clear that if the function $u \in C^{k+1}\left(\overline{\Omega}\right)$, then $u \in C^{k+\alpha}\left(\overline{\Omega}\right), \forall\alpha \in (0,1)$ and

$$|u|_{k+\alpha} \leq C\|u\|_{C^{k+1}\left(\overline{\Omega}\right)},\ \forall u \in C^{k+1}\left(\overline{\Omega}\right),$$

where $C = C(\Omega,\alpha) > 0$ is a constant independent on the function u.

Notations and Definitions

Consider the Sobolev space $H^k(\Omega)$ of all functions with the norm defined as

$$\|u\|^2_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_{\Omega} |D^\alpha u|^2 \, dx < \infty,$$

where $D^\alpha u$ are weak derivatives of the function u. By the definition $H^0(\Omega) = L_2(\Omega)$. It is well known that $H^k(\Omega)$ is a Hilbert space with the inner product defined as

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_{\Omega} D^\alpha u D^\alpha v dx.$$

Notations and Definitions

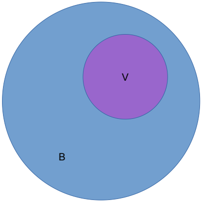
Let $T > 0$ and $\Gamma \subseteq \partial\Omega$ be a part of the boundary $\partial\Omega$ of the domain Ω. We will use the following notations

$$Q_T = \Omega \times (0, T), S_T = \partial\Omega \times (0, T), \Gamma_T = \Gamma \times (0, T), D_T^{n+1} = \mathbb{R}^n \times (0, T).$$

The space $C^{2k,k}\left(\overline{Q}_T\right)$ is defined as the set of all functions $u(x,t)$ having derivatives $D_x^\alpha D_t^\beta u \in C\left(\overline{Q}_T\right)$ with $|\alpha| + 2\beta \leq 2k$ and with the following norm

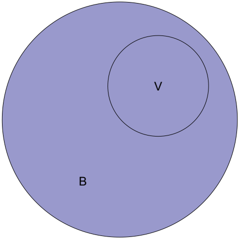
$$\|u\|_{C^{2k,k}\left(\overline{Q}_T\right)} = \sum_{|\alpha|+2\beta\leq 2k} \max_{\overline{Q}_T} \left| D_x^\alpha D_t^\beta u(x,t) \right|.$$

Notations and Definitions



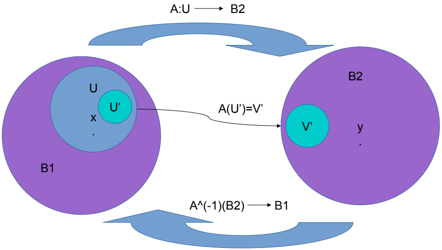
Definition 1. Let B be a Banach space. The set $V \subset B$ is called *precompact* set if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence (i.e., the Cauchy subsequence).
Although by the Cauchy criterion the subsequence in this Definition 1 converges to a certain point, there is no guarantee that this point belongs to the set V. If we consider the closure of V, i.e. the set $\overline{V}$, then all limiting points of all convergent sequences in V would belong to $\overline{V}$.

Notations and Definitions



Definition 2. Let B be a Banach space. The set $V \subset B$ is called *compact* set if V is a closed set, $V = \overline{V}$, every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence and the limiting point of this subsequence belongs to the set V.

Notations and Definitions



Definition 3. Let B_1 and B_2 be two Banach spaces, $U \subseteq B_1$ be a set and $A : U \to B_2$ be a continuous operator. The operator A is called a *compact operator* or *completely continuous* operator if it maps any bounded subset $U' \subseteq U$ in a precompact set in B_2. Clearly if U' is a closed set, then $A(U')$ is a compact set.

Notations and Definitions. Ascoli-Archela theorem

The following theorem is well known under the name of Ascoli-Archela theorem (More general formulations of this theorem can also be found).

