Stabilized P1 finite element method for electromagnetic inverse problems

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Coefficient Inverse Problems

Coefficient of some PDE on FEM mesh.

- Breast cancer, land mines and “invisible materials” can all be computed using different types of wave equations: acoustic, elastic or electromagnetic.

- In the mathematical literature, these questions are called the coefficient inverse problems. A coefficient inverse problem for a given partial differential equation (PDE) aims at estimating a spatially distributed coefficient of the model PDE using measurements taken on the boundary of the domain of interest.

- Software package WavES (waves24.com) is developed for efficient solution of time-dependent wave equations (acoustic, elastic and electromagnetic) and CIPs for them in C++/PETSc.
Domain decomposition for the Maxwell’s equations

$\Omega = \Omega_1 \cup \Omega_2$

$\Omega_1$

$\Omega_2$

$\varepsilon = 1$

$\varepsilon \in [1, d_1]$

$\Omega_1$

$\Omega_2$

$\varepsilon = 1$

Figure: Domain decomposition in $\Omega$ for the Maxwell’s equations

$\varepsilon \partial_{tt} \mathbf{e} + \nabla \times \nabla \times \mathbf{e} = \mathbf{f}$ for the electric field $\mathbf{e} = (e_1, e_2, e_3)$. 
Domain decomposition for the Maxwell’s equations

\[ \varepsilon \in [1, \kappa_1] \]

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ \Omega = \Omega_1 \cup \Omega_2 \]

b) \( \Omega_2 \)

**Figure:** Domain decomposition in \( \Omega \).

The forward problem is the Maxwell’s equations for the electric field \( \mathbf{e} = (e_1, e_2, e_3) \) in a bounded domain \( \Omega \) of \( \mathbb{R}^3 \) with boundary \( \partial \Omega \)

\[
\varepsilon \partial_{tt} \mathbf{e} + \nabla \times \nabla \times \mathbf{e} = 0 \quad \text{in } \Omega \times (0, T),
\]

\[
\mathbf{e}(\cdot, 0) = \mathbf{e}_0(\cdot), \text{ and } \partial_t \mathbf{e}(\cdot, 0) = \mathbf{e}_1(\cdot) \quad \text{in } \Omega,
\]

\[
\partial_n \mathbf{e} = -\partial_t \mathbf{e} \quad \text{on } \partial \Omega \times (0, T),
\]

\[
\nabla \cdot (\varepsilon \mathbf{e}) = 0 \quad \text{in } \Omega.
\]

(1)
An explicit P1 finite element scheme for Maxwell’s equations

- Complete convergence study of the approximation of Maxwell’s equations (2) by means of standard linear P1 finite elements for the space discretization, combined with a well-known explicit finite-difference scheme for the time discretization. The analysis applies to the particular case where the electric permittivity has a constant value in the boundary neighbourhood. Optimal convergence results are derived under reasonable assumptions, provided a classical CFL condition holds.


- The numerical validation of the scheme above is presented in

  L. Beilina, V. Ruas, Numerical validation of an explicit P1 finite-element scheme for Maxwell’s equations in a polygon with variable permittivity away from its boundary, arXiv:1905.03619
The forward problem is the Maxwell’s equations for the electric field \( e = (e_1, e_2, e_3) \) in a bounded domain \( \Omega \) of \( \mathbb{R}^3 \) with boundary \( \partial \Omega \)

\[
\varepsilon \partial_{tt} e + \nabla \times \nabla \times e = 0 \quad \text{in } \Omega \times (0, T),
\]

\[
e(\cdot, 0) = e_0(\cdot), \text{ and } \partial_t e(\cdot, 0) = e_1(\cdot) \quad \text{in } \Omega,
\]

\[
\partial_n e = -\partial_t e \quad \text{on } \partial \Omega \times (0, T),
\]

\[
\nabla \cdot (\varepsilon e) = 0 \quad \text{in } \Omega.
\]

(2)

Then requiring that \( e|_{t=0} = e_0 \) and \( \{\partial_t e\}|_{t=0} = e_1 \), we write for all \( v \in [H^1(\Omega)]^3 \),

\[
(\partial_{tt} e, v)_\varepsilon + (\nabla e, \nabla v) + (\nabla \cdot e, \nabla \cdot v) - (\nabla \cdot e, \nabla \cdot v) + (\partial_t e, v)_{\partial \Omega} = 0 \quad \forall t \in (0, T).
\]

(3)

Problem (3) is equivalent to Maxwell’s equations (2), see [BR].

