Introduction

The objective of this book is to give a detailed account of some aspects of the multivariable residue theory that was initiated by the works of Liebermann, Coleff, and Herrera in the mid seventies. The core is to study various questions of ideals of analytic functions, polynomials, etc, by representating ideal sheaves as annahilators of so-called *residue currents*, and the $\bar{\partial}$-equation on nonsmooth analytic spaces.

Currents are analytic objects that in many aspects behave like differential forms, so, e.g., they may fit inte integral formulas. On the other hand currents also have a geometric nature. For instance, closed positive, and more generally normal, currents are natural generalizations of Lelong currents, that can be identified by analytic varieties. Most of the currents that occur in residue theory are *pseudomeromorphic*. Such currents can be seen as generalizations of, possibly non-reduced, varieties, and they share important geometric properties with the normal currents, such as the dimension principle, see below.

In one variable, the local theory is quite simple; in fact, each local ideal is principal, so deciding whether a given function belongs to the local ideal or not at a given point just amounts to checking its vanishing order at the point. Therefore, residue theory in one variable is mainly used for global questions, e.g., to find the value of an integral by summing up all residues of a meromorphic form in an open set. In the several variable case already the local residue theory is challenging; nevertheless as soon as it is accessible, many global questions can be handled as well. For instance, membership problems for polynomial ideals, existence of sections of vector bundles, etc, are discussed. The residue theory also extends to singular spaces, and in that case the local theory is non-trivial already in the one-dimensional case.

Residue theory is intimately related to integral formulas, and indeed, integral formulas plays an important role in this book. For instance, membership in an ideal can often be expressed by an integral formula. There are are, however, a lot of important aspects that are not at all touched upon. For instance there is a deep and close connection to $\mathcal{D}$-module theory that is not discussed.

No previous knowledge of residue theory or integral representation is assumed. However, we use the books of Demailly, [58], Eisenbud, [62, 63], Lazarsfeld, [**?**], Hörmander, [**?**, 67] (complex analys, distribution), Circa, [**?**], and Gunning-Rossi, [65], ??, Griffiths-Harris, ngn med upplosningar, as general references. Only at a few occasions we include a proof that can be found in some of these books; it is only when some idea in the proof will be referred to later on.

1. A glimse of multivariable residue calculus

Let $\mathcal{J} = \mathcal{J}_0$ be an ideal in the local ring $\mathcal{O} = \mathcal{O}_0(\mathbb{C}^n)$. We can always find a finite number of generators, i.e., (germs of) functions $f_1, \ldots, f_p$, such that $\phi \in \mathcal{J}$ if and only if $\phi = \psi_1 f_1 + \cdots + \psi_p f_p$ for some $\psi_j \in \mathcal{O}$. However, for many purposes such a representation of an ideal in terms of a set of generators is not useful. For example, the fact that $\mathcal{J}$ is topologically closed in $\mathcal{O}$ is not at all obvious from this representation; whereas it is immediate if $\mathcal{J}$ is represented as the kernel of a continuous mapping.

Assume that $\mathcal{J}$ is generated by one single function f (not vanishing identically). From the classical theory there is a Schwartz distribution U such that

$$fU = 1. \tag{1.1}$$

Then $R = \bar\partial U$ is a $(0,1)$-current, and since $U = 1/f$ outside the sero set $Z = Z(f)$ of f, it follows that R has support on Z; we say that R is a *residue current*. It has the following important property:

If $\phi \in \mathcal{O}$, then $\phi \in \mathcal{J}$ if and only if $\phi R = 0$[1].

In fact, by (1.1) we have that $\phi R = \phi\bar\partial U = \bar\partial(\phi U)$ so that $\phi R = 0$ if and only if $h := \phi U$ is holomorphic, and in view of (1.1) this holds if and only if $fh = \phi$ for a holomorphic h. Thus we have expressed $\mathcal{J}$ as the *annihilator* of the current R. It is now clear that $\mathcal{J}$ is closed. Notice that U is neither unique nor explicit. However one can define the principal-value current $[1/f]$ as

$$\Big\langle \Big[\frac{1}{f}\Big], \xi \Big\rangle = \lim_{\epsilon\to 0}\int_{|f|^2>\epsilon} \frac{\xi}{f}, \quad \xi \in \mathcal{D}_{n,n}.$$

The existence of this limit is highly non-trivial; the proof relies on the possibility to resolve singularities, Hironaka's theorem. Given the existence it is however clear that $f(1/f) = 1$ and so $\mathcal{J}$ will be the annihilator of $\bar\partial(1/f)$. Notice that

$$\Big\langle \bar\partial\Big[\frac{1}{f}\Big], \xi \Big\rangle = \lim_{\epsilon\to 0}\int_{|f|^2=\epsilon} \frac{\xi}{f}, \quad \xi \in \mathcal{D}_{n,n-1},$$

where the limit is taken over all regular values of $|f|^2$. If g is another generator of $\mathcal{J}$, then $g = af$, where a is non-vanishing, and it turns out, although not at all obvious, that $a\bar\partial(1/g) = \bar\partial(1/f)$; thus this current associated to $\mathcal{J}$ is essentially canonical. As we will see the definition of the principal value current $1/f$ is robust in the sense that any reasonable limit procedure will do. For instance if χ is any smooth approximand of the characteristic function for the interval $[1,\infty)$, and v is any smooth strictly positive function, then

$$\Big\langle \bar\partial\Big[\frac{1}{f}\Big], \xi \Big\rangle = \lim_{\epsilon\to 0}\int \chi(v|f|^2/\epsilon)\frac{\xi}{f}, \quad \xi \in \mathcal{D}_{n,n-1}.$$

For this reason we can unambiguously denote this current simply by $1/f$, and this step to consider it as an object in its own without referring to a particular limit procedure [2], has great notational as well as conceptual advantages and will be of fundamental importance in this book.

If $n = 1$, then the condition $\phi\bar\partial(1/f) = 0$ means that

$$\lim_{\epsilon\to 0}\int_{|f|^2=\epsilon} \frac{\phi h dz}{f}$$

vanishes for all smooth h. If we restrict to holomorphic h, then by Cauchy's theorem we can omit the limit since all the intergrals coincide (as soon as ϵ is small enough), and the meaning then is that

$$\mathrm{R}es_0(\phi h dz/f) = 0$$

for each $h \in \mathcal{O}$. Already this weaker condition on ϕ implies that ϕ/f is holomorphic, i.e., that $\phi \in \mathcal{J}$. A similar weaker formulation exists in the several variable case.

[1] Observe that ϕR is the current R multiplied by the smooth function ϕ.

[2] This is called *reification* in mathematics education.

However, it turns out that for many purposes it is much more convenient to work with the robust current $\bar\partial(1/f)$ rather than the one whose action is only determined on a subspace of all test forms.

A main theme in this book is to discuss analogues for ideals that are not principal. If $\mathcal{J}$ is generated by $f_1, \ldots, f_p$ as before, and $Z = Z(\mathcal{J})$ has codimension p, then one can form the so-called Coleff-Herrera product

$$\mu^f = \bar\partial\Big[\frac{1}{f_p} \wedge \ldots \wedge \bar\partial\frac{1}{f_1}\Big];$$

for various definitions and historical remarks, see Ch 3, Section ??. This current has support on Z and we again have that $\phi\mu^f = 0$ if and only if $\phi \in \mathcal{J}$. It turns out that also μ^f is canonical, i.e., up to a non-vanishing holomorphic function it only depends onthe ideal $\mathcal{J}$. For a general ideal $\mathcal{J}$ we will associate a (vector-valued) current $R^{\mathcal{J}}$ that has support on $Z(\mathcal{J})$ and whose annihilator ideal coincides with J. This current R^J is also *explicit*, in the sense that it is obtained from generators of the ideal and all its syzygies by a limit procedure. Moreover, it fits into integral formulas. Most important are interpolation-division formulas like

$$\phi(z) = f(z) \cdot \int_\zeta A(\zeta, z)\phi(\zeta) + \int_\zeta B(\zeta, z)R^{\mathcal{J}}(\zeta)\phi(\zeta), \tag{1.2}$$

where A and B are kernels that are holomorphic in z; for convenience we here make use of the analytic side of currents and write integrals in (1.2) although formally it is actually currents acting on test forms. In particular, if ϕ is in the ideal $\mathcal{J}$, then the second term vanishes so (1.2) indeed provides a realization of the membership. In a sense that will be made precise in Ch 4 the current $R^{\mathcal{J}}$ is also essentially *unique*.

In many situations it is convenient to consider a residue current R^f obtained from the generators f_j of $\mathcal{J}$ whose annihilator ideal is at least contained in $\mathcal{J}$ and with the advantage that it is much simpler and more explicit than R^J. In such a case we still have a representation like (1.2).

As suggested above, in the multivariable residue theory much attention is paid to the question to decide whether a given current μ (like $\phi R^{\mathcal{J}}$) vanishes, which is a purely local question. To this end we have some basic tools: Most currents that occur are *pseudomeromorphic*. The sheaf $\mathcal{PM}$ of pseudomeromorphic currents has several useful properties. It is closed under $\bar\partial$; if V is a subvariety of X and μ is psudomeromorphic, then the natural restriction of μ to the open set $X \setminus V$ has a natural extension to X that we denote $\mathbf{1}_{X\setminus V}\mu$ and this current is again pseudomeromorphic. It follows that

$$\mathbf{1}_V := \mu - \mathbf{1}_{X\setminus V}\mu$$

has support on V. In this way μ can be put into pieces, and the vanishing of μ can be proved in different ways on different pieces. The geometric nature of pseudomeromorphic currents is reflected by the *dimension principle* (a similar principle holds for positive closed (or normal) (q, q)-currents):

If μ is pseudomeromorphic has bidegree $(*, q)$ is pseudomeromorphic and has support on an analytic subvariety of codimension larger than q, then $\mu = 0$.

Multiplication by $\mathbf{1}_V$ should be considered as an operatation $\mu \mapsto \mathbf{1}_V\mu$ on the sheaf $\mathcal{PM}$ and $\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V\cap W}\mu = \mathbf{1}_W\mathbf{1}_V\mu$. We also have some other important operatations on $\mathcal{PM}$: Given a pseudomeromorphic current μ and a holomorphic function f we define new pseudomeromorphic currents

$$\Big[\frac{1}{f}\Big]\mu := \frac{|f|^{2\lambda}}{f}\wedge\mu\big|_{\lambda=0}, \quad \bar\partial\Big[\frac{1}{f}\Big]\wedge\mu := \frac{\bar\partial|f|^{2\lambda}}{f}\wedge\mu\big|_{\lambda=0}, \tag{1.3}$$

of course the existence of the necessary analytic continuations is part of a theorem, and then the Leibniz rules

$$\bar\partial\Big(\Big[\frac{1}{f}\Big]\mu\Big) = \bar\partial\Big[\frac{1}{f}\Big]\wedge\mu + \Big[\frac{1}{f}\Big]\bar\partial\mu, \quad \bar\partial\Big(\bar\partial\Big[\frac{1}{f}\Big]\wedge\mu\Big) = -\bar\partial\Big[\frac{1}{f}\Big]\wedge\bar\partial\mu,$$

hold. Again, as in the case for the simple principal value current $1/f$ above, any reasonable limit procedure can be used here, so these operations are robust.

Although we write the expressions in (1.3) as multiplications, and sometimes think of them in this way, formally they are operators acting on μ. Thus if f and g are two holomorphic functions then in general

$$\bar\partial\Big[\frac{1}{g}\Big]\wedge\bar\partial\Big[\frac{1}{f}\Big] \tag{1.4}$$

will change (by more than a minus sign) if f and g are interchanged. For instance, one can verify that

$$\bar\partial\Big[\frac{1}{zw}\Big]\wedge\bar\partial\Big[\frac{1}{z}\Big] = 0,$$

wheras

$$\bar\partial\Big[\frac{1}{z}\Big]\wedge\bar\partial\Big[\frac{1}{zw}\Big] = \bar\partial\Big[\frac{1}{z}\Big]\wedge\bar\partial\Big[\frac{1}{w}\Big] \neq 0.$$

If f and g form a complete intersection, i.e., $\operatorname{codim}\{f = g = 0\}$ is 2, then (1.4) just changes sign when f and g are interchanged. Let us prove this and at the same time illustrate the usefulness of the dimension principle: It is not hard to see that $\alpha\bar\partial[1/f] = \bar\partial[1/f]\cdot\alpha$ if α is a smooth function. Therefore the pseudomeromorphic current

$$\mu = \Big[\frac{1}{g}\Big]\bar\partial\Big[\frac{1}{f}\Big] - \bar\partial\Big[\frac{1}{f}\Big]\cdot\Big[\frac{1}{g}\Big]$$

vanishes outside the zero set of g. However, μ s certainly has support on the zero set of f, so its support is in fact contained in a variety of codimension 2. By the dimension principle therefore $\mu = 0$. Applying Leibniz' rules we find that

$$\bar\partial\Big[\frac{1}{g}\Big]\wedge\bar\partial\Big[\frac{1}{f}\Big] + \bar\partial\Big[\frac{1}{f}\Big]\wedge\bar\partial\Big[\frac{1}{g}\Big] = 0.$$

2. Summary of content

In Ch.1 we discuss integral representation in general, but specifically focused on constructions for applications in later chapters.

In Ch.2 we discuss basic limit procedures in residue theory. We introduce the sheaf $\mathcal{PM}$ of pseudomeromorphic currents. We discuss the Coleff-Herrera product, the somewhat more general notion of Coleff-Herrera currents, a certain uniqueness property. The chapter contains a quite long discussion about the history and various definitions of the Coleff-Herrera product.

For the construction of more general currents we need the concept of super structure. Chapter 3 contains ...... fundamental principle, etc etc etc

Ch 4 Division problems (and interpolation).

Ch 5 Interpolation-division integral formulas.

Ch 6 The $\bar\partial$-equation on a reduced analytic space.

Chapter 1

Some prerequisites

In this chapter we collect some results that will be used throughout this book and which are not so easily accessible in the literature.

1. Functional calculus for forms of even degree

Let E be an m-dimensional vector space and recall that $\Lambda^k E$ consists of all alternating multilinear forms on the dual space E^*. If $v \in E^*$ we define contraction (or interior multiplication) with v, $\delta_v \colon \Lambda^{k+1}E \to \Lambda^k E$, by

$$(\delta_v \omega)(u_1, \ldots, u_k) = \omega(v, u_1, \ldots, u_k).$$

It is readily checked that this is an alternating form and therefore an element in $\Lambda^k E$. Clearly δ_v is complex-linear in v.

To get a more hands-on idea how δ_v acts, let us choose a basis e_j for E, with dual basis e_j^*, such that $v = e_1^*$. Then $\delta_v(e_1 \wedge e_J) = e_J$ if $1 \notin J$. Thus

$$\delta_v(\alpha \wedge \beta) = \delta_v \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \delta_v \beta, \tag{1.1}$$

if $\alpha = e_J$ and $\beta = e_K$. By linearity, then (1.1) holds for arbitrary forms. One says that δ_v is an *anti-derivation*.

Now let $\omega_1, \ldots, \omega_m$ be even forms, i.e., in $\oplus_\ell \Lambda^{2\ell} E$, and let $\omega_j = \omega_j' + \omega_j''$ be the decomposition in components of degree zero and positive degree, respectively. Notice that $\wedge$ is commutative for even forms. Thus if $p(z) = \sum_\alpha c_\alpha z^\alpha = \sum_\alpha c_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ is a polynomial, then we have a natural definition of $p(\omega)$ as the form $\sum_\alpha c_\alpha \omega_1^{\alpha_1} \wedge \ldots \wedge \omega_m^{\alpha_m}$. However, it is often convenient to use more general holomorphic functions.

Now $\omega' = (\omega_1', \ldots, \omega_m')$ is a point in $\mathbb{C}^m$ and for f holomorphic in some neighborhood of ω' we define

$$f(\omega) = \sum_\alpha f^{(\alpha)}(\omega')(\omega'')_\alpha, \tag{1.2}$$

where we use the convention that

$$w_\alpha = \frac{w_1^{\alpha_1} \wedge \ldots \wedge w_m^{\alpha_m}}{\alpha_1! \cdots \alpha_m!}.$$

Thus $f(\omega) = f(\omega' + \omega'')$ is defined as the formal power series expansion at the point ω'. Since the sum is finite, $f(\omega)$ is a well-defined form, and if ω depends continuously (smoothly, holomorphically) on some parameter(s), $f(\omega)$ will do as well.

If $f(z) - g(z) = \mathcal{O}((z - \omega')^M)$ for a large enough M, then $f(\omega) = g(\omega)$.

Lemma 1.1. *Suppose that $f_k \to f$ in a neighborhood of $\omega' \in \mathbb{C}^m$ and that $\omega_k \to \omega$. Then $f_k(\omega_k) \to f(\omega)$.*

Proof. In fact, by the Cauchy estimates, $f_k^{(\alpha)} \to f^{(\alpha)}$ uniformly for each α in a slightly smaller neighborhood . Therefore, $f_k^{(\alpha)}(\omega_k') \to f^{(\alpha)}(\omega_k')$ for each α. It follows that

$$f_k(\omega_k) - f(\omega) = f_k(\omega_k) - f(\omega_k) + f(\omega_k) - f(\omega) \to 0$$

since only a finite number of derivatives come into play. □

Clearly

$$(af + bg)(\omega) = af(\omega) + bg(\omega), \quad a, b \in \mathbb{C},$$

and moreover we have

Proposition 1.2. *If p is a polynomial, then the definition above of $p(\omega)$ coincides with the natural one. If f, g are holomorphic in a neighborhood of ω', then*

$$(fg)(\omega) = f(\omega) \wedge g(\omega). \tag{1.3}$$

If f is holomorphic in a neighborhood of ω' (possibly $\mathbb{C}^r$-valued) and h is holomorpic in a neighborhood of $f(\omega')$, then

$$(h \circ f)(\omega) = h(f(\omega)). \tag{1.4}$$

If v is in E^*, then

$$\delta_v f(\omega) = \sum_1^m \frac{\partial f}{\partial z_j}(\omega) \wedge \delta_v \omega_j, \tag{1.5}$$

and if ω depends on a parameter, then

$$df(\omega) = \sum_1^m \frac{\partial f}{\partial z_j}(\omega) \wedge d\omega_j. \tag{1.6}$$

Proof. For the first statement, with no loss of generality, we may assume that $\omega' = 0$, and $p(z) = z^\beta$. Then $p^{(\alpha)}(0)(\omega'')^\alpha$ vanishes for $\alpha \neq \beta$ and equals $(\omega'')^\beta$ for $\alpha = \beta$. By linearity the first statement follows.

Now (1.3) clearly holds for polynomials, and since we can approximate f, g with polynomials f_k, g_k in $\mathcal{O}(\{0\})$, the general case follows from Lemma 1.1. One can obtain (1.4) in a similar way, noting that if $\tau_k = f_k(\omega)$ and $h_k \to h$ in a neighborhood of $f(\omega')$, then $h_k(\tau_k) \to h(\tau) = h(f(\omega))$, and $h_k(\tau_k) = (h_k \circ f_k)(\omega) \to (h \circ f)(\omega)$.

The remaining statements also clearly hold for polynomials and hence in general. □

Example 1.3. Since

$$\frac{1}{1-z} = 1 + z + z^2 +$$

in a the unit disk, if ω is an even form and $|\omega'| < 1$, we have

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \omega^3 + \cdots,$$

and $(1-\omega)[1/(1-\omega)] = 1$. In fact, the partial sums S_N converge in a neighborhood of ω'. If in addition $\omega' = 0$ we have

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \omega^3 + \cdots \omega^m.$$

□

Example 1.4. If ω_1 and ω_2 are even forms, then

$$e^{\omega_1 + \omega_2} = e^{\omega_1} \wedge e^{\omega_2}. \tag{1.7}$$

In fact, if $f(z_1, z_2) = z_1 + z_2$, $\pi_j(z_1, z_2) = z_j$, then $(\exp \circ f) = (\exp \pi_1)(\exp \pi_2)$. By (1.3) and (1.4) hence

$$e^{\omega_1 + \omega_2} = \exp(f(\omega_1, \omega_2)) = (\exp \circ f)(\omega_1, \omega_2) = \\ \exp \pi_1(\omega_1, \omega_2) \wedge \exp \pi_2(\omega_1, \omega_2) = e^{\omega_1} \wedge e^{\omega_2},$$

since (clearly) $\pi_j(\omega_1, \omega_2) = \omega_j$. □

Of course, in both examples one can easily check the statements directly as well.

2. Integral operators

Let $\alpha(\zeta, z)$ be any form of on $\mathbb{C}^n \times \mathbb{C}^n$ with compact support in ζ. We define

$$\int_\zeta \alpha(\zeta, z) \tag{2.1}$$

as the form in z such that

$$\int_z \phi(z) \wedge \int_\zeta \alpha(\zeta, z) = \int_{z,\zeta} \phi(z) \wedge \alpha(\zeta, z)$$

for test forms forms ϕ. The right hand side is well-defined since $\mathbb{C}^n$ has even real dimension so the orientation (volume form) on $\mathbb{C}^n \times \mathbb{C}^n$ is unambiguously defined. A moment of thought reveals that in practice the definition means that one first moves all differentials of ζ to the right (or to the left) and then perform the integration with respect to ζ. For instance, if $\psi(\zeta, z)$ is a function, then

$$\int_\zeta \psi(\zeta, z) d\zeta \wedge dz \wedge d\bar{\zeta} = -\Big[\int_\zeta \psi(\zeta, z) d\zeta \wedge d\bar{\zeta}\Big] dz.$$

Clearly, only components of α that have bidegree (n, n) in ζ can give any contribution in (2.1). We have the Fubini theorem

$$\int_z \int_\zeta \alpha(\zeta, z) = \int_\zeta \int_z \alpha(\zeta, z) \tag{2.2}$$

if α has bidegree $(2n, 2n)$.

3. Interior multiplication by a holomorphic vector field

Let

$$\xi = \xi_1 \frac{\partial}{\partial \zeta_1} + \cdots + \xi_n \frac{\partial}{\partial \zeta_n}$$

be a holomorphic vector field, and let δ_ξ denote interior multiplication (contraction) by ξ, cf., Section 1 above, so that δ_ξ is a mapping

$$\delta_\xi \colon \mathcal{E}_{p,q} \to \mathcal{E}_{p-1,q}, \quad \mathcal{C}_{p,q} \to \mathcal{C}_{p-1,q}.$$

Recall that δ_ξ is an anti-derivation, cf., (1.1). We claim that

$$\delta_\xi \bar{\partial} f = -\bar{\partial} \delta_\xi f. \tag{3.1}$$

In fact, by linearity it is enough to check for f of the form $f = \phi\gamma$, where ϕ is a function and $\gamma = dz_I \wedge d\bar{z}_J$. We have $\bar{\partial}\delta_\xi f = \bar{\partial}(\phi\delta\gamma) = \bar{\partial}\phi \wedge \delta_\xi\gamma$ since $\delta_\xi\gamma$ is $\bar{\partial}$-closed, and $\delta_\xi \bar{\partial} f = \delta_\xi(\bar{\partial}\phi \wedge \gamma) = -\bar{\partial}\phi \wedge \delta_\xi \gamma$, since $\bar{\partial}\phi$ is a $(0,1)$-form.

4. Convolution of forms and currents in $\mathbb{C}^n$

Let $\nu\colon \mathbb{C}^n_\zeta\times\mathbb{C}^n_z\to\mathbb{C}^n$ be the mapping $\nu(\zeta,z)=z-\zeta$. If $f(\nu)$ is a form in $\mathbb{C}^n$, then ν^*f is a form in $\mathbb{C}^n_z\times\mathbb{C}^n_\zeta$ that we simple write as $f(z-\zeta)$. In practice this means that each occurence of ν_j in $B(\nu)$ shall be replaced by $z_j-\zeta_j$, each ocurrence of $d\nu_j$ shall be replaced by $d(z_j-\zeta_j)$ etc.

Given forms f,g in the Schwartz class $\mathcal{S}=\mathcal{S}(\mathbb{C}^n)$ (i.e., their coefficients when expressed in the standard coordinates are in $\mathcal{S}$) we can define the convolution

$$(f*g)(z)=\int_\zeta f(z-\zeta)\wedge g(\zeta).$$

In principle this definition is real; however we will profit from the fact that our underlaying space $\mathbb{C}^n$ has *even* real dimension, and leave it to the interested reader to find out what happens in the odd-dimensional case. Since the convolution is just (up to a sign) the ordinary convolution of certain components of f and g, it follows that $f*g$ is again a form with coefficients in $\mathcal{S}$. Notice that if ψ is in $\mathcal{S}$, then

$$\int_z (f*g)(z)\wedge\psi(z)=\int_z\int_\zeta f(z)\wedge g(\zeta)\wedge\psi(z+\zeta); \tag{4.1}$$

this is seen by making the change of coordinates $\zeta'=\zeta,\ z'=z-\zeta$ on $\mathbb{C}^{2n}$, and as in the usual case we can take (4.1) as the definion of $f*g$ when they are currents, and one of them has compact support. The following facts are easily verified:

$$\begin{gathered}\deg f*g=\deg f+\deg g-2n,\\ \text{if } f\in\mathcal{S}_{p,q} \text{ and } g\in\mathcal{S}_{p',q'},\ \text{then } f*g\in\mathcal{S}_{p+p'-n,q+q'-n},\\ f*g=(-1)^{\deg f\cdot\deg g}g*f,\\ (f*g)*h=f*(g*h),\\ f*[0]=f,\\ d(f*g)=df*g+(-1)^{\deg f}f*dg,\\ \bar\partial(f*g)=\bar\partial f*g+(-1)^{\deg f}f*\bar\partial g.\end{gathered} \tag{4.2}$$

Let us just verify the last three of them: If $f=[0]$, then in view of (4.1) we have that

$$\int_z (f*[0])(z)\wedge\psi(z)=\int_z f(z)\wedge\int_\zeta[0](\zeta)\wedge\psi(z+\zeta)=\int_z f(z)\wedge\psi(z)$$

for any form ψ in $\mathcal{S}$. The next to last equality follows in the following way:

$$\begin{aligned}\int d(f*g)\wedge\psi&=(-1)^{1+\deg f+\deg g}\int f*g\wedge d\psi=\\ &\int\int f(z)\wedge g(\zeta)\wedge d\psi(\zeta+z)=\int\int d(f(z)\wedge g(\zeta))\wedge\psi=\\ &\int\int df(z)\wedge g(\zeta)\wedge\psi+(-1)^{\deg f}\int\int f(z)\wedge dg(\zeta)\wedge\psi=\\ &\int df*g\wedge\psi+(-1)^{\deg f}\int f*dg\wedge\psi.\end{aligned}$$

The last equality follows by identifying terms of relevant bidegree.

Example 4.1 (Approximate identity). Let ϕ be a (n,n)-form with compact support such that $\int\phi = 1$, and let $\phi_\epsilon(z) = \phi(z/\epsilon)$. Then $\phi_\epsilon \to [0]$ in the current sense. □

5. Double complexes

The following special case of a general spectral sequence will be encountered over and over in this book ????. We refer to any basic text on homological algebra for proofs.

Let $M_{\ell,k}$ be modules over a ring R and assume that for each fixed k have a complex

$$\cdots \stackrel{d'}{\to} M_{\ell-1,k} \stackrel{d'}{\to} M_{\ell,k} \stackrel{d'}{\to} M_{\ell+1,k} \stackrel{d'}{\to} \cdots \tag{5.1}$$

and for each fixed ℓ have a complex

$$\cdots \stackrel{d''}{\to} M_{\ell,k-1} \stackrel{d''}{\to} M_{\ell,k} \stackrel{d''}{\to} M_{\ell,k+1} \stackrel{d''}{\to} \cdots \tag{5.2}$$

such that $d'd'' = -d''d'$. We then have a *double complex*. If

$$M_j = \oplus_{\ell+k=j} M_{\ell,k} \tag{5.3}$$

and $d = d' + d''$, then

$$\cdots \stackrel{d}{\to} M_{j-1} \stackrel{d}{\to} M_j \stackrel{d}{\to} M_{j+1} \stackrel{d}{\to} \cdots$$

is a complex, called the *total complex* associated with $M_{\ell,k}$. The double complex $M_{\ell,k}$ is *bounded* if for each j only a finite number of $M_{\ell,k}$ with $k+\ell = j$ are nonzero, i.e., all the sums (5.3) are finite.

Lemma 5.1. *Assume that $M_{\ell,k}$ is bounded and that, for each k, the complex* (5.1) *is exact except at $\ell = 0$ where we have the cohomology group (module)*

$$A_k = \frac{\mathrm{Ker}\,(M_{0,k} \to M_{1,k})}{\mathrm{Im}\,(M_{-1,k} \to M_{0,k})}.$$

Then we get induced mappings $d''\colon A_j \to A_{j+1}$ so that

$$\cdots \stackrel{d''}{\to} A_{k-1} \stackrel{d''}{\to} A_k \stackrel{d''}{\to} A_{k+1} \stackrel{d'}{\to} \cdots$$

is a complex, and moreover, the natural mappings

$$H^k(A_\bullet) \to H^k(M_\bullet) \tag{5.4}$$

are isomorphisms.

In particular, if (5.1) is exact for each k, then $A_k = 0$ and hence $H^k(M_\bullet) = 0$.

Let us describe the mapping (5.4). If $\phi \in H^j(A_\bullet)$, then it is represented by an element $\phi_{0,k} \in M_{0,k}$ such that $d''\phi_{0,k} = d'\phi_{-1,k+1}$ for some $\phi_{-1,k+1} \in M_{-1,k+1}$. By the anti-commutativity, $d'd''\phi_{-1,k+1} = -d''d'\phi_{-1,k+1} = 0$ and hence by the exactness, $d''\phi_{-1,k+1} = d'\phi_{-2,k+2}$ for some $\phi_{-2,k+2}$, etc. By the boundedness, this procedure will terminate, and thus we get an element $\tilde\phi = \phi_{0,k} + \phi_{-1,k+1} + \phi_{-2,k+2} + \cdots + \phi_{-N,k+N}$ such that $d\tilde\phi = 0$. Thus $\tilde\phi$ defines an element in $H^k(M_\bullet)$. The lemma states that this procedure induces a well-defined mapping (5.4) that is an an isomorphism.

In many cases we will meet, $M_{\ell,k} = 0$ for $\ell < 0$ and then $A_k = \mathrm{Ker}\,(M_{0,k} \to M_{1,k})$. In such a case, the mapping (5.4) is much simpler, since then $\phi \in H^k(A_\bullet)$ is just represented by $\tilde\phi_{0,k}$.