Theorem *The set of functions $\mathcal{M} \subset C\left(\overline{\Omega}\right)$ is a compact set if and only if it is uniformly bounded and equicontinuous. In other words, if the following two conditions are satisfied:*

1. There exists a constant $M > 0$ such that

$$\|f\|_{C(\overline{\Omega})} \leq M, \ \forall f \in \mathcal{M}.$$

2. For any $\varepsilon > 0$ there exists $\delta = \delta\left(\varepsilon\right) > 0$ such that

$$|f\left(x\right) - f\left(y\right)| < \varepsilon, \quad \forall x, y \in \{|x - y| < \delta\} \cap \overline{\Omega}, \ \forall f \in \mathcal{M}.$$

Classical Correctness and Conditional Correctness

The notion of the classical correctness is called sometimes *Correctness by Hadamard*.

Definition. Let B_1 and B_2 be two Banach spaces. Let $G \subseteq B_1$ be an open set and $F : G \to B_2$ be an operator. Consider the equation

$$F(x) = y, \; x \in G. \tag{1}$$

The problem of solution of equation (2) is called *well-posed by Hadamard,* or simply *well-posed,* or *classically well-posed* if the following three conditions are satisfied:

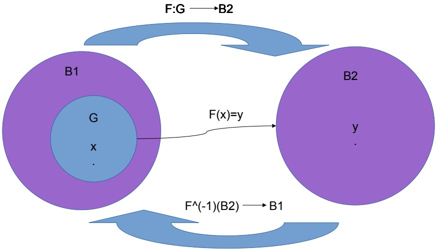
1. For any $y \in B_2$ there exists a solution $x = x(y)$ of equation (2) (existence theorem).

2. This solution is unique (uniqueness theorem).

3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

If equation (2) does not satisfy to at least one these three conditions, then the problem (2) is called *ill-posed*.

Classical Correctness



The problem is *classically well-posed* if:
1. For any $y \in B_2$ there exists a solution $x = x(y)$ of $F(x) = y$.
2. This solution is unique (uniqueness theorem).
3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

Classical Correctness and Conditional Correctness

We say that the right hand side of equation

$$F(x) = y, \; x \in G. \tag{2}$$

is given with an error of the level $\delta > 0$ (small) if $\|y^* - y\|_{B_2} \leq \delta$, where y^* is the exact value.

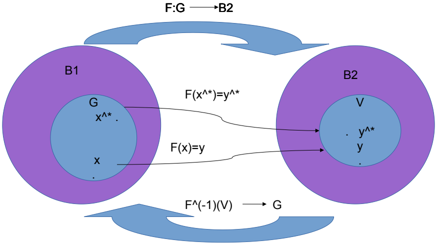
Definition Let B_1 and B_2 be two Banach spaces. Let $G \subset B_1$ be an *a priori* chosen set of the form $G = \overline{G}_1$, where G_1 is an open set in B_1. Let $F : G \to B_2$ be a continuos operator. Suppose that $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here y^* is the ideal noiseless data, y_δ is noisy data . The problem (2) is called *conditionally well-posed on the set* G, or *well-posed by Tikhonov* on the set G if the following three conditions are satisfied:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

Conditional Correctness



The problem (2) is called *conditionally well-posed on the set* G if:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

The Fundamental Concept of Tikhonov

This concept consists of the following three conditions which should be in place when solving the ill-posed problem (2):

1. One should *a priori* assume that there exists an ideal exact solution x^* of equation (2) for an ideal noiseless data y^*.

2. The correctness set G should be chosen *a priori*, meaning that some *a priori* bounds imposed on the solution x of equation (2) should be imposed.

3. To construct a stable numerical method for the problem (2), one should assume that there exists a family $\{y_\delta\}$ of right hand sides of equation (2), where $\delta > 0$ is the level of the error in the data with $\|y^* - y_\delta\|_{B_2} \leq \delta$. Next, one should construct a family of approximate solutions $\{x_\delta\}$ of equation (2), where x_δ corresponds to y_δ. The family $\{x_\delta\}$ should be such that

$$\lim_{\delta \to 0^+} \|x_\delta - x^*\| = 0.$$

Ill-posed problem: differentiation of a function given with a noise

Suppose that the function $f(x), x \in [0,1]$ is given with a noise, i.e. suppose that instead of $f(x) \in C^1[0,1]$ the following function $f_\delta(x)$ is given

$$f_\delta(x) = f(x) + \delta f(x), x \in [0,1],$$

where $\delta f(x)$ is the noisy component. Let $\delta > 0$ be a small parameter such that $\|\delta f\|_{C[0,1]} \leq \delta$. Let us show that the problem of calculating the derivative $f_\delta'(x)$ is unstable.