Let $V_h$ be the usual $P_1$ FE-space of continuous functions related to a mesh $T_h$ fitting $\Omega$, consisting of triangles with maximum edge length $h$, belonging to a quasi-uniform family of meshes. Setting $V_h := [V_h \cap H^1_0(\Omega)]^2$ we define $e_{0h}$ (resp. $e_{1h}$) to be the usual $V_h$-interpolate of $e_0$ (resp. $e_1$). Then the semi-discretized problem is: 

\begin{align*}
\text{Find } e_h \in V_h \text{ such that } \forall v \in V_h \\
(\partial_{tt} e_h, v)_{\varepsilon} + (\nabla e_h, \nabla v) + (\nabla \cdot [\varepsilon e_h], \nabla \cdot v) \\
- (\nabla \cdot e_h, \nabla \cdot v) + (\partial_t e_h, v)_{\partial \Omega} = 0, \\
e_h(\cdot, 0) = e_{0h}(\cdot) \text{ and } \partial_t e_h(\cdot, 0) = e_{1h}(\cdot) \text{ in } \Omega.
\end{align*}

(4)
Given a number $N$ of time steps we define the time increment $\tau := T/N$. Then we approximate $e_h(k\tau)$ by $e_h^k \in V_h$ for $k = 1, 2, \ldots, N$ according to the following scheme for $k = 1, 2, \ldots, N - 1$:

\[
\begin{aligned}
\left( \frac{e_h^{k+1} - 2e_h^k + e_h^{k-1}}{\tau^2}, v \right) + (\nabla e_h^k, \nabla v) + (\nabla \cdot e_h^k, \nabla \cdot v) - (\nabla \cdot e_h^k, \nabla \cdot v) &
\end{aligned}
\]

\[
\begin{aligned}
+ \left( \frac{e_h^{k+1} - e_h^{k-1}}{2\tau}, v \right)_{\partial \Omega} &= 0 \quad \forall v \in V_h,
\end{aligned}
\]

$e_h^0 = e_{0h}$ and $e_h^1 = e_h^0 + \tau e_{1h}$ in $\Omega$.

(5)

$e_h^{k+1}$ cannot be determined explicitly by (5) at every time step. In order to enable an explicit solution we use the classical mass-lumping technique.
For a constant $\varepsilon$ this consists of replacing on the left hand side of (5) the inner product $(\mathbf{u}, \mathbf{v})_\varepsilon$ by an inner product $(\mathbf{u}, \mathbf{v})_{\varepsilon,h}$, using the trapezoidal rule to compute the integral of $\int_K \varepsilon \mathbf{u}_|K \cdot \mathbf{v}_|K \, dx$ (resp. $\int_{K \cap \partial \Omega} \mathbf{u}_|K \cdot \mathbf{v}_|K \, dS$), for every element $K$ in $\mathcal{T}_h$, where $\mathbf{u}$ stands for $\mathbf{e}_{h}^{k+1} - 2\mathbf{e}_{h}^{k} + \mathbf{e}_{h}^{k-1}$.

In this case the matrix associated with $(\varepsilon \mathbf{e}_{h}^{k+1}, \mathbf{v})_h$ for $\mathbf{v} \in \mathbf{V}_h$, is a diagonal matrix.

In our case $\varepsilon$ is not constant, but the same property will hold if we replace in each element $K$ the integral of $\varepsilon \mathbf{u}_|K \cdot \mathbf{v}_|K$ in a triangle $K \in \mathcal{T}_h$ as follows:

$$\int_K \varepsilon \mathbf{u}_|K \cdot \mathbf{v}_|K \, dx \approx \varepsilon(G_K) \text{area}(K) \sum_{i=1}^{3} \frac{\mathbf{u}(S_K,i) \cdot \mathbf{v}(S_K,i)}{3},$$

where $S_{K,i}$ are the vertexes of $K$, $i = 1, 2, 3$, $G_K$ is the centroid of $K$. 
Convergence results

Now we assume that $\tau$ satisfies the following CFL-condition:

$$\tau \leq h/\nu \text{ with } \nu = C(1 + 3\|\varepsilon - 1\|_{\infty})^{1/2}$$

where $C$ is a mesh-independent constant. Assume that the solution of Maxwell's equations (2) is such that $\mathbf{e} \in [H^4\{\Omega \times (0, T)\}]^2$.