If in addition (5.2) is exact for each ℓ except at $k = 0$, and the cohomology there is B_ℓ, then we have natural isomorphisms

$$H^k(B_\bullet) \simeq H^k(M_\bullet) \simeq M^k(A_\bullet).$$

6. Forms and currents on a reduced analytic space

Let X be a reduced analytic space of pure dimension n. Locally there is an embedding

$$i\colon X \to \Omega \subset \mathbb{C}^N.$$

We say $i^*\xi = 0$ for a smooth form $\xi \in \mathcal{E}^\Omega$ if $i^*\xi$ vanishes on the regular part X_{reg} of X, and we let

$$\mathcal{E}^X = \mathcal{E}^\Omega / \mathcal{K}er\, i^*$$

be the sheaf of smooth forms on X. We shall see that this definition is independent of the embedding i. To begin with, two minimal embeddings are biholomorphically equivalent and hence give rise to the same sheaf. If i is an arbitrary embedding of X, then after possibly shrinking Ω one can factorize i as

$$X \xrightarrow{j} \widehat{\Omega} \xrightarrow{\iota} \widehat{\Omega} \times \mathcal{B} = \Omega$$

where j is a minimal embedding and $\mathcal{B}$ is a ball in $\mathbb{C}^M$. Since $j^*\iota^*\xi = (\iota \circ j)^*\xi$, we have a natural injective mapping

$$A\colon \iota^*\colon \mathcal{E}^\Omega / \mathcal{K}er\, i^* \to \mathcal{E}^{\widehat{\Omega}} \mathcal{K}er\, j^*,$$

and it is enough to see that it is an isomorphism. Let $\pi\colon \Omega = \widehat{\Omega} \times \mathcal{B} \to \widehat{\Omega}$ be the natural projection. Then $\pi \circ \iota$ is the identity on $\widehat{\Omega}$ and hence $\iota^*\pi^*\eta = \eta$. Thus A is surjective, and hence an isomorphism. Clearly, the wedge product on $\mathcal{E}^\Omega$ induces a wedge product on $\mathcal{E}^X$ and, we have a mapping $i^*\colon \mathcal{E}^\Omega \to \mathcal{E}^X$ such that $i^*\xi \wedge i^*\xi' = i^*(\xi \wedge \xi')$.

We define the sheaf of currents, $\mathcal{C}^X$, as the dual of the compactly supported smooth forms. This means concretely that the currents τ on X can be identified with the currents τ' in Ω such that $\tau'.\xi = 0$ for all ξ in $\mathcal{K}er\, i^*$. It is natural to write $\tau' = i_*\tau$. We say that τ has bidegree (p, q) if $i_*\tau$ has bidegree $(N - n + p, N - n + q)$. Notice in particular that

$$i_*1 = [X],$$

the Lelong current associated with X in Ω.

Remark 6.1. Notice that if μ is a current in Ω, then $\mu.\xi = 0$ for all ξ such that $i^*\xi = 0$ if and only if $\eta \wedge \mu = 0$ for all η such that $i^*\eta = 0$. □

Let X, Y be reduced analytic spaces and $f\colon Y \to X$ a proper holomorphic mapping. We than have the pullback $f^*\colon \mathcal{E}^X \to \mathcal{E}^Y$ and hence the push-forward $f_*\colon \mathcal{C}^Y \to \mathcal{C}^X$. We will frequently use the following simple lemma.

Lemma 6.2. *If α is a smooth form, then*

$$\alpha \wedge f_*\tau = f_*(f^*\alpha \wedge \tau). \tag{6.1}$$

Proof. By definition there are local embeddings $i\colon X \to \Omega$ and $j\colon Y \to \Omega'$ such that f extends to a mapping $F\colon \Omega' \to \Omega$, and $i_* f_* \tau = F_* j_* \tau$. It is therefore enough to check (3.1) in case X, Y are smooth. Now

$$\langle \alpha \wedge f_*\tau, \xi\rangle = \pm\langle f_*\tau, \alpha\wedge\rangle\xi = \pm\langle \tau, f^*(\alpha\wedge\xi)\rangle = \\ \pm\langle \tau, f^*\alpha\wedge f^*\xi\rangle = \langle f^*\alpha\wedge\tau, f^*\xi\rangle = \langle f_*(f^*\alpha\wedge\tau), \xi\rangle,$$

since $f_*\tau$ has odd degree if and only if τ has. □

7. Principalization of an ideal sheaf

Let X be a reduced analytic space, let $\mathcal{J} \to X$ be an ideal sheaf and let $\pi'\colon X' \to X$ be the blow-up of X along $\mathcal{J}$. Then the pullback of $\mathcal{J}$ to X' is principal, i.e., locally generated by one single holomorphic function. More precisely, there is a line bundle $L' \to X'$ and a global section h of L' that generates $(\pi')^*\mathcal{J}$. We say that the blow-up is a *principalization* of $\mathcal{J}$. The divisor defined by h is called the exceptional divisor. In general X' is not normal, so it is convenient to let $X' \to X$ be the normalization if the blow-up. In a normal space the singular locus has at least codimension 2, and so for instance each divisor has a well-defined order.

If $f_1, \dots, f_m$ is a tuple of holomorphic functions on X that generates $\mathcal{J}$, then it follows that $(\pi')^* f_j = h f'_j$, where f'_j is a section of L^{-1}. Moreover, the tuple $f' = (f'_1, \dots, f'_m)$ is non-vanishing (in fact, at a given point $x \in X'$, $f^0 f'_j$ define the sheaf $(\pi')^*\mathcal{J}$, and by a standard fact for local rings the ideal must be generated by one of them.)

The pullback of a principal ideal is certainly principal. Thus if we compose by the normalization $X'' \to X'$ we get a normal principalization $X'' \to X$ of $\mathcal{J}$. It is very important since it is unique; however, in general X'' is not smooth, and in many situations it is convenient with a smooth principalization.

By Hironaka's theorem we can find a smooth modification $\widetilde{X} \to X'$ such that the exceptional divisor $D = \alpha_1 D_1 + \cdots + \alpha_\nu D_\nu$ has simple normal crossings, cf., ???, above. The composed modification $\pi\colon \widetilde{X} \to X$ is called a *log resolution* for $\mathcal{J}$. If f^0 is a section of the line bundle $L_D \to \widetilde{X}$ that defines $\pi^*\mathcal{J}$ then locally in $\widetilde{X}$ one can choose coordinates and a local frame such that f^0 is a monomial. If $\mathcal{J}$ is generated by $f_1, \dots, f_m$, then as before we have

$$\pi^* f = f^0 f'$$

where f' is a non-vanishing tuple of sections of L_D^{-1}.

8. The Koszul complex superstructures

Chapter 2

Integral representation in domains in $\mathbb{C}^n$

Integral representation of a holomorphic functions f means that f is expressed as a superposition of other functions, preferably functions that are simple in some sense. For instance, by the Cauchy integral formula in one variable a functions in a domain D is written as a superposition of simple rational functions $z \mapsto 1/(z-\zeta)$, where $\zeta \in \partial D$.

1. The one-variable case

For fixed $z \in \mathbb{C}$,

$$\omega_{\zeta-z} = \frac{1}{2\pi i}\frac{d\zeta}{\zeta - z}$$

is the Cauchy kernel with pole at z. It is holomorphic in $\mathbb{C}\setminus\{z\}$ and locally integrable in $\mathbb{C}$. It is well-known that if ϕ is a C^1-function in $\mathbb{C}$ with compact support, then we have

$$\int \omega_{\zeta-z} \wedge \bar\partial\phi = \phi(0), \tag{1.1}$$

which can be rephrased as saying that

$$\bar\partial\omega_{\zeta-z} = [z] \tag{1.2}$$

in the current sense, where $[z]$ denotes the $(1,1)$-current point evaluation at z, see Lemma 2.2 below. This equation leads to, or is more or less equivalent to, Cauchy-Green's formula.

Proposition 1.1 (Cauchy-Green's formula)**.** *If f is C^1 in Ω and $D \subset \Omega$ is bounded and has smooth boundary (or at least some reasonable regularity, like piecewise C^1) then*

$$f(z) = \int_{\partial D} \omega_{\zeta-z} f + \int_D \omega_{\zeta-z} \wedge \bar\partial f, \quad z \in D. \tag{1.3}$$

Notice that

$$\omega_{\zeta-z} \wedge \bar\partial f = \frac{1}{2\pi i}\frac{\partial f}{\partial \bar z}\frac{dz\wedge d\bar z}{\zeta - z} = -\frac{1}{\pi}\frac{\partial f}{\partial \bar z}\frac{dV(z)}{\zeta - z},$$

where dV is the planar volume measure.

Proof. Notice that (1.3) is just (1.1) if f has compact support in D. Now suppose that f vanishes identically in a neighborhood of the point z. Then $d(\omega_{\zeta-z}f) = -\omega_{\zeta-z}\wedge\bar\partial f$ and hence (1.3) follows from Stokes' theorem. For the general case, let χ be a smooth cutoff function that has compact support in D and is identically 1 in a neighborhood of the point z. Then $f = \chi f + (1-\chi)f = f_1 + f_2$ where f_1 has compact support in D and f_2 vanishes in a neighborhood of z. Since (1.3) is linear in f, the general case now follows. □

As an immediate corollary we have, for a holomorphic function f, the Cauchy formula,

$$f(z) = \int_{\partial D} f(\zeta)\omega_{\zeta-z}, \quad z \in D.$$

This formula is a corner stone in the theory of one complex variable and probably one of the most remarkable formulas in analysis, with regard to beauty and importance.

We will now consider multivariable analogues of this formula, whereas generalizations of (1.3) are postponed to Section 7. We let D denote a bounded open with a boundary regular enough so that Stokes' theorem holds.

2. The Cauchy-Fantappiè-Leray formula

Let $s = s_1 d\zeta_1 + \cdots + s_n d\zeta_n$ be a $(1,0)$-form such that

$$2\pi i\langle \zeta, s\rangle = 2\pi i(\zeta_1 s_1 + \cdots + \zeta_n s_n) = 1. \tag{2.1}$$

Such a form s always exists outside 0; for instance one can take

$$b = \frac{\partial|\zeta|^2}{2\pi i|\zeta|^2} = \frac{\sum_1^n \bar\zeta_j d\zeta_j}{2\pi i|\zeta|^2}. \tag{2.2}$$

Lemma 2.1. *If* (2.1) *holds in the open set V, then*

$$d\big(s\wedge(\bar\partial s)^{n-1}\big) = \bar\partial\big(s\wedge(\bar\partial s)^{n-1}\big) = 0$$

in V.

Proof. Since $s\wedge(\bar\partial s)^{n-1}$ has bidegree $(n, n-1)$, the first equality is immediate. It is clear that

$$\bar\partial\big(s\wedge(\bar\partial s)^{n-1}\big) = (\bar\partial s)^n$$

so we have to verify that

$$(\bar\partial s)^n = 0. \tag{2.3}$$

From (2.1) we have that $0 = \bar\partial(1/2\pi i) = \zeta_1\bar\partial s_1 + \cdots + \zeta_n\bar\partial s_n$, so the 1-forms $\bar\partial s_1, \ldots, \bar\partial s_n$ are linearly dependent at each point in V and thus $\bar\partial s_1\wedge\bar\partial s_2\wedge\ldots\wedge\bar\partial s_n = 0$. It follows that

$$(\bar\partial s)^n = \big(\sum_1^n \bar\partial s_j\wedge d\zeta_j\big)^n = n!\bar\partial s_1\wedge\ldots\wedge\bar\partial s_n\wedge d\zeta_n\wedge\ldots\wedge d\zeta_1 = 0.$$

□

If $n = 1$, then the only possible choice of s that satisfies (2.1) is the Cauchy kernel with pole at 0.

The form $B_{n,n-1} = b\wedge(\bar\partial b)^{n-1}$ is called the Bochner-Martinelli kernel. Notice that

$$B_{n,n-1} = \frac{1}{(2\pi i)^n}\frac{\partial|\zeta|^2}{|\zeta|^2}\wedge\Big(\bar\partial\frac{\partial|\zeta|^2}{|\zeta|^2}\Big)^{n-1} = \frac{1}{(2\pi i)^n}\frac{\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{n-1}}{|\zeta|^{2n}}; \tag{2.4}$$

here we use the fact that $\partial|\zeta|^2\wedge\partial|\zeta|^2 = 0$. Therefore, $B_{n,n-1}$ is $\mathcal{O}(|\zeta|^{-2n+1})$ and thus locally integrable. We have the following multivariable analog of (1.1).

Lemma 2.2. *The Bochner-Martinelli kernel satifies*

$$dB_{n,n-1} = \bar\partial B_{n,n-1} = [0]. \tag{2.5}$$

Proof. Let ξ be a test form (function). Outside the origin $d(B_{n,n-1}\xi) = -B_{n,n-1}\wedge d\xi = -B_{n,n-1}\wedge\bar\partial\xi$. By Stokes' formula and Lemma 2.1,

$$-\int_{|\zeta|>\epsilon} B_{n,n-1}\wedge\bar\partial\xi = \int_{|\zeta|=\epsilon}\xi\wedge B_{n,n-1} =$$

$$\frac{1}{(2\pi i)^n}\frac{1}{\epsilon^{2n}}\int_{|\zeta|=\epsilon}\xi\partial|\zeta|^2(\bar\partial\partial|\zeta|^2)^{n-1} = \frac{1}{(2\pi i)^n}\frac{1}{\epsilon^{2n}}\int_{|\zeta|<\epsilon}\big[\xi(\bar\partial\partial|\zeta|^2)^n + \mathcal{O}(|\zeta|)\big].$$

In the second term, the sphere is considered as the boundary of the ϵ-ball. The right hand side tends to $\xi(0)$ when $\epsilon \to 0$ since

$$(\frac{i}{2}\partial\bar\partial|\zeta|^2)^n = \Big(\frac{i}{2}\sum_1^n d\zeta_j \wedge d\bar{\zeta_j}\Big)^n =$$

$$n!\Big(\frac{i}{2}\Big)^n d\zeta_1 \wedge d\bar{\zeta_1} \wedge \ldots \wedge d\zeta_n \wedge d\bar{\zeta_n} = n! dV(\zeta)$$

and the volume of the unit ball is $\pi^n/n!$, cf., Example 3.7. □

We have the following classical global formula.

Proposition 2.3 (The Cauchy-Fantappiè-Leray formula)**.** *Assume $D \subset\subset \Omega$, and that σ is a smooth $(1,0)$-form on ∂D such that $2\pi i\langle\sigma, \zeta - z\rangle = 1$ for some $z \in D$. Then*

$$f(z) = \int_{\partial D} \sigma \wedge (\bar\partial\sigma)^{n-1} f, \quad f \in \mathcal{O}(\Omega).$$

To interprete the integral, let σ denote any smooth extension to a neighborhood of ∂D. Since $\sigma \wedge (\bar\partial\sigma)^{n-1} = \sigma \wedge (d\sigma)^{n-1}$ for degree reasons, the pull-back to ∂D is an intrinsically defined form.

Proof. With no loss of generality we may assume that $z = 0$. Let σ be a smooth extension to a neighborhood of ∂D. By continuity $\langle\zeta, \sigma\rangle \neq 0$ close to ∂D and if χ is an appropriate cutoff function, then

$$s = (1-\chi)\frac{\sigma}{2\pi i\langle\zeta, \sigma\rangle} + \chi b$$

satisfies (2.1) outside the origin and is equal to b close to the origin. We claim that then

$$d\big(s \wedge (\bar\partial s)^{n-1}\big) = [0]. \tag{2.6}$$

In fact, outside the origin (2.6) follows from Lemma 2.1 and near the origin $s \wedge (\bar\partial s)^{n-1}$ is equal to $B_{n,n-1}$ and so it follows from Lemma 2.2. Now the proposition follows from (2.6) and Stokes' theorem, noting that $s \wedge (\bar\partial s)^{n-1} = \sigma \wedge (\bar\partial\sigma)^{n-1}$ as forms on ∂D. □

For any domain D and $z \in D$ we can use $\sigma(\zeta) = b(\zeta - z)$ and thus obtain a representation formula for holomorphic functions, generalizing the Cauchy formula for $n = 1$. When $n > 1$ unfortunately it will not depend holomorphically on the variable z.

If there is a form $s(\zeta, z)$ for $\zeta \in \partial D$ depending holomorphically on z such that $\langle s(\zeta, z), \zeta - z\rangle = 1$, such an s is called a holomorphic support function, then D must be pseudoconvex. In general, though, pseudoconvexity is not sufficient, even if we assume that D has smooth boundary. However, if D is strictly pseudoconvex one can always find a holomorphic support function, cf., Example 8.5 below. It is also true for a large class of weakly pseudoconvex domains, e.g., all convex domains, see Example 2.5 below.

Example 2.4. Let $\mathbb{B} = \{\zeta;\ |\zeta| < 1\}$ be the unit ball. For $z \in \mathbb{B}$ we can take

$$\sigma = \frac{\partial|\zeta|^2}{2\pi i(1 - \bar\zeta \cdot z)}.$$

By a similar argument as for the equality (2.4) we get

$$f(z) = \int_{|\zeta|=1} \frac{f(\zeta)d\nu(\zeta)}{(1-\bar\zeta\cdot z)^n}, \tag{2.7}$$

where

$$d\nu(\zeta) = \frac{1}{(2\pi i)^n}\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{n-1}.$$

Since $|\zeta|$ is invariant under rotations of $\mathbb{C}^n$, $d\nu(\zeta)$ is as well, and since furthermore the integral of $d\nu$ over the unit sphere is 1, just take $f = 1$ in (2.7), it follows that $d\nu$ is normalized surface measure. The representation formula (2.7) is called the Szegö integral. □

Example 2.5. More generally, let $D = \{\rho < 0\}$ be a convex domain in $\mathbb{C}^n$ with defining function ρ, i.e, $d\rho \neq 0$ on ∂D; it is not necessary to assume that ρ is a convex function. Then for any $z \in D$,

$$2\mathrm{Re}\,\langle\partial\rho(\zeta), \zeta - z\rangle > 0, \quad \zeta\in\partial D. \tag{2.8}$$

In fact, the left hand side is the real scalar product of the gradient of ρ and the vector $\zeta-z$, and by the convexity of D it must be strictly positive when z is an interior point. We can thus use $s(\zeta, z) = \partial\rho(\zeta)/\langle\partial\rho(\zeta), \zeta - z\rangle 2\pi i$ and get the classical representation formula

$$f(z) = \frac{1}{(2\pi i)^n}\int_{\partial D}\frac{f(\zeta)\partial\rho\wedge(\bar\partial\partial\rho)^{n-1}}{\langle\partial\rho(\zeta), \zeta - z\rangle^n}, \quad z\in D.$$

□

In the proof of (2.3) we used some multilinear algebra. There is an even slicker argument: Let $\delta_{\zeta-z}\colon \mathcal{E}_{p,q}(U)\to\mathcal{E}_{p-1,q}(U)$ be contraction with the vector field

$$2\pi i\sum_1^n(\zeta_k - z_k)\frac{\partial}{\partial\zeta_k}.$$

Then (2.1) precisely means that $\delta_\zeta s = 1$. Thus, cf., (3.1), $\delta_\zeta\bar\partial s = -\bar\partial\delta_\zeta s = -\bar\partial 1 = 0$, and therefore

$$(\bar\partial s)^n = \delta_\zeta\big(s\wedge(\bar\partial s)^n\big) = 0,$$

since $s\wedge(\bar\partial s)^n$ vanishes for degree reasons. This type of arguments will permeate this paper. It might be instructive at this stage to prove by straight-forward brutal computation that the right hand side of (2.4) is closed. For the more complicated situations that we will encounter, the necessary computations are in practice impossible to perform without multilinear algebra techniques.

3. A general Cauchy-Fantappiè-Leray formula

We have already seen that the Cauchy-Fantappiè-Leray formula formula is sort of a substitute for the Cauchy formula in several variables. We shall now see that the proper generalization of the Cauchy kernel is a representative of a certain cohomology class.

For any integer m, let $\mathcal{L}^m(U) = \oplus_{k=0}^n\mathcal{C}_{k,k+m}(U)$. For instance, $u\in\mathcal{L}^{-1}(U)$ can be written $u = u_{1,0} + \ldots + u_{n,n-1}$, where the indices denote bidegree in $d\zeta$. We let $\mathcal{L}^m_{\mathcal{E}}(U)$ denote the subspace of smooth forms in $\mathcal{L}^m(U)$.

Fix a point $z \in \mathbb{C}^n$. Since $\delta_{\zeta-z}\bar\partial f = -\bar\partial\delta_{\zeta-z} f$, cf., (3.1), $\mathcal{E}_{p,q}$ is a double complex with mappings $\bar\partial$ and $\delta_{\zeta-z}$. If $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar\partial$, then $\nabla^2_{\zeta-z} = 0$, and we have the associated total complex

$$\cdots \xrightarrow{\nabla_{\zeta-z}} \mathcal{L}^m \xrightarrow{\nabla_{\zeta-z}} \mathcal{L}^{m+1} \xrightarrow{\nabla_{\zeta-z}} \cdots .$$

The usual wedge product extends to a mapping $\mathcal{L}^m(U)\times\mathcal{L}^{m'}(U)\to\mathcal{L}^{m+m'}(U)$, such that $g\wedge f = (-1)^{mm'} f\wedge g$, (assuming that one of the factors is smooth) and $\nabla_{\zeta-z}$ satisfies the same formal rules as the usual exterior differentiation, i.e., $\nabla_{\zeta-z}$ is a anti-derivation,

$$\nabla_{\zeta-z}(f\wedge g) = \nabla_{\zeta-z} f\wedge g + (-1)^m f\wedge\nabla_{\zeta-z} g, \quad f\in\mathcal{L}^m(U). \tag{3.1}$$

In order to generalize Cauchy's formula to higher dimensions we will look for $u\in\mathcal{L}^{-1}(U)$ such that

$$\nabla_{\zeta-z} u(\zeta) = 1 - [z]. \tag{3.2}$$

If $n=1$ the Cauchy kernel with pole at z, $u(\zeta) = d\zeta/2\pi i(\zeta - z)$, is the only possible solution. If $n>1$, (3.2) means that

$$\delta_{\zeta-z} u_{1,0} = 1, \quad \delta_{\zeta-z} u_{k+1,k} - \bar\partial u_{k,k-1} = 0,\ 1\le k\le n-2, \quad \bar\partial u_{n,n-1} = [z]. \tag{3.3}$$

However, we first look for smooth u such that $\nabla_{\zeta-z} u = 1$ outside z.

Example 3.1. Let $s = \sum_1^n s_j dz_j$ be a $(1,0)$-form in U such that $\delta_{\zeta-z} s = 2\pi i\sum_j s_j(\zeta_j - z_j) = 1$. Then clearly $z\notin U$. Since the component of zero degree of the form $\nabla_{\zeta-z} u$ is nonvanishing, we can define the form

$$u = \frac{s}{\nabla_{\zeta-z} s}, \tag{3.4}$$

and we claim that

$$\nabla_{\zeta-z} u = 1 \tag{3.5}$$

in U. In fact, by the functional calculus for forms we have

$$\nabla_{\zeta-z} u = \frac{\nabla_{\zeta-z} s}{\nabla_{\zeta-z} s} - \frac{s}{(\nabla_{\zeta-z} s)^2}\nabla^2_{\zeta-z} s = 1$$

since $\nabla^2_{\zeta-z} = 0$. More explicitly,

$$u = \frac{s}{\nabla_{\zeta-z} s} = \frac{s}{1-\bar\partial s} = s + s\wedge\bar\partial s + s\wedge(\bar\partial s)^2 + \cdots + s\wedge(\bar\partial s)^{n-1}. \tag{3.6}$$

The sceptical reader can of course verify that the right hand side of (3.6) fulfills (3.5) by a straight-forward computation. However, in more involved situations the compact formalism is indispensable. Notice that highest order term is precisely the Cauchy-Fantappiè-Leray form in Proposition 2.3. □

Notice that (3.4) is unaffected if s is replaced by ξs for a nonvanishing function ξ, since $\nabla(\xi s) = \xi\nabla s - \bar\partial\xi\wedge s$, and that the last term here cannot give any contribution, due to the factor s in the nominator. Therefore it is enough that $\delta_{\zeta-z} s \neq 0$ in the previous example, and then we get

$$u = \frac{s}{\nabla_{\zeta-z} s} = \frac{s}{\delta_{\zeta-z} s} + \frac{s\wedge\bar\partial s}{(\delta_{\zeta-z} s)^2} + \cdots + \frac{s\wedge(\bar\partial s)^{n-1}}{(\delta_{\zeta-z} s)^n}.$$

Proposition 3.2. *If b is the form in* (2.2)*, then*

$$B = \frac{b}{\nabla_\zeta b} = \frac{b}{1-\bar\partial b} = b \wedge \sum_1^n (\bar\partial b)^{k-1},$$

is locally integrable in $\mathbb{C}^n \setminus \{0\}$ and satisfies (3.2) *(with $z=0$).*

We will refer to B as the (full) Bochner-Martinelli form.

Proof. A simple computation yields

$$B_{k,k-1} = b \wedge (\bar\partial b)^{k-1} = \frac{\partial|\zeta|^2 \wedge (\bar\partial\partial|\zeta|^2)^{k-1}}{(2\pi i)^k |\zeta|^{2k}},$$

so $B_{k,k-1} = \mathcal{O}(|\zeta|^{-(2k-1)})$ and hence locally integrable. We already know that $\bar\partial B_{n,n-1} = [0]$, so it remains to verify that

$$-\int \bar\partial\phi \wedge B_{k,k-1} = \int \phi \wedge \delta_\zeta B_{k+1,k}, \quad \phi \in \mathcal{D}_{n-k,n-k}(\mathbb{C}^n). \tag{3.7}$$

However,

$$-\int_{|\zeta|>\epsilon} \bar\partial\phi \wedge B_{k,k-1} = \int_{|\zeta|=\epsilon} \phi \wedge B_{k,k-1} + \int_{|\zeta|>\epsilon} \phi \wedge \delta_\zeta B_{k+1,k}.$$

Moreover, since $k<n$, $B_{k,k-1} = \mathcal{O}(|\zeta|^{-(2k-3)})$, and hence the boundary integral tends to zero when $\epsilon \to 0$. Thus (3.7) follows. □

Suppose that $z \notin U$. If f is any form in U such that $\nabla_{\zeta-z} f = 0$, then there is a form w such that $\nabla_{\zeta-z} w = f$. In fact, $u(\zeta) = B(\zeta - z)$ is smooth in U and $\nabla_{\zeta-z} u = 1$, and thus $\nabla_{\zeta-z}(u \wedge f) = f$.

We are now ready to prove the main result of this section, stating that the proper generalization of the Cauchy form from one variable is a certain cohomology class $\omega_{\zeta-z}$.

Proposition 3.3. *Suppose that $z \in D$ and $z \notin U \supset \partial D$. If $u \in \mathcal{L}^{-1}_{\mathcal{E}}(U)$ and $\nabla_{\zeta-z} u = 1$, then $\bar\partial u_{n,n-1} = 0$. All such forms $u_{n,n-1}$ define the same Dolbeault cohomology class $\omega_{\zeta-z}$ in U and any representative for $\omega_{\zeta-z}$ occurs in this way. For any representative we have that*

$$\phi(z) = \int_{\partial D} \phi(\zeta) u_{n,n-1}, \quad \phi \in \mathcal{O}(\overline{D}). \tag{3.8}$$

Proof. If $\nabla_{\zeta-a} u = 1$ then $\bar\partial u_n = 0$. If u' is another solution then $\nabla_{\zeta-a}(u-u') = 0$ and since $a \notin U$ there is a solution to $\nabla_{\zeta-a} w = u - u'$, and hence $\bar\partial w_{n,n-2} = u'_{n,n-1} - u_{n,n-1}$. If u is a fixed solution and ψ is a $(n,n-2)$ form, then $u' = u - \nabla_{\zeta-a}\psi$ is another solution, and $u'_{n,n-1} = u_{n,n-1} + \bar\partial\psi$.

If $u'_{n,n-1} - u_{n,n-1} = \bar\partial w_{n,n-2}$ in U and ϕ is holomorphic, then

$$d(\phi w_{n,n-2}) = \phi u'_{n,n-1} - \phi u_{n,n-1}.$$

Therefore, Stokes' theorem, applied to the compact manifold ∂D, implies that the integral in (3.8) is unchanged if $u_{n,n-1}$ is replaced by $u'_{n,n-1}$. Since (3.8) holds for $u^a(\zeta) = B(\zeta - a)$ it therefore holds for any u. □

With the choice of u from Example 3.1, (3.8) is just the Cauchy-Fantappiè-Leray formula. However, there are other possibilities.

Example 3.4. If we have several $(1,0)$-forms $s^1, \dots, s^n$ such that $\delta_{\zeta-z}s^j = 1$ we can get a solution u to $\nabla_{\zeta-z}u = 1$ by letting $u_1 = s^1$, $u_{k+1} = s^{k+1} \wedge \bar\partial u_k$. Thus $s^n \wedge \bar\partial s^{n-1} \wedge \dots \wedge \bar\partial s^1$ is a representative for $\omega_{\zeta-z}$. □

We have the following analogue of (3.8).

Proposition 3.5. *With the same notation as in the Proposition 3.3, let u be a current solution to $\nabla_{z-a}u = 1$ in U. If χ is a cutoff function that is 1 in a neighborhood of z and such that the support of $\bar\partial\chi$ is contained in U, then*

$$\phi(z) = -\int \bar\partial\chi \wedge \phi u_{n,n-1}, \quad \phi \in \mathcal{O}(U). \tag{3.9}$$

We leave the proof as an exercise for the reader.

Example 3.6. Let us define the current $v = v_{1,0} + \dots + v_{n,n-1}$ in $\mathbb{C}^n$ by

$$v_1 = \frac{1}{2\pi i}\frac{d\zeta_1}{\zeta_1 - z_1}, \quad v_k = \frac{1}{(2\pi i)^k}\frac{d\zeta_k}{\zeta_k - z_k} \wedge \bar\partial v_{k-1}.$$

The products are well-defined since they are just tensor products of distributions. From Proposition 3.5 we get the representation formula

$$f(z) = -(2\pi i)^{-1}\int_{\zeta_n} \bar\partial_{\zeta_n}\chi(\dots, z_{n-1}, \zeta_n) \wedge f(\dots, z_{n-1}, \zeta_n)\frac{d\zeta_n}{\zeta_n - z_n}.$$

Of course, this formula follows immediately from the one-variable Cauchy formula. □

Example 3.7 (Volume of the unit ball). Let B be the (full) Bochner-Martinelli form and let v be the current from the previous example. Let χ be a cutoff function that is 1 in a neighborhood of 0. Then $B_{n,n-1} - v_{n,n-1} = d(B \wedge v)_{n.n-2}$ on the support of $\bar\partial\chi$, and thus

$$\int_{\partial\mathbb{B}} B_{n,n-1} = \int B_{n,n-1} \wedge \bar\partial\chi = \int v_{n,n-1} \wedge \bar\partial\chi = \int \chi\bar\partial v_{n,n-1} = \int [0] = 1.$$

It follows now from the proof of Lemma 2.2 that the volume of the unit ball is $\pi^n/n!$. □

If we have a solution to $\nabla_{\zeta-z}u = 1$ in $\Omega \setminus \{z\}$ that has a current extension across z it is natural to ask whether (3.2) holds.

Proposition 3.8. *Suppose that $u \in \mathcal{L}^{-1}_{\mathcal{E}}(\Omega \setminus \{z\})$ solves $\nabla_{\zeta-z}u = 1$ in $\Omega \setminus \{z\}$ and that $|u_k| \lesssim |\zeta - z|^{-(2k-1)}$. Then u is locally integrable and (3.2) holds.*

Proof. If u^1 and u^2 both satisfy the growth condition, then $u^1 \wedge u^2 = \mathcal{O}(|\zeta - z|^{-(2n-2)})$ and $\nabla_{\zeta-a}(u^1 \wedge u^2) = u^2 - u^1$ pointwise outside a. Hence it holds in the current sense. If (3.2) holds for one of them it thus holds for both; taking one of them as the Bochner-Martinelli form, the proposition follows. □

Example 3.9. Let $s(\zeta)$ be a smooth $(1,0)$-form in Ω such that $|s(\zeta)| \le C|\zeta - z|$ and $|\delta_{\zeta-z}s(\zeta)| \ge C|\zeta - z|^2$. Then $u = s/\nabla_{\zeta-z}s$ satisfies the hypothesis in Proposition 3.8 and therefore (3.2) holds. □

4. Weighted representation formulas

We will now consider formulas that allow representation of functions with growth at the boundary, unbounded domains, and, which is most important for us, provide division with a tuple of functions, and interpolation. To this end we introduce so-called *weights*. Let z be a fixed point in $\Omega \subset \mathbb{C}^n$. A smooth form $g \in \mathcal{L}^0(\Omega)$ such that $\nabla_{\zeta-z} g = 0$ and $g_{0,0}(z) = 1$ is called a smooth *weight with respect to z*.