Examples of ill-posed problems. Differentiation of a function given with a noise

For example, take

$$\delta f(x) = \frac{\sin(n^2 x)}{n},$$

where $n > 0$ is a large integer. Then the $C[0,1]$-norm of the noisy component is small,

$$\|\delta f\|_{C[0,1]} \leq \frac{1}{n}.$$

However, the difference between derivatives of noisy and exact functions

$$f_\delta'(x) - f'(x) = \delta f'(x) = n \cos n^2 x$$

is not small in any reasonable norm.

Ill-posed problem: differentiation of a function given with a noise

A simple regularization method of stable calculation of derivatives is that the step size h in the corresponding finite difference discretization should be connected with the level of noise δ.

$$f'_\delta(x) \approx \frac{f(x+h) - f(x)}{h} + \frac{\delta f(x+h) - \delta f(x)}{h}. \tag{3}$$

The first term in the right hand side of (3) is close to the exact derivative $f'(x)$, if h is small enough. The second term, however, comes from the noise and we need to balance these two terms via an appropriate choice of $h = h(\delta)$.

$$\left| f'_\delta(x) - \frac{f(x+h) - f(x)}{h} \right| = \left| \frac{\delta f(x+h) - \delta f(x)}{h} \right| \leq \frac{2\delta}{h}.$$

Hence, we should choose $h = h(\delta)$ such that

$$\lim_{\delta \to 0} \frac{2\delta}{h(\delta)} = 0.$$

Ill-posed problem: differentiation of a function given with a noise

For example, let $h(\delta) = \delta^{\mu}$, where $\mu \in (0,1)$. Then

$$\lim_{\delta\to 0}\left| f'_{\delta}(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h} = \frac{2\delta}{\delta^{\mu}} \leq\leq \lim_{\delta\to 0}\left(2\delta^{1-\mu}\right) = 0.$$

Hence, the problem becomes stable for this choice of the grid step size $h(\delta) = \delta^{\mu}$. This means that $h(\delta)$ is the regularization parameter.

Ill-posed problem: integral equation of the first kind

Let $\Omega \subset \mathbb{R}^n$ is a bounded domain and the function $K(x,y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Recall that the equation

$$g(x) + \int_{\Omega} K(x,y) f(y)\,dy = p(x), x \in \Omega, \tag{4}$$

where $p(x)$ is a bounded function, is called *integral equation of the second kind*. These equations are considered quite often in the classic theory of PDEs. The classical Fredholm theory works for these equations. Next, let $\Omega' \subset \mathbb{R}^n$ be a bounded domain and the function $K(x,y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Unlike (4), the equation

$$\int_{\Omega} K(x,y) f(y)\,dy = p(x), x \in \Omega' \tag{5}$$

is called the integral equation of the first kind. The Fredholm theory does not work for such equations. The problem of solution of equation (5) is an ill-posed problem.

Ill-posed problem: integral equation of the first kind

Consider equation (5):

$$\int_{\Omega} K(x,y) f(y)\, dy = p(x), x \in \Omega'$$

The function $K(x,y)$ is called *kernel* of the integral operator. Equation (5) can be rewritten in the form

$$Af = p, \tag{6}$$

where $A : C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}'\right)$ is the integral operator in (5). It is well known from the standard Functional Analysis course that A is a compact operator. We now show that the problem (6) is ill-posed.

Ill-posed problem: integral equation of the first kind

Let $\Omega = (0,1), \Omega' = (a,b)$. Let $f_n(x) = f(x) + \sin nx$. Then for $x \in (0,1)$

$$\int_0^1 K(x,y) f_n(y)\, dy = \int_0^1 K(x,y) f(y)\, dy + \int_0^1 K(x,y) \sin ny dy = g_n(x), \tag{7}$$

where $g_n(x) = p(x) + p_n(x)$ and

$$p_n(x) = \int_0^1 K(x,y) \sin ny dy.$$

By the Lebesque lemma

$$\lim_{n\to\infty} \|p_n\|_{C[a,b]} = 0.$$

However, it is clear that

$$\|f_n(x) - f(x)\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$$

is not small for large n.