Provided the CFL condition (6) holds and $\tau \leq 1/2|\eta|$, $\eta := 2 + |\varepsilon|_{1,\infty} + 2|\varepsilon|_{2,\infty}$, there exists a constant $C$ depending only on $\Omega$, $\varepsilon$ and $T$ such that,

$$\max_{1 \leq m \leq N-1} \left\|\partial_t (\mathbf{e}_h - \mathbf{e})^{m+1/2}\right\| + \max_{2 \leq m \leq N} \left\|\nabla (\mathbf{e}_h^m - \mathbf{e}^m)\right\|$$

$$\leq C(\tau + h + h^2/\tau) \left\{ \|\mathbf{e}\|_{H^4[\Omega \times (0, T)]} + |\mathbf{e}_0|_2 + |\mathbf{e}_1|_2 \right\}.$$  

(7) means that, as long as $\tau$ varies linearly with $h$, it holds the first order convergence of the FEM P1 scheme in terms of either $\tau$ or $h$.

*L. Beilina, V. Ruas, Numerical validation of an explicit P1 finite-element scheme for Maxwell’s equations in a polygon with variable permittivity away from its boundary, arXiv:1905.03619*
Our goal is to find $\varepsilon_r$ by minimizing the Tikhonov functional:

$$F(\varepsilon_r) = F(E, \varepsilon_r) := \frac{1}{2} \int_{S_T} (E - \tilde{g})^2 z_\delta(t) dx dt + \frac{1}{2} \gamma \int_{G} (\varepsilon_r - \varepsilon_{r,\text{glob}})^2 \, dx,$$

where $\gamma > 0$ is the regularization parameter, and $\varepsilon_{r,\text{glob}}(x)$ is the computed coefficient via the globally convergent method.

We can’t get better accuracy of the solution than $\delta$. This means that all other parameters in the reg. process are much larger than $\delta$. Let $\mu \in (0, 1)$.

Since $\lim_{\delta \to 0} \left( \frac{\delta^{2\mu}}{\delta^2} \right) = \infty$, then there exists a sufficiently small number $\delta_0(\mu) \in (0, 1)$ such that $\delta^{2\mu} > \delta^2, \forall \delta \in (0, \delta_0(\mu))$. Hence, we choose

$$\gamma(\delta) = \delta^{2\mu}, \mu \in (0, 1).$$
Minimization is performed via Lagrangian
\[ L(E, \lambda, \varepsilon_r, \varepsilon_r^g) = F(E, \varepsilon_r) + \int_{\Omega_T} \lambda(\varepsilon_r \frac{\partial^2 E}{\partial t^2} + \nabla(\nabla \cdot E) - \nabla \cdot (\nabla E) - s \nabla(\nabla \cdot (\varepsilon_r E)))dxdt. \]

Then we search for a stationary point \( w \in U^1 \) such that
\[ L'(w)(\overline{w}) = 0, \quad \forall \overline{w} \in U^1, \]
\[ U^1 = H_{E}^1(\Omega_T) \times H_{\lambda}^1(\Omega_T) \times B(\Omega), \]

where \( B(\Omega) \) is the space of functions bounded on \( \Omega \) with the norm
\[ \|f\|_{B(\Omega)} = \sup_{\Omega} |f|. \]

To find the Fréchet derivative \( L'(w) \), we consider
\[ L(w + \overline{w}) - L(w), \quad \forall \overline{w} \in U^1 \] and single out the linear, with respect to \( \overline{w} \), part to get for \( x \in \Omega \)
\[ L'(w)(x) = \gamma(\varepsilon_r - \varepsilon_r^g)(x) - \int_{0}^{T} \frac{\partial \lambda}{\partial t} \frac{\partial E}{\partial t}(x, t) dt + s \int_{0}^{T} (\nabla \cdot E)(\nabla \cdot \lambda)(x, t) dt. \]
We present a posteriori error estimate for three kinds of errors:

- For the error $|L(u) - L(u_h)|$ in the Lagrangian with $u = (E, \lambda, \varepsilon_r), u_h = (E_h, \lambda_h, \varepsilon_h)$ [BJ1, B, BJ2].
- For the error $|F(\varepsilon_r) - F(\varepsilon_h)|$ in the Tikhonov functional [BOOK].
- For the error $|\varepsilon_r - \varepsilon_h|$ in the reg. solution of this functional. [BOOK].