If g, g' are weights, then $g \wedge g'$ is a weight as well. This follows from (3.1) and the simple observation that $(g \wedge g')_{0,0}(z) = g_{0,0}(z) g'_{0,0}(z) = 1$. More generally, if $g^1, \ldots, g^m$ are weights and $G(\lambda_1, \ldots, \lambda_m)$ is a holomorphic function, defined on the image of $\zeta \mapsto (g^1_{0,0}, \ldots, g^m_{0,0})$, and such that $G(1, \ldots, 1) = 1$, then $G(g^1, \ldots, g^m)$ is a weight. This follows from Proposition 1.2

Example 4.1. Let w be a smooth $(1,0)$-form and assume that $G(\lambda)$ is holomorphic on the image of $\delta_{\zeta-z} w$ and that $G(0) = 1$. Then

$$G(\nabla_{\zeta-z} w) = G(\langle w, \zeta - z\rangle - \bar\partial w) = \sum_0^n G^{(k)}(\langle w, \zeta - z\rangle) \frac{(-\bar\partial w)^k}{k!}$$

is a weight in Ω. □

Proposition 4.2. *If g is a smooth weight in Ω, $z \in D \subset\subset \Omega$, and $\nabla_{\zeta-z} u = 1$ in a neighborhood U of ∂D, then*

$$\phi(z) = \int_D g_{n,n}\phi + \int_{\partial D} (g \wedge u)_{n,n-1}\phi, \qquad \phi \in \mathcal{O}(\overline{D}). \tag{4.1}$$

Proof. Let us first assume that u' is a form such that (3.2) holds. Then

$$\nabla_{\zeta-z}(u' \wedge g) = (1 - [z]) \wedge g = g - g_{0,0}(z)[z] = g - [z]$$

for degree reasons. In particular we have that

$$d((u' \wedge g)_{n,n-1}\phi) = \bar\partial(u' \wedge g)_{n,n-1}\phi = \phi(z)[z] - g_{n,n}\phi$$

and so (4.1) follows by Stokes' theorem but with u' rather than u. Now $\nabla(u' \wedge u) = u - u'$ and hence

$$d((u' \wedge u)_{n,n-2}\phi) = \bar\partial(u' \wedge u)_{n,n-2}\phi = u'_{n,n-1}\phi - u_{n,n-1}\phi.$$

Now (4.1) follows for u by Stokes' theorem since ∂D hasno boundary. □

Alternatively one can proceed as in the proof of Proposition 2.3 above, and extend u to a form that satisfies (3.2).

For instance, if D is the ball and we take $u = \sigma/\nabla_{\zeta-z}\sigma$, with the σ from Example 2.4. Then u depends holomorphically on z. If $g = 1$ we get back (2.7).

As we have seen, the Cauchy-Fantappiè-Leray formula represents a holomorphic function in a domain D in terms of its values on ∂D; provided of course that the function has some reasonable boundary values. To admit representation for a larger class of functions we can use an appropriate weight. Here we will exemplify with the ball.

Example 4.3 (Weighted Bergman projections in the ball). Notice that

$$1 + \nabla_{\zeta-z} \frac{\partial|\zeta|^2}{2\pi i(1 - |\zeta|^2)} = \frac{1 - \bar\zeta \cdot z}{1 - |\zeta|^2} - \frac{1}{2\pi i}\bar\partial \frac{\partial|\zeta|^2}{1 - |\zeta|^2}.$$

Therefore, as long as $z,\zeta\in\mathbb{B}$, for any complex α,

$$g=\Big(\frac{1-\bar\zeta\cdot z}{1-|\zeta|^2}-\frac{1}{2\pi i}\bar\partial\frac{\partial|\zeta|^2}{1-|\zeta|^2}\Big)^{-\alpha},$$

is welldefined, and in fact a weight. If $\operatorname{Re}\alpha$ is large (in fact >1 is enough), then g vanishes on $\partial\mathbb{B}$ (for fixed z of course) and is then a weight with compact support. More specifically, if

$$\omega=\frac{i}{2}\partial\frac{\bar\partial|\zeta|^2}{1-|\zeta|^2},\quad \omega_k=\omega^k/k!,$$

then

$$g_{n,n}=c_\alpha\Big(\frac{1-|\zeta|^2}{1-\bar\zeta\cdot z}\Big)^{\alpha+n}\omega_n,$$

where

$$c_\alpha=(-1)^n n!\frac{1}{\pi^n}\frac{\Gamma(-\alpha+1)}{\Gamma(n+1)\Gamma(-\alpha-n+1)}.$$

Using that $\Gamma(n+1)=n!$ and $\Gamma(\tau+1)=\tau\Gamma(\tau)$ we get

$$c_\alpha=\frac{1}{\pi^n}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}.$$

We claim that

$$\omega_n=\frac{dV(\zeta)}{(1-|\zeta|^2)^{n+1}}. \tag{4.2}$$

To see this it is enough to see that both sides coincide after application by $\delta=\delta_\zeta/2\pi i$. Notice that

$$\delta\partial\bar\partial|\zeta|^2=\bar\partial|\zeta|^2,\quad \delta\omega=\frac{i}{2}\frac{\bar\partial|\zeta|^2}{(1-|\zeta|^2)^2}.$$

Thus

$$\delta\omega_n=\delta\omega\wedge\omega_{n-1}=\Big(\frac{i}{2}\Big)^n\frac{\bar\partial|\zeta|^2}{(1-|\zeta|^2)^2}\wedge\partial\frac{\bar\partial|\zeta|^2}{(1-|\zeta|^2)^2}=\Big(\frac{i}{2}\Big)^n\frac{\bar\partial|\zeta|^2\wedge(\partial\bar\partial|\zeta|^2)^{n-1}}{(1-|\zeta|^2)^{n+1}}$$

$$\Big(\frac{i}{2}\Big)^n\delta\frac{(\partial\bar\partial|\zeta|^2)^n}{(1-|\zeta|^2)^{n+1}}=\delta\frac{dV}{(1-|\zeta|^2)^{n+1}},$$

and thus (4.2) holds. It is clear that all terms in g will vanish on the boundary if $\operatorname{Re}\alpha$ is large. Summing up we get, for $\operatorname{Re}\alpha$ large the representation

$$\phi(z)=\frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}\int_{|\zeta|<1}\frac{\phi(\zeta)dV_\alpha(\zeta)}{(1-\bar\zeta\cdot z)^{n+\alpha}},\quad \phi\in\mathcal{O}(\overline{\mathbb{B}}), \tag{4.3}$$

where

$$dV_\alpha=(1-|\zeta|^2)^{\alpha-1}dV.$$

However, $(1-|\zeta|^2)=(1+|\zeta|)(1-|\zeta|)\sim 2(1-|\zeta|)$, i.e., roughly speaking the distance from ζ to the boundary, and hence the right hand side has meaning as a convergent integral for all α with $\operatorname{Re}\alpha>0$. Moreover, it depends holomorphically on α, and by the uniqueness theorem the equality must hold for all α with $\operatorname{Re}\alpha>0$.

One can verify that the integral in (4.3) converges for any $\phi\in L^2(dV_\alpha)$. Since the kernel is self-adjoint it follows that the integral operator P_α so defined must be the orthogonal projection $L^2(dV_\alpha)\to\mathcal{O}(\mathbb{B})\cap L^2(dV_\alpha)$, the so-called Bergman projection. $\square$

There is a similar formula in a strictly pseudoconvex domain, cf., Example 8.5. Let us also mention a representation formula that works for polynomials.

Example 4.4. In $\mathbb{C}^n$ we can take the weight

$$g = \Big(1 - \nabla_{\zeta - z} \frac{\partial|\zeta|^2}{2\pi i(1+|\zeta|^2)}\Big)^{m+n} = \Big(\frac{1+\bar\zeta\cdot z}{1+|\zeta|^2} + \frac{1}{\pi}\Omega\Big)^{m+n}$$

for positive integers m, where

$$\Omega = \frac{i}{2}\partial\bar\partial \log(1+|\zeta|^2).$$

We then get the representation

$$f(z) = \int f(\zeta) \frac{(m+n)!}{\pi^n m!} \Big(\frac{1+\bar\zeta\cdot z}{1+|\zeta|^2}\Big)^m \Omega_n,$$

where

$$\Omega_n = \Omega^n/n! = \frac{dV}{(1+|\zeta|^2)^{n+1}},$$

for polynomials f of degree $\leq m$. In fact, for fixed z and big R we have, in view of (4.1), the equality

$$f(z) = \int_{|\zeta|<R} f(\zeta) \frac{(m+n)!}{m!} \Big(\frac{1+\bar\zeta\cdot z}{1+|\zeta|^2}\Big)^m \Omega_n + \int_{|\zeta|=R} f(\zeta) u \wedge g,$$

where $u(\zeta) = B(\zeta - z)$ is the Bochner-Martinelli form with pole at z. It is not too hard to check that the boundary integral tends to zero when $R \to \infty$. As in the previous example, the associated projection onto the polynomials is orthogonal. $\square$

5. Singular weights

A main interest for us is formulas for division and interpolation. This leads us to consider weights that are non-smooth.

Lemma 5.1. *If g a smooth form in $\mathcal{L}^0(\Omega)$ such that $\nabla_{\zeta-z} g = 0$, then g is a weight with respect to z if and only if*

$$\int g \wedge g' = 1 \tag{5.1}$$

for each smooth weight g' with respect to z with compact support in Ω.

It is in fact enough to check the condition for just one weight with compact support as will be clear from the proof.

Proof. If g is a weight, then $g \wedge g'$ is a weight with compact support, and so (5.1) follows from (4.1). Assume that $h \in \mathcal{L}^0(\Omega)$ has compact support and $\nabla_{\zeta-z} h = 0$. Then it follows from the proof of Proposition 4.2 that

$$\int h = h_{0,0}.$$

If we know that $\nabla_{\zeta-z} g = 0$ and g' is aweight with compact support, therefore the integral in (5.1) is equal to $(g \wedge g')_{0,0}(z) = g_{0,0}(z)$, and thus $g_{0,0}(z) = 1$ so that g is a weight. $\square$

If g is any current in $\mathcal{L}^0(\Omega)$ such that $\nabla_{\zeta-z}g=0$, then we say that g is a *(singular) weight with respect to* z if (5.1) holds for any smooth weight with compact support. In view of Lemma 5.1 this definition is consistent with the previous one when g is smooth. For instance, $[z]$ is a weight with respect to z.

Proposition 5.2. *If g is a singular weight with respect to z with compact support in Ω and ϕ is holomorphic, then*

$$\int g\phi = \phi(z). \tag{5.2}$$

More generally, if $\phi \in \mathcal{L}^0(\Omega)$ and $\nabla_{\zeta-z}\phi = 0$, then

$$\int g\wedge\phi = \phi_{0,0}(z). \tag{5.3}$$

Proof. Let χ be a curoff function that is 1 in a neighborhood of z and let u be a form that fulfills (3.2) and is smooth outside z. Then $g'=\chi-\bar\partial\chi\wedge u$ is a smooth weight with compact support, and so

$$\int g\wedge g' = 1.$$

Notice that $w=(\chi-1)u$ is smooth and $\nabla_{\zeta-z}w = g'-1$. Thus $\nabla(g\wedge w) = g\wedge g' - g$. Now (5.2) follows from Stokes' theorem.

If $\phi_{0,0}=1$, then then $g'\wedge\phi$ is a smooth weight with compact support, and $g\wedge\phi = g\wedge g'\wedge\phi$, so by definition (5.2) holds. If $\phi_{0,0}(0)\neq 0$ we apply the result to $\phi/\phi_{0,0}(0)$. If $\phi_{0,0}(0)=0$, we apply the same argument to $\phi+1$. □

Notice that $g'=\chi_D-\bar\partial\chi_D\wedge u$ is a singular weight with compact support. Thus Proposition 4.2 follows from (5.2).

When we deal with singular weights we must avoid formulas like (4.1) with boundary integrals. Instead we use smooth regularizations of χ_D. That is, we choose the smooth weight

$$g'=\chi-\bar\partial\chi\wedge u$$

with compact support and get the representation

$$\phi(z)=\int\chi g\phi-\int\bar\partial\chi\wedge u\chi g\phi$$

as the analoge of (4.1). In general we also want a holomorphic dependence of z. Let us first consider the ball again.

Example 5.3. Notice that if $|z|<|\zeta|$, then

$$s=\frac{1}{2\pi i}\frac{\partial|\zeta|^2}{|\zeta|^2-\bar\zeta\cdot z}$$

is defined, and $\delta_{\zeta-z}s=1$. Let K be the closed unit ball and assume that χ is a cutoff function the the ball $\mathbb{B}(0,R)$, $R>1$, that is 1 in a neighborhood of K. Then for each z in a neighborhood of K,

$$g'=\chi+\bar\partial\chi\wedge s/\nabla_{\zeta-z}s$$

is a weight, with respect to z, with compact support in Ω, that depends holomorphically on z. If g is any weight with respect to z we thus have the representation

$$\phi(z)=\int g'\wedge g\phi.$$

Notice that

$$\frac{s}{\nabla_{\zeta-z}} = \sum_{k=1}^{n} s\wedge(\bar\partial s)^{k-1} = \sum_{k=1}^{n}\frac{1}{(2\pi i)^k}\frac{\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{k-1}}{(|\zeta|^2-\bar\zeta\cdot z)^k}.$$

□

There is a similar weight for any Stein compact K.

Example 5.4. Let $K \subset\subset \Omega$ be a Stein compact and χ a cutoff function in Ω that is 1 in a neighborhood of K. Then there is a $(1,0)$-form $s(\cdot, z)$ on the support of $\bar\partial\chi$, depending holomorphically on z in a neighborhood of K, such that $\delta_{\zeta-z}s = 1$. In fact, to begin with can find, for a fixed ζ', a form a, depending holomorphically on z, such that $\delta_{\zeta-z}a = 1$ at $\zeta = \zeta'$ (since the tuple $\zeta'_j - z_j$ is non-vanishing on the Stein compact K there are holomorphic functions a_j in a neighborhood of K such that $a_1(\zeta'_1 - z_1) + \cdots + a_n(\zeta'_n - z_n) = 1/2\pi i$). Then for ζ close to ζ', $s = a/\delta_{\zeta-z}a$ is holomorphic in a neighborhood of K and $\delta_{\zeta-z}s = 1$. By a partition of unity we obtain the desired form. Now $g = \chi+\bar\partial\chi\wedge s/\nabla_{\zeta-z}s$ is a weight with respect to each z in a neighborhood of K, with compact support in Ω, and depending holomorphically on z. □

Lemma 5.5. *If w is any current in $\mathcal{L}^{-1}(\Omega)$, then*

$$g = 1 + \nabla_{\zeta-z}w \tag{5.4}$$

is a weight with respect to z.

Proof. In fact, if g' is a smooth weight with compact support, then

$$\int g\wedge g' = \int g' + \int \nabla_{\zeta-z}(w\wedge g') = 1$$

since the the last term in the expression in the middle vanishes by Stokes' theorem. □

Remark 5.6. In fact, any weight g has the form (5.4), and if g is smooth, then w can be chosen to be smooth. Here is a sketch of a proof: Take $h = g - 1$. Assume that $z = 0$ and write $\nabla = \nabla_\zeta$. If $\nabla w' = h$ in a neighborhood of 0 and $\nabla w'' = h$ outside 0 and χ is a suitable cutoff function, then

$$w = \chi w' + (1-\chi)w'' + \bar\partial\chi\wedge u\wedge(w' - w'')$$

solves $\nabla w = h$ globally. Outside 0 we can take $w = u \wedge h$. It is thus enough to solve $\nabla w = h$ in a neighborhood of 0. By solving a sequence of $\bar\partial$-equations in this neighborhood, starting with $\bar\partial v_{n,n-1} = h_{n,n}$, we find that

$$h = \alpha + \nabla v,$$

where α is holomorphic. The assumption

$$\int h\wedge g' = 0$$

for smooth weights with compact support, implies that

$$0 = \int \alpha g' = \alpha(0).$$

It is now possible to find a holomorphic β such that $\nabla\beta = \delta_z\beta = \alpha$. Thus $w = v + \beta$ will do. □

Remark 5.7. In view of the previous remark it is easy to check that if $g \in \mathcal{L}^0(\Omega)$ and $\nabla_{\zeta-z} g = 0$, then g is a weight if and only if $\int g \wedge g'$ for some smooth weight with compact support. □

Remark 5.8. Remark 5.6 can be put in a more general context. The first argument shows that the cohomology of the $\nabla_{\zeta-z}$-complex

$$\stackrel{\nabla_{\zeta-z}}{\to} \mathcal{L}^k(\Omega) \stackrel{\nabla_{\zeta-z}}{\to} \mathcal{L}^{k+1}(\Omega) \stackrel{\nabla_{\zeta-z}}{\to}$$

coincides with the cohomology of the germs at z, i.e., $H^k(\mathcal{L}^\bullet(\Omega)) = H^k(\mathcal{L}^\bullet_z)$. Next notice that $\mathcal{L}^\bullet_z, \nabla_{\zeta-z}$ is the total complex associated with the double complex $(M_{\ell,k}.d', d'') = (\mathcal{C}_{k,-\ell}, \delta_{\zeta-z}, \bar\partial)$. Since it is exact with respect to $\bar\partial$ except at level 0, where the (co)homology is $\mathcal{O}_z^{-\ell}$, the module of germs of holomorphic $-\ell$-forms, it follows from Lemma 5 that

$$H^k(\mathcal{L}^\bullet_z) = H^{-k}(\mathcal{O}^\bullet_z).$$

It is well-known, cf., ???, that the right hand side vanishes except at $k = 0$, where it is $\mathbb{C}$. Precisely the same argument works for smooth forms.

Check details!!! □

6. A glimse of division-interpolation formulas

We shall now consider a simple example of a division-interpolation formula. Let f be holomorphic in the ball $\mathbb{B}(0, R)$ and let h be a holomorphic $(1, 0)$-form such that $\delta_{\zeta-z} H = f(z) - f$.

Remark 6.1. Such a form h is called a Hefer form for f. It can be obtained elementarily in a convex domain like the ball. In fact,

$$f(z) - f(\zeta) = \int_0^1 d_t f(z + t(\zeta - z)) = \sum_1^n (\zeta_j - z_j) \int_0^1 \frac{\partial f}{\partial w_j}(z + t(\zeta - a))dt$$

so we can take $h = h_1(\zeta, z)d\zeta_1 + \cdots + h_n(\zeta, z)d\zeta_n$ where

$$h_j(\zeta, z) = \int_0^1 \frac{\partial f}{\partial w_j}(z + t(\zeta - a))dt.$$

□

Let us assume that f is not identically zero. From distribution theory it is known that there is a distribution U such that such that $fU = 1$. Before long we will discuss a canonical choice but for the moment any such U will do. Notice that $R = \bar\partial U$ is a $(0, 1)$-current. We claim that if ϕ is holomorphic, then $\phi = fv$ for a holomorphic function v if and only if $\phi R = 0$. In fact, $\phi = fv$ for a holomorhic v if and only if $\phi U = fvU = v$ is holomorhic, which holds if and only if $\bar\partial(\phi U) = \phi R = 0$.

In view of Lemma 5.5, $g' = 1 + \nabla_{\zeta-z} hU$ is a weight with respect to z. A simple computation reveals that

$$g' = f(z)U + h \wedge R.$$

Let g be a weight with compact support in $\mathbb{B}(0, R)$ and depending holomorphically on z in a neighborhood of the closed unit ball K. If $\Phi \in \mathcal{O}(\mathbb{B}(0, R))$ we thus have the representation

$$\Phi(z) = f(z) \int U g\Phi + \int h \wedge R \wedge g\Phi. \tag{6.1}$$

Notice that the second term is holomorphic in a neighborhood of K. It is the remainder when trying to divide Φ by f. If ϕ is any section of the quotient sheaf $\mathcal{O}/(f)$ over $\mathbb{B}(0,R)$ it is the image of a global Φ (since the ball is Stein) or in other words, Φ is an extension of ϕ. It follows from (6.1) that

$$\Psi(z) = \int h \wedge R \wedge g\phi$$

is a holomorphic extension of ϕ to a neighborhood of of K.

Remark 6.2. In this argument we relied on the a priori existence of a global extension Φ of ϕ. One can prove directly that Ψ is an extension. To this end fix a point z^0 and let Φ be a local extension at z^0. Choose a weight $\tilde{g}$ with support close to z^0. Then by the argument above

$$\tilde{\Phi}(z) = \int h \wedge R \wedge g \wedge \tilde{g}\phi$$

is a local extentsion of ϕ. Now, $1-\tilde{g} = \nabla w$ where w is smooth, and $\nabla(h\wedge R) = f(z)R$, so

$$H \wedge R \wedge g \wedge (1-\tilde{g})\phi = \nabla(h \wedge R \wedge g \wedge (1-\tilde{g})\phi) + f(z)R \wedge g \wedge w\phi.$$

By Stokes' theorem we have, for z close to z^0, that

$$\Psi(z) - \tilde{\Phi}(z) = \int h \wedge R \wedge g \wedge (1-\tilde{g})\phi = f(z)\int R \wedge g \wedge w\phi.$$

We conclude that also Ψ is a local extension of ϕ at z^0. □

Let now ϕ be a smooth function such that $\phi = fa$ where a is smooth. Then $\partial^\alpha_{\bar\zeta}\phi = f\partial^\alpha_{\bar\zeta}a$ and hence

$$\big(\partial^\alpha_{\bar\zeta}\phi\big)R = 0 \tag{6.2}$$

for all multiindices α. Indeed we have

Theorem 6.3. *If f is holomorphic and ϕ is smooth, then $\phi = fa$ where a is smooth if and only if* (6.2) *holds for all α.*

First proof. Assume that (6.2) holds and let $a := \phi U$. Then $fa = \phi f U = \phi$, so we have to prove that a is smooth. The hypothesis implies that

$$\partial^\alpha_{\bar z}a = (\partial^\alpha_{\bar z}\phi)U \tag{6.3}$$

for all α. Since R has some finite order N' it belongs to some Sobolev space W^{-N}. Moreover, if $\bar\partial\psi \in W^r$ then $\psi \in W^{r+1}$. From (6.3) we can thus conclude that $a \in W^k$ for all k, which implies that a is smooth. The last statement follows for instance from the Fourier transform. □

We shall now give a proof based on integral representation that we shall generalize, later on, to ideals generated by more than one function f. Since the statement is local, assume we are in a ball $X \subset \mathbb{C}^n$, identify X with the set $\{(\zeta,\bar\zeta) \in \mathbb{C}^{2n};\ \zeta \in X\}$ and let $\widetilde{X}$ be an open neighborhood of X in $\mathbb{C}^{2n}_{\zeta,\omega}$. Assume that ϕ is smooth in a neighborhood of the closure of X and consider

$$\tilde\phi(\zeta,\omega) = \sum_\alpha (\partial^\alpha_{\bar\zeta}\phi)(\zeta)\frac{(\omega-\bar\zeta)^\alpha}{\alpha!}\chi(\lambda_{|\alpha|}(\omega-\bar\zeta)), \tag{6.4}$$

where χ is a cutoff function in $\mathbb{C}^n$ which is 1 in a neighborhood of 0, and λ_k are positive numbers. If $\lambda_k \to \infty$ fast enough, the series converges to a smooth function in $\widetilde{X}$ such that

$$\tilde{\phi}(\zeta, \bar{\zeta}) = \phi(\zeta),$$

and

$$\bar{\partial}\tilde{\phi}(\zeta, \omega) = \mathcal{O}(|\omega - \bar{\zeta}|^\infty).$$

Such a function $\tilde{\phi}$ is called an almost holomorphic extension of ϕ. If ϕ is realanalytic one can take $\lambda_k = 1$ for all k and then $\tilde{\phi}$ is the holomorphic extension of ϕ.

Lemma 6.4. *Let v^z denote the Bochner-Martinelli form in $\widetilde{X}$ with respect to the point $(z, \bar{z})$, and let*

$$\Phi^z(\zeta, \omega) = \tilde{\phi}(\zeta, \omega) - \bar{\partial}\tilde{\phi} \wedge v^z.$$

Then Φ^z is smooth in ζ, ω, z and $\nabla_{(\zeta,\omega)-(z,\bar{z})}\Phi^z = 0$. Moreover, if (6.2) *holds for all α, then $\Phi^z \wedge (R \otimes 1) = 0$.*

Proof. Since

$$v^z = \frac{b}{\nabla_{(\zeta,\omega)-(z,\bar{z})} b},$$

where $b = \sum_1^n (\zeta_j - z_j) d\zeta_j + \sum_1^n (\omega_j - \bar{z}_j) d\omega_j$, we have that

$$\Phi^z(\zeta, \omega) = \tilde{\phi}(\zeta, \omega) + \sum_{\ell=1}^{2n} \frac{\mathcal{O}(|\omega - \bar{\zeta}|^\infty)}{(|\zeta - z|^2 + |\omega - \bar{z}|^2)^{\ell - 1/2}},$$

if ϕ is smooth, and thus Φ^z is smooth. If (6.2) holds for all α, then also $(\partial_{\bar{\zeta}}^\alpha \bar{\partial}\phi) \wedge R = 0$ for all α and therefore $\tilde{\phi} \wedge R \otimes 1 = 0$ and $\bar{\partial}\tilde{\phi} \wedge R \otimes 1 = 0$. □

Second proof of Theorem 6.3. Notice that $f \otimes 1 \cdot U \otimes 1 = 1$ in $\widetilde{X}$. From Lemma 5.2 and Lemma 6.4 we have the representation

$$\Phi^z_{0,0}(z, \bar{z}) = f(z) \int_{\widetilde{X}} U(\zeta) g \wedge \Phi^z(\zeta, \omega) + \int_{\widetilde{X}} \Phi^z(\zeta, \omega) \wedge h \wedge R(\zeta).$$

If now (6.2) holds for all α, then the second integral vanishes. The first integral depends smoothly on z, and $\Phi^z_{0,0} = \tilde{\phi}(z, \bar{z}) = \phi(z)$ and so Theorem 6.3 is proved. □

7. Koppelman formulas in domains in $\mathbb{C}^n$

If $f = f_1 d\zeta$ is a smooth $(0,1)$-form in $D \subset \mathbb{C}$ such that

$$\int_D |f_1| dV < \infty,$$

then

$$u(z) = \int_D \omega_{\zeta - z} \wedge f$$

is a smooth solution to $\bar{\partial} u = f$ in D; if f has compact support it follows from (4.2) and (1.2), the general case follows by writing $f = \chi f + (1 - \chi) f$. We shall now consider multivariable analogues.

Let $B_{n,n-1}(\eta)$ be the Bochner-Martinelli form and let $\eta\colon \mathbb{C}^n_\zeta \times \mathbb{C}^n_z \to \mathbb{C}^n_\eta$ be the mapping $\eta(\zeta, z) = \zeta - z$. We let $B_{n,n-1}(\zeta - z) := \eta^* B_{n,n-1}$ in $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z$. It is a smooth form outside the diagonal Δ in $\mathbb{C}^n \times \mathbb{C}^n$ that is locally integrable on $\mathbb{C}^n \times \mathbb{C}^n$ in view of (2.4). Notice that η is minus the mapping ν in Section 4, but notice also that $B(\zeta - z) = B(z - \zeta)$. Notice also that $B(\zeta - z)$ is locally integrable on $\mathbb{C}^n \times \mathbb{C}^n$. Since $\bar\partial B_{n,n-1} = [0]$ it follows from (4.2) that

$$\bar\partial B_{n,n-1}(z-\zeta) = [\Delta]. \tag{7.1}$$

It is instructive, however, to give a direct argument.

Proof of (7.1). We have to check that

$$\int_\zeta \int_z B_{n,n-1}(z-\zeta) \wedge \bar\partial \psi(\zeta, z) = \int_z \psi(z,z) \tag{7.2}$$

for any form $\psi(\zeta, z)$ of total bidegree (n, n). Here $\psi(z, z)$ means the pullback of ψ to the diagonal $\Delta = \{(z, z);\ z \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n$, i.e., $i^*\psi$, where $i\colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$, $z \mapsto (z, z)$. Notice that $d = d_\zeta + d_z$ and $\bar\partial = \bar\partial_\zeta + \bar\partial_z$. Since $\bar\partial$ commutes with pullbacks of holomorphic mappings, by a complex-linear change of variables on $\mathbb{C}^n \times \mathbb{C}^n$, keeping in mind that the orientation is preserved, and that Fubini's theorem holds, the integral on the left hand side of (7.2) becomes

$$-\int_z \int_\zeta \bar\partial\psi(\zeta + z, \zeta) \wedge B_{n,n-1}(\zeta) =$$
$$\int_z \Big[\int_\zeta \bar\partial_\eta \psi(\zeta + z, z) \wedge B_{n,n-1}(\zeta)\Big] + \int_\zeta \Big[\int_z \bar\partial_z \psi(\zeta + z, z)\Big] \wedge B_{n,n-1}(\zeta).$$

In the first inner integral, for degree reasons only components of ψ which have bidegree $(0, 0)$ in ζ can give a contribution, and in view of (2.2) the inner integral therefore becomes $-\psi(z, z)$. In the inner integral in the second term for degree reasons one can replace $\bar\partial_z$ by d_z, and then the integral vanishes by Stokes' theorem. □

Proposition 7.1 (Koppelman's formula)**.** *Let K be a form in $\Omega \times \Omega$ of bidegree $(n, n-1)$ that is locally integrable and smooth outside the diagonal, and such that $\bar\partial K = [\Delta]$. Let $K_{p,q}$ be the component of bidegree (p, q) in z, and consequently $(n-p, n-q-1)$ in ζ. If f is a smooth (p, q)-form, then*

$$f(z) = \bar\partial_z \int_D K_{p,q-1} \wedge f + \int_D K_{p,q} \wedge \bar\partial f + \int_{\partial D} K_{p,q} \wedge f, \quad z \in D.$$

It is clear that if we can make the boundary integral disappear, then for each f such that $\bar\partial f = 0$, we get a solution to $\bar\partial u = f$.