Ill-posed problem: the case of a general compact operator

Let H_1 and H_2 be two Hilbert spaces with $\dim H_1 = \dim H_2 = \infty$. Remind that a sphere in an infinitely dimensional Hilbert space is not a compact set.

Theorem *Let* $G = \left\{ \|x\|_{H_1} \leq 1 \right\} \subset H_1$. *Let* $A : G \to H_2$ *be a compact operator and let* $R(A) := A(G)$ *be its range. Consider an arbitrary point* $y_0 \in R(A)$. *Let* $\varepsilon > 0$ *be a number and* $U_\varepsilon(y_0) = \left\{ y \in H_2 : \|y - y_0\|_{H_2} < \varepsilon \right\}$. *Then there exists a point* $y \in U_\varepsilon(y_0) \diagdown R(A)$. *If, in addition, the operator* A *is one-to-one, then the inverse operator* $A^{-1} : R(A) \to G$ *is not continuous. Hence, the problem of the solution of the equation*

$$A(x) = z, x \in G, z \in R(A) \tag{8}$$

is unstable, i.e. this is an ill-posed problem.

Tikhonov's theorem

Theorem (Tikhonov, 1943). *Let B_1 and B_2 be two Banach spaces. Let $U \subset B_1$ be a compact set and $F : U \to B_2$ be a continuous operator. Assume that the operator F is one-to-one. Let $V = F(U)$. Then the inverse operator $F^{-1} : V \to U$ is continuous.*

Proof. Assume the opposite: that the operator F^{-1} is not continuous on the set V. Then there exists a point $y_0 \in V$ and a number $\varepsilon > 0$ such that for any $\delta > 0$ there exists a point y_δ such that although $\|y_\delta - y_0\|_{B_2} < \delta$, still $\left\|F^{-1}(y_\delta) - F^{-1}(y_0)\right\|_{B_1} \geq \varepsilon$. Hence, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \delta_n = 0^+$ and the corresponding sequence $\{y_n\}_{n=1}^{\infty} \subset V$ such that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n, \| \underbrace{F^{-1}(y_n)}_{x_n} - \underbrace{F^{-1}(y_0)}_{x_0} \|_{B_1} \geq \varepsilon. \tag{9}$$

Denote

$$x_n = F^{-1}(y_n), x_0 = F^{-1}(y_0). \quad (10)$$

Then by (9) we have

$$\|x_n - x_0\|_{B_1} \geq \varepsilon. \quad (11)$$

Since U is a compact set and all points $x_n \in U$, then one can extract a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ from the sequence $\{x_n\}_{n=1}^{\infty}$. Let $\lim_{k\to\infty} x_{n_k} = \overline{x}$. Then $\overline{x} \in U$. Since $F(x_{n_k}) = y_{n_k}$ and the operator F is continuous, then by (9) and (10) we have:

$$\begin{aligned} x_n &= F^{-1}(y_n) \Rightarrow F(x_n) = y_n, \\ x_0 &= F^{-1}(y_0) \Rightarrow F(x_0) = y_0; \\ F(\bar{x}) &= \lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = y_0 \end{aligned} \quad (12)$$

So, we obtained, that $F(x_0) = y_0$ and $F(\bar{x}) = y_0$. Since the operator F is one-to one, we should have $\overline{x} = x_0$. However, by (11) $\|\overline{x} - x_0\|_{B_1} \geq \varepsilon$. We got a contradiction. $\square$

Model inverse problems

We will consider now following model inverse problems:

- Elliptic inverse problems
  - Elliptic CIPs
  - Cauchy problem
  - Inverse source problem
  - Inverse spectral problem
- Hyperbolic CIPs
- Parabolic CIPs

Elliptic inverse problems

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ be a domain with a boundary Γ.
We will present several inverse problems for a second order elliptic PDE

$$- \nabla \cdot (a(x) \nabla u) + b(x) \cdot \nabla u + p(x) u = f(x), \quad x \in \Omega \tag{13}$$