For (1) and (2) we have:

$$L(u) - L(u_h) = L'(u_h)(u - u_h) + R(u, u_h),$$
$$F(\varepsilon_r) - F(\varepsilon_h) = F'(\varepsilon_h)(\varepsilon_r - \varepsilon_h) + R(\varepsilon_r, \varepsilon_h),$$  \hspace{1cm} (11)

where $R(u, u_h), R(\varepsilon_r, \varepsilon_h)$ are the second order remainders terms. We assume that $\varepsilon_h$ is located in the small neighborhood of the regularized solution $\varepsilon_r$. Thus, the terms $R(u, u_h), R(\varepsilon_r, \varepsilon_h)$ are small and we can neglect them.


We use the Galerkin orthogonality principle

\[ L'(u_h)(\bar{u}) = 0 \quad \forall \bar{u} \in U_h, \]
\[ F'(\varepsilon_h)(b) = 0 \quad \forall b \in V_h, \]  

(12)

together with the splitting

\[ u - u_h = (u - u_h^l) + (u_h^l - u_h), \]
\[ \varepsilon_r - \varepsilon_h = (\varepsilon_r - \varepsilon_h^l) + (\varepsilon_h^l - \varepsilon_h), \]  

(13)

where \( u_h^l \in U_h \) is the interpolant of \( u \), and \( \varepsilon_h^l \in V_h \) is the interpolant of \( \varepsilon_r \), and get the following error representation:

\[ L(u) - L(u_h) \approx L'(u_h)(u - u_h^l), \]
\[ F(\varepsilon_r) - F(\varepsilon_h) \approx F'(\varepsilon_h)(\varepsilon_r - \varepsilon_h^l). \]  

(14)

In a posteriori error estimate (14)

- Terms \( L'(u_h) \) and \( F'(\varepsilon_h) \) represents residuals.
- Terms \( u - u_h^l \) and \( \varepsilon_r - \varepsilon_h^l \) are weights.
The two-stage numerical procedure for solution of CIP

**Stage 1.** Approximately globally convergent numerical method provides a good approximation for the exact solution $\varepsilon_{glob}$.

**Stage 2.** Adaptive Finite Element Method refines it via minimization of the corresponding Tikhonov functional with $\varepsilon_0 = \varepsilon_{glob}$:

$$J(u, \varepsilon) = \frac{1}{2} \int_\Gamma \int_0^T (u - \tilde{u})^2 z_\delta(t) ds dt + \frac{1}{2} \gamma \int_\Omega (\varepsilon - \varepsilon_0)^2 \, dx. \quad (15)$$

where $\tilde{u}$ is the observed wave field in the model PDE (for example, acoustic wave equation), $u$ satisfies this model PDE and thus depends on $\varepsilon$. 

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In 2008-2013 a new approach have been developed: approximately globally convergent numerical method for CIPs for a hyperbolic and parabolic PDE [BOOK,BK]. This method uses idea of the convexification method of [KT].

This approach works with single measurement data.
- **Single measurement** means that either only a single position of the point source or only a single direction of the incident plane wave is considered.


We have verified the two-stage numerical procedure on experimental data used in military applications.

Non-local reconstruction methods were developed in the past only for the case of multiple measurements: Belishev, Kabanikhin, Isaacson, Mueller, Novikov, Siltanen.
The two-stage numerical procedure in the reconstruction of dielectrics from experimental data

a) The rectangular prism depicts our computational domain Ω. Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. The signal was measured with the time interval 20 picoseconds with total time 12.3 nanoseconds. b) Schematic diagram of locations of detectors on the bottom side of the prism Ω. The distance between neighboring detectors was 10 mm.


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The two-stage numerical procedure

**Stage 1.** Approximately globally convergent numerical method provides a good approximation for the exact solution.

**Stage 2.** Adaptive Finite Element Method refines it.

\[ \varepsilon_r^{(5,2)} = 3.9, \quad n^{(5,2)} = 1.97 \]

\[ \varepsilon_r, h \approx 4.2, \quad n_{glob} = \sqrt{\varepsilon_r, h} \approx 2.05 \]

a) A sample of the reconstruction result of the dielectric cube No. 1 (4 cm side) via the first stage. b) Result after applying the adaptive stage (2-nd stage). The side of the cube is 4 cm=1.33 wavelength.
Results of the two-stage procedure, cube nr.2 (big)

\[ \varepsilon_r(5, 5) = 3.19, \quad n(5, 5) = 1.79 \]

\[ \varepsilon_{r, h} \approx 3.0, \quad n_{\text{glob}} = \sqrt{\varepsilon_{r, h}} \approx 1.73 \]

a) Reconstruction of the dielectric cube No. 2 (6 cm side) via the first stage.  
b) The final reconstruction result after applying the adaptive stage (2-nd stage). The side 6 cm=2 wavelength.