Proof. Let us first assume that $f(\zeta)$ has compact support and let $\psi(z)$ be a test form of bidegree $(n-p, n-q)$. Now,

$$\int_z \psi(z) \wedge f(z) = -\int\int \bar\partial(\psi(z) \wedge f(\zeta)) \wedge K =$$
$$-\int_z \bar\partial_z \psi(z) \wedge \int_\zeta f \wedge K - (-1)^{p+q} \int_z \psi(z) \wedge \int_\zeta \bar\partial f \wedge K.$$

In the first term we can integrate by parts in the z-integral. After moving f and $\bar\partial f$ to the right in the ζ-integrals we then get the equality

$$\int \psi(z) \wedge f(z) = \int_z \psi(z) \wedge \bar\partial_z \int_\zeta K \wedge f + \int_z \psi(z) \wedge \int_\zeta K \wedge \bar\partial f$$

which is equivalent to the theorem in case f has compact support. The general case can now be deduced, e.g., by replacing f by $\chi_k f$ where $\chi_k \nearrow \chi_D$ and take limits, or by mimicking the proof of Proposition 1.1. □

Certainly $K = \eta^* B$ fulfills the hypothesis in this proposition. We will now look at more interesting choices. Let $b(\zeta - z) = \eta^* b(\zeta, z) = \sum_1^n \bar\eta_j d\eta_j / 2\pi i |\eta|^2$ and let

$$s(\zeta, z) = \sum_1^n s_j(\zeta, z) d(\zeta_j - z_j)$$

be a form in $\Omega \times \Omega$ such that $2\pi i \sum s_j(\zeta_j - z_j) = 1$ outside $\Delta \subset \Omega \times \Omega$, and $s(\zeta, z) = b(\zeta - z)$ in a neighborhood of Δ. Let us temporarily call such a form *admissible.* Thus $0 = \sum_j \eta_j \bar\partial s_j$ and precisely as in the proof of Lemma 2.1 we conclude that

$$K = s \wedge (\bar\partial s)^{n-1}$$

is $\bar\partial$-closed outside Δ. Since $K = B_{n,n-1}(z - \zeta)$ in a neighborhood of Δ we thus have, cf., (7.1), that $\bar\partial K = [\Delta]$.

Lemma 7.2. *If f is a smooth (p, q)-form, then*

$$\int_{\zeta \in D} K_{p,q-1}(\zeta, z) \wedge f(\zeta)$$

is a smooth $(p, q-1)$-form in D.

Proof. Fix a point z^0. If ω is a small enough neighborhood of z^0, then $s = b$ for all $\zeta \in \omega$ and z close to z^0. Take a cutoff function χ in ω such that $\chi = 1$ in a neighborhood of z^0, and consider the decomposition

$$u(z) = \int_D (1 - \chi) K \wedge f + \int \chi K \wedge f.$$

The first term is smooth in a neighborhood of z^0 since there is no singularity in the integral. On the other hand, for z close to z^0 the second integral is

$$\int (\chi f)(\zeta) \wedge B_{n,n-1}(z - \zeta) = (B_{n,n-1} * (\chi f))(z),$$

i.e., convolution of $B_{n,n-1}$ with a test form, and thus it is smooth. □

Example 7.3 (The Dolbeault-Grothendieck lemma). Assume that f is a smooth (p, q)-form that is $\bar\partial$-closed in a the unit ball $\mathbb{B}$ and $q \geq 1$. Then there is a smooth $(p, q-1)$-form u in $r\mathbb{B}$, $r < 1$, such that $\bar\partial u = f$.

We will use the notation $\bar\zeta \cdot d(\zeta - z)$ for $\sum_1^n \bar\zeta_j d(\zeta_j - z_j)$, etc. Let χ be a cutoff function in $\mathbb{B}$ that is identically 1 in a neighborhood of the closure of $r\mathbb{B}$, $r < 1$. Then

$$s(\zeta, z) = \chi(\zeta) b(\zeta - z) + (1 - \chi(\zeta)) \frac{\bar\zeta \cdot d(\zeta - z)}{2\pi i(|\zeta|^2 - \bar\zeta \cdot z)}$$

is an admissible form for z in $r\mathbb{B}$, and for ζ close to $\partial\mathbb{B}$ it is holomorphic in z. (One can extend it to an admissible form for $z \in \mathbb{B}$ as well by taking $\tilde\chi(z)s + (1 - \tilde\chi(z))b(\zeta - z)$ where $\tilde\chi$ is identically 1 in a neighborhood of the support of χ; but this is immaterial for us, since we just bother about z in $r\mathbb{B}$.)

If $q \geq 1$ it follows that $K_{p,q} = 0$ if $z \in r\mathbb{B}$ and ζ is close to $\partial\mathbb{B}$, since then no $d\bar z$ can occur. Therefore the boundary integral vanishes and we get

$$f(z) = \bar\partial_z \int_{\mathbb{B}} K_{p,q-1} \wedge f + \int_{\mathbb{B}} K_{p,q} \wedge \bar\partial f, \quad z \in r\mathbb{B}.$$

If in addition $\bar\partial f = 0$ in $\mathbb{B}$ we thus get a solution in $r\mathbb{B}$. □

Remark 7.4. With the notation in Koppelman's formula one can define the kernel K as $s \wedge (ds)^{n-1}$ instead. It is then still true that $dK = 0$ outside Δ. One can then prove the slightly more general Koppelman formula

$$f(z) = d_z \int_D K \wedge f + \int_D K \wedge df + \int_{\partial D} K \wedge f. \tag{7.3}$$

When restricting to the component K' of K that is $(n, n-1)$ in $dz, d\zeta$ we get back the previous Koppelman formula, but (7.3) also contains other relations that sometimes are useful. □

Example 7.5 (The Dolbeault-Grothendieck lemma for currents). Assume that f is a $\bar\partial$-closed (p,q)-current in the unit ball $\mathbb{B}$. If $q \geq 1$, then there is a current u in $r\mathbb{B}$ such that $\bar\partial u = f$. If $q = 0$, then f is holomorphic. Multiplying with a cutoff function we may assume that f has compact support in $\mathbb{B}$ and is $\bar\partial$-closed in $r'\mathbb{B}$, where $r < r' < 1$. Let $B_{n,n-1}$ be the Bochner-Martinelli form so that $\bar\partial B_{n,n-1} = [0]$. Since f has compact support, thus $\bar\partial B_{n,n-1} * f = f - B_{n,n-1} * \bar\partial f$ by (4.2). Moreover, since $\bar\partial f = 0$ in $r'\mathbb{B}$,

$$B_{n,n-1} * \bar\partial f(z) = \int_\zeta B_{n,n-1}(z - \zeta) \wedge \bar\partial f(\zeta)$$

for $z \in r'\mathbb{B}$ and is smooth in z there. Furthermore, it is $\bar\partial$-closed there since both the other terms are. Thus we can solve $\bar\partial v = B_{n,n-1} * \bar\partial f$ in $r\mathbb{B}$, cf., Example 7.3, and hence $\bar\partial(B_{n,n-1} * f + v) = f$ in $r\mathbb{B}$. □

8. Weighted Koppelman formulas

Let Λ be the exterior algebra over the subbundle of $T^*(X \times X)$ spanned by $T^*_{0,1}(X \times X)$ and the differentials $d\eta_1, \ldots, d\eta_n$. In this section all forms will take values in Λ. We let δ_η denote formal interior multiplication with

$$2\pi i \sum_1^n \eta_j \frac{\partial}{\partial \eta_j},$$

on this subbundle, i.e., such that $(\partial/\partial\eta_j) d\eta_k = \delta_{jk}$. Moreover, we let $\nabla_\eta = \delta_\eta - \bar\partial$. Now $\bar\partial$ acts on both variables ζ and z. Let

$$b = \frac{\eta \cdot d\eta}{2\pi i |\zeta|^2} = \frac{\sum_j (\bar\zeta_j - \bar z_j) d\eta_j}{2\pi i |\zeta - z|^2} = \frac{\partial |\zeta - z|^2}{2\pi i |\zeta - z|^2}.$$

and consider the Bochner-Martinelli form

$$\eta^* B = \frac{b}{\nabla_\eta b} = b + b \wedge (\bar\partial b) + \cdots + b \wedge (\bar\partial b)^{n-1}.$$

Notice that

$$\eta^* B_{k,k-1} = b \wedge (\bar\partial b)^{k-1} = \frac{1}{(2\pi i)^k} \frac{\partial |\eta|^2 \wedge (\bar\partial \partial |\eta|^2)^{k-1}}{|\eta|^{2k}}$$

so that

$$B_{k,k-1} = \mathcal{O}(1/|\eta|^{2k-1}). \tag{8.1}$$

Proposition 8.1. *The form $u = b/\nabla_\eta b$ is locally integrable in $\mathbb{C}^n \times \mathbb{C}^n$ and solves*

$$\nabla_\eta u = 1 - [\Delta]. \tag{8.2}$$

We already know that $\bar\partial u_{n,n-1} = [\Delta]$. The rest of the proof is completely analogous to the proof of Proposition 3.2.

Proposition 8.2. *If u is any smooth form in $X \times X \setminus \Delta$, with values in Λ, such that $\nabla_\eta u = 1$ and such that* (8.1) *holds locally at the diagonal. Then u is locally integrable in $X \times X$ and* (8.2) *holds.*

This is verified precisely as Proposition 3.8.

Example 8.3. Assume that $s(\zeta, z)$ is a smooth form in $X \times X$ such that

$$|s| \le C|\zeta|, \qquad |\langle s, \eta\rangle| \ge C|\eta|^2 \tag{8.3}$$

uniformly locally at the diagonal. Then

$$u = \frac{s}{\nabla_\eta s} = \frac{s}{2\pi i\langle s, \eta\rangle} + \cdots + \frac{s \wedge (\bar\partial s)^{n-1}}{(2\pi i)^n \langle s, \eta\rangle^n}$$

fulfills the hypothesis in Proposition 8.2. □

We say that a (smooth) form $g = g_{0,0} + \cdots + g_{n,n}$ in $\Omega \times \Omega$ with values in Λ is a *weight* if $\nabla_\eta g = 0$ and $g_{0,0} = 1$ on the diagonal Δ. As before, if g, g' are weights, then $g \wedge g'$ is a weight. If $w = w_{1,0} + \cdots + w_{n,n-1}$ is a smooth form, then $g = 1 + \nabla_\eta w$ is a weight.

If g is a weight and u is a locally integrable form that satisfies (8.2), then

$$\nabla_\eta (g \wedge u) = g - [\Delta].$$

If we let $K = (g \wedge u)_{n,n-1}$ and $P = g_{n,n}$ we thus have

$$\bar\partial K = [\Delta] - P$$

which leads to the weighted Koppelman formula

$$f(z) = \bar\partial \int_D K_{p,q-1} \wedge f + \int_D K_{p,q} \wedge \bar\partial f + \int_{\partial D} K_{p,q} \wedge f + \int_D P_{p,q} \wedge f, \quad f \in \mathcal{E}_{p,q}(\overline{D}), \tag{8.4}$$

cf., Proposition 7.1. Again $K_{p,q}$ denotes the component of bidegree (p, q) in z and hence $(n-p, n-q-1)$ in ζ.

In order to obtain a solution formula for $\bar\partial$ we must get rid of the last two terms. If $g = G(\nabla_\eta w_{1,0})$, $G(0) = 1$ and $w_{1,0}$ depends holomorphically on z, then $P_{p,q} = 0$ for $q > 0$, and so we get rid of the last term in the Koppelman formula. If in addition the weight vanishes on the boundary, then also the boundary integral vanishes. Let us consider a couple of examples.

Example 8.4. Let ρ be a convex function in $\mathbb{C}^n$. A Taylor expansion at the point ζ gives $\rho(z) = \rho(\zeta) + 2\mathrm{Re}\,\langle \partial\rho(\zeta), z - \zeta\rangle + Q_2$, where $Q_2 \ge 0$. Thus

$$-2\mathrm{Re}\,\langle \partial\rho(\zeta), \eta\rangle \le \rho(z) - \rho(\zeta). \tag{8.5}$$

Let $H = \sum \partial_{\zeta_j} d\eta_j/\pi i$ and form the weight

$$g = e^{-\nabla_\eta H} = e^{-2\langle \partial\rho(\zeta), \eta\rangle} e^{i\partial\bar\partial\rho}.$$

Let $u = \eta^* B$ be the Bochner-Martinelli form. If f is a smooth $(0, q)$-form in $\mathbb{C}^n$ such that

$$\int |f| e^{-\rho} (i\partial\bar\partial\rho)^{n-k} < \infty, \quad k = 0, \ldots, n-k,$$

then

$$v(z) = \int K_{0,q-1} \wedge f$$

converges and is a solution to $\bar\partial v = f$ in $\mathbb{C}^n$. In fact,

$$K_{0,q-1} \wedge f = e^{-2\langle \partial\rho(\zeta),\eta\rangle} \sum_{k=0}^{n-q} u_{n-k,n-k-1} \wedge (i\partial\bar\partial\rho)^k \wedge f,$$

if $u_{n-k,n-k-1}$ here is the component of u that has bidegree $(0, q-1)$ in z, and in view of (8.5) thus

$$|K_{0,q-1} \wedge f| \leq Ce^{\rho(z)} \sum_{0}^{n-q} \frac{1}{|\zeta - z|^{2(n-k-1)-1}} (i\partial\bar\partial\rho)^k |f(\zeta)| e^{-\rho(\zeta)}.$$

It also follows that the solution v roughly speaking has a growth like $\exp(z)$. □

In a similar way one can use the weight in Example 4.4 and get solutions to $\bar\partial v = f$ when f has polynomial growth in $\mathbb{C}^n$.

Example 8.5 (Weighted Henkin-Ramirez formulas). Let $D = \{\rho < 0\}$ be strictly pseudoconvex with smooth boundary and assume that ρ is a defining function that is strictly plurisubharmonic in a neighborhood of $\overline{D}$. It follows from [**?**] that there is a smooth n-tuple $H(\zeta, z)$ in a neighborhood of $\overline{D \times D}$, holomorphic in z, such that

$$2\mathrm{Re}\, \langle H(\zeta, z), \eta\rangle \geq \rho(\zeta) - \rho(z) + \delta|\eta|^2.$$

If

$$\Phi(\zeta, z) = \langle H(\zeta, z), \eta\rangle - \rho(\zeta)$$

it follows that

$$2\mathrm{Re}\, \Phi \leq -\rho(\zeta) - \rho(z) + \delta|\eta|^2. \tag{8.6}$$

Let $h = H \cdot d\eta/2\pi i$. We form the weight

$$g = \Big(1 - \nabla_\eta \frac{h}{\rho}\Big)^{-\alpha} = \Big(\frac{\Phi}{-\rho} + \bar\partial \frac{h}{\rho}\Big)^{-\alpha}.$$

If we choose $\alpha > 0$ we get, cf., Example 4.3, then $P = g_{n,n}$ behaves like $(-\rho(\zeta))^{\alpha-1}$ for fixed $z \in D$. We get representation formulas, holomorphic in z that are very similar to the ones on the ball.

Moreover, assume that f is smooth and $\bar\partial$-closed in D and has a growth at the boundary at most as a power of $-\rho$. For an appropriate choice of $\alpha > 0$, and with $K = (g \wedge u)_{n,n-1}$, then

$$v = \int_D K_{0,q-1} \wedge f$$

is a solution to $\bar\partial v = f$, since the boundary integral in Koppelman's formula vanishes.

For optimal estimates of the solution, however, it turns out that one should replace the Bochner-Martinelli form u by a form that is better adapted to the local geometry at the boundary. Let $M(\zeta, z) = -H(z, \zeta)$ and let

$$s = \overline{\langle M, \eta\rangle} M \cdot d\eta/2\pi i - \rho(z)|\eta|^2.$$

Then s satisfies (8.3) and thus $u = s/\nabla_\eta s$ will do just as well. Let us compute the boundary values of the resuling solution when f is a $(0, 1)$-form. Notice that when $\rho(z) = 0$, then s is parallell to $M \cdot \eta/\Phi(z, \zeta) = m/\Phi(z, \zeta)$, which is holomorphic in ζ, and hence this term itself is the only contribution to $K_{0,0}$ from u. Thus we just get the term $g_{n-1,n-1}$ from the weight g, which is, cf., the computaion in Example 4.3,

$$c_\alpha\Big(\frac{-\rho}{\Phi}\Big)^{n+\alpha-1}\Big(\bar\partial\frac{h}{-\rho}\Big)^{n-1}.$$

The second factor is

$$\Big(\frac{\bar\partial h}{-\rho}+\frac{\bar\partial\rho\wedge h}{(-\rho)^2}\Big)^{n-1}=\frac{-\rho(\bar\partial h)^{n-1}+(n-1)(\bar\partial h)^{n-2}\wedge\bar\partial\rho\wedge h}{(-\rho)^n}.$$

Thus

$$v(z)=\int_D\frac{(-\rho)^{\alpha-1}m\wedge(\rho\bar\partial h-(n-1)\bar\partial\rho\wedge h)\wedge(\bar\partial h)^{n-2}\wedge f}{\Phi(\zeta,z)^{\alpha+n-1}\Phi(z,\zeta)}.$$

One can check that $|\Phi(z,\zeta)|\sim|\Phi(\zeta,z)|$. If z is a fixed point at the boundary, then $\{\zeta;\ |\Phi(\zeta,z)|<\epsilon\}$ is a so-called Koranyi tent with center at ζ. It has length $\sim\sqrt{\epsilon}$ in the complex-tangential directions, and $\sim\epsilon$ in the last to ones, so that the volume is like ϵ^{n+1}. One can also check that

$$\int_{z\in\partial D}\frac{1}{|\Phi|^{n+\gamma}}\sim\frac{1}{(-\rho(\zeta))^\gamma}$$

if $\gamma>0$. Notice morover that $m=h$ on the diagonal so that $|m\wedge h|\le C|\eta|$. From (8.6) we have that $|\eta|\le\sqrt{|\Phi|}$. Combining, if $\alpha>1/2$, we get the estimate

$$\int_{\partial D}|v|\le C\int_D|f|+(1/\sqrt{-\rho})|\bar\partial\phi\wedge f|.$$

It was proved independently by Henkin and Skoda, and it was the first triumph of weighted integral solution formulas. The previously known solution formula, the Henkin-Ramirez formula, roughly speaking a formula corresponding to $\alpha=0$, does not admit this estimate. $\square$

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Remark 8.6. The statements about Φ follows from the fact that $-d_\zeta\Phi|_{\zeta=z}=d_z\Phi|_{z=\zeta}=\partial\rho(z)$ so that

$$d_\zeta\mathrm{Re}\,\Phi|_{\zeta=z}=-d\rho,\quad d_\zeta\mathrm{Im}\,\Phi|_{\zeta=z}=cd^c\rho,$$

and similarily for d_z, where the constant c depends on the normalization of d^c. Notice that in any case the annihilator of $d\rho$ and $d^c\rho$ is the complex tangent space at ζ. If we choose real coordinates $s_1,\dots,s_{2n}$ such that $ds_1=d\rho$ and $ds_2=d^c\rho$, then $|\Phi|\sim|s_1|+|s_2|+\sum_{j\ge3}s_j^2$. $\square$

Example 8.7. Notice that in the ball,

$$\Phi(\zeta,z)=1-\bar\zeta\cdot z$$

and

$$\langle s,\eta\rangle=|1-\bar\zeta\cdot z|^2-(1-|\zeta|^2)(1-|z|^2).$$

$\square$

Chapter 2

Multivariable residue currents

1. The one-variable case

Given a holomorphic function f in an open domain X in the complex plane, not vanishing identically, we want to define the principal value current (distribution)

$$\Big\langle\Big[\frac{1}{f}\Big],\xi dz\wedge d\bar z\Big\rangle=\lim_{\epsilon\to 0}\int_{|f|>\epsilon}\frac{1}{f}\xi dz\wedge d\bar z. \tag{1.1}$$

Given that this limit exists, clearly $f[1/f]=1$ in X. We then also have residue current

$$\Big\langle\bar\partial\Big[\frac{1}{f}\Big],\xi dz\Big\rangle=\lim_{\epsilon\to 0}\int_{|f|=\epsilon}\frac{1}{f}\xi dz. \tag{1.2}$$

However, the zero set of f is discrete and the attempted definition of $[1/f]$ is local so we may assume that $z=0$ is the only zero on X.

Remark 1.1. If the test form ξdz is holomorphic, the integrals in the right hand side of (1.2) are all equal for small ϵ by Cauchy's theorem and we interpret this number as the residue at 0 (times $2\pi i$) of the meromorphic form $\xi dz/f$. If we fix the holomorphic coordinate z we get the classical notion of residue

$$\text{Res}_{\,z=0}\frac{\xi}{f}:=\frac{1}{2\pi i}\int_{|z|=\epsilon}\frac{\xi}{f}dz.$$

□

Notice that $f(z)=z^m g(z)$ where g is nonvanishing, so locally $f(z)=(z\phi(z))^m$, and we can take $w=z\phi(z)$ as a new holomorphic coordinate. Thus it is enough to consider $f(z)=z^m$. The following proposition is fundamental for residue theory.

Proposition 1.2. *For each integer m and test function $\xi\in\mathcal{D}(\mathbb{C})$ the limit*

$$\Big\langle\Big[\frac{1}{z^m}\Big],\xi dz\wedge d\bar z\Big\rangle=\lim_{\epsilon\to 0}\int_{|z|^2>\epsilon}\frac{\xi dz\wedge d\bar z}{z^m} \tag{1.3}$$

exists and defines a current. We have the following equalities:

$$z\Big[\frac{1}{z^{m+1}}\Big]=\Big[\frac{1}{z^m}\Big], \tag{1.4}$$

$$\frac{\partial}{\partial z}\Big[\frac{1}{z^m}\Big]=-m\Big[\frac{1}{z^{m+1}}\Big],\quad m\ge 1, \tag{1.5}$$

$$\Big\langle\bar\partial\Big[\frac{1}{z^m}\Big],\xi dz\Big\rangle=\lim_{\epsilon\to 0}\int_{|z|^2=\epsilon}\frac{\xi dz}{z^m}=\frac{2\pi i}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}\xi(0),\qquad m\ge 1, \tag{1.6}$$

$$\bar z\bar\partial\Big[\frac{1}{z^m}\Big]=0,\qquad d\bar z\wedge\bar\partial\Big[\frac{1}{z^m}\Big]=0,\quad m\ge 1. \tag{1.7}$$

$$\partial\Big[\frac{1}{z^m}\Big]=-m\Big[\frac{1}{z^{m+1}}\Big]dz,\qquad \partial\bar\partial\Big[\frac{1}{z^m}\Big]=m\bar\partial\Big[\frac{1}{z^{m+1}}\Big]\wedge dz,\quad m\ge 1 \tag{1.8}$$

$$\bar\partial\Big[\frac{1}{z^m}\Big]\wedge\frac{dz^m}{2\pi i}=m[0]. \tag{1.9}$$

Here, as before, $[0]$ denotes the $(1,1)$-current evaluation 0, i.e., $\langle [0], \xi \rangle = \xi(0)$ for test functions ξ, cf., (1.2).

Proof. If $m = 1$, then $1/z$ is locally integrable and (1.3) holds by dominated convergence. By Taylor's formula,

$$\xi(z) = p_m\xi(z) + r_m\xi(z),$$

where $p_m\xi(z)$ is a polynomial of degree $m-1$ and $r_m\xi(z) = \mathcal{O}(|z|^m)$. Consider

$$I = \int_{\epsilon<|z|^2<1} \frac{z^\ell \bar{z}^k dz \wedge d\bar{z}}{z^m}.$$

By the change of variables $z \mapsto \lambda z$ with $|\lambda| = 1$ we find that $I = \lambda^{\ell-k-m} I$, and hence $I = 0$ if $\ell + k < m$. We may assume that ξ has support in the unit disk. Then

$$\int_{|z|^2>\epsilon} \frac{\xi(z) dz \wedge d\bar{z}}{z^m} = \int_{\epsilon<|z|^2<1} \frac{\xi(z) dz \wedge d\bar{z}}{z^m} = \int_{\epsilon<|z|^2<1} \frac{r_m\xi(z) dz \wedge d\bar{z}}{z^m}.$$

Since $r_m\xi(z) = \mathcal{O}(|z|^m)$, we see that the limit in (1.3) exists and that

$$\Big\langle \Big[\frac{1}{z^m}\Big], \xi dz \wedge d\bar{z} \Big\rangle = \int_{|z|<1} \frac{r_m\xi(z) dz \wedge d\bar{z}}{z^m}. \tag{1.10}$$

It is clear from the definition that (1.4) holds. By Stokes' theorem we have that

$$m \int_{|z|^2>\epsilon} \frac{\xi dz \wedge d\bar{z}}{z^{m+1}} = \int_{|z|^2>\epsilon} \frac{(\partial\xi/\partial z) dz \wedge d\bar{z}}{z^m} \pm \int_{|z|^2=\epsilon} \frac{\xi d\bar{z}}{z^m}.$$

In the last integral we can replace ξ by $r_m\xi$ for similar symmetry reasons as above, and then it becomes clear that it is $\mathcal{O}(\epsilon)$. Now (1.5) follows. The first equality in (1.6) follows by Stokes' formula (notice the orientation!). When $m = 1$, the second equality holds because then the integral is just the mean value of ξ (times $2\pi i$) over the circle with radius ϵ. When $m > 1$ we can replace ξ by $p_m\xi$ and for symmetry reasons again only the z^{m-1}-term gives a contribution. Its coefficient is precisely $\partial^{m-1}\xi/\partial z^{m-1}(0)/(m-1)!$, so the equality follows from the case $m = 1$. The first equality in (1.7) follows, e.g., from (1.6), whereas the second one is obvious for degree reasons. Finally, (1.8) follows from (1.6) and (1.7), and (1.9) follows from (1.4) and (1.6). □

It is often conceptually convenient to treat currents as (generalized) differential forms and write

$$\int_z \frac{\xi dz \wedge d\bar{z}}{z^m}$$

rather than

$$\Big\langle \Big[\frac{1}{z^m}\Big], \xi dz \wedge d\bar{z} \Big\rangle.$$

In particular we identify the principal value current with the associated semi-meromorphic form.

Corollary 1.3. *For a function ϕ that is holomorphic in a neighborhood of* 0 *the following are equivalent:*

(*i*) $\qquad \phi \in (z^m)$,

(*i*) $\qquad \phi \bar{\partial}\Big[\frac{1}{z^m}\Big] = 0$, *(iii)* $\frac{\partial^\ell \phi}{\partial z^\ell}(0) = 0,\ \ell = 0, \ldots, m-1.$

Notice that *(ii)* means that the *current* $\phi\bar\partial[1/z^m]$ vanishes, not to be mixed up with current acting on a test form! Thus we can represent the ideal (z^m) either by a generator z^m, as the annihilator of a residue current, or by so-called Noetherian differential operators.

Proof. If (i) holds, then $\phi = \psi z^m$ where ψ is holomorphic, and so

$$\phi\bar\partial\Big[\frac{1}{z^m}\Big] = \psi\bar\partial z^m\Big[\frac{1}{z^m}\Big] = \psi\bar\partial 1 = 0,$$

according to (1.4), and thus (ii) holds. If $\langle\bar\partial[1/z^m], \phi\xi dz\rangle = 0$ for all test forms ξdz, then in view of (1.6), (iii) must hold. Thus (ii) implies (iii). Finally (iii) implies (i) by Taylor's formula. □

In the several variable case we will often rely on another way to define the currents $[1/z^m]$ and $\bar\partial[1/z^m]$:

Lemma 1.4. *Let ξ be a test function in $\mathbb{C}$ and m a positive integer. Then*

$$\lambda \mapsto \int |z|^{2\lambda}\xi(z)\frac{dz\wedge d\bar z}{z^m}$$

and

$$\lambda \mapsto \int \bar\partial|z|^{2\lambda}\wedge\xi(z)\frac{dz}{z^m},$$

a priori defined when $\mathrm{Re}\,\lambda \gg 0$*, both have analytic continuations to* $\mathrm{Re}\,\lambda > -1/2$*, and the values at* $\lambda = 0$ *are* $\langle[1/z^m], \xi dz\wedge d\bar z\rangle$ *and* $\langle\bar\partial[1/z^m], \xi dz\rangle$*, respectively.*

Proof. We may assume again that ξ has support in the unit disk. With the same notation as in the proof of Proposition 1.2 we write $\xi = p_m\xi + r_m\xi$. If $\mathrm{Re}\,\lambda \gg 0$, then for similar symmetry reasons as before, we have

$$\int |z|^{2\lambda}\frac{\xi dz\wedge d\bar z}{z^m} = \int_{|z|<1}\frac{|z|^{2\lambda}\xi dz\wedge d\bar z}{z^m} = \int_{|z|<1}\frac{|z|^{2\lambda}r_m\xi(z)dz\wedge d\bar z}{z^m}. \tag{1.11}$$

Hence the proposed analytic continuation to $\mathrm{Re}\,\lambda > -1/2$ exists and when $\lambda = 0$ it is equal to $\langle[1/z^m], \xi dz\wedge d\bar z\rangle$ in view of (1.10). The second integral in the lemma is

$$\big\langle\bar\partial|z|^{2\lambda}/z^m, \xi dz\big\rangle = \big\langle -|z|^{2\lambda}/z^m, \bar\partial\xi\wedge dz\big\rangle$$

for large $\mathrm{Re}\,\lambda$, and from the first part of the lemma and the uniqueness of analytic continuation, the value at $\lambda = 0$ is $\langle -[1/z^m], \bar\partial\xi\wedge dz\rangle = \langle\bar\partial[1/z^m], \xi dz\rangle$. □

It is often convenient to suppress the test form and say: The functions $\lambda \mapsto |z|^{2\lambda}v^\lambda/z^m$ and $\bar\partial(|z|^{2\lambda}v^\lambda)/z^m$, a priori just defined for $\mathrm{Re}\,\lambda \gg 0$ have current-valued analytic continuations to $\mathrm{Re}\,\lambda > -1/2$, and the values at $\lambda = 0$ are precisely the principal value current $1/z^m$ and the residue current $\bar\partial(1/z^m)$, respectively. Thus

$$\Big[\frac{1}{z^m}\Big] = \lim_{\epsilon\to 0}\chi_{|z|^2>\epsilon}\frac{1}{z^m} = \frac{|z|^{2\lambda}}{z^m}\Big|_{\lambda=0}.$$

Remark 1.5. Notice that

$$\int |z|^{2\lambda}\Big[\frac{1}{z^m}\Big]\xi dz\wedge d\bar z = \int \frac{|z|^{2\mu}}{z^m}|z|^{2\lambda}\xi dz\wedge d\bar z\Big|_{\mu=0} = \int |z|^{2\lambda}\frac{1}{z^m}\xi dz\wedge d\bar z,$$

if $\mathrm{Re}\,\lambda \gg 0$. By the uniqueness of analytic continuation it must hold in general, i.e., $|z|^{2\lambda}[1/z^m]|_{\lambda=0} = [1/z^m]$, cf., also Proposition 4.1 below. □

2. Tensor products

We will now consider tensor products of one-variable principal-value currents.

Definition 1. Given strictly positive integers $m_1, \dots, m_r$, $r \le n$, we define the current

$$\Big[\frac{1}{z^m}\Big] = \Big[\frac{1}{z_1^{m_1} \cdots z_r^{m_r}}\Big]$$

in $\mathbb{C}^n$ as the tensor product of the currents $[1/z_1^{m_1}], \dots, [1/z_r^{m_r}]$.

It follows from Proposition 1.2 that

$$\Big\langle \Big[\frac{1}{z_1^{m_1} \cdots z_r^{m_r}}\Big], \xi dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \Big\rangle = \lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_r \to 0} \int_{|z_1|^2 > \epsilon_1, \dots, |z_r|^2 > \epsilon_r} \frac{\xi dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}{z_1^{m_1} \cdots z_r^{m_r}} \tag{2.1}$$

if ξ is a test form in $\mathbb{C}^n$. We shall now consider other ways to represent this principal value current.

We say that a function χ on the real line is a *smooth approximand of the characteristic function* $\chi_{[1,\infty)}$ of the interval $[1, \infty)$ if χ is smooth, equal to 0 in a neighborhood of 0 and 1 in a neighborhood of ∞. In the sequel the notation

$$\chi \sim \chi_{[1,\infty)}$$

means that χ is *either* $\chi_{[1,\infty)}$ or a smooth approximand.