We can consider the equation (13) with suitable b.c. (for example, Dirichlet or Neumann b.c.).
Functions $a(x), b(x)$ and $p(x)$ are known as

- $a(x)$ - conductivity or diffusion coefficient
- $b(x)$ - convection coefficient
- $p(x)$ potential coefficient
- $f(x)$ - the source term

The Elliptic Coefficient Inverse Problem

$$-\nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u + p(x)u = f(x), \quad x \in \Omega \tag{14}$$

Let the function $u(x) \in C^2$ satisfies to the (14) and

$$u|_\Gamma = p\left(x\right), \quad \frac{\partial u}{\partial n}|_\Gamma = q\left(x\right). \tag{15}$$

The Elliptic Coefficient Inverse Problem. Suppose that one of coefficients in equation (14) is unknown inside of the domain Ω and is known outside of it. Assume that all other coefficients in (14) are known. Determine that unknown coefficient inside of Ω, assuming that the functions $p\left(x\right)$ and $q\left(x\right)$ in (15) are known.

Cauchy problem

Cauchy problem arises, for example, in electrocardiography and geophysical prospectation. This problem is severally ill-posed and lacks a continuous dependence on data.

Let Γ_c and $\Gamma_i = \Gamma \setminus \Gamma_c$ be two disjoint parts of the boundary Γ. Here,

- Γ_c - observation boundary
- Γ_i - boundary, where observations are not taken

The Cauchy problem reads: given the Cauchy data g and h on the boundary Γ_c, find the function u on the boundary Γ_i, or:

$$
\begin{aligned}
-\nabla \cdot (a(x)\nabla u) &= 0, x \in \Omega, && (16)\\
u &= g, \quad x \in \Gamma_c, && (17)\\
a\frac{\partial u}{\partial n} &= h, \quad x \in \Gamma_c. && (18)
\end{aligned}
$$

Hadamard's example for the Cauchy problem

This example shows that the Cauchy problem for Laplace equation does't depends continuously on data. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and the boundary $\Gamma_c = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Consider the solution $u = u_n, n = 1, 2, ...$ to the Cauchy problem

$$\triangle u = 0, \quad x \in \Omega, \tag{19}$$

$$u = 0, \quad x \in \Gamma_c, \tag{20}$$

$$\frac{\partial u}{\partial n} = -n^{-1} \sin nx_1, \quad x \in \Gamma_c. \tag{21}$$

The function

$$u_n = n^{-2} \sin nx_1 \sinh nx_2 = n^{-2} \sin nx_1 (e^{nx_2} - e^{-nx_2})/2$$

is the solution of the problem (19) and it is a unique solution (by Holmgren's theorem for Laplace equation).
We observe that on Γ_c we have $\lim_{n\to\infty} \frac{\partial u_n}{\partial n} = 0$. However, for all $x_2 > 0$ the solution $\lim_{n\to\infty} u_n(x_1, x_2) = \lim_{n\to\infty} n^{-2} \sin nx_1 \sinh nx_2 = \lim_{n\to\infty} n^{-2} \sin nx_1 (e^{nx_2} - e^{-nx_2})/2 = \infty$.

Inverse source problem

The classical linear inverse problem is to recover a source function f in the equation

$$-\triangle u = f, \quad x \in \Omega \tag{22}$$

from the Cauchy data (g, h) on the boundary Γ:

$$u = g, \quad x \in \Gamma, \tag{23}$$

$$\frac{\partial u}{\partial n} = h, \quad x \in \Gamma. \tag{24}$$

Applications of this problem are in electroencephalography to determine electrical activities of brain from electrodes placed on a head, and electrocardiography to determine heart's electrical activity from body-surface potential distribution. This problem does't has unique solution.

This can be proved if we add one compactly supported function and obtain a different source on the rhs of (22) with the same Cauchy data which is not changed.