We use time-dependent experimental backscattered data measured at the Optoelectronics and Optical Communications Center at UNCC, USA.

Our goal in experimental verification was to reconstruct different dielectric and metallic targets. For metallic targets we determine the effective or appearing dielectric constant such that

$$\varepsilon_r \text{ (metal)} \in [10, 30]. \quad (16)$$

for metals. The set of admissible coefficients for the function $\varepsilon_r(x)$ in $\Omega$ is

$$M_{\varepsilon_r} = \{\varepsilon_r(x) : \varepsilon_r(x) \in [1, 25], \varepsilon_r(x) = 1 \ \forall x \in \mathbb{R}^3 \setminus \Omega.\}$$

We compute refractive indexes $n^{\text{comp}}$ of inclusions as

$$\varepsilon_r^{\text{comp}} = \max_{\Omega} \varepsilon_r \left(\frac{N}{\Omega}\right)(x), \ n^{\text{comp}} = \sqrt{\varepsilon_r^{\text{comp}}}. \quad (17)$$
Data collection scheme

a) A photograph explaining our data collection process. The distance between the target (wooden block) and the measurement plane is about 0.8 m, which is about 26 wave lengths. b) Picosecond Pulse Generator which generates electric pulses. It produced one component of the electrical field with the wavelength 0.03 meter every 10 picoseconds while tektronix oscilloscope registered backscattered data. The pulse goes to the transmitter which is a horn antenna (source). c) Detected signal is recorded by Textronix Oscilloscope which produces a digitized time resolved signal with step size in time 10 picoseconds (10 \( \times \) 10\(^{-12}\) sec.). The total time of measurements for one pulse is 10 nanoseconds (10 \( \times \) 10\(^{-9}\) sec) = 10\(^4\) picoseconds = 10\(^{-8}\) seconds which corresponds to 1000 timesteps.
Experimental verification of the two-stage numerical procedure. Names of targets

<table>
<thead>
<tr>
<th>Target number</th>
<th>Specification of the target</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a piece of oak, rectangular prism</td>
</tr>
<tr>
<td>2</td>
<td>a piece of pine</td>
</tr>
<tr>
<td>3</td>
<td>a metallic sphere</td>
</tr>
<tr>
<td>4</td>
<td>a metallic cylinder</td>
</tr>
<tr>
<td>5</td>
<td>a piece of oak</td>
</tr>
<tr>
<td>6</td>
<td>a metallic rectangular prism</td>
</tr>
<tr>
<td>7</td>
<td>a wooden doll, air inside, heterogeneous target</td>
</tr>
<tr>
<td>8</td>
<td>a wooden doll, metal inside, heterogeneous target</td>
</tr>
<tr>
<td>9</td>
<td>a wooden doll, sand inside, heterogeneous target</td>
</tr>
<tr>
<td>Target number</td>
<td>1</td>
</tr>
<tr>
<td>---------------</td>
<td>-------</td>
</tr>
<tr>
<td><strong>Measured n, error</strong></td>
<td>2.11, 19%</td>
</tr>
<tr>
<td><strong>n in glob.conv, error</strong></td>
<td>1.92, 9%</td>
</tr>
<tr>
<td><strong>n, coarse mesh, error</strong></td>
<td>1.94, 8%</td>
</tr>
<tr>
<td><strong>n, 1 time ref. mesh, error</strong></td>
<td>1.94, 8%</td>
</tr>
<tr>
<td><strong>n, 2 times ref.mesh, error</strong></td>
<td>1.84, 0%</td>
</tr>
<tr>
<td><strong>n, 3 times ref.mesh, error</strong></td>
<td>1.89, 0%</td>
</tr>
</tbody>
</table>

**Table:** Computed \(n(ta\text{rge})\) and directly measured refractive indices of dielectric targets together with both measurement and computational errors as well as the average error.