Let $a_1, \dots, a_\rho$ be strictly positive integers, $r \le \rho \le n$, and let us write $z^a = z_1^{a_1} \cdots z_\rho^{a_\rho}$ and $z^m = z_1^{m_1} \cdots z_r^{m_r}$. Moreover, let $dz \wedge d\bar{z} = dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$

Lemma 2.1. *Let v be a smooth strictly positive function and let $\chi \sim \chi_{[1,\infty)}$. For any test function ξ,*

$$\lim_{\epsilon \to 0} \int \chi(|z^a|^2 v/\epsilon) \frac{\xi dz \wedge d\bar{z}}{z^m} = \Big\langle \Big[\frac{1}{z^m}\Big], \xi dz \wedge d\bar{z} \Big\rangle. \tag{2.2}$$

Moreover,

$$\lambda \mapsto \int |z^a|^{2\lambda} v^\lambda \frac{\xi dz \wedge d\bar{z}}{z^m},$$

a priori defined for $\operatorname{Re} \lambda \gg 0$*, has an analytic continuation to* $\operatorname{Re} \lambda > -\epsilon$ *and*

$$\int |z^a|^{2\lambda} v^\lambda \frac{\xi dz \wedge d\bar{z}}{z^m} \Big|_{\lambda = 0} = \Big\langle \Big[\frac{1}{z^m}\Big], \xi dz \wedge d\bar{z} \Big\rangle. \tag{2.3}$$

Proof. We first consider the case $v = 1$. Notice that

$$(\lambda_1, \dots, \lambda_\rho) \mapsto \varphi(\lambda_1, \dots, \lambda_\rho) = \prod_{j=1}^{\ell} |z_j^{a_j}|^{2\lambda_j} \frac{1}{z_j^{m_j}},$$

with $m_j = 0$ for $j > r$, is holomorphic in the product set $\{\operatorname{Re} 2a_j\lambda_j > -1\}$. In particular, $\lambda \mapsto \varphi(\lambda, \dots, \lambda)$ is holomorphic for $\operatorname{Re} \lambda > -\epsilon$. Now (2.3) follows from Lemma 1.4.

Recall that Taylor's formula with remainder term of order m for a function ψ of one complex variable w can be written

$$\psi(w) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum_{j+\ell=k} w^j \bar{w}^\ell \frac{\partial^k \psi}{\partial w^j \partial \bar{w}^\ell}(0) +$$

$$\sum_{j+\ell=m} w^j \bar{w}^\ell \frac{1}{(m-1)!} \int_0^1 \frac{\partial^m \psi}{\partial w^j \partial \bar{w}^\ell}(tw)(1-t)^{m-1} dt.$$

Let us first apply it to $z_1 \mapsto \xi(z_1, \ldots, z_n)$ with remainder term of order m_1. Then we apply the same formula to

$$z_2 \mapsto \frac{\partial^m \xi}{\partial z_1^j \partial \bar{z}_1^\ell}(tz_1, z_2, \ldots, z_n)$$

with remainder term of order m_2 and plug the result into the first formula. Proceeding in this way we end up with a smooth decomposition

$$\xi = p_m \xi + r_m \xi,$$

where $r_m \xi = \mathcal{O}(|z_1|^{m_1} \cdots |z_r|^{m_r})$ and $p_m \xi$ has the following property: For each term τ_ℓ in $p_m \xi$ there is an index $j = j(\ell)$ such that τ_ℓ is a monomial in z_j of degree at most $m_j - 1$, for each fixed value of the other variables. By a symmetry argument as in the proof of Proposition 1.2 it follows that

$$\int_{|z_j|<1} |z_j^{a_j}|^{2\lambda} \frac{\tau_\ell dz_j \wedge d\bar{z}_j}{z_j^{m_j}} = 0 \tag{2.4}$$

if $\operatorname{Re} \lambda \gg 0$.

We may assume that ξ has support in the unit polydisk $\Delta^n = \{|z_j| < 1\}$. For $\operatorname{Re} \lambda \gg 0$ we now have, in view of (2.4) and Fubini's theorem, that

$$\int |z^a|^{2\lambda} \frac{\xi dz \wedge d\bar{z}}{z^m} = \int_{\Delta^n} |z^a|^{2\lambda} \frac{\xi dz \wedge d\bar{z}}{z^m} = \int_{\Delta^n} |z^a|^{2\lambda} \frac{r_m \xi dz \wedge d\bar{z}}{z^m}.$$

In the last integral, the integrand is bounded, so by (2.3) we get

$$\left\langle \frac{1}{z^m}, \xi dz \wedge d\bar{z} \right\rangle = \int_{\Delta^n} \frac{r_m \xi dz \wedge d\bar{z}}{z^m}. \tag{2.5}$$

In the same way,

$$\int \chi(|z^a|^2/\epsilon) \frac{\xi dz \wedge d\bar{z}}{z^m} = \int_{\Delta^n} \chi(|z^a|^2/\epsilon) \frac{\xi dz \wedge d\bar{z}}{z^m} = \int_{\Delta^n} \chi(|z^a|^2/\epsilon) \frac{r_m \xi dz \wedge d\bar{z}}{z^m}.$$

Therefore,

$$\lim_{\epsilon \to 0} \int \chi(|z^a|^2/\epsilon) \frac{\xi dz \wedge d\bar{z}}{z^m} = \int_{\Delta^n} \frac{r_m \xi dz \wedge d\bar{z}}{z^m}. \tag{2.6}$$

Combining (2.5) and (2.6) we get (2.2) in case $v = 1$.

Now suppose that v is smooth and strictly positive. Notice that $p_m(v^\mu \xi)$ and hence also $r_m(v^\mu \xi)$ are entire functions of μ. It follows that

$$(\lambda, \mu) \mapsto \int |z^a|^{2\lambda} v^\mu \frac{\xi dz \wedge d\bar{z}}{z^m} = \int_{\Delta^n} |z^a|^{2\lambda} \frac{r_m(v^\mu \xi) dz \wedge d\bar{z}}{z^m}$$

is holomorphic in the set $\{\operatorname{Re} \lambda > -\epsilon\} \times \mathbb{C}_\mu$, and taking $\lambda = \mu = 0$ we get (2.3).

Close to 0 we can make the smooth non-holomorphic change of coordinates $w_1 = v^{1/2m_1} z_1, w_j = z_j$, $j = 2, \ldots, n$. After scaling we may assume that this works in the unit polydisk. Then

$$\int \chi(|z^a|^2 v/\epsilon) \frac{\xi dz \wedge d\bar{z}}{z^m} = \int \chi(|w^a|^2/\epsilon) \frac{\tilde{\xi}(w) dw \wedge d\bar{w}}{w^m} \tag{2.7}$$

for a certain smooth function $\tilde{\xi}(w)$. From (2.2) with $v = 1$ we know that

$$\lim_{\epsilon \to 0} \int \chi(|w^a|^2/\epsilon) \frac{\tilde{\xi}(w) dw \wedge d\bar{w}}{w^m} = \Big\langle \frac{1}{w^m}, \tilde{\xi} dw \wedge d\bar{w} \Big\rangle. \tag{2.8}$$

However, we also have that

$$\int |z^a|^{2\lambda} v^\lambda \frac{\xi(z) dz \wedge d\bar{z}}{z^m} = \int_w |w^a|^{2\lambda} \frac{\tilde{\xi}(w) dw \wedge d\bar{w}}{w^m},$$

and in view of (2.3) we conclude that

$$\langle [1/z^m], \xi dz \wedge d\bar{z} \rangle = \langle [1/w^m], \tilde{\xi} dw \wedge d\bar{w} \rangle.$$

The general case follows from this equality in combination with (2.8) and (2.7). □

3. The principal value current $[1/f]$

Let f be a holomorphic function on the analytic space X. We are now going to define the principal value current $1/f$.

Recall that a proper holomorphic mapping $\pi \colon \widetilde{X} \to X$ is a *modification* if there is an analytic subset $V \subset X$ such that the restriction of π to $\widetilde{X} \setminus \pi^{-1}V$ is a biholomorphism onto $X \setminus V$. By Hironaka's theorem there is modification (resolution of singularities) $\pi \colon \widetilde{X} \to X$ such that the zero set of $\pi^* f$ has *normal crossings*. This means that in a neighborhood of each point in $\widetilde{X}$ one can choose local coordinates in which $\pi^* f$ is a monomial.

Let v be a smooth strictly positive function and consider, for λ such that $\operatorname{Re}\lambda \gg 0$,

$$\int_X \frac{|f|^{2\lambda} v^\lambda \xi}{f} = \int_{\widetilde{X}} \frac{|\pi^* f|^{2\lambda} \pi^* v^\lambda \pi^* \xi}{\pi^* f}. \tag{3.1}$$

Since π is proper and ξ has compact support, $\pi^*\xi$ has compact support in $\widetilde{X}$. Thus we have a finite open cover $\mathcal{U}_k$ such that $\pi^* f$ is a monomial in each $\mathcal{U}_k$ for appropriate local coordinates. If ρ_k is a partition of unity ρ_k subordinate to $\mathcal{U}_k$ we have

$$\int_X \frac{|f|^{2\lambda} v^\lambda \xi}{f} = \sum_k \int \frac{|\pi^* f|^{2\lambda} \pi^* v^\lambda \pi^* \xi}{\pi^* f} \rho_k.$$

Fix a k and assume that $\pi^* f = s_1^{m_1} \cdots s_r^{m_r} =: s^m$ in $\mathcal{U}_k$. Thus

$$\int \frac{|\pi^* f|^{2\lambda} \pi^* v^\lambda \pi^* \xi}{\pi^* f} \rho_k = \int_s |s^m|^{2\lambda} v^\lambda \frac{\alpha}{s^m}, \tag{3.2}$$

where α is a smooth form with compact support. It now follows from Lemma 2.1 that the analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ exists, and that the value at $\lambda = 0$ is independent of v.

Definition 2. We define

$$\Big\langle \Big[\frac{1}{f}\Big], \xi \Big\rangle := \int_X \frac{|f|^{2\lambda} v^\lambda \xi}{f}\Big|_{\lambda=0}.$$

Remark 3.1. If follows now from (3.1) and the definition that

$$\Big\langle \Big[\frac{1}{f}\Big], \xi \Big\rangle = \Big\langle \Big[\frac{1}{\pi^* f}\Big], \pi^*\xi \Big\rangle$$

which means that

$$\Big[\frac{1}{f}\Big] = \pi_*\Big(\Big[\frac{1}{\pi^* f}\Big]\Big).$$

□

Example 3.2. Notice that if a is a non-vanishing holomorphic function, then

$$a\Big[\frac{1}{af}\Big] = \Big[\frac{1}{f}\Big].$$

□

Example 3.3. If f is a holomorphic section of a Hermitian line bundle $L \to X$, then in a local frame $|f|_L^2 = |f|^2 v$ for some smooth strictly positive function v. It follows that

$$\Big[\frac{1}{f}\Big] := \frac{|f|_L^{2\lambda}}{f}\Big|_{\lambda=0}$$

is a well-defined L^{-1}-valued current. □

With the same notation as before we have, if $\chi \sim \chi_{[1,\infty)}$,

$$\int \chi(|f|^2 v/\epsilon)\frac{\xi}{f} = \int \chi(|\pi^* f|^2 \pi^* v/\epsilon)\frac{\pi^*\xi}{\pi^* f} = \sum_k \int \chi(|\pi^* f|^2 \pi^* v/\epsilon)\frac{\pi^*\xi}{\pi^* f}\rho_k.$$

Moreover,

$$\int \chi(|\pi^* f|^2 \pi^* v/\epsilon)\frac{\pi^*\xi}{\pi^* f}\rho_k = \int \chi(|s^m|^2 \pi^* v/\epsilon)\frac{\alpha}{s^m}.$$

From Lemma 2.1 we conclude that the limit when $\epsilon \to 0$ exists and is equal to (2.7). Thus we have proved

Proposition 3.4. *Assume that v is smooth and strictly positive on X and that $\chi \sim \chi_{[1,\infty)}$. Then*

$$\Big[\frac{1}{f}\Big] = \lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f}.$$

We shall now see that one can replace $\chi(|f|^2 v/\epsilon)$ by more general regularizations.

Lemma 3.5. *Let ψ be a smooth function on $[0,\infty]$ such that $\psi(\infty) = 1$ and $\psi(0) = 0$. Then $(d/dt)\psi(t/\epsilon) \to \delta_0$ as measures on $[0,\infty)$.*

Here "smooth at ∞" means that $\tilde{\psi}(s) = \psi(1/s)$ is smooth at 0.

Proof. First notice that $\tilde{\psi}'(s) = -\psi'(1/s)/s^2$, and thus $\psi'(t) = \mathcal{O}(1/t^2)$ as $t \to \infty$. If ϕ is continuous with compact support on $[0,\infty)$, therefore

$$|\phi(\epsilon t)\psi'(t)| \le \frac{C}{(1+t)^2}$$

where C is independent of ϵ. By the dominated convergence theorem we have

$$\int_0^\infty \frac{d}{dt}\psi(t/\epsilon)\phi(t)dt = \int_0^\infty \frac{d}{d\tau}\psi(\tau)\phi(\epsilon\tau)d\tau \to \phi(0)\int_0^\infty \frac{d}{d\tau}\psi(\tau)d\tau = \phi(0).$$

□

Proposition 3.6. *Let ψ be as in the lemma and let f be a holomorphic function on X and v smooth and strictly positive. Then for any $k \geq 1$,*

$$\lim_{\epsilon\to 0}\int \psi(|f|^2 v/\epsilon)^k \frac{\xi}{f} = \Big\langle \frac{1}{f^k}, \xi\Big\rangle, \quad \xi \in \mathcal{D}_{n,n}(X).$$

Proof. On the set $\Omega = \{(z,t) \in \mathbb{C}^n \times (0,\infty);\ |f(\zeta)|^2 v > t\}$ we have, for each fixed ϵ, that

$$\Big|\frac{1}{f}\xi \frac{d}{dt}\psi(t/\epsilon)^k\Big| \leq C\frac{1}{\sqrt{t}}.$$

Hence we have an integrable singularity on Ω and by Fubini's theorem we get

$$\int_0^\infty \frac{d}{dt}\psi(t/\epsilon)^k \int_{|f|^2v>t}\frac{\xi}{f^k}dt = \int \frac{\xi}{f^k}\int_0^{|f|^2v}\frac{d}{dt}\psi(t/\epsilon)^k dt = \int \frac{\psi(|f|^2v/\epsilon)^k\xi}{f^k}.$$

However,

$$J(t) = \int_{|f|^2v>t}\frac{\xi}{f^k}$$

is a continuous function with compact support on $[0,\infty)$ such that $J(0) = \langle 1/f^k, \xi\rangle$ according to Proposition 3.4, with $\chi = \chi_{[1,\infty)}$, and so Proposition 3.6 follows from Lemma 3.5 applied to ψ^k instead of ψ. □

Proposition 3.6 is more general than Proposition 3.4 and for instance allows us to take $\psi(t) = t/(1+t)$.

Example 3.7. If v is smooth and strictly positive and $k \geq 1$, then

$$\lim_{\epsilon\to 0}\Big(\frac{v\bar{f}}{v|f|^2+\epsilon}\Big)^k = \Big[\frac{1}{f^k}\Big].$$

□

Example 3.8. Even if $f_t(z)$ belongs holomorphically on a parameter t it is not true that $[1/f_t]$ and thus neither $\bar\partial[1/f_t]$ are necessarily even continuous in t. Take $f_t(z) = z^2 - t^2$. Since

$$\frac{1}{z^2-t^2} = \frac{1}{2t}\Big(\frac{1}{z-t} - \frac{1}{z+t}\Big)$$

it follows that

$$\Big\langle \bar\partial\Big[\frac{1}{z^2-t^2}\Big], \xi dz\Big\rangle = \frac{\xi(t)-\xi(-t)}{2t}$$

for $t \neq 0$, and if for instance $\xi(z) = \bar{z}$ in a neighborhood of 0, then the limit when $t \to 0$ does not exist. □

4. Elementary pseudomeromorphic currents

It turns out that many of the currents that appear in multivariable residue theory are *pseudomeromorphic*. These currents have several geometric features. For instance, a pseudomeromorphic current μ of bidegree $(*,p)$ must vanish if its support is contained in a variety with codimension larger than p. In the next sections we will see that if μ has support on the variety V with codimension p and $\bar\partial\mu = 0$, then μ must be a so-called Coleff-Herrera current[3]. We shall also see that if V is a

[3]Notice the analogy with normal currents. If a normal (p,p)-current μ has support on V and $\operatorname{codim} V > p$ then $\mu = 0$. If $\operatorname{codim} V = p$, then μ is (the Lelong current associated with) an analytic cycle with support on V, see, e.g., [58, ???].

subvariety of X, then the restriction of a pseudomeromorphic current μ to the open set $X \setminus V$ has a natural extension as a pseudomeromorphic current on X. To begin with we discuss elementary pseudomeromorphic currents.

Let t_j be coordinates in $\mathbb{C}^n$ and let α be a smooth form with compact support. We know that

$$\tau = \alpha \wedge \frac{1}{t_1^{m_1}} \cdots \frac{1}{t_k^{m_k}} \bar\partial \frac{1}{t_{k+1}^{m_{k+1}}} \wedge \ldots \wedge \bar\partial \frac{1}{t_r^{m_r}} \tag{4.1}$$

is a well-defined current, since it is the tensor product of one-variable currents (times α). We say that τ is an *elementary pseudomeromorphic current*, and we refer to $1/t_j^{m_j}$ and $\bar\partial(1/t_\ell^{m_\ell})$ as its *principal value factors* and *residue factors*, respectively. It is clear that (4.1) is commuting in the principal value factors and anti-commuting in the residue factors. We say the the affine set $\{t_{k+1} = \cdots = t_r = 0\}$ is its *elementary support.* Clearly the support of τ is contained in the intersection of the elementary support and the support of α.

It is readily shown that if τ is elementary as in (4.1), then

(4.2)

$$\bar\partial\tau = \sum_{j=1}^{k} (-1)^{\deg\alpha} \alpha \wedge \frac{1}{t_1^{m_1}} \cdots \frac{1}{t_{j-1}^{m_{j-1}}} \frac{1}{t_{j+1}^{m_{j+1}}} \cdots \frac{1}{t_k^{m_k}} \bar\partial \frac{1}{t_j^{m_j}} \wedge \bar\partial \frac{1}{t_{k+1}^{m_{k+1}}} \wedge \ldots \wedge \bar\partial \frac{1}{t_r^{m_r}} + \bar\partial\alpha \wedge \frac{1}{t_1^{m_1}} \cdots \frac{1}{t_k^{m_k}} \bar\partial \frac{1}{t_{k+1}^{m_{k+1}}} \wedge \ldots \wedge \bar\partial \frac{1}{t_r^{m_r}},$$

and thus a finite sum of elementary pseudomeromorphic currents. In the same way we see, cf., (1.8), that $\partial\tau$ is a finite sum of elementary pseudomeromorphic currents. It is clear that $\gamma \wedge \tau$ is elementary if γ is a smooth form.

Lemma 4.1. *If $k+1 \le \ell \le p$, i.e., ℓ corresponds to any of the residue factors, then*

$$\bar t_\ell \tau = 0, \qquad d\bar t_\ell \wedge \tau = 0. \tag{4.3}$$

If t^a is any monomial, v is smooth and strictly positive, and $\chi \sim \chi_{[1,\infty)}$, then the analytic continuation

$$|t^a|^{2\lambda} v^\lambda \tau|_{\lambda=0}$$

and the the limit

$$\lim_{\epsilon\to 0} \chi(|t^a|^2 v/\epsilon)\tau$$

both exist. If t^a contains a coordinate corresponding to any of the residue factors in τ, then they both vanish, and otherwise they are both equal to τ.

It follows from the proof that $\chi(|t^a|^2 v/\epsilon)\tau$ is a well-defined current even if $\chi = \chi_{[1,\infty)}$ if $\epsilon > 0$ is small enough. If χ is smooth this problem does not appear.

Proof. The equalities in (4.3) follow immediately from (1.7). If t^a contains a factor t_j that corresponds to a residue factor in τ, i.e., $k+1 \le j \le p$, then $\chi(|t^a|^2 v/\epsilon)$ vanishes in a neighborhood of the support of τ so $\chi(|t^a|^2 v/\epsilon)\tau = 0$ for all $\epsilon > 0$. Moreover, t^a vanishes on the support of τ so $|t^a|^{2\lambda} v^\lambda \tau = 0$ if $\operatorname{Re}\lambda \gg 0$. Thus the analytic continuation trivially exists and is 0.

Now assume that t^a has no factor corresponding to any residue factor; say t^a consists of the variables $t_1, \ldots, t_\nu$, $\nu \le k$. In short hand notation

$$\tau = \alpha \wedge \frac{1}{t^{m'}} \frac{1}{t^{m''}} \bar\partial \frac{1}{t^{m'''}},$$

where $t^{m'}=t_1^{m_1}\cdots t_\nu^{m_\nu}$, $t^{m''}=t_{\nu+1}^{m_{\nu+1}}\cdots t_k^{m_k}$, and

$$\bar\partial\frac{1}{t^{m'''}}=\bar\partial\frac{1}{t_{k+1}^{m_{k+1}}}\wedge\ldots\wedge\bar\partial\frac{1}{t_r^{m_r}}.$$

If η is a test form, then

$$\langle |t^a|^{2\lambda}v^\mu\tau,\eta\rangle=\Big\langle |t^a|^{2\lambda}\frac{1}{t^{m'}}\frac{1}{t^{m''}}\bar\partial\frac{1}{t^{m'''}},v^\mu\alpha\wedge\eta\Big\rangle; \tag{4.4}$$

thus a tensor products of currents acting on the test form $v^\mu\alpha\wedge\eta$. It now follows from (the proof of) Lemma 2.1 that (4.4) is holomorphic for $\operatorname{Re}\lambda>-\epsilon,\mu\in\mathbb{C}$, and that the value at $\lambda=0$ is

$$\Big\langle\frac{1}{t^{m'}}\frac{1}{t^{m''}}\bar\partial\frac{1}{t^{m'''}},v^\mu\alpha\wedge\eta\Big\rangle=\langle v^\mu\tau,\eta\rangle.$$

Letting $\mu=0$ we get $\langle\tau,\eta\rangle$.

Notice now that

$$\langle\chi(|t^a|^2v/\epsilon)\tau,\eta\rangle=\Big\langle\chi(|t^a|^2v/\epsilon)\frac{1}{t^{m'}}\frac{1}{t^{m''}}\bar\partial\frac{1}{t^{m'''}},\alpha\wedge\eta\Big\rangle. \tag{4.5}$$

If $v=1$ we have again a tensor product of currents acting on the test form $\alpha\wedge\eta$, and by Lemma 2.1, $\chi(|t^a|^2/\epsilon)(1/t^{m'})\to 1/t^{m'}$, so (4.5) tends to $\langle\tau,\eta\rangle$ as claimed. If v is arbitrary, we first make a non-holomorphic change of variables in (4.5) as in the proof of Lemma 2.1, and then we are back to the case $v=1$. $\square$

Notice also that

$$\bar\partial(|t^a|^{2\lambda}v^\lambda)\wedge\tau|_{\lambda=0}$$

and the the limit

$$\lim_{\epsilon\to0}\bar\partial\chi(|t^a|^2v/\epsilon)\wedge\tau$$

both exist and are equal to a certain sum of elementary pseudomeromorphic currents. This follows from the equality

$$\bar\partial|t^a|^{2\lambda}v^\lambda\wedge\tau=\bar\partial\big(|t^a|^{2\lambda}v^\lambda\tau\big)-|t^a|^{2\lambda}v^\lambda\wedge\bar\partial\tau,$$

the analogous one for $\bar\partial\chi(|t^a|^2v/\epsilon)\wedge\tau)$, the proposition, and (4.2).

Assume now that t^b is a monomial with the same coordinate factors as t^a. It follows that

$$\Big[\frac{1}{t^b}\Big]\tau:=\frac{|t^a|^{2\lambda}}{t^b}v^\lambda\tau|_{\lambda=0} \tag{4.6}$$

exists and defines an elementary pseudomeromorphic current. In fact, if t^a contains one of the residue factors, then we get 0; otherwise we get

$$\alpha\wedge\frac{1}{t^{m'+a}}\frac{1}{t^{m''}}\bar\partial\frac{1}{t^{m'''}},$$

with the notation from the proof of the lemma. Again we also have

$$\Big[\frac{1}{t^b}\Big]\tau=\lim_{\epsilon\to0}\chi(|t^a|^2v/\epsilon)\frac{1}{t^b}\tau \tag{4.7}$$

5. The sheaf of pseudomeromorphic currents

Fix a point $x \in X$. We say that a germ μ of a current at x is *pseudomeromorphic* at x, $\mu \in \mathcal{PM}_x$, if it is a finite sum of currents of the form $\pi_*\tau = \pi^1_* \cdots \pi^m_* \tau$, where $\mathcal{U}$ is a neighborhood of x,

$$\mathcal{U}_m \xrightarrow{\pi^m} \cdots \xrightarrow{\pi^2} \mathcal{U}_1 \xrightarrow{\pi^1} \mathcal{U}_0 = \mathcal{U}, \tag{5.1}$$

each $\pi^j \colon \mathcal{U}_j \to \mathcal{U}_{j-1}$ is either a modification, a simple projection $\mathcal{U}_{j-1} \times Z \to \mathcal{U}_{j-1}$, or an open inclusion (i.e., $\mathcal{U}_j$ is an open subset of $\mathcal{U}_{j-1}$), and τ is elementary on $\mathcal{U}_m$.

By definition the union $\mathcal{PM} = \cup_x \mathcal{PM}_x$ is an open subset of the sheaf $\mathcal{C} = \mathcal{C}^X$ and hence it is a subsheaf, the sheaf of *pseudomeromorphic* currents, of $\mathcal{C}$. A section μ of $\mathcal{PM}$ over an open set $\mathcal{V} \subset X$, $\mu \in \mathcal{PM}(\mathcal{V})$, is then a locally finite sum

$$\mu = \sum (\pi_\ell)_* \tau_\ell, \tag{5.2}$$

where each π_ℓ is a composition of mappings as in (5.1) (with $\mathcal{U} \subset \mathcal{V}$) and τ_ℓ is elementary. For simplicity we will often suppress the subscript ℓ in π_ℓ.

If ξ is a smooth form, then, cf., Section 6, $\xi \wedge \pi_*\tau = \pi_*\big(\pi^*\xi \wedge \tau\big)$. Thus $\mathcal{PM}$ is closed under exterior multiplication by smooth forms. Since $\bar\partial$ commutes with push-forwards it follows, in view of (4.2), that $\mathcal{PM}$ is closed under $\bar\partial$.

Lemma 5.1. *Assume that $p\colon Y \to X \subset\subset \mathbb{C}^n$ is a modification and τ is an elementary pseudomeromorphic current in X (with respect to the standard coordinates in $\mathbb{C}^n$). Then there is a modification $\tilde p\colon \widetilde Y \to Y$ such that*

$$\tau = p_*\tilde p_* \sum_\ell \tau_\ell,$$

where the sum is finite and each τ_ℓ is elementary with respect to some local coordinates in $\widetilde Y$. If h is holomorphic in Y we may assume as well that $\tilde p^* h$ is a monomial times a nonvanishing factor with respect to the same local coordinate systems.

Proof. Let us first assume that p is a modification and that τ is elementary with respect to the coordinates t_j in X, say of the form (4.1). Notice that p^*t_j are global holomorphic functions in Y. There is a smooth modification $\tilde p\colon \widetilde Y \to Y$ and an open cover $\mathcal{U}_\ell$ of $\tilde Y$ such that, for each ℓ, all the functions $\tilde p^* p^* t_j$ are monomials (with respect to the same local coordinates s) times a nonvanishing holomorphic factor in $\mathcal{U}_\ell$. Take a partition of unity χ_ℓ subordinate to $\mathcal{U}_\ell$. If

$$\tau^\lambda := \tau^{\lambda_1,\ldots,\lambda_r} := \frac{\bar\partial |t_1|^{2\lambda_1}}{t_1^{a_1}} \wedge \ldots \wedge \frac{\bar\partial |t_k|^{2\lambda_r}}{t_k^{a_k}} \wedge \alpha \frac{|t_{k+1}|^{2\lambda_{k+1}}}{t_{k+1}^{a_{k+1}}} \cdots \frac{|t_r|^{2\lambda_r}}{t_r^{a_r}},$$

where $r \le n$, then

$$\tau = \tau^{\lambda_1,\ldots,\lambda_r}|_{\lambda_r=0} \cdots |_{\lambda_1=0}.$$

Let $\pi = \tilde p \circ p$. For $\lambda \gg 0$ we have that

$$\pi^*\tau^\lambda = \sum_\ell \chi_\ell \pi^* \tau^\lambda.$$

By repeated applications of Lemma 4.1 it follows, for each ℓ, that

$$\chi_\ell \pi^* \tau^\lambda |_{\lambda_N=0} \cdots |_{\lambda_1=0} \tag{5.3}$$

exists and is a finite sum $\tilde{\tau}_\ell$ of elementary currents in $\mathcal{U}_\ell$. Since $\tau^\lambda = \pi_*\pi^*\tau^\lambda$ when $\mathrm{Re}\,\lambda \gg 0$, we conclude that

$$\tau = \pi_* \sum_\ell \tilde{\tau}_\ell = p_* \tilde{p}_* \sum_\ell \tilde{\tau}_\ell.$$

The last statement about h follows from the proof. □

Proposition 5.2. *Assume that $\mu \in \mathcal{PM}$ has support on the zero set V of the holomorphic function h. Then $\bar{h}\mu = d\bar{h} \wedge \mu = 0$.*

This intuitively means that the current τ only involves holomorphic derivatives of test forms.

Proof. Starting with any representation (5.2) of μ, by repeated use of Lemma 5.1 we can obtain a new representation (5.2) such that π^*h is a monomial for each ℓ. Let us take such a representation and decompose it as

$$\mu = \sum_\ell \pi_* \tau'_\ell + \sum_\ell \pi_* \tau''_\ell,$$

where τ'_ℓ are those elementary pseudomeromorphic currents that have a residue factor corresponding to a coordinate factor in π^*h. In other words, those τ_ℓ whose elementary supports are contained in $\pi^{-1}V$. Since h vanishes on the support of μ,

$$0 = \chi(|h|^2/\epsilon)\mu = \sum_\ell \pi_*\big(\chi(|\pi^*h|^2/\epsilon)\tau'_\ell\big) + \sum_\ell \pi_*\big(\chi(|\pi^*h|^2/\epsilon)\tau''_\ell\big)$$

for $\epsilon > 0$. It follows from Proposition 4.1 that the limit of the right hand side is equal to

$$\sum_\ell \pi_* \tau''_\ell$$

so we can conclude that

$$\mu = \sum_\ell \pi_* \tau'_\ell.$$

Now

$$\bar{h}\mu = \sum_\ell \pi_*\big(\overline{\pi^*h}\tau'_\ell\big), \quad d\bar{h} \wedge \mu = \sum_\ell \pi_*\big(d\overline{\pi^*h} \wedge \tau'_\ell\big),$$

and from Proposition 4.1 we have that $\overline{\pi^*h}\tau'_\ell = d\overline{\pi^*h} \wedge \tau'_\ell = 0$. Thus Proposition 5.2 follows. □

Remark 5.3. In this proof it was advantegous to use the regularization with ϵ rather than λ since it is obvious that $\chi(|h|^2/\epsilon)\mu = 0$ if μ has support on the zero set of h. A posteriori it is clear that also $|h|^{2\lambda}\mu = 0$. This can also be concluded directly from [**?**, Theorem 2.3.11], since V is Whitney regular. □

We now get

Theorem 5.4 (Dimension principle)**.** *If $\mu \in \mathcal{PM}$ has bidegree (k,p) and support on a variety V with codim$V > p$, then $\mu = 0$.*

Proof. Locally V_{reg} is on the form $\{w_1 = \ldots = w_{p+\ell} = 0\}$ for some $\ell \geq 1$ and suitable coordinates $(z_1, \ldots, z_{n-p-\ell}, w_1, \ldots, w_{p+\ell})$. From Proposition 5.2 we have that

$$d\bar{w}_j \wedge \mu = 0, \quad j = 1, \ldots, p+\ell. \tag{5.4}$$

However, if (5.4) holds, then μ must be of the form

$$\mu = \mu' \wedge d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_{p+\ell}.$$

Since $\ell \geq 1$ therefore $\mu = 0$. We can conclude that $\mu = 0$ on V_{reg} so the support must be contained in $V \setminus V_{reg}$ which has codimension at least $p+2$. By finite induction we find that $\mu = 0$. □

Proposition 5.5. *If $p\colon X' \to X$ is a modification, then*

$$p_*\colon \mathcal{PM}(X') \to \mathcal{PM}(X)$$

is surjective.