Inverse source problem: example of non-uniqueness

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ is a bounded domain with the boundary Γ. Let $\omega_i, i = 1, 2$ be two balls which have different radius r_i, respectively, centered at the origin o, and these balls are inside the domain Ω. Choose the scalars λ_i such that $\lambda_1 r_1^d = \lambda_2 r_2^d$. Let the source has the form $f_i = \lambda_i \xi_{\omega_i}$, where ξ denotes the characteristic, or indicator function of the set S in the Laplace equation for $i = 1, 2$

$$-\triangle u_i = f_i, \quad x \in \Omega, \tag{25}$$
$$u_i = g, \quad x \in \Gamma. \tag{26}$$

In other words,

$$\xi = \xi_S(x) := \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S \end{cases} \tag{27}$$

Inverse source problem: example of non-uniqueness

Then for $\forall v \in H(\Omega) = \{v \in H^2(\Omega) : \triangle v = 0\}$ the variational formulation of (25) will be:

$$
\begin{aligned}
-(\triangle u_i, v)_\Omega &= -[(v, \frac{\partial u_i}{\partial n})_\Gamma - (\nabla u_i, \nabla v)_\Omega] && (28)\\
&= -(v, \frac{\partial u_i}{\partial n})_\Gamma + (u_i, \frac{\partial v}{\partial n})_\Gamma - (u_i, \triangle v)_\Omega && (29)\\
&= (u_i, \frac{\partial v}{\partial n})_\Gamma - (v, \frac{\partial u_i}{\partial n})_\Gamma = (f_i, v), && (30)
\end{aligned}
$$

where $(\cdot, \cdot)$ is the standard L_2 inner product.

Inverse source problem: example of non-uniqueness

Since $u_i = g, i = 1,2, \quad x \in \Gamma$ then the equation above can be rewritten as

$$(f_i, v)_\Omega = (g, \frac{\partial v}{\partial n})_\Gamma - (v, \frac{\partial u_i}{\partial n})_\Gamma, i = 1,2. \tag{31}$$

By the mean value theorem for harmonic functions we have

$$(f_i, v)_\Omega = |\omega_i| v(O), \quad i = 1,2. \tag{32}$$

By construction of f_i and using (31), (32) we get
$\forall v \in H(\Omega) = \{v \in H^2(\Omega) : \triangle v = 0\}$

$$(v, \frac{\partial u_1}{\partial n})_\Gamma - (v, \frac{\partial u_2}{\partial n})_\Gamma = 0, i = 1,2. \tag{33}$$

Since $\forall h \in H^{1/2}(\Gamma)$ there exists the harmonic function $v \in H(\Omega)$ such that $v = h$ on Γ and $H^{1/2}$ is dense in $L^2(\Gamma)$ we conclude that $(\frac{\partial u_1}{\partial n})_\Gamma = (\frac{\partial u_2}{\partial n})_\Gamma$ what means that two different sources have identical Cauchy data.

How to solve inverse source problems

- In practical applications it is often requered minimum-norm sources or harmonic sources such that $\triangle f = 0$
- Often are considered localized sources modeled by monopoles, dipoles or their combinations. In the case of combinations of monopoles and dipoles can be obtained unique recovery of the source function via Holmgren's theorem.
- Direct algorithms for location of monopoles and dipoles are developed in
  El Badia, Ha-Duong, An inverse source problem in potential analysis, *Inverse Problems*, 16, pp.651–663, 2000.

Inverse spectral problem

The forward problem is

$$Au = \lambda u, \tag{34}$$

where A is an elliptic operator, λ is eigenvalue and u is respective eigenfunction.
The inverse spectral problem is to recover the coefficients in the operator A or the geometry of the domain Ω from partial or multiple spectral data (knowledge of eigenvalues and eigenfunctions).