<table>
<thead>
<tr>
<th>Target number</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varepsilon_r(ta\text{rge})) of glob.conv.</td>
<td>14.4</td>
<td>15.0</td>
<td>25</td>
<td>13.6</td>
</tr>
<tr>
<td>(\varepsilon_r(ta\text{rge})) coarse mesh</td>
<td>14.4</td>
<td>17.0</td>
<td>25</td>
<td>13.6</td>
</tr>
<tr>
<td>(\varepsilon_r(ta\text{rge})) 1 time ref.mesh</td>
<td>14.5</td>
<td>17.0</td>
<td>25</td>
<td>13.6</td>
</tr>
<tr>
<td>(\varepsilon_r(ta\text{rge})) 2 times ref.mesh</td>
<td>14.6</td>
<td>17.0</td>
<td>25</td>
<td>13.7</td>
</tr>
<tr>
<td>(\varepsilon_r(ta\text{rge})) 3 times ref.mesh</td>
<td>14.6</td>
<td>17.0</td>
<td>25</td>
<td>14.0</td>
</tr>
<tr>
<td>(\varepsilon_r(ta\text{rge})) 4 times ref.mesh</td>
<td>14.6</td>
<td>17.0</td>
<td>17.0</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Computed appearing dielectric constants \(\varepsilon_r(ta\text{rge})\) of metallic targets with numbers 3,4,6 as well as of target number 8 which is a metal covered by a dielectric.
Initial guesses $\varepsilon_0 = \varepsilon_{glob}$ obtained on the first stage

a) piece of oak  
b) metallic sphere  
c) metallic cylinder  
d) wooden doll, air  
e) wooden doll, metall  
f) wooden doll, sand

**Figure:** Reconstructions of some targets obtained on the first stage of our two-stage numerical procedure.
Reconstruction using AFEM

Figure: Reconstructions of some targets obtained on the second stage via adaptive finite element method.

a) piece of oak  b) metallic sphere  c) metallic cylinder

d) wooden doll, air  e) wooden doll, metall  f) wooden doll, sand
Reconstruction using AFEM (zoomed views)

a) piece of oak  
b) metallic sphere  
c) metallic cylinder  
d) wooden doll, air  
e) wooden doll, metall  
f) wooden doll, sand

**Figure:** Reconstructions of some targets obtained on the second stage via adaptive finite element method.
Imaging of objects buried in a dry sand

Figure: (a): Experimental setup; (b) Schematic diagram of our data acquisition.

L. Beilina, Nguyen Trung Thành, M. V. Klibanov and J. B. Malmberg, Reconstruction of shapes and refractive indices from backscattering experimental data using the adaptivity, Inverse Problems, 30, 105007, 2014
Table: Result of the globally convergent algorithm: the refractive indices \( n = \sqrt{\epsilon} \) and the burial depths of non-metallic targets. Obj.11: N/A: diffuse scattering.