Proof. Assume that $\mu = \pi_*\tau$, where π is a composed mapping as in (5.1) and τ is elementary in $\mathcal{U}_m$. It is enough to see that $\mu = p_*\mu'$ for some $\mu' \in \mathcal{PM}(\mathcal{V})$ where $\mathcal{V} = p^{-1}\mathcal{U}$. The proposition then follows since a general global section is a locally finite sum of such μ and p is proper.

We claim that (5.1) can be extended to a commutative diagram

$$(5.5)\qquad \begin{array}{ccccccccccc} \widetilde{\mathcal{V}} & = & \mathcal{V}_m & \xrightarrow{\tilde\pi_m} & \cdots & \xrightarrow{\tilde\pi_2} & \mathcal{V}_1 & \xrightarrow{\tilde\pi_1} & \mathcal{V}_0 & = & \mathcal{V} \\ & & \downarrow p_m & & & & \downarrow p_1 & & \downarrow p & & \\ \widetilde{U} & = & \mathcal{U}_m & \xrightarrow{\pi_m} & \cdots & \xrightarrow{\pi_2} & \mathcal{U}_1 & \xrightarrow{\pi^1} & \mathcal{U}_0 & = & \mathcal{U} \end{array}$$

so that each vertical map is a modification and each $\tilde\pi_j$ is either a modification, a simple projection, or an open inclusion. To see this, assume that this is done up to level k. It is well-known that if $\pi_{k+1}\colon \mathcal{U}_{k+1} \to \mathcal{U}_k$ is a modification, then there are modifications $\tilde\pi_{k+1}\colon \mathcal{V}_{k+1} \to \mathcal{V}_k$ and $p_{k+1}\colon \mathcal{V}_{k+1} \to \mathcal{U}_{k+1}$ such that

$$\begin{array}{ccc} \mathcal{V}_{k+1} & \xrightarrow{\tilde\pi_{k+1}} & \mathcal{V}_k \\ \downarrow p_{k+1} & & \downarrow p_k \\ \mathcal{U}_{k+1} & \xrightarrow{\pi^{k+1}} & \mathcal{U}_k \end{array}$$

commutes. If instead $\mathcal{U}_{k+1} = \mathcal{U}_k \times Z$ then we simply take $\mathcal{V}_{k+1} = \mathcal{V}_k \times Z$. Finally, if $i\colon \mathcal{U}_{k+1} \to \mathcal{U}_k$ is an open inclusion, then we take $\mathcal{V}_{k+1} = p_k^{-1}\mathcal{U}_{k+1}$.

By Lemma 5.1 there is a pseudomeromorphic current $\tilde\tau$ with compact support in $\mathcal{V}_m$ such that $p_m\tilde\tau = \tau$. If $\tilde\pi$ is the composed mapping in the upper line, it follows that $\mu' = \tilde\pi_*\tilde\tau$ is pseudomeromorphic in $\mathcal{V}$ such that $p_*\mu' = \mu$.

□

6. Restrictions of pseudomeromorphic currents

Assume that μ is pseudomeromorphic and V is a subvariety. We shall now see that the restriction of μ to the open set $X \setminus V$ has a natural pseudomeromorphic extension $\mathbf{1}_{X\setminus V}\mu$ to X.

Lemma 6.1. *Let μ be pseudomeromorphic, h a holomorphic function, and v a smooth strictly positive function. The function $\lambda \mapsto |h|^{2\lambda}v^\lambda\mu$ (a priori defined for $Re\,\lambda \gg 0$) has a current-valued analytic continuation to $Re\,\lambda > -\epsilon$. If $\chi \sim \chi_{[1,\infty)}$, then*

$$\mathbf{1}_{X\setminus V}\mu = \lim_{\delta\to 0^+} \chi(|h|/\delta)\mu \qquad (6.1)$$

exists and is equal to $|h|^{2\lambda}v^\lambda\mu|_{\lambda=0}$.

It follows from the proof that currents in the limit exist for small enough $\epsilon > 0$ even if $\chi = \chi_{[1,\infty)}$.

Proof. With the setup and notation from the proof of Proposition 5.2 it follows, in view of Lemma 4.1, that both the limit and the analytic continuation exist, and both are equal to $\sum \pi_* \tau''_\ell$. □

In particular, we see that the limit only depends on the zero set of h and not on the particular choice of h.

Let V be the germ of a subvariety at x and choose a tuple f of holomorphic functions whose common zero set is precisely V. We claim that for each germ of a pseudomeromorphic current μ at x,

$$\mathbf{1}_{X\setminus V} := |f|^{2\lambda}\mu|_{\lambda=0}$$

exists and is independent of the choice of f. In fact, X ne a suitable neighborhood of x and let $p\colon X' \to X$ be a principalization so that $p^*f = f^0 f'$ as above. If $\mu = p_*\mu'$, then $= |f|^{2\lambda}\mu = p_*(|f^0|^{2\lambda}|f'|^{2\lambda}\mu')$ an so it follows from Lemma 6.1 that the analytic continuation exists. If g is another such tuple, then we can find a common principalization, and it follows from the lemma that the value at $\lambda = 0$ only depends on the set $p^{-1}V$, and so it only depends on V. It also follows that we also have

$$\mathbf{1}_{X\setminus V}\mu = \lim_{\delta\to 0} \chi(|f|/\delta)\mu.$$

It is clear that $\mathbf{1}_{X\setminus V}$ coincides with μ in the open set $X \setminus V$, and hence

$$\mathbf{1}_V\mu := \mu - \mathbf{1}_{X\setminus V}\mu$$

is pseudomeromorphic and has support on V. In particular, $\mathbf{1}_V\mu = \mu$ if μ has support on V.

Notice that if $\mathfrak{p}\colon X' \to X$ is any composition of modifications, simple projections, and open inclusions, and $\mu = p_*\mu'$, then $|f|^{2\lambda}\mu = p_*(|p^*f|^{2\lambda}\mu')$, and hence

$$\mathbf{1}_V\mu = p_*(\mathbf{1}_{p^{-1}V}\mu'). \tag{6.2}$$

Moreover, notice that if α is a smooth form, then

$$\mathbf{1}_V(\alpha\wedge\mu) = \alpha\wedge\mathbf{1}_V\mu. \tag{6.3}$$

Let $\mu = \pi_*\tau$ where τ is elementary. If the elementary support H of τ is contained in $\pi^{-1}V$, then $\mathbf{1}_{\pi^{-1}V}\tau = \tau$. Notice that H is a linear subspace. If H has codimension q, then $\tau = \alpha\wedge\tau'$, where α is smooth and τ' has bidegree $(0,q)$. If H is not contained in $\pi^{-1}V$, then, since H is irreducible, $H\cap\pi^{-1}V$ has codimension at least $q+1$. By (6.3) and the dimension principle we now have that $\mathbf{1}_{\pi^{-1}V}\tau = \alpha\wedge\mathbf{1}_{\pi^{-1}V}\tau' = 0$. Thus

Lemma 6.2. *If μ has the form* (5.2) *then*

$$\mathbf{1}_V\mu = \sum \pi_*\tau'_\ell,$$

where τ'_ℓ are those elementary pseudomeromorphic currents whose elementary supports are contained in $\pi^{-1}V$.

As an immediate consequence we get

Lemma 6.3. *If V, W are analytic sets, then*

$$\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V\cap W}\mu = \mathbf{1}_W\mathbf{1}_V\mu. \tag{6.4}$$

Later on we will see that the mapping $(V, \mu) \mapsto \mathbf{1}_V \mu$ extends to all *constructible sets* V, i.e., all sets in the Boolean algebra generated by the analytic sets.

Remark 6.4. We can now strengthen Proposition 5.2: *If α is a holomorphic form that vanishes on V and the support of the pseudomeromorphic current μ is contained in V, then $\bar\alpha \wedge \mu = 0$.* In fact, we know that we can write μ on the from (5.2) where all τ_ℓ have their elementary support contained in $\pi^{-1}V$. Now

$$\bar\alpha \wedge \mu = \sum \pi_*\big(\pi^*\bar\alpha \wedge \tau_\ell\big) = \sum \pi_*\big(\overline{\pi^*\alpha} \wedge \tau_\ell\big),$$

and $\pi^*\bar\alpha$ is an anti-holomorphic form that vanishes on $\pi^{-1}V$. Thus it is enough to prove that $\bar\gamma \wedge \tau = 0$ if γ is a holomorphic form that vanishes on the elementary support H of τ. However, if $H = \{s_1 = \cdots = s_r = 0\}$, then such a γ must have the form[4] $s_1\gamma_1 + \cdots + s_r\gamma_r + ds_1 \wedge \gamma_1' + \cdots + ds_r \wedge ds_r$, and so $\bar\gamma \wedge \tau = 0$ in view of Lemma 4.1. □

7. Another basic operation on $\mathcal{PM}^X$

We now consider another fundamental operation on $\mathcal{PM}^X$. Given a holomorphic function h we define

$$\Big[\frac{1}{h}\Big]T := \frac{|h|^{2\lambda}}{h}T\Big|_{\lambda=0}, \qquad \bar\partial\Big[\frac{1}{h}\Big] \wedge T := \frac{\bar\partial|h|^{2\lambda}}{h} \wedge T\Big|_{\lambda=0}. \tag{7.1}$$

The existence of the necessary analytic continuations, and that the result is pseudomeromorphic, follows as in the first part of the proof of Proposition 5.2 in combination with (4.6). Notice that the support of the second current in (7.1) is contained in the intersection of the support of T and $V(h)$.

Lemma 7.1. *The formal Leibniz rules*

$$\bar\partial\Big(\frac{1}{h}T\Big) = \bar\partial\frac{1}{h} \wedge T + \frac{1}{h}\bar\partial T, \quad \bar\partial\Big(\bar\partial\frac{1}{h} \wedge T\Big) = -\bar\partial\frac{1}{h} \wedge \bar\partial T \tag{7.2}$$

hold. If α is a smooth form, then

$$\alpha \wedge \frac{1}{h}T = \frac{1}{h}\alpha \wedge T, \quad \alpha \wedge \bar\partial\frac{1}{h} \wedge T = (-1)^{deg\,\alpha}\bar\partial\frac{1}{h} \wedge \alpha \wedge T. \tag{7.3}$$

Proof. If $\operatorname{Re}\lambda \gg 0$, then

$$\bar\partial\Big(\frac{|h|^{2\lambda}}{h}T\Big) = \frac{\bar\partial|h|^{2\lambda}}{h} \wedge T + \frac{|h|^{2\lambda}}{h}\bar\partial T, \qquad \bar\partial\Big(\frac{\bar\partial|h|^{2\lambda}}{h} \wedge T\Big) = -\frac{\bar\partial|h|^{2\lambda}}{h} \wedge \bar\partial T.$$

Now (7.2) follows by the uniqueness of analytic continuation. In a similar way,

$$\alpha \wedge \frac{|h|^{2\lambda}}{h}T = \frac{|h|^{2\lambda}}{h}\alpha \wedge T, \qquad \alpha \wedge \frac{\bar\partial|h|^{2\lambda}}{h} \wedge T = \frac{\bar\partial|h|^{2\lambda}}{h} \wedge \alpha \wedge T,$$

so (7.3) follows as well. □

Example 7.2. Let f a be meromorphic $(k, 0)$-form on X, i.e., (locally) $f = g/h$ where h is a holomorphic function that does not vanish identically on any irreducible component of X and g is a holomorphic $(k, 0)$-form (i.e., given a local embedding of X in a smooth Ω, g is obtained from a holomorphic $(k.0)$-form G in Ω). By definition

[4] First write $\gamma = \gamma' + \gamma''$ where γ'' has no factor ds_j, $j \le r$. Since $s_{r+1}, \dots, s_n$ is a coordinate system on H, the various ds_I in γ'' are independent, so all coefficients a_I must vanish on H, and thus of the form $s_1\alpha_1 + \cdots + s_r\alpha_r$.

$f = g'/h'$ if and only if $g'h - gh'$ vanishes outside a set V of positive codimension. In that case

$$g\Big[\frac{1}{h}\Big] = g'\Big[\frac{1}{h'}\Big] \tag{7.4}$$

outside $V \cup V(h) \cup V(h')$ which has positive codimension. By the dimension principle, thus (7.4) holds as currents. Thus there is a well-defined principal value current associated with f, and this current is pseudomeromorphic. □

Example 7.3. If f and g are any two holomorphic functions it follows that

$$\frac{1}{g}\frac{1}{f} = \frac{1}{f}\frac{1}{g}$$

by the dimension principle. By Leibniz' rule we conclude that

$$\bar\partial\frac{1}{g}\cdot\frac{1}{f} + \frac{1}{g}\bar\partial\frac{1}{f} = \bar\partial\frac{1}{f}\cdot\frac{1}{g} + \frac{1}{f}\cdot\bar\partial\frac{1}{g}.$$

However, it is not true in general that

$$\bar\partial\frac{1}{g}\cdot\frac{1}{f} = \frac{1}{f}\cdot\bar\partial\frac{1}{g}$$

as the next example shows. □

Example 7.4. Let z be the standard coordinate in $\mathbb{C}$ and let a, b be positive integers. It follows directly from the definition that

$$\frac{1}{z^a}\bar\partial\frac{1}{z^b} = 0,$$

whereas

$$\bar\partial\frac{1}{z^b}\cdot\frac{1}{z^a} = \bar\partial\frac{1}{z^{b+a}}.$$

□

Example 7.5. Assume that f, g are holomorphic and codim $\{f = g = 0\}$ is at least 2. Then

$$\Big[\frac{1}{f}\Big]\bar\partial\Big[\frac{1}{g}\Big] = \bar\partial\Big[\frac{1}{g}\Big]\cdot\Big[\frac{1}{f}\Big]$$

by the dimension principle, and by Leibniz rule hence

$$\bar\partial\Big[\frac{1}{g}\Big]\wedge\bar\partial\Big[\frac{1}{f}\Big] = -\bar\partial\Big[\frac{1}{f}\Big]\wedge\bar\partial\Big[\frac{1}{g}\Big].$$

□

Formally one should think of $T \mapsto (1/h)T$ and $T \mapsto \bar\partial(1/h)\wedge T$ as operators on $\mathcal{PM}$. Notice also that

$$h\frac{1}{h}T = \mathbf{1}_{X\setminus V(h)}T$$

which in general is *not* equal to T[5].

From (the proof of) Proposition **??** and (4.7) it follows that we can replace the analytic continuation with a limit with cutoff functions. That is, if $\chi \sim \chi_{[1,\infty)}$ we have

$$\mathbf{1}_{X\setminus V}T = \lim_{\epsilon\to 0}\chi(|h|^2 v/\epsilon)T \tag{7.5}$$

[5]We have not even excluded the possibility that h is identically 0 on some (or all) irreducible components of X

if $V(h) = V$. $\epsilon > 0$ is small enough. In the same way,

$$\frac{1}{h}T = \lim_{\epsilon\to 0}\frac{\chi(|h|^2 v/\epsilon)}{h}T, \qquad \bar\partial\frac{1}{h}\wedge T = \lim_{\epsilon\to 0}\frac{\bar\partial\chi(|h|^2 v/\epsilon)}{h}\wedge T,$$

although in the second equality one should let χ be smooth, in order to avoid unnecessary interpretation problems, cf., also Example 7.6 below.

In most cases it is a matter of taste if one use the analytic continuation or the limit with cutoff functions. In most situations we stick to the analytic continuation since we find it more practical. However, at a few occasions, as in the proof of ???? above, it is convenient to have smooth approximands, like the smooth regularization $\bar\partial\chi(|h|^2 v/\epsilon)/h$ of $\bar\partial(1/h)$.

Example 7.6. Let V be a subvarity of an open set $X \subset \mathbb{C}^n$ of pure codimension p, and let f, h be holomorphic in X, such that $V(f) \supset V(h) \cup V_{sing}$, but f does not vanish identically on any irreducible component of V.

It is proved in Section **??** below that the Lelong current $[V]$ is pseudomeromorphic. Thus $(1/h)[V]$ is a well-defined pseudomermorphic current. By the dimension principle we have that $\mathbf{1}_{V(f)}(1/h)[V] = 0$ and thus

$$\frac{1}{h}[V] = \mathbf{1}_{X\setminus V(f)}\frac{1}{h}[V] = \lim_{\epsilon\to 0}\chi(|f|^2/\epsilon)\frac{1}{h}[V]$$

according to (7.5). Since $\chi(|f|^2/\epsilon)(1/h)$ has support outside V_{sing}, we have that

$$\Big\langle\frac{1}{h}[V], \xi\Big\rangle = \lim_{\epsilon\to 0}\int_V \chi(|f|^2/\epsilon)\frac{1}{h}\wedge\xi,$$

for test forms ξ. Taking $\bar\partial$ we get

$$\Big\langle\bar\partial\frac{1}{h}\wedge[V], \xi\Big\rangle = \lim_{\epsilon\to 0}\int_V \bar\partial\chi(|f|^2/\epsilon)\wedge\frac{1}{h}\wedge\xi,$$

as long as χ is smooth. If $\chi = \chi_{[1,\infty)}$, then for almost all small $\epsilon > 0$ we can apply Stokes' theorem, and so we get

$$\Big\langle\bar\partial\frac{1}{h}\wedge[V], \xi\Big\rangle = \lim_{\epsilon\to 0}\int_{V\cap\{|f|^2=\epsilon\}}\frac{1}{h}\wedge\xi.$$

□

Example 7.7. The currents $\log|z|^2$ and $\mu := d\bar z/\bar z = \bar\partial\log|z|^2$ are *not* pseudomeromorphic. In fact, if μ is pseudomeromorphic , then we can form $\tau = (dz/z)\wedge\mu$ and then τ is equal to $dz\wedge d\bar z/|z|^2$ outside the origin. In view of (7.5) the limit $\lim\chi(|z|^2/\delta)\tau$ would exist, but this is certainly not true. □

Example 7.8 (The Poincaré-Leray residue formula). Let g be a holomorphic function in $X \subset \mathbb{C}^n$ such that $dg \neq 0$ on V_{reg}, where $V = V(g)$. We claim that there is a unique meromorphic $(n-1, 0)$-form ω on V, cf., Example 7.2 above, such that

$$i_*\omega = \bar\partial\Big[\frac{1}{g}\Big]\wedge dz \tag{7.6}$$

if ω here denotes the associated principal value current, cf., Example 7.2. The form ω is called the Poincaré-Leray residue of the meromorphic form dz/g in Ω.

Clearly ω (or equivalently, the associated principal value current) is unique if it exists since i_* is injective. Given a point $x \in V$ we can find a form ω' in a neighborhood in Ω such that

$$dg \wedge \omega' = 2\pi i. \tag{7.7}$$

In fact, for some j,

$$\omega' = \frac{1}{\partial g/\partial z_j}\widehat{dz_j} \tag{7.8}$$

will do at x. Then

$$\int_\Omega i_* i^* \omega' \wedge \xi = \int_V i^*\omega' \wedge i^*\xi = \int_\Omega [V]\wedge\omega'\wedge\xi = \int_\Omega \bar\partial\Big[\frac{1}{g}\Big]\wedge\frac{dg}{2\pi i}\wedge\omega'\wedge\xi = \int_\Omega \bar\partial\Big[\frac{1}{g}\Big]\wedge dz\wedge\xi$$

so (7.6) holds for $\omega = i^*\omega'$. Here we have used the Poincaré-Lelong formula, cf., Example 0.16, $\bar\partial(1/g)\wedge dg/2\pi i = [V]$. Possibly after a linear change of coordinates we may assume that $\partial g/\partial z_j$ is generically nonvanishing on each irreducible component of V. Thus $\omega = i^*\omega'$ is in fact a meromorphic form on V and so it defines a pseudomeromorphic current that we also denote by ω, cf., Example 7.2. Since (7.8) holds outside V_{reg} it must hold across V_{sing} by the dimension principle, since both sides are pseudomeromorphic.

Let h be any tuple such that $V(h)$ contains V_{sing} but no irreducible component of V. Then $\omega = \lim \chi(|h|/\delta)\omega$. If $h = \partial g/\partial z_j$ as above and $\chi = \chi_{[1,\infty)}$ we get

$$\Big\langle \bar\partial\frac{1}{g}\wedge dz, \xi\Big\rangle = \lim_{\epsilon\to 0}\int_{V\cap\{|\partial g/\partial z_1|^2>\epsilon\}} \frac{2\pi i dz_2\wedge\ldots\wedge dz_n\wedge\xi}{\partial g/\partial z_1}.$$

We can also take $h = (\partial g/\partial z_1, \ldots, \partial g/\partial z_n)$. Since Since also

$$\omega' = \frac{2\pi i\sum_j \overline{(\partial g/\partial z_j)}\widehat{dz_j}}{|dg|^2}$$

satisfies (7.7) outside V_{sing} we get the formula

$$\Big\langle \bar\partial\frac{1}{g}\wedge dz, \xi\Big\rangle = \lim_{\epsilon\to 0}\int_{V\cap\{|dg|^2>\epsilon\}} \frac{2\pi i\sum_j \overline{(\partial g/\partial z_j)}\widehat{dz_j}\wedge\xi}{|dg|^2}.$$

□

The form ω on V and analogues for general varieties we be of basic importance in Section **??**.

8. Comments to Chapter ??

The definition here is from [10] and it is in turn a slight elaboration of the definition introduced in [13].

9. The standard extension property, SEP

Let Z be a pure-dimensional subvariety of X. We say that a pseudomeromorphic current μ on X with support on Z has the *standard extension property*, SEP, on Z if $\mathbf{1}_V\mu = 0$ for each $V \subset Z$ of positive codimension on Z. We let $\mathcal{W}_Z^X$ denote the subsheaf of $\mathcal{PM}^X$ of currents with the SEP on Z. Instead of $\mathcal{W}_X^X$ we just write $\mathcal{W}^X$. We shall first discuss a special case of global pseudomeromorphic currents with the SEP on X.

10. Almost semi-meromorphic currents

Recall that a current is semi-meromorphic if is a principal value current of the form α/f, where α is a smooth form and f is a holomorphic function. We shall now discuss a far-reaching generalization of the operation $\tau \mapsto (1/f) \wedge \tau$.

Let X be a puredimensional analytic space. We say that a current a is *almost semi-meromorphic* in X, $a \in ASM(X)$, if there is a modification $\pi\colon X' \to X$ such that

$$a = \pi_*(\omega/f), \tag{10.1}$$

where f is a holomorphic section of a line bundle $L \to X'$, not vanishing identically on any irreducible component of X', and ω is a smooth section of L. We say that a is *almost smooth* in X, $a \in AS(X)$, if one can choose $f = 1$.

We can assume that X' is smooth because otherwise we take a smooth modification $\pi'\colon X'' \to X'$ and consider the pullback of f and ω to X''. If nothing else is said we always tacitly assume that X' is smooth.

Assume that $a \in ASM(X)$ and that V has positive codimension in X. Since $\pi^{-1}V$ has positive codimension in X' we have that $\mathbf{1}_V a = \pi_*(\mathbf{1}_{\pi^{-1}V}(\omega/f)) = 0$. Thus $ASM(X)$ is a subspace of $\mathcal{W}(X)$.

Example 10.1. Let $b = \partial|z|^2/2\pi|z|^2$ in $\mathbb{C}^n$ and let $\pi\colon X \to \mathbb{C}^n$ be the blow-up at 0. Then $b = \alpha/s$ where s is a section that defines the exceptional divisor and α is smooth. Now

$$\bar\partial b = \pi_*\Big(\frac{\bar\partial\alpha}{s}\Big) + \pi_*\Big(\bar\partial\frac{1}{s} \wedge \alpha\Big) = \pi_*\Big(\frac{\bar\partial\alpha}{s}\Big),$$

since the second term must vanish by the dimension principle. It follows that

$$b \wedge (\bar\partial b)^{k-1} = \pi_*\Big(\frac{\alpha \wedge (\bar\partial\alpha)^{k-1}}{s^k}\Big)$$

are almost semimeromorphic for $k \le n$. □

Remark 10.2. One can of course introduce a notion of locally almost semimeromorphic currents and consider the associated sheaf. However, will have no immediate need for this notion. □

Given two modifications $X_1 \to X$ and $X_2 \to X$ there is a modification $\pi\colon X' \to X$ that factorizes over both X_1 and X_2, i.e., we have $X' \to X_j \to X$ for $j = 1, 2$. Therefore, given $a_1, a_2 \in ASM(X)$ we can assume that $a_j = \pi_*(\omega_j/f_j)$, $j = 1, 2$. It follows that

$$a_1 + a_2 = \pi_*\Big(\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2}\Big) = \pi_*\frac{f_2\omega_1 + f_2\omega_2}{f_1 f_2},$$

so that $a_1 + a_2$ is in $ASM(X)$ as well. Moreover, $A := \pi_*(\omega_1 \wedge \omega_2/f_1 f_2)$ is an almost semimeromorphic current that coincides with $a_1 \wedge a_2$ outside the set $\pi(\text{sing}\,(\pi) \cup \pi V(f_1) \cup V(f_2)$. If we had two other representations of a_j we would get an almost semimeromorphic A' that coincides generically with $a_1 \wedge a_2$ on X. Because of the SEP thus $A = A'$. Thus we can define $a_1 \wedge a_2$ as A. It is readily verified that

$$a_2 \wedge a_1 = (-1)^{\deg a_1 \deg a_2} a_1 \wedge a_2$$

as usual.

Let $\text{sing}\,(\pi)$ be the (analytic) set where π is not a biholomorphism. By definition of modification it has positive codimension. Let $Z \subset X'$ be the zero set of f. By assumption also Z has positive codimension. Notice that a is smooth outside

$\pi(Z \cup \text{sing}\,(\pi))$ which has positive codimension in X. It follows that the smallest Zariski-closed set $V = ZSS(a)$ such that a is smooth outside V, the *Zariski-singular support* of a, has positive codimension in X.

Example 10.3. Notice that if Z is empty, then a is almost smooth, and $ZSS(a) \subset \pi(\text{sing}\,(\pi))$. However, this inclusion may be strict. Notice that if a is smooth, i.e., $ZSS(a)$ is empty, then $\omega = \pi^* a$ outside $\text{sing}\,(\pi)$. Since both sides are smooth across $\text{sing}\,(\pi)$, by continuity also the equality must hold everywhere in X'. □

Lemma 10.4. *If a is almost semi-meromorphic in X, then it has a representation* (10.1) *such that f is non-vanishing in $X' \setminus \pi^{-1} ZSS(a)$, and $\omega = f\pi^* a$ there.*

Proof. Let $V = ZSS(a)$ and let us assume that we have a representation (10.1) and that X' is smooth. Let Z' be an irreducible component of $Z = Z(f)$ such that Z' is not fully contained in $\pi^{-1}V$. Since X' is smooth, Z' is a Cartier divisor, and so there is a section s of a line bundle $L' \to X'$ that defines Z'. Since Z' is irreducible, f has a fixed order r along Z' and it follows that $f = f'g$ where $f' = s^r$ and g holomorphic and non-vanishing on $Z' \cap Z_{reg}$. Outside $\text{sing}\,(\pi) \cup Z = \text{sing}\,(\pi) \cup Z \cup \pi^{-1}V$ we have that $\omega = f\pi^* a$ and hence

$$\omega = f\pi^* a = f' g \pi^* a \tag{10.2}$$

there. By continuity it follows that (10.2) must hold in $X' \setminus \pi^{-1}V$ since both sides are smooth there.

We now claim that ω/f' is smooth in X'. Taking this for granted, the lemma now follows by a finite induction over the number of irreducible components of Z not fully contained in $\pi^* a$. Thus we have to prove the claim.

It is a local statement in X' so given a point in X' we can choose local coordinates s in a neighborhood U of that point and consider each coefficient of the form ω with respect to these coordinates. Thus we may assume that ω is a function. Then still $\omega = f'\gamma$ where γ is smooth. For all multiindices α we thus have that

$$\frac{\partial^\alpha \omega}{\partial \bar{t}^\alpha} \bar{\partial} \frac{1}{f'} = 0 \tag{10.3}$$

in $X' \setminus \pi^{-1}V$. By assumption $Z' \cap \pi^{-1}V$ has positive codimension on Z'. By the dimension principle it follows that (10.3) holds in X' for all α. From Theorem 6.3 we conclude that ω/f' is smooth in U. It follows that it is smooth in X' □

Theorem 10.5. *Assume that $a \in ASM(X)$. For each $\tau \in \mathcal{PM}(X)$, there is a unique pseudomeromorphic current $A\tau$ in X that coincides with $a \wedge \tau$ in $X \setminus ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)} A\tau = 0$.*

Let h be a tuple (locally) such that $Z(h) = V := ZSS(a)$. If the extension $A\tau$ exists, then $A\tau = \mathbf{1}_{X \setminus V}$ and thus

$$A\tau = \lim_{\epsilon \to 0} \chi(|h|/\epsilon) a \wedge \tau. \tag{10.4}$$

In particular, the extension must be unique. It is natural to denote this extension by $a \wedge \tau$ as well.

Conversely, if the limit in (10.4) exists as a pseudomeromorphic current in X, then $A\tau$ must coincide with $a \wedge \tau$ in $X \setminus V$. Moreover, $\chi(|h|/\epsilon)A\tau = \chi(|h|/\epsilon)a \wedge \tau$ and hence $\mathbf{1}_{X \setminus ZSS(a)} A\tau = A\tau$, i.e., $\mathbf{1}_V A\tau = 0$. To prove the theorem it is thus enough to verify the the limit in (10.4) exists as a pseudomeromorphic current.

Proof. In view of Lemma 10.4 we may assume that a has the form (10.1), where $Z = Z(f)$ is contained in $\pi^{-1}V$ and that $\omega/f = \pi^* a$ in $X' \setminus \pi^{-1}V$. Let $\chi_\epsilon = \chi(|h|/\epsilon)$, so that $\pi^*\chi_\epsilon = \chi(|\pi^* h|/\epsilon)$.