Example of an inverse spectral problem

Let the operator A is applied to u as:

$$Au = -u''(t) + q(t)u(t), \quad t \in (0,1) \tag{35}$$

where $q(t)$ is the potential. Then the classical **Sturm-Liouville problem** reads: given a potential $q(t)$ and constants $h, H > 0$ find eigenvalues $\{\lambda_k\}$ and eigenfunctions $\{u_k\}$ such that

$$-u''(t) + q(t)u(t) = \lambda u(t), t \in (0,1), \tag{36}$$

$$u'(0) - hu(0) = 0, \tag{37}$$

$$u'(1) + Hu(1) = 0. \tag{38}$$

The set of eigenvalues $\{\lambda_k\}$ is real and countable.
The **inverse Sturm-Liouville problem** is to recover the potential $q(t), h, H$ from the knowledge of spectral data $(\{\lambda_k\}, \{u_k\})$. This data can take different forms.
Numerical solution of these problems is presented in
M. T. Chu, G. H. Golub, *Inverse eigenvalue problems*, Oxford University Press, Oxford, New York, 2005.

The Hyperbolic Coefficient Inverse Problem

Let us assume that the domain Ω is a ball,
$\Omega = \{|x| < R\} \subset \mathbb{R}^n, R = const. > 0$. Let $T = const. > 0$. Denote
$Q_T^{\pm} = \Omega \times (-T, T), S_T^{\pm} = \partial\Omega \times (-T, T)$.
Let the function $u(x,t) \in C^2\left(\overline{Q}_T\right)$ satisfies to the

$$c(x)\, u_{tt} = \Delta u + \sum_{|\alpha| \leq 1} a_\alpha(x)\, D_x^\alpha u, \text{ in } Q_T, \tag{39}$$

$$u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), \tag{40}$$

$$u|_{S_T} = p(x,t), \quad \frac{\partial u}{\partial n}|_{S_T} = q(x,t), \tag{41}$$

where functions $a_\alpha, c \in C\left(\overline{Q}_T\right)$ and $c \geq 1$.

The Hyperbolic Coefficient Inverse Problem. Suppose that one of coefficients in equation (39) is unknown inside of the ball Ω and is known outside of it. Assume that all other coefficients in (42) are known and conditions (40) are satisfied. Determine that unknown coefficient inside of Ω, assuming that the functions $p(x,t)$ and $q(x,t)$ in (41) are known.

The Coefficient Inverse Problem for a parabolic equation

Consider the Cauchy problem for the following forward parabolic equation

$$c(x)\,u_t = \Delta u + \sum_{|\alpha|\leq 1} a_\alpha(x)\, D_x^\alpha u, \ \text{ in } D_T^{n+1} = \mathbb{R}^n \times (0, T), \tag{42}$$

$$u(x,0) = f_0(x), \tag{43}$$

$$c, a_\alpha \in C^\beta(\mathbb{R}^n), f_0 \in C^{2+\beta}(\mathbb{R}^n), \beta \in (0,1), c(x) \geq 1. \tag{44}$$

Given conditions (44), this problem has unique solution $u \in C^{2+\beta,1+\beta/2}\left(\overline{D}_T^{n+1}\right)$. Assume that $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$. Let $\Gamma \subseteq \partial\Omega$ be a part of the boundary of the domain Ω and $T = const. > 0$.

The Parabolic Coefficient Inverse Problem. Suppose that one of coefficients in equation (42) is unknown inside of the ball Ω and is known outside of it. Assume that all other coefficients in (42) are known and conditions (43), (44) are satisfied. Determine that unknown coefficient inside of Ω, assuming that the following functions $p(x,t)$ and $q(x,t)$ are known

$$u\mid_{\Gamma_T} = p(x,t), \quad \frac{\partial u}{\partial n}\mid_{\Gamma_T} = q(x,t). \tag{45}$$

CIP for a parabolic equation is an ill-posed problem

Let the function $a(x) \in C^{\alpha}(\mathbb{R}^n), \alpha \in (0,1)$ and $a(x) = 0$ outside of the bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^3$. Consider the following Cauchy problem

$$u_t = \Delta u + a(x)\, u, \ \ (x,t) \in D_T^{n+1}, \tag{46}$$

$$u(x,0) = f(x). \tag{47}$$

Here the function $f(x) \in C^{2+\alpha}(\mathbb{R}^n)$ has a finite support in $\mathbb{R}^n$. Another option for the initial condition is

$$f(x) = \delta(x - x_0), \quad x_0 \notin \overline{\Omega} \tag{48}$$

The inverse problem is: assume that the function $a(x)$ is unknown inside of the domain Ω. Determine this function for $x \in \Omega$ assuming that the following function $g(x,t)$ is known

$$u\mid_{S_T} = g(x,t). \tag{49}$$

CIP for a parabolic equation is an ill-posed problem

Let us show that this CIP is an ill-posed problem. Let the function u_0 be the fundamental solution of the heat equation $u_{0t} = \Delta u_0$,

$$u_0(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \exp\left(-\frac{|x|^2}{4t}\right).$$