<table>
<thead>
<tr>
<th>Object #</th>
<th>Material</th>
<th>Comp. depth</th>
<th>Exact depth</th>
<th>Comp. ( n ) Test 1</th>
<th>Comp. ( n ) Test 2</th>
<th>Measured ( n(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Water</td>
<td>3.6</td>
<td>4.0</td>
<td>4.7</td>
<td>4.9</td>
<td>4.88</td>
</tr>
<tr>
<td>4</td>
<td>Wet wood</td>
<td>5.5</td>
<td>9.8</td>
<td>4.4</td>
<td>4.5</td>
<td>4.02</td>
</tr>
<tr>
<td>9</td>
<td>Teflon</td>
<td>2.9</td>
<td>2.5</td>
<td>1.0</td>
<td>1.18</td>
<td>1.0</td>
</tr>
<tr>
<td>10</td>
<td>Ceramic</td>
<td>4.0</td>
<td>5.0</td>
<td>1.0</td>
<td>1.23</td>
<td>1.39</td>
</tr>
<tr>
<td>11</td>
<td>Wood with metal screws</td>
<td>4.6</td>
<td>4.0</td>
<td>1.0</td>
<td>1.46</td>
<td>1.89 (wood) N/A</td>
</tr>
<tr>
<td>13</td>
<td>Rock</td>
<td>2.0</td>
<td>2.3</td>
<td>1.0</td>
<td>1.34</td>
<td>1.34</td>
</tr>
<tr>
<td>14</td>
<td>Coffee grounds</td>
<td>2.0</td>
<td>2.5</td>
<td>1.0</td>
<td>1.46</td>
<td>1.11</td>
</tr>
<tr>
<td>15</td>
<td>Ceramic</td>
<td>2.6</td>
<td>2.5</td>
<td>1.0</td>
<td>1.51</td>
<td>1.39</td>
</tr>
<tr>
<td>19</td>
<td>Water</td>
<td>7.5</td>
<td>9.5</td>
<td>4.5</td>
<td>5.2</td>
<td>4.88</td>
</tr>
<tr>
<td>22</td>
<td>Wet wood</td>
<td>2.9</td>
<td>3.0</td>
<td>4.8</td>
<td>5.3</td>
<td>4.02</td>
</tr>
<tr>
<td>23</td>
<td>Wet wood</td>
<td>5.7</td>
<td>7.5</td>
<td>4.0</td>
<td>4.1</td>
<td>4.02</td>
</tr>
<tr>
<td></td>
<td>Empty bottle</td>
<td>missed</td>
<td>missed</td>
<td>missed</td>
<td>missed</td>
<td>1.0</td>
</tr>
<tr>
<td>24</td>
<td>Wet wood</td>
<td>5.1</td>
<td>6.8</td>
<td>3.67</td>
<td>3.0</td>
<td>4.02</td>
</tr>
</tbody>
</table>
Reconstructions of superresolution objects

Figure: The resolution limit which follows from the Born approximation, i.e. the diffraction limit, is $\lambda/2$, where $\lambda$ is the wavelength of the signal. In our experimental device $\lambda = 4.5$ centimeters (cm). We have resolved two targets at the distance of $1 \text{ cm} = \lambda/4.5$ between their surfaces. At the same time, the backscattering signal was measured at the distance of about $80 \text{ cm} \approx 18$ wavelengths off the targets, i.e. in the far field zone. Superresolution can occur because of nonlinear scattering.

The experimental setup for back-scattered data collected in the field by a radar of the US Army Research Laboratory

1. Radar collects data
2. Process data to form 2D image
3. Extract out image chip of likely target for analysis
4. Take a downrange cut through image
5. Downrange cut produces a single time waveform
6. Use this waveform to estimate dielectric contrast

Quantitative Reconstruction of Small Inclusions

a) \( \tilde{\Omega} = \Omega_{\text{FEM}} \cup \Omega_{\text{FDM}} \)

Comp. set-up: \( \tilde{\Omega} = (-0.8, 0.8)^3 \), and \( \Omega_{\text{FEM}} = (-0.7, 0.7)^3 \) (scale in decimeters), the mesh size \( h_0 = 0.05 \), reg. parameter = \( \gamma = 0.01 \), time interval \( T = [0, 3] \), time step \( \tau = 0.006 \), noise \( \sigma = 3\% \), 10\% in data, \( \omega = 40 \) in (19). We initialize only \( E_2 \) in \( (E_1, E_2, E_3) \) on \( S_T \)

\[
f(t) = \begin{cases} 
\sin (\omega t) & \text{if } 0 < t < 2\pi / \omega, \\
0 & \text{if } t > 2\pi / \omega.
\end{cases}
\]  

(18)

Test 1: The goal of this numerical test is to reconstruct a smooth function $\varepsilon$ only inside $\Omega_{FEM}$. We define this function for $x \in \Omega_{FEM}$ as

$$
\varepsilon(x) = 1.0 + 1.0e^{-|x-x_1|^2/0.2} + 1.0e^{-|x-x_2|^2/0.2},
$$

$$
x_1 = (0.3, 0.0, 0.0) \in \Omega_{FEM}, x_2 = (-0.4, 0.2, 0.0) \in \Omega_{FEM}.
$$

Test 2: In this test we reconstruct three small inclusions of diameter $d = 2$ mm with the centers of the inclusions at $(-0.3, 0.0, -0.25)$, $(0.3, 0.2, -0.25)$ and $(0.3, -0.2, -0.25)$, respectively, and $\varepsilon = 2.0$ inside the inclusions.

Test 3: In this test we reconstruct four small inclusions of diameter $d = 2$ mm with the centers of the inclusions at $(-0.3, 0.0, 0.25)$, $(0.0, 0.2, 0.25)$, $(0.0, -0.2, -0.25)$, and $(0.3, -0.2, -0.25)$, respectively, and $\varepsilon = 2.0$ inside the inclusions.