By proposition 4.10 there is $\tau' \in \mathcal{PM}(X')$ such that $\pi_*\tau' = \tau$. Thus

$$\chi_\epsilon a \wedge \tau = \chi_\epsilon a \wedge \pi_*\tau' = \pi_*\big(\pi^*\chi_\epsilon \pi^* a \wedge \tau'\big) = \pi_*\big(\pi^*\chi_\epsilon \frac{\omega}{f} \wedge \tau'\big).$$

Notice that

$$\pi^*\chi_\epsilon \frac{\omega}{f} \wedge \tau' \to \mathbf{1}_{X'\setminus \pi^{-1}V} \frac{\omega}{f} \wedge \tau'$$

when $\epsilon \to 0$. In particular, this is a pseudomeromorphic current. Thus the limit in (10.4) exists and is pseudomeromorphic. □

Notice that if W is any analytic set, then

$$\mathbf{1}_W(a \wedge \tau) = a \wedge 1_W \tau. \tag{10.5}$$

In fact, the equality holds in the open set $X \setminus ZSS(a)$ since a is smooth there. On the other hand are both sides zero on $ZSS(a)$ since $\mathbf{1}_{ZSS(a)}\mathbf{1}_W(a\wedge\tau) = \mathbf{1}_W\mathbf{1}_{ZSS(a)}(a \wedge \tau) = 0$.

Assume now that X is smooth, z is a coordinate system and let $dz := dz_1 \wedge \ldots \wedge dz_n$.

Lemma 10.6. *If $\mu\wedge dz$ is almost semi-meromorphic then μ is almost semi-meromorphic as well.*

Proof. Assume that $\mu \wedge dz = \pi_*(\omega/f)$. If f is a section of $L \to X'$, then ω must be a section of $L \otimes K_{X'}$. Now $g = \pi^* dz$ is a genericallynon-vanishing section of $K_{X'}$. Thus $\mu' = \pi_*(\omega/fg)$ is almost semimeromorphic in X, and $\mu' \wedge dz = \pm\pi_*(g\omega/fg) = \pm\mu\wedge dz$. It follows that $\mu = \pm\mu'$ and thus μ is in $ASM(X)$. □

Lemma 10.7. *If a is an almost semi-meromorphic $(p,*)$-current on a smooth X and z is a coordinate system, then $(\partial a/\partial z_\ell)\mu$ is almost semi-meromorphic as well.*

Proof. Assume that

$$a = \sum_{|I|=p} a_I \wedge dz_I.$$

Fix a multiindex J and let J^c be the complementary index. Then

$$a \wedge dz_{J^c} = \pm a_I \wedge dz.$$

Un view of Lemma 10.6 thus a_I is in $ASM(X)$. Moreover,

$$\frac{\partial a}{\partial z_1} = \sum_{|I|=p} \frac{\partial a_I}{\partial z_1} \wedge dz_I.$$

Therefore it is enough to consider the case when a has bidegree $(0,q)$. Assume that $a = \pi_*(\omega/f)$. Let $D = D' + \bar\partial$ be a Chern connection on L. Then

$$\partial a = \pi_*(\partial \frac{\omega}{f}) = \pi_* \frac{D'\omega \cdot f - \omega D' f}{f^2}$$

which is thus in $ASM(X)$. Therefore $\partial a/\partial z_1 \wedge dz = \partial a \wedge dz_2 \wedge \ldots dz_n$ is in $ASM(X)$. From Lemma10.6 we conclude that $\partial a/\partial z_1$ is in $ASM(X)$. □

Notice that if a_1, a_2 are almost smooth, then $a_1 \wedge a_2$ is almost smooth. Moreover, $\bar\partial a_j$ are almost smooth and $\bar\partial(a_1 \wedge a_2) = \bar\partial a_1 \wedge a_2 + (-1)^{\deg a_1} a_1 \wedge \bar\partial a_2$.

Ar det sant att $\partial a/\partial z_1$ ar almost smooth ???

Example 10.8. Let W be a hypersurface in X. We claim that if $\alpha \in \mathcal{PM}_{k,0}(X)$ and the restriction α' to $X \setminus W$ is holomorphic, then α is meromorphic on X. In fact, by assumption, α' has a current extension to X, so if we have an embedding $i\colon X \hookrightarrow \Omega$, then the current $i_*\alpha'$, a priori defined in $\Omega \setminus W$, has a current extension to Ω. By [23, Theorem 1], α' has a meromorphic extension $\tilde{\alpha}$ to X, and since both α and $\tilde{\alpha}$ are in $\mathcal{PM}^X_{k,0}$, $\alpha = \tilde{\alpha}$ by the dimension principle.

Let a be a meromorphic form in Ω such that $\alpha = i^* a$. Then $i_*\alpha = a \wedge [X]$, where $[X]$ is the Lelong current associated with X in Ω, so $\bar\partial\alpha = 0$ on X precisely means that $\bar\partial(a \wedge [X]) = 0$ in Ω. This in turn by the definition in [23] means that α is in the sheaf (that we denote) $\mathcal{B}^X_k$ of Barlet-Henkin-Passare holomorphic $(k,0)$-forms. We conclude that $\mathcal{B}^X_k$ is the subsheaf of $\bar\partial$-closed currents in $\mathcal{PM}^X_{k,0}$. □

11. Some further properties of $\mathcal{PM}$ and $\mathcal{W}$

Notice that if τ is an elementary pseudomeromorphic current in $\mathbb{C}^n_z$ and z^α is a monomial, then there is an elementary current τ' such that $z^\alpha \tau' = \tau$. In fact, by induction it is enough to assume that the monomial is z_1. If z_1 is a residue factor or a principal value factor in τ then we just raise the power of z_1 in that factor one unit. Otherwise we take $\tau' = (1/z_1)\tau$.

We shall now see that this observation holds in more generality.

Proposition 11.1. *Assume that $\mu \in \mathcal{PM}_x$ where $x \in X$ and X is smooth.*
(i) If $h \in \mathcal{O}_x$ is not identically zero, then there is $\mu' \in \mathcal{PM}_x$ such that $h\mu' = \mu$.
(ii) If

$$\mu = \sum_{|I|=p}{}' \mu_I \wedge dz_I,$$

then each μ_I is in $\mathcal{PM}_x$.

(iii) $(\partial/\partial z_\ell) \neg \mu$ is in $\mathcal{PM}_x$

(iv) $(\partial/\partial z_\ell)\mu$ (Lie derivative) is in $\mathcal{PM}_x$.

The same statement holds with $\mathcal{W}^X_x$ instead of $\mathcal{PM}^X_x$.

By a partition of unity we get a global μ' such that $h\mu' = \mu$. Also (ii) and (iv) hold globally if, say, $\partial/\partial/z_\ell$ is replaced by a global holomorphic vector field.

Notice that (i) is not true if h is anti-holomorphic. In fact, if $\bar z\mu' = 1$, then $(1/z)\mu'$ is equal to $1/|z|^2$ outside 0. Thus $\lim \chi(|z|^2/\delta)\mu'$ does not exist, and hence μ' cannot be pseudomeromorphic. Moreover, neither (*iii*) or (*iv*) is true for $\partial/\partial \bar z \ell$. For example, the current

$$\tau = \frac{\partial}{\partial z}\frac{1}{z} = \frac{\partial}{\partial z} \neg \bar\partial \frac{1}{z}$$

is nonzero but with support at 0 so, in view of the dimension principle, it cannot be pseudomeromorphic.

Proof. We know that there is a modification $\pi\colon \widetilde{X} \to X$ such that

$$\mu = \sum_\ell \tau_\ell,$$

where τ_ℓ are elementary and $\pi^* h$ is locally a monomial in $\widetilde{X}$, cf., Lemma 5.1. As noted above we can find elementary τ'_ℓ such that $\pi^* h \tau'_\ell = \tau_\ell$. Thus

$$\mu = \pi_* \sum_\ell \pi^* h \tau'_\ell = \pi_* \big(\pi^* h \sum \tau'_\ell\big) = h \pi_* \sum \tau'_\ell =: h\mu'.$$

Thus (i) is proved.

We now consider (ii). We first assume that μ has bidegree $(n,*)$ so that $\mu = \hat{\mu} \wedge dz$ and show that $\hat{\mu}$ is pseudomeromorphic. We may assume that $\mu = \pi_*(\tau \wedge ds)$ where τ is elementary. Since π is generically surjective, we may assume, at least locally in $\widetilde{X}$, that $s = (s', s'')$ where $h = \det(\partial \pi / \partial s') = \det(\partial z / \partial s')$ is not vanishing identically. Let us assume that this holds on the support of τ; ortherwise we use a partition of unity in $\widetilde{X}$. By (i) there is τ' such that $h\tau' = \tau$ in $\widetilde{X}$. Now

$$\hat{\mu} \wedge dz = \pi_*(\tau \wedge ds) = \pi_*(\tau' \wedge h ds' \wedge ds'') = \pi_*(\tau' \wedge \pi^* dz \wedge ds'') = \pm \pi_*(\tau' \wedge ds'') \wedge dz.$$

Thus $\hat{\mu} = \pm \pi_*(\tau' \wedge ds'')$ is pseudomeromorphic.

In general, $\mu_I \wedge dz = \pm \mu \wedge dz_{I^c}$, where I^c is the complementary multiindex of I. It follows that μ_I is pseudomeromorphic. Thus (ii) is proved.

Now (iii) immediately follows. It is enough to prove (iv) for μ of bidegree $(0,*)$. We notice that

$$\frac{\partial}{\partial z_\ell} \mu = \frac{\partial}{\partial z_\ell} \neg \partial \mu$$

and thus (iv) follows from (iii) since $\partial \mu$ is pseudomeromorphic.

Let us now consider the case with $\mathcal{W}$. If $\mu \in \mathcal{W}$, then $f[1/h]\mu = \mathbf{1}_{X \setminus V(h)} \mu = \mu$ so (i) follows. To see (ii) just notice that

$$\mathbf{1}_V \mu = \sum_{|I|=p}{}' (\mathbf{1}_V \mu_I) \wedge dz_I,$$

and hence μ has the SEP if and only if each μ_I has. Now (iii) follows directly. For (iv) we need the following simple but useful lemma.

Lemma 11.2. *A current $\mu \in \mathcal{PM}_x$ is in $\mathcal{W}_x$ if and only if it has a representation*

$$\mu = \sum_\ell \pi_* \tau_\ell$$

where no τ_ℓ has elementary support contained in any set $\pi^{-1} V$, where V has positive codimension in X_x.

This lemma is a simple consequence of Lemma 6.2 above. Now (iv) follows for $\mu \in \mathcal{W}_x$ just noting that if the elementary support cannot decrease under the action of ∂. □

12. Tensor products and direct images under simple projections

Lemma 12.1. *If $T \in \mathcal{PM}^X$ and $T' \in \mathcal{PM}^{X'}$, then $T \otimes T' \in \mathcal{PM}(X \times X')$.*

Proof. It is enough to consider $T = \pi_* \tau$ and $T' = \pi'_* \tau'$, where τ and τ' are elementary and π and π are as in (3.2). However, the mapping $\pi \otimes \pi' \colon \widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}'$ is a composition of modifications, simple projections, and open inclusions. To see this, just notice that if $p \colon Y \to X$ is a modification, a simple projection, or an open inclusion, then the same holds for $p \otimes I \colon Y \times Z \to X \times Z$. Now $\tau \otimes \tau'$ is elementary in $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}'$ and $T \otimes T' = (\pi \otimes \pi')_* \tau \otimes \tau'$. □

It is easy to verify that

$$\mathbf{1}_{V\times V'}T\otimes T' = \mathbf{1}_V T\otimes \mathbf{1}_{V'}T'. \tag{12.1}$$

Lemma 12.2. *Assume that* $p\colon Z\times W\to Z$ *is a simple projection. If* μ *is in* $\mathcal{PM}^{Z\times W}$ *and* $p^{-1}K\cap \text{supp}\,\mu$ *is compact for each compact set* $K\subset Z$*, then* $p_*\mu$ *is in* $\mathcal{PM}^Z$*.*

Proof. Since being pseudomeromorphic is a local property, multiplying μ if necessary by a suitable cutoff function we can assume that μ has compact support. By compactness and a partition of unity we then have a finite representation $\mu=\sum_\ell \pi_*\tau_\ell$. Now the lemma follows from the very definition of $\mathcal{PM}$. □

Example 12.3. If p is a simple projection $X\times X'\to X$, we can take any test form χ in X' with total integral 1. Then the tensor product $\tau\otimes\chi$ is en elementary current in $X\times X'$ such that $p_*(\tau\otimes\chi)=\tau$. □

It follows from Example 10.1, tensorizing with 1, and a linear change of coordinates that $b\wedge(\bar\partial b)^{k-1}$ is almost semi-meromorphic in $\mathbb{C}^n\times\mathbb{C}^n$ if $b=\partial|\zeta-z|^2/2\pi i|\zeta-z|^2$. If X is a domain in $\mathbb{C}^n$ then weighted integral kernel like $K=(g\wedge u)_{n,n-1}$ in as in Ch?. Section **??** is almost semi-meromorphic in $X\times X$. Now let μ be pseudomeromorphic with compact support in a domain $X\subset\mathbb{C}^n$. Then $\mu\otimes 1$ is pseudomeromorphic in $X\times X$ and thus $K\wedge\mu:=K\wedge(\mu\otimes 1)$ is pseudomeromorphic in $X\times X$. Notice that this product is unproblematic since, after a linear change of variables locally, it is a tensor product. Since K is almost semimeromorphic it is the limit of $\chi(|\zeta-z|/\epsilon)K$ and hence the product $K\wedge\mu$ considered as a tensor product coincides with it considered as product of an almost semi-meromorphic current and a pseudomeromorphic current in $X\times X$ as in Theorem **??**.

It follows that we can apply the simple projection $p\colon X_\zeta\times X_z\to Z_z$ and get a pseudomeromorphic current $K\mu$ that is precisely the bla from Section ???. We thus have the Koppelman formula

$$\mu=\bar\partial K\mu+K\bar\partial\mu+P\mu,$$

and that all terms are pseudomeromorphic.

In particular it follows, cf., the proof of ??? (the case with general currents) that

$$0\to\mathcal{O}\to\mathcal{PM}_{0,0}\xrightarrow{\bar\partial}\mathcal{PM}_{0,1}\to$$

is a fine resolution of $\mathcal{O}^X$ when X is smooth.

Easy to see that can "divide" by h: Say more: $K\mu$ is in $\mathcal{W}$.

Proposition 12.4. *The integral operators* K,P *map pseudomeromorphic currents with compact support into* $\mathcal{W}$ *and any pseudomeromorphic into* $\mathcal{W}$ *if* K,P *have compact support with respect to* ζ*.*

Proof. Assume that $\mu\in\mathcal{PM}$ has compact support and that V has positive codimension. In view of (10.5) and (12.1) we have

$$\mathbf{1}_V K\mu=\mathbf{1}_V p_*(K\wedge\mu\otimes 1)=p_*(\mathbf{1}_{\mathbb{C}^n\times V}(K\wedge\mu\otimes 1))=$$
$$p_*(K\wedge\mathbf{1}_{\mathbb{C}^n\times V}(\mu\otimes 1))=p_*(K\wedge\mathbf{1}_{\mathbb{C}^n}\mu\otimes\mathbf{1}_V 1)=0,$$

since $\mathbf{1}_V 1=0$. The argument for P is even simpler. □

We can now provide a completely different proof of Proposition 11.1.

A new proof of Proposition 11.1. (This proof is not elaborated in detail!) First assume that μ is in $\mathcal{W}$. As in the previous proof we then have that $h[1/h]\mu = \mu$. Notice also that $h\bar\partial[1/h]\mu = \bar\partial h[1/h]\mu = \bar\partial\mu$ so (i) is proved for $\tau = \bar\partial\mu$ where μ isin $\mathcal{W}$. In the general case we may assume that μ has compact support in $\mathbb{C}^n$. Let K be the Bochner-Martinelli integral operator. Then

$$\mu = \bar\partial K\mu + K\bar\partial\mu.$$

Now (i) follows in view of Proposition 12.4.

Notice that if μ is pseudomeromorphic and $\mu\wedge dz$ is in $\mathcal{W}$ then μ is in $\mathcal{W}$. Now take a general μ in $\mathcal{PM}_{0,*}$ with compact support and apply Koppelman's formula again. It is not hard to verify that

$$K(\mu\wedge dz) = K\mu\wedge dz$$

if K is the Bochner-Martinelli integral operator and μ is any current with compact support. If now $\mu\wedge dz$ is in $\mathcal{PM}$, then $\bar\partial(\mu\wedge dz)) = \bar\partial\mu\wedge dz$ is in $\mathcal{PM}$ and hence $K(\bar\partial(\mu\wedge dz)) = K(\bar\partial\mu)\wedge dz$ is in $\mathcal{W}$ and in particular in $\mathcal{PM}$. In the same way, $K(\mu\wedge dz) = K\mu\wedge dz$ is in $\mathcal{W}$ and hence $K\mu$ is in $\mathcal{W}$. We conclude that if $\mu\wedge dz$ is in $\mathcal{PM}$, then μ is in $\mathcal{PM}$.

man fixar sedan allmanna fallet emd dz_I och (iii)

(iv) follows as before from (ii). However we want to give a completely different proof:

(iv) Let τ be any current on X and consider the current

$$\tau' = \tau\otimes\bar\partial\frac{dw}{2\pi i w^2}$$

on the manifold $X' = X\times\mathbb{C}_w$. Clearly τ' has support on X and we claim that it has the SEP with respect to X. In fact, $\tau' = \mathbf{1}_X\tau'$ so that if π is the projection $(z,w)\mapsto z$, then

$$\mathbf{1}_V\tau' = \mathbf{1}_{\pi^{-1}V\cap X}\tau' = \mathbf{1}_{\pi^{-1}V}\mathbf{1}_X\tau' = \mathbf{1}_{\pi^{-1}V}\tau'.$$

Moreover, if $h(z)$ cuts out V in X, then

(12.2)

$$\mathbf{1}_{V\times\mathbb{C}_w}\tau' = \lim(1-\chi(|h(z)|/\epsilon)\tau' = \lim(1-\chi(|h(z)|/\epsilon)\tau\otimes\bar\partial\frac{dw}{w^2} = \mathbf{1}_V\tau\otimes\bar\partial\frac{dw}{w^2} = 0.$$

Now let us make the change of variables

$$z_1 = \zeta_1 - w, \quad z_j = \zeta_j,\ j = 2,\ldots,n, \quad w = \omega,$$

and let p be the natural projection $(\zeta,\omega)\mapsto\zeta$. Since

$$\bar\partial\frac{dw}{2\pi i w^2}.\xi(w) = \frac{\partial\xi}{\partial w}(0)$$

it is readily verified that $p_*\tau' = \partial\tau/\zeta_1$. Now,

$$\mathbf{1}_V(\partial\tau/\zeta_1) = \mathbf{1}_V p_*\tau' = p_*(\mathbf{1}_{p^{-1}V}\tau')) = p_*0 = 0,$$

cf., (12.2), and thus $\partial\tau/\zeta_1$ is in $\mathcal{W}^X$.

$\square$

13. Local representation of $\mathcal{W}_Z^X$

Assume that X is smooth. Let $Z \subset X$ be a smooth submanifold of codimension p and let us choose local coordinates (z, w) such that $Z = \{w_1 = \cdots = w_p = 0\}$.

Lemma 13.1. *Each $\mu \in \mathcal{PM}_Z^X$ of bidegree $(0,k)$ has a unique representation as a finite sum*

$$\mu = \sum_{|\alpha|=p} \mu_\alpha \otimes \bar\partial \frac{1}{w^{\alpha+\mathbf{1}}} \tag{13.1}$$

where μ_α are in $\mathcal{PM}^Z_{0,k-p}$. Moreover, $\mu \in \mathcal{W}_Z^X$ if and only if each μ_α is in $\mathcal{W}^Z$.

Here

$$\bar\partial \frac{1}{w^{\alpha+\mathbf{1}}} = \bar\partial \frac{1}{w_1^{\alpha_1+1}} \wedge \ldots \wedge \bar\partial \frac{1}{w_p^{\alpha_p+1}}.$$

Proof. ???

□

14. Pseudomeromorphic currents on reduced subvarieties

Theorem 14.1. *Assume that $i\colon X \to Y$ is an embedding of a reduced pure-dimensional space X into a smooth manifold Y.*

(i) If τ is in $\mathcal{PM}^X$, then $i_*\tau$ is in $\mathcal{PM}^Y$, and if τ is in $\mathcal{W}^X$ then $i_*\tau$ is in $\mathcal{W}_X^Y$.

(ii) If τ is in $\mathcal{C}^X$ and $i_*\tau$ is in $\mathcal{PM}^Y$, and in addition,

$$\mathbf{1}_{X_{sing}} i_*\tau = 0, \tag{14.1}$$

then τ is in $\mathcal{PM}^X$. If $i_*\tau$ is in $\mathcal{W}_X^Y$, then τ is in $\mathcal{W}^X$.

That is, we have the natural mappings

$$i_*\colon \mathcal{PM}^X \to \mathcal{PM}^Y, \quad i_*\colon \mathcal{W}^X \to \mathcal{W}_X^Y.$$

Notice that the condition (14.1) in (ii) is automatically fulfilled if $i_*\tau$ is in $\mathcal{W}_X^Y$.

As already mentioned the proof of Theorem 14.1 relies on the existence of a *strong desingularization*, see, e.g., [**?**] and the refererences given there. This means that there is a smooth modification $p\colon \widetilde{Y} \to Y$ that is a biholomorphism outside X_{sing} and such that the strict transform $\widetilde{X}$ of X is a smooth submanifold of $\widetilde{Y}$ and the restriction p' of p to $\widetilde{X}$ is a modification $p'\colon \widetilde{X} \to X$ of X. Thus we have a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{i}} & \widetilde{Y} \\ \downarrow p' & & \downarrow p \\ X & \xrightarrow{i} & Y \end{array} . \tag{14.2}$$

Proof of Theorem 14.1. First assume that X is a smooth submanifold. The statement (i) is local so we may assume that $Y = X_z \times \mathbb{C}^r$ and $i(z) = (z, 0)$. It is easily checked that $i_*\tau$ is equal to the tensor product

$$\mu := \tau \wedge [w = 0] \tag{14.3}$$

where $[w=0]$ means the point evalutation at $0 \in \mathbb{C}^r$. In view of Lemma **??** it is then pseudomeromorphic since $[w=0] = \bar\partial \frac{1}{w} \wedge dw(2\pi i)^{-r}$ is. For a test form $\xi = \xi(z,w)$, we can write $\xi = \xi' + \xi''$, where ξ' contains no occurrences of dw_j or $d\bar w_j$. Then

$$i_*\tau.\xi = \tau.i^*\xi = \tau.i^*\xi' = \tau.\xi'(\cdot, 0) = \mu.\xi,$$

cf., (14.3), and hence $i_*\tau = \mu$ is pseudomeromorphic in Y. Now assume that $i\colon X \to Y$ is arbitrary and consider (14.2). Any $\tau \in \mathcal{PM}(X)$ can be written $p'_*\tilde\tau$ for some $\tilde\tau \in \mathcal{PM}(\widetilde X)$ according to Proposition 4.10. By the first part we now that $\tilde i_*\tilde\tau$ is pseudomeromorphic in $\widetilde Y$. Thus $i_*\tau = i_*p'_*\tilde\tau = p_*\tilde i_*\tilde\tau$ is pseudomeromorphic in Y, and so the first part of (i) is proved.

Assume that $V \subset X$ has positive codimension. Since $i^{-1}V = V$ we have, cf., (**??**), that $\mathbf{1}_V i_*\tau = i_*\mathbf{1}_V\tau$. Thus $i_*\tau$ is in $\mathcal{W}_X^Y$ if (and only if) τ is in $\mathcal{W}^X$, and so the second part of (i) follows.

We now consider (ii). Again assume first that X is smooth. Again the statement is local so we may assume that $Y = X_z \times \mathbb{C}^r_w$. Let $\pi\colon Y \to X_z$ be the projection $(z,w) \mapsto z$. Since $i_*\tau$ is pseudomeromorphic by assumption also $p_*i_*\tau$ is pseudomeromorphic. Now,

$$p_*i_*\tau.i^*\xi = i_*\tau.p^*i^*\xi = i_*\tau.\xi'(\cdot,0) = \tau.i^*\xi,$$

for all test forms ξ, and hence $p_*i_*\tau$. We conclude that τ is in $\mathcal{PM}^X$. Thus (ii) holds in case $X \subset Y$ is smooth.

Now assume that $i\colon X \to Y$ is general, $\mu := i_*\tau \in \mathcal{PM}(Y)$, and consider (14.2). We claim that $\mu = p_*\tilde\mu$, where $\tilde\mu \in \mathcal{PM}(\widetilde Y)$, $\tilde\mu$ has support on $\widetilde X$, and $\mathbf{1}_{p^{-1}X_{sing}}\tilde\mu = 0$. To begin with $\mu = p_*\hat\mu$ for some $\hat\mu \in \mathcal{PM}(\widetilde Y)$ according to Proposition 4.10. Since

$$0 = \mathbf{1}_{Y\setminus X}p_*\hat\mu = p_*(\mathbf{1}_{\widetilde Y\setminus p^{-1}X}\hat\mu),$$

cf., (**??**), we have that $\mu = p_*\mu'$ where $\mu' := \mathbf{1}_{p^{-1}X}\hat\mu$ has support on $p^{-1}X$. Notice that this set is in general much larger than the strict transform $\widetilde X$ of X. Now

$$\mu' = \mathbf{1}_{p^{-1}X_{sing}}\mu' + \mathbf{1}_{p^{-1}(X\setminus X_{sing})}\mu'$$

and, by assumption (14.1), $0 = \mathbf{1}_{X_{sing}}\mu = p_*\mathbf{1}_{p^{-1}X_{sing}}\mu'$, and thus $\mu = p_*\tilde\mu$ where

$$\tilde\mu := \mathbf{1}_{p^{-1}(X\setminus X_{sing})}\mu'$$

has support on the closure of $p^{-1}(X \setminus X_{sing})$ which is (contained in) $\widetilde X$. Thus the claim is proved.

Next we claim that $\tilde\mu = \tilde i_*\tilde\tau$ for a current $\tilde\tau$ on $\widetilde X$. In fact, let ξ is a test form on $\widetilde Y$ such that $\tilde i^*\xi = 0$. Since p is a biholomorphism outside $p^{-1}X_{sing}$, $\xi \wedge \tilde\mu = 0$ there since $\mu = i_*\tau$ there. Since $\tilde\mu$ has support on $\widetilde X$ it follows that $\xi \wedge \tilde\mu = 0$ outside $\widetilde X \cap p^{-1}X_{sing}$, and hence $\xi \wedge \tilde\mu = 0$ by continuity. Thus the claim follows.

From the smooth case we know that $\tilde\tau$ is pseudomeromorphic and therefore $p'_*\tilde\tau$ is pseudomeromorphic as well. Finally, $i_*p'_*\tilde\tau = p_*\tilde i_*\tilde\tau = p_*\tilde\mu = \mu = i_*\tau$ and thus $p'_*\tilde\tau = \tau$. Thus τ is pseudomeromorphic. The second part of (ii) is verified as the second part of (i). □

15. Comments to Section ??

Repeated limits very similar to Coleff-Herrera's original definition.

Eller att detta i en remark.

Can prove that indeed coincide, see Larkang-Samuelsson

The sheaves $\mathcal{B}_k^X$ were defined in this way in [23] but introduced earlier by Barlet, [15], in a different way, see [23, Remark 5].

Chapter 3

Coleff-Herrera currents

We shall now consider an important sheaf of residue currents whose annihilator ideals

The prototype is the CH product which is a

We shall now generalize Example ???.

0.1. **The Coleff-Herrera product.** We say that the tuple $f = (f_1, \ldots, f_m)$ of holomorphic functions on X is a *complete intersection* on X if $\operatorname{codim} V(f) = m$, where

$$V(f) = \{f_m = \cdots = f_1 = 0\}.$$

If f is defined in a neighborhood of $x \in X$ we say that it is a *complete intesection at* x if the germ of $V(f)$ at x has codimension m. This holds if and only if f_k is a regular sequence in the local ring $\mathcal{O}_x$. If f_k is a complete intersection at x, then also each subset of f_j is a complete intersection (a regular sequence in $\mathcal{O}_x$). Notice that f_j is a complete intersection on X if and only if it is a complete intersection at each point $x \in V(f)$.

Theorem 0.1. *Assume that* $(f_1, \ldots, f_m)$ *is a complete intersection at* x. *Then*

$$\mu^{f_m,\ldots,f_1} := \bar\partial\frac{1}{f_m}\wedge\ldots\wedge\bar\partial\frac{1}{f_1} \tag{0.1}$$

is $\bar\partial$*-closed, has support on* $V(f)$ *and is anti-commuting in* f_j. *Moreover,*

$$f_m\frac{1}{f_m}\bar\partial\frac{1}{f_{m-1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1} = \bar\partial\frac{1}{f_{m-1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1} \tag{0.2}$$

and

$$f_m\bar\partial\frac{1}{f_m}\wedge\bar\partial\frac{1}{f_{m-1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1} = 0. \tag{0.3}$$

The (germ of a) current μ^f is called the *Coleff-Herrera product* defined by the tuple $(f_1, \ldots, f_m)$ at x.

Proof. Since each subset of f_j is a complete intersection at x we can proceed by induction over the number m of factors. The theorem is clearly true if $m = 1$. Suppose it is proved for k and consider

$$T = \frac{1}{f_{k+1}}\bar\partial\frac{1}{f_k}\wedge\ldots\wedge\bar\partial\frac{1}{f_1} - \bar\partial\frac{1}{f_k}\wedge\ldots\wedge\bar\partial\frac{1}{f_1}\cdot\frac{1}{f_{k+1}}.$$

By the induction hypothesis $\mu^{f_1,\ldots,f_k}$ has support on $V(f_1, \ldots, f_k)$. In view of (7.3), the pseudomeromorphic current T must have support on $V(f_{k+1}, f_1, \ldots, f_k)$. Since this set has codimension $k+1$ and T has bidegree $(0, k)$ it follows from the dimension principle that $T = 0$. By Leibniz' rule (7.2) we get

$$\mu^{f_{k+1},f_k\ldots,f_1} = (-1)^k\mu^{f_k,\ldots,f_1,f_{k+1}}.$$

It follows now (using the induction hypothesis again) that $\mu^{f_{k+1},f_k,\ldots,f_1}$ is anti-commuting in f_j. Notice that

$$\mu^{f_k,\ldots,f_1} - f_{k+1}\frac{1}{f_{k+1}}\mu^{f_k,\ldots,f_1} = \mathbf{1}_{V(f_{k+1})}\mu^{f_k,\ldots,f_1}$$

has bidegree $(0,k)$ and support on $V(f_{k+1}, f_k, \ldots, f_1)$ so again by the dimension principle it must vanish, i.e., (0.2) holds. Finally we get (0.3) from (0.2) and Leibniz' rule. □

One can define μ^f in almost any reasonable way if one just avoids limits by cutoff functions $\chi_{[1,\infty)}$, see the discussion in Section **??** below.