It is well known that by [LSU]

$$u(x,t) = \int\limits_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi)\, d\xi + \int\limits_0^t \int\limits_\Omega u_0(x-\xi,t-\tau)\, a(\xi)\, u(\xi,\tau)\, d\tau. \tag{50}$$

Because of the presence of the integral $\int\limits_0^t (\cdot)\, d\tau$ the integral (50) is a Volterra-like integral equation of the second kind.

[LSU] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, R.I., 1968.

Hence, it can be solved as in [LSU]

$$u(x,t) = \underbrace{\int\limits_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi)\, d\xi}_{u_0^f} + \sum_{n=1}^{\infty} u_n(x,t), \tag{51}$$

$$u_n(x,t) = \int\limits_0^t \int\limits_{\Omega} u_0(x-\xi,t-\tau)\, a(\xi)\, u_{n-1}(\xi,\tau)\, d\tau.$$

One can prove that each function $u_n \in C^{2+\alpha,1+\alpha/2}\left(\overline{D}_T^{n+1}\right)$ and using [LSU]

$$\left| D_x^{\beta} D_t^k u_n(x,t) \right| \leq \frac{(Mt)^n}{n!}, \quad |\beta| + 2k \leq 2, \tag{52}$$

where $M = \|a\|_{C^{\alpha}(\overline{\Omega})}$. In the case when $f = \delta(x - x_0)$ the first term in the right hand side of (51) should be replaced with $u_0(x - x_0, t)$.

Let $u_0^f(x,t)$ be the first term of the right hand side of (51) and
$v(x,t) = u(x,t) - u_0^f(x,t)$.
Using (52), one can rewrite (51) as

$$v(x,t) = \int_0^t \int_\Omega u_0(x-\xi, t-\tau)\left(a(\xi)\, u_0^f(\xi,\tau) + P(a)(\xi,\tau)\right) d\xi d\tau, \tag{53}$$

where $P(a)$ is a nonlinear operator applied to the function a. It is clear from (51)-(53) that the operator $P : C^\alpha\left(\overline{\Omega}\right) \to C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_T\right)$ is continuous. Setting in (53) $(x,t) \in S_T$, recalling (49) and denoting denoting $\overline{g}(x,t) = g(x,t) - u_0^f(x,t)$, we obtain a nonlinear integral equation of the first kind with respect to the unknown coefficient $a(x)$

$$\underbrace{\int_{S_T} u_0(x-\xi, t-\tau)\left(u_0^f(\xi,\tau)\, a(\xi) + P(a)(\xi,\tau)\right) d\xi d\tau}_{A(a)} = \overline{g}(x,t), \quad (x,t) \in S_T. \tag{54}$$

Let $A(a)$ be the operator in the left hand side of (54). Let $H_1 = L_2(\Omega)$ and $H_2 = L_2(S_T)$. Consider now the set U of functions defined as

$$U = \left\{ a : a \in C^{\alpha}\left(\overline{\Omega}\right), \|a\|_{C^{\alpha}\left(\overline{\Omega}\right)} \leq M \right\} \subset H_1.$$

Since the $L_2(\Omega)$ −norm is weaker than the $C^{\alpha}\left(\overline{\Omega}\right)$ −norm, then U is a bounded set in H_1 and $A : U \to C(S_T)$ is a compact operator by Theoirem 1.1 of [BK]. Since the norm in $L_2(S_T)$ is weaker than the norm in $C(S_T)$, then $A : U \to H_2$ is also a compact operator. Hence, from the Theorem about an ill-posed problem (Theorem 1.2 of [BK]) follows that the problem of solution of the equation

$$A(a) = g, a \in U \subset H_1, g \in H_2$$

is an ill-posed problem.