Test 4: The inclusions of this test are the same as in Test 3, but here the data consists of measurements of two backscattered wave fields: one backscattered field initiated at the front boundary $\partial_1\tilde{\Omega}$, and another one at the back boundary $\partial_2\tilde{\Omega}$. 
Table 1. *Results obtained on the coarse mesh.* We present reconstructions of the maximal contrast $\tilde{\varepsilon} = \max_{\Omega_{FEM}} \varepsilon_{h_0}^{M_0}$ together with computational errors in percents. Here, $M_0$ is the final number of iteration in CGM on the coarse mesh.

<table>
<thead>
<tr>
<th>$\sigma = 3%$</th>
<th>$\tilde{\varepsilon}$</th>
<th>er., %</th>
<th>$M_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>1.93</td>
<td>3.5</td>
<td>2</td>
</tr>
<tr>
<td>Test 2</td>
<td>2.94</td>
<td>47</td>
<td>2</td>
</tr>
<tr>
<td>Test 3</td>
<td>1.77</td>
<td>11.5</td>
<td>2</td>
</tr>
<tr>
<td>Test 4</td>
<td>1.9</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma = 10%$</th>
<th>$\tilde{\varepsilon}$</th>
<th>er., %</th>
<th>$M_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>1.94</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Test 2</td>
<td>2.81</td>
<td>40.5</td>
<td>2</td>
</tr>
<tr>
<td>Test 3</td>
<td>2.04</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Test 4</td>
<td>2.03</td>
<td>1.5</td>
<td>2</td>
</tr>
</tbody>
</table>

Results on the adaptively refined meshes

Table 2. Results obtained on \( k_{\text{rec}} \) times adaptively refined mesh. We present reconstructions of the maximal contrast \( \tilde{\varepsilon} = \max_{\Omega_{\text{FEM}}} \varepsilon_{\text{rec}} \) together with computational errors in percents.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \tilde{\varepsilon} )</th>
<th>er., %</th>
<th>( M_{k_{\text{rec}}} )</th>
<th>( k_{\text{rec}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>2.04</td>
<td>2</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Test 2</td>
<td>1.99</td>
<td>0.5</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Test 3</td>
<td>1.55</td>
<td>22.5</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Test 4</td>
<td>1.9</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>( \tilde{\varepsilon} )</th>
<th>er., %</th>
<th>( M_{k_{\text{rec}}} )</th>
<th>( k_{\text{rec}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>1.97</td>
<td>1.5</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Test 2</td>
<td>1.92</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Test 3</td>
<td>1.88</td>
<td>6</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Test 4</td>
<td>2.15</td>
<td>7.5</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>

Test 1: isosurfaces of rec. on a coarse mesh, $\sigma = 10\%$.

\[ x \in \Omega_{\text{FEM}} : \varepsilon_h(x) = 1.2 \quad x \in \Omega_{\text{FEM}} : \varepsilon_h(x) = 1.5 \quad x \in \Omega_{\text{FEM}} : \varepsilon_h(x) = 1.8 \]
Finally refined mesh

$x_1x_2$-view

$x_1x_3$-view

$x_2x_3$-view

Five times adaptively refined mesh when the level of the noise in the data was $\sigma = 10\%$. 
Test 1: isosurfaces of rec. on the 5 ref. mesh, $\sigma = 10\%$.

$$x \in \Omega_{\text{FEM}} : \varepsilon_{h_0}(x) = 1.2 \quad x \in \Omega_{\text{FEM}} : \varepsilon_{h_0}(x) = 1.5 \quad x \in \Omega_{\text{FEM}} : \varepsilon_{h_0}(x) = 1.8$$
Test 4: isosurfaces of rec., $\sigma = 10\%$.

Test 4. Reconstruction obtained with two plane waves. We present reconstruction of the four inclusions of diameter $d = 2\text{mm}$ (in red color) obtained on the coarse mesh (upper figures) and on the two times adaptively refined mesh (lower figures). The level of noise in the data is $\sigma = 10\%$. 

On coarse mesh $x \in \Omega_{\text{FEM}} : \varepsilon_{h_0}(x) = 2.03$

On final mesh $x \in \Omega_{\text{FEM}} : \varepsilon_{\text{rec}}(x) = 2.15$
References