0.2. **The Koszul complex.** Let $f = (f_1, \ldots, f_m)$ be a tuple of holomorphic functions on X. Let E be a trivial vector bundle of rank m with global frame $e_1, \ldots, e_m$ and let e_j^* be its dual frame for the dual bundle E^*. If we consider $f = \sum_j f_j e_j^*$ as a section of E^*, it induces a mapping δ_f on the exterior algebra ΛE. We will also consider differential forms and currents with values in Λ. For instance $\mathcal{E}_{0,k}(\Lambda^\ell E)$ is the sheaf of smooth $(0,k)$-forms with values in $\Lambda^\ell E$ which we consider as a subsheaf of the sheaf of the bundle $\Lambda(E \oplus T^*(X))$. Thus a section v of $\mathcal{E}_{0,k}(\Lambda^\ell E)$ is just a formal expression

$$v = \sum_{|I|=\ell}{}' v_I \wedge e_I,$$

where v_I are smooth $(0,k)$-forms, and with the convention that $d\bar{z}_j \wedge e_j = -e_j \wedge d\bar{z}_j$ etc. In the same way we have the sheaf $\mathcal{C}_{q,k}(\Lambda^\ell E)$ of $\Lambda^\ell E$-valued (q,k)-currents, etc. Notice that both $\bar\partial$ and δ_f act as anti-derivations on these sheaves, i.e.,

$$\bar\partial(v \wedge w) = \bar\partial v \wedge w + (-1)^{\deg v} v \wedge \bar\partial w$$

if at least one of v and w is smooth, and similarly for δ_f. Moreover, it is straight forward to check that

$$\delta_f \bar\partial = -\bar\partial \delta_f. \tag{0.4}$$

If we let

$$\nabla_f := \delta_f - \bar\partial.$$

it follows from (0.4) that

$$\nabla_f^2 = 0. \tag{0.5}$$

Notice that ∇_f is also an anti-derivation. If

$$\mathcal{L}^k := \oplus_j \mathcal{C}_{0,j+k}(\Lambda^j E),$$

we get the complex

$$\stackrel{\nabla_f}{\to} \mathcal{L}^{k-1} \stackrel{\nabla_f}{\to} \mathcal{L}^k \stackrel{\nabla_f}{\to} \mathcal{L}^{k+1} \stackrel{\nabla_f}{\to} .$$

For instance, a section of $\mathcal{L}^{-1}$ is of the form $v = v_1 + \cdots + v_m$, where v_k is a $(0,k-1)$-current with values in $\Lambda^k E$. More formally, (0.4) means that $\mathcal{C}_{0,k}(\Lambda^\ell E)$ is so-called double complex, and $\mathcal{L}^k$ with the mappings ∇_f is the associated total complex.

The Dolbeault-Grothendieck lemma for currents means that

$$0 \to \mathcal{O} \to \mathcal{C}^{0,0} \stackrel{\bar\partial}{\to} \mathcal{C}^{0,1} \stackrel{\bar\partial}{\to} \tag{0.6}$$

is exact. Thus the double complex $\mathcal{C}^{0,k}(\Lambda^\ell E)$ is exact in the k-direction except at $k=0$, where we have the cohomology sheaves $\mathcal{O}(\Lambda^\ell E)$. By standard homological algebra it follows that the natural mapping

$$\frac{\operatorname{Ker}(\mathcal{O}(\Lambda^\ell E) \stackrel{\delta_f}{\to} \mathcal{O}(\Lambda^{\ell-1} E))}{\operatorname{Im}(\mathcal{O}(\Lambda^{\ell+1} E) \stackrel{\delta_f}{\to} \mathcal{O}(\Lambda^\ell E))} \simeq \frac{\operatorname{Ker}(\mathcal{L}^{-\ell} \stackrel{\nabla_f}{\to} \mathcal{L}^{\ell+1})}{\operatorname{Im}(\mathcal{L}^{-\ell-1} \stackrel{\nabla_f}{\to} \mathcal{L}^{-\ell})} \tag{0.7}$$

is an isomorphism, cf., ???? Ch.0. We can just as well replace $\mathcal{C}$ by $\mathcal{E}$. In particular, the case $k=0$ has the following two useful implications; let $e = e_1 \wedge \ldots \wedge e_m$, and let $\mathcal{J} = \mathcal{J}(f)$ be the ideal sheaf generated by f_j, i.e., the image of $\delta_f \colon \mathcal{O}(E) \to \mathcal{O}$.

Lemma 0.2. *(i) If there is a current v in $\mathcal{L}^{-1}$ such that $\nabla_f v = \phi$, then ϕ belongs to $\mathcal{J}$.*
(ii) If $\mu \wedge e \in \mathcal{C}_{0,p}(\Lambda^p E)$ and $\nabla_f(\mu \wedge e) = 0$, then there is a function $\psi \in \mathcal{O}$, unique in $\mathcal{O}/\mathcal{J}$, and a current v in $\mathcal{L}^{-1}$ such that $\nabla_f v = \psi - \mu \wedge e$.

For the reader's convenience, and for further reference, we supply a direct proof of the lemma.

Proof. Let $v = v_1 + \ldots + v_m$, where $v_k \in \mathcal{C}_{0,k-1}(\Lambda^k E)$. Then $\bar\partial v_m = 0$ and since (0.6) is exact we can solve $\bar\partial w_m = n_m$ locally. Now, $\bar\partial[v_{m-1} + \delta_f w_m] = \bar\partial v_{m-1} - \delta_f \bar\partial w_m = \bar\partial v_{m-1} - \delta_f v_m = 0$ and so we can solve $\bar\partial w_{m-1} = v_{m-1} + \delta_f w_m$. Continuing in this way we finally get that $\psi = v_1 + \delta_f w_2$ is a holomorphic solution to $\delta_f \psi = \phi$. The second statement is verified in a similar way. □

Let $\sigma = \sum_j (\bar f_j/|f|^2) e_j$ in $X \setminus V(f)$ and notice that $\delta_f \sigma = 1$ there. Since

$$\nabla_f \sigma = 1 - \bar\partial\sigma \tag{0.8}$$

has even degree and the scalar term is nonvanishing, cf., Ch 0 ???, we can form

$$u = \frac{\sigma}{\nabla_f \sigma} \tag{0.9}$$

and by the functional calculus, using (0.5), we have that

$$\nabla_f u = 1 \tag{0.10}$$

in $X \setminus V(f)$. From (0.8) we get the more explicit representation

$$u = \sigma + \sigma \wedge \bar\partial\sigma + \sigma \wedge (\bar\partial\sigma)^2 + \cdots + \sigma \wedge (\bar\partial\sigma)^{m-1},$$

so one can verify (0.10) directly as well. This form u in $\mathcal{L}^{-1}$ will be of fundamental importance later on.

0.3. **Duality theorem for the Coleff-Herrera product.** Assume now that $f = (f_1, \ldots, f_m)$ is a complete intersection at x and let μ^f be the associated Coleff-Herrera product. It follows from Theorem 0.1 that $f_j \mu^f = 0$, i.e., $\phi\mu^f = 0$ for all ϕ in the ideal $\mathcal{J}(f)_x$. On the other hand it is clear that the annihilator $\operatorname{ann}\mu^f$, i.e., the set of functions in $\mathcal{O}_x$ such that the current $\phi\mu^f$ vanishes at x, is an ideal in the local ring $\mathcal{O}_x$. We shall now see that this ideal is in fact equal to $\mathcal{J}(f)_x$.

Consider the current

$$v = \frac{1}{f_1} e_1 + \frac{1}{f_2}\bar\partial\frac{1}{f_1} \wedge e_1 \wedge e_2 + \frac{1}{f_3}\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1} \wedge e_1 \wedge e_2 \wedge e_3 + \cdots = \frac{e_1}{f_1} + \frac{e_2}{f_2} \wedge \bar\partial\frac{e_1}{f_1} + \frac{e_3}{f_3} \wedge \bar\partial\frac{e_2}{f_2} \wedge \bar\partial\frac{e_1}{f_1} + \cdots . \tag{0.11}$$

A simple computation, using Theorem 0.1, yields that

$$\nabla_f v = 1 - \mu^f \wedge e. \tag{0.12}$$

Proposition 0.3. *Let f be a complete intersection at x and assume that there is a current U such that $\nabla_f U = 1 - \mu \wedge e$. Then $\operatorname{ann}\mu = \mathcal{J}(f)_x$ at x.*

Proof. If $\phi \in \operatorname{ann}\mu$, then $\nabla_f U\phi = \phi - \phi\mu\wedge e = \phi$ and hence $\phi \in \mathcal{J}(f)_x$ by Lemma 0.2. Conversely, if $\phi \in \mathcal{J}(f)_x$, then there is a holomorphic ψ such that $\phi = \delta_f\psi = \nabla_f\psi$ and hence $\phi\mu = \nabla_f\psi \wedge \mu = \nabla_f(\psi \wedge \mu) = 0$. □

In view of (0.12) we get

Theorem 0.4 (Duality theorem)**.** *If f is a complete intersection at x, then* $\operatorname{ann}\mu^f = \mathcal{J}(f)_x$.

Example 0.5. We also get a simple proof of the well-known fact that if f is a complete intersection on X, then the sheaf complex

$$0 \to \mathcal{O}(\Lambda^m E) \stackrel{\delta_f}{\to} \mathcal{O}(\Lambda^{m-1} E) \stackrel{\delta_f}{\to} \ldots \stackrel{\delta_f}{\to} \mathcal{O}(\Lambda^1 E) \stackrel{\delta_f}{\to} \mathcal{O} \to \mathcal{O}/\mathcal{J} \to 0$$

is exact.

In fact, if $x \in V(f)$, then f is a complete intersection there. Let ϕ be a section of $\mathcal{O}(\Lambda^k E)$, $k \geq 1$, such that $\delta_f\phi = 0$. If v is the current in (0.12), then

$$\nabla_f(v \wedge \phi) = (1 - \mu^f \wedge e) \wedge \phi = \phi,$$

since $e \wedge \phi = 0$ for degree reasons. By (0.7) we get a holomorphic solution to $\delta_f\psi = \phi$. On the other hand, if x is outside $V(f)$, then $f_j \neq 0$ for some f_j. Given ϕ such that $\delta_f\phi = 0$ we can then take $\psi = e_j \wedge \phi/f_j$. □

0.4. **Coleff-Herrera currents.** The Coleff-Herrera product is the model for a slightly more general kind of currents called *Coleff-Herrera currents*.

Definition 3. Let V be an analytic variety in X of pure codimension p. A $(0,p)$-current μ with support on V is a Coleff-Herrera current on V, $\mu \in \mathcal{CH}_V$, if it is $\bar\partial$-closed,

$$\bar{I}_V \mu = 0, \tag{0.13}$$

and it has the following property: For any holomorphic function h that does not vanish identically on any irreducible component of V,

$$\lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)\mu = \mu \tag{0.14}$$

if $\chi \sim \chi_{[1,\infty)}$.

It is clear that $\mathcal{CH}_V$ is a sheaf of $\mathcal{O}$-modules. The property (0.13) means that $\bar{h}\mu = 0$ for any holomorphic h that vanishes on V. The last property is called the *standard extension property*, SEP, (with respect to V) and means that μ is determined by its values on $V \setminus Y$ for any hypersurface Y not containing any irreducible component of V.

Example 0.6. If $\mu \in \mathcal{PM}_{0,p}$ has support on V, then $\mathbf{1}_{V(h)}\mu = 0$ by the dimension principle, which can be expressed as (0.14), cf., (7.5). Moreover, from Proposition 5.2 it follows that (0.13) is fulfilled. If in addition $\bar\partial\mu = 0$ therefore μ is in $\mathcal{CH}_V$. In particular, if f is a p-tuple such that $V(f)$ has codimension p and is contained in V, then the Coleff-Herrera product μ^f is in $\mathcal{CH}_V$. □

The sheaf $\mathcal{CH}_V$ is important for several reasons. For instance, each element in the local (moderate) cohomology sheaves $\mathcal{H}^p_{[V]}$ has a unique representative in $\mathcal{CH}_V$, i.e., the natural mapping $\mathcal{CH}_V \to \mathcal{H}^p_{[V]}$ is an isomorphism, see Section 4.11 below. Another reason is that there is a close connection between Coleff-Herrera currents and Noetherian differential operators. This will be discussed in Section 0.6.

0.5. **Basic properties of Coleff-Herrera currents.**

Lemma 0.7. *If μ is in $\mathcal{CH}_V$ and for each neighborhood ω of V there is a current w with support in ω such that $\bar\partial w = \mu$, then $\mu = 0$.*

The proof will also provide a description of μ locally on V_{reg}. Later on we will see that a similar description holds even across the singular part.

Proof. Locally on V_{reg} we can choose coordinates (z, w) such that $V = \{w = 0\}$. We claim that there is a natural number M such that

$$\mu = \sum_{|\alpha|\le M-p} a_\alpha(z)\bar\partial\frac{1}{w_1^{\alpha_1+1}}\wedge\ldots\wedge\bar\partial\frac{1}{w_p^{\alpha_p+1}}, \tag{0.15}$$

where a_α are the push-forwards of $\mu\wedge w^\alpha dw/(2\pi i)^p$ under the projection $(z,w)\mapsto z$. In fact, since $\bar w_j\mu = 0$ and $\bar\partial\mu = 0$ it follows that $d\bar w_j\wedge\mu = 0$, $j = 1,\ldots,p$, and hence $\mu = \mu_0 d\bar w_1\wedge\ldots\wedge d\bar w_p$. Therefore it is enough to check (0.15) for test forms of the form $\xi(z,w)dw\wedge d\bar z\wedge dz$. Since $\bar w_j\mu = 0$ we have by a Taylor expansion in w (the sum is finite since μ has finite order), cf., (1.6), that

$$\int_{z,w}\mu\wedge\xi dw\wedge d\bar z\wedge dz = \sum_\alpha\int_{z,w}\mu\wedge\frac{\partial^\alpha\xi}{\partial w^\alpha}(z,0)\frac{w^\alpha}{\alpha!}dw\wedge d\bar z\wedge dz =$$
$$\sum_\alpha\int_z a_\alpha(z)\frac{\partial^\alpha\xi}{\partial w^\alpha}(z,0)\frac{1}{\alpha!}dw\wedge d\bar z\wedge dz(2\pi i)^p =$$
$$\sum_\alpha\int_z a_\alpha(z)\int_w\bar\partial\frac{1}{w^{\alpha+1}}\wedge\xi(z,w)dw\wedge d\bar z\wedge dz.$$

Since μ is $\bar\partial$-closed it follows that a_α are holomorphic. It follows from Corollary 1.9 that

$$\bar\partial\frac{1}{w_p^{\beta_p}}\wedge\ldots\wedge\bar\partial\frac{1}{w_1^{\beta_1}}\wedge dw_1^{\beta_1}\wedge\ldots\wedge dw_p^{\beta_p}/(2\pi i)^p = \beta_1\cdots\beta_p[w=0],$$

where $[w = 0]$ denote the current of integration over V_{reg}.

Now assume that $\bar\partial\gamma = \mu$ and γ has support close to V. We have, for $|\beta| = M$, that

$$\bar\partial(\gamma\wedge dw^\beta) = (2\pi i)^p a_{\beta-1}(z)\beta_1\cdots\beta_p[w=0].$$

If ν is the component of $\gamma\wedge dw^\beta$ of bidegree $(p, p-1)$ in w, thus

$$d_w\nu = \bar\partial_w\nu = (2\pi i)^p a_{\beta-1}\beta_1\cdots\beta_p[w=0].$$

Integrating with respect to w we get that $a_{\beta-1}(z) = 0$. By finite induction we can conclude that $\mu = 0$ locally on V_{reg}. Thus μ vanishes on V_{reg} and by the SEP it follows that $\mu = 0$. □

We have the following uniqueness theorem:

Theorem 0.8. *Let $f = (f_1,\ldots,f_p)$ be a complete intersection at x. If there is a current solution $v\in\mathcal{L}^{-1}$ to $\nabla_f v = \tau\wedge e$ and $\tau\in\mathcal{CH}_{V(f)}$, then $\tau = 0$.*

Proof. Let ω be any neighborhood of V and take a cutoff function χ that is 1 in a neighborhood of V and with support on ω. Let u be the smooth form (0.9) in $X\setminus V(f)$ such that $\nabla_f u = 1$ there. Then

$$g = \chi - \bar\partial\chi\wedge u$$

is a smooth form in ω and $\nabla_f g = 0$. Moreover, $g_0 = 1$ and $g_k = 0$, $k \geq 1$, in a neighborhood of V. Therefore,

$$\nabla_f(g \wedge v) = g_0 \tau \wedge e = \tau \wedge e,$$

and thus $w = -(g \wedge v)_{p-1}$ is a current solution to $\bar\partial w = \tau$ with support in ω. From Lemma 0.7 we conclude that $\tau = 0$. □

Corollary 0.9. *Let f be a complete intersection at x. If $\nabla_f U = 1 - \mu \wedge e$ and $\mu \in \mathcal{CH}_{V(f)}$, then μ is equal to the Coleff-Herrera product μ^f.*

Proof. Take v as in (0.11) such that $\nabla_f v = 1 - \mu^f \wedge e$. Then $\nabla_f(v - U) = (\mu - \mu^f) \wedge e$, and by Theorem 0.8 thus $\mu = \mu^f$. □

Corollary 0.10. *Let f be a complete intersection at x. If $\mu \in \mathcal{CH}_{V(f)}$ and $\mathcal{J}(f)\mu = 0$, then there is a holomorphic function ψ, unique in $\mathcal{O}/\mathcal{J}(f)$, such that $\mu = \psi\mu^f$.*

Proof. From the assumptions it follows that $\nabla_f \mu \wedge e = 0$. In view of Lemma 0.2 there is a function ψ, unique in $\mathcal{O}/\mathcal{J}(f)$, such that $\psi - \mu \wedge e = \nabla_f U$ for some current $U \in \mathcal{L}_x^{-1}$. Now take $v \in \mathcal{L}_x^{-1}$ such that $\nabla_f v = 1 - \mu^f \wedge e$. Then $\nabla_f(\psi v) = \psi - \psi\mu^f \wedge e$. It then follows from Corollary 0.9 that $\mu = \psi\mu^f$ at x. □

The corollary can be expressed more algebraically as saying that the mapping $\phi \mapsto \phi\mu^f \wedge e$ induces an isomorphism

$$\mathcal{O}/\mathcal{J}(f) \simeq \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), \mathcal{CH}_{V(f)}(\Lambda^p E))$$

at x, where $\mathcal{CH}_{V(f)}(\Lambda^p E)$ denotes the sheaf of currents in $\mathcal{CH}_{V(f)}$ with values in the vector bundle $\Lambda^p E$.

This isomorphism only depends on the section f of E, and not on the choice of frame, i.e., the current $\mu^f \wedge e$ is independent of the frame. In fact, let e'_j be another frame of E, let $(e'_j)^*$ be its dual frame, and let f'_j be the corresponding functions so that

$$f'_1(e'_1)^* + \cdots + f'_p(e'_p)^* = f_1 e_1^* + \cdots + f_p e_p^*$$

and let $\mu^{f'}$ denote the associated Coleff-Herrera product $\bar\partial(1/f'_1) \wedge \ldots \wedge \bar\partial(1/f'_p)$. Then since δ_f is an invariant operation on ΛE, we have a current solution to $\nabla_f V' = 1 - \mu^{f'} \wedge e'$ if $e' := e'_1 \wedge \ldots \wedge e'_p$. By Corollary 0.9 we conclude that

$$\mu^f \wedge e = \mu^{f'} \wedge e'. \tag{0.16}$$

This equality can be rephrased as the so-called transformation law for the Coleff-Herrera product:

Corollary 0.11 (Transformation law)**.** *Assume that f_j is a complete intersection at x. If g is a holomorphic invertible $p \times p$ matrix and $f' = gf$, then*

$$\bar\partial\frac{1}{f'_p} \wedge \ldots \wedge \bar\partial\frac{1}{f'_1} = \det g \bar\partial\frac{1}{f_p} \wedge \ldots \wedge \bar\partial\frac{1}{f_1}. \tag{0.17}$$

We will see later on that the same formula holds for any g such that also f' is a complete intersection.

Proof. Let e' be the frame such that $e^* f = (e')^* gf$. Then $e' = eg^T$ and thus $e'_1 \wedge \ldots \wedge e'_p = \det g^T e_1 \wedge \ldots \wedge e_p$. Now (0.17) follows from (0.16). □

We can just as well consider the sheaf

$$\mathcal{CH}_V^k := \mathcal{CH}_V(\Lambda^k T^*_{1,0}(X))$$

of (k,p)-currents in $\mathcal{CH}_V$. If $\mu \in \mathcal{CH}_V^k$ and $\mathcal{J}(f)\mu = 0$, then in view of Corollary 4.3 there is then a holomorphic $(k,0)$-form ψ such that $\mu = \mu^f \psi$.

Example 0.12. We claim that the Lelong current $[V]$ is a section of $\mathcal{CH}_V^p$. In fact, it is a (p,p)-current that has support on V and is even d-closed. Moreover, it is clearly annihilated by $\bar{I}_V$. To see the SEP, let h be a holomorphic function that does not vanish identically on (any irreducible component of) V, and let $\pi\colon \tilde{V} \to V$ be a smooth modification. Then

$$\int_V \chi(|h|^2/\epsilon)\xi = \int_{\tilde{V}} \chi(|\pi^* h|^2/\epsilon)\pi^*\xi \to \int_{\tilde{V}} \pi^*\xi = \int_V \xi$$

for test forms ξ, by the dominated convergence theorem since the zero set of $\pi^* h$ is a set of measure zero on $\tilde{V}$.

If z are local coordinates at x we thus have that

$$[V] = \sideset{}{'}\sum_{|I|=p} \tau_I \wedge dz_I,$$

where τ_I are in $\mathcal{CH}_V$ at x. Since certainly $\mathcal{J}(f)[V] = 0$ it follows from Corollary ?? that there are holomorphic functions a_I such that $\tau_I = a_I \mu^f$, i.e.,

$$[V] = \mu^f \wedge A = \mu^f \wedge \sideset{}{'}\sum_{|I|=p} a_I dz_I.$$

We will see later on that one can choose

$$A = \alpha df_1 \wedge \ldots \wedge df_p,$$

where α is a suitable holomorphic function, that is constant on each irreducible component V_ℓ of V at x. □

We have the following structure result for Coleff-Herrera currents.

Corollary 0.13 (Structure theorem for $\mathcal{CH}_V$). *Let V be any variety of pure codimension p. Any $\mu \in \mathcal{CH}_V^k$ is locally of the form $\psi \wedge \mu^g$ where g is complete intersection and ψ is a holomorphic $(k,0)$-form.*

Proof. Any V of pure codimension p is locally a subset of $V(f)$ for a complete intersection $f = (f_1, \ldots, f_p)$; this follows from the local parametrization theorem structure ?????. For sufficiently large M, $g = (f_1^M, \ldots, f_p^M)$ will annihilate $\mu \in \mathcal{CH}_V$ and hence $\mu = \psi\mu^g$ according to Corollary 4.3. □

Corollary 0.14. *If μ is a Coleff-Herrera current, then its annihilator sheaf $\operatorname{ann}\mu$ is coherent.*

Proof. Locally we have that $\operatorname{ann}\mu$ is the ideal of ϕ in $\mathcal{O}$ such that $\phi\psi$ is in the ideal $\mathcal{J}(g)$. Since $\mathcal{O}/\mathcal{J}(g)$ is coherent it follows that $\operatorname{ann}\mu$ is coherent. □

It follows from Corollary 0.13 that any Coleff-Herrera current is pseudomeromorphic. In view of Example 0.6 we therefore have

Corollary 0.15 (Characterization of $\mathcal{CH}_V$). *The sheaf $\mathcal{CH}_V$ is precisely the subsheaf of $\mu \in \mathcal{W}_V^{*,p}$ such that $\bar{\partial}\mu = 0$.*

In particular, in view of Example 0.12, all Lelong currents $[V]$ are pseudomeromorphic.

Example 0.16 (Poincaré-Lelong's formula). Let g be a holomorphic function in X with multiplicity α_ℓ on the irreducible component V_ℓ of $V = V(g)$. Then

$$\bar\partial\frac{1}{g}\wedge dg/2\pi i = \sum_\ell \alpha_\ell[V_\ell]. \tag{0.18}$$

In fact, in a neighborhood $\Omega \subset X$ of any point on $V_\ell \backslash V_{sing}$ we can choose holomorphic coordintes z such that $g = z_1^{\alpha_\ell}$. Noting that $[z_1 = 0]$ considered as a current in $\mathbb{C}^n$ is the tensor product of the current $[z_1 = 0]$ in $\mathbb{C}$ and the function 1 in $\mathbb{C}^{n-1}$, we now have from the one-variable case Prop **??** ??????? (SKA jmf med beviset av lma 5.2 har!!) that

$$\bar\partial\frac{1}{g}\wedge dg/2\pi i = \bar\partial\frac{1}{z_1^{\alpha_\ell}}\wedge dz_1^{\alpha_\ell}/2\pi i = \alpha_\ell[z_1 = 0]$$

in Ω. Thus (0.18) holds in $X \setminus V_{sing}$. However, since both sides are pseudomeromorphic (p,p)-currents, and V_{sing} has codimension at least $p+1$, it follows from the dimension principle that (0.18) holds in X. □

Corollary 0.17. *Suppose that V has pure codimension p and V' is a subvariety of V of the same dimension, i.e., a union of irreducible components of V. Then $\mathcal{CH}_{V'}$ is precisely the currents in $\mathcal{CH}_V$ that have support on V'.*

Let A be a hypersurface such that $V \cap A$ has positive codimension and let $\mathcal{CH}_V(A)$ denote the sheaf of Coleff-Herrera currents on V with possible poles at A. This means that $\mu \in \mathcal{CH}_V(A)$ if and only if there is a holomorphic function h with $V(h) \cap V \subset A$ such that $\tilde\mu = h\mu$ is in $\mathcal{CH}_V$. Notice then that

$$\mu = \frac{1}{h}\tilde\mu. \tag{0.19}$$

In fact, (0.19) holds outside A, and hence it holds across by the dimension principle. It follows that

$$\bar\partial\mu = \bar\partial\frac{1}{h}\wedge\tilde\mu$$

is a $\bar\partial$-closed current in $\mathcal{PM}_{*,p+1}$ with support on $V \cap A$ and thus it is in $\mathcal{CH}_{V\cap A}$ in view of ???.

Proposition 0.18. *The operator $\bar\partial$ re is a well-defined mapping*

$$\bar\partial\colon \mathcal{CH}_V(A) \to \mathcal{CH}_{V\cap A}.$$

0.6. **Noetherian differential operators.** Ev flytta allt om BM-strommar till section 6 ????

Let $\mu \in \mathcal{CH}_V, x$ and let $J = \operatorname{ann}\mu$, i.e., $J/\mathcal{O}_x$ is the ideal of all ϕ such that $\phi\mu = 0$. We shall now see that J is described by so-called Noetherian differential operators.

Theorem 0.19 (Björk). *Let V be a germ of an analytic variety of pure codimension p at $0 \in \mathbb{C}^n$. There is a neighborhood Ω of 0 such that for each $\mu \in \mathcal{CH}_V(E_0^*)$ in Ω, there are holomorphic differential operators $\mathcal{L}_1, \dots, \mathcal{L}_\nu$ in Ω such that for any $\phi \in \mathcal{O}(E_0)$, $\mu\phi = 0$ if and only if*

$$\mathcal{L}_1\phi = \cdots = \mathcal{L}_\nu\phi = 0 \ \text{on} \ V. \tag{0.20}$$

Proof. It follows from the local normalization parametrization?? theorem that one can find holomorphic functions $f_1, \dots, f_p$ in an open neighborhood Ω, forming a complete intersection, such that V is a union of irreducible components of $V(f) = \{f = 0\}$, and such that

$$df_1 \wedge \dots \wedge df_p \neq 0$$

on $V \setminus W$ where W is a hypersurface not containing any component of V_f. By a suitable choice of coordinates $(\zeta, \omega) \in \mathbb{C}^{n-p} \times \mathbb{C}^p$ we may assume that W is the zero set of

$$h = \det \frac{\partial f}{\partial \omega}.$$

Let

$$z = \zeta, \quad w = f(\zeta, \omega).$$

Since

$$\frac{d(z,w)}{d(\zeta,\omega)} = \det \begin{bmatrix} I & 0 \\ \partial f/\partial\zeta & \partial f/\partial\omega \end{bmatrix} = \det \frac{\partial f}{\partial \omega},$$

locally outside W, (z, w) is a local holomorphic coordinate system. Take $\mu \in \mathcal{O}(E_0^*)$. From Corollary 4.3 we know that there is an M and a holomorphic function A such that

$$\mu = A\bar\partial\frac{1}{f_1^{M+1}} \wedge \dots \bar\partial\frac{1}{f_p^{M+1}}.$$

Since (z, w) are coordinates locally in $\Omega \setminus W$, by a Taylor expansion of $w \mapsto A(z, w)$ we see that

$$\mu = \int_{w=0} \sum_{0\le\alpha\le M} \frac{\partial^{M-\alpha}A(z,0)}{\partial w^{M-\alpha}} \frac{1}{(M-\alpha)!} \bar\partial\frac{1}{w_p^{\alpha_p}} \wedge \dots \wedge \bar\partial\frac{1}{w_1^{\alpha_p}}$$

there. In view of (1.6) therefore

$$\langle \mu, \hat\xi dz \wedge dw \wedge d\bar w\rangle = \int_{w=0} \sum_{0\le\alpha\le M} c_\alpha \frac{\partial^{M-\alpha}A(z,0)}{\partial w^{M-\alpha}} \frac{\partial^\alpha \hat\xi}{\partial w^\alpha}. \tag{0.21}$$

Notice now that $\phi\mu = 0$ in $\Omega \setminus W$ if and only if for all $\hat\xi$ with support in $\Omega \setminus W$,

$$0 = \langle \phi\mu, \hat\xi dz \wedge dw \wedge d\hat w\rangle = \int_{w=0} \sum_{0\le\ell\le M} Q_\ell \phi \frac{\partial^\ell \hat\xi}{\partial w^\ell}, \tag{0.22}$$

where

$$Q_\ell = \sum_{\ell\le\alpha\le M} \frac{\partial^{M-\alpha}A}{\partial w^{M-\alpha}} \frac{\partial^{\alpha-\ell}}{\partial w^{\alpha-\ell}}.$$

However, by applying to $\xi = w^\alpha \eta$ for appropriate α (induction downwards) it follows that (0.22) holds for all $\xi \in \mathcal{D}(\Omega \setminus W)$ if and only if $Q_\ell \phi = 0$ on $V \cap (\Omega \setminus W)$ for all $\ell \le M$.

Notice now that

$$\begin{bmatrix} \frac{\partial\zeta}{\partial z} & \frac{\partial\zeta}{\partial w} \\ \frac{\partial\omega}{\partial z} & \frac{\partial\omega}{\partial w} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \frac{\partial f}{\partial\zeta} & \frac{\partial f}{\partial\omega} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -(\frac{\partial f}{\partial\omega})^{-1}\frac{\partial f}{\partial\zeta} & (\frac{\partial f}{\partial\omega})^{-1} \end{bmatrix}$$

and hence

$$\frac{\partial\omega}{\partial w} = (\frac{\partial f}{\partial\omega})^{-1} = \frac{1}{h}\gamma,$$

where γ is a holomorphic matrix. It follows that

$$\frac{\partial}{\partial w_j}=\sum_k \frac{\partial \omega_k}{\partial w_j}\frac{\partial}{\partial \omega_k}=\frac{1}{h}\sum_k \gamma_{jk}\frac{\partial}{\partial \omega_k}. \tag{0.23}$$

It follows from (0.23) that Q_ℓ are (semi-)global differential operators of the form $Q_\ell=\mathcal{L}_\ell/h^N$, where $\mathcal{L}_\ell$ are holomorphic. Now, $\phi\mu=0$ if and only if $\phi\mu=0$ on $X\setminus W$ by the SEP, and this holds as we have seen if and only if $\mathcal{L}_\ell\phi=0$ on $V\setminus W$ which by continuity holds if and only if $\mathcal{L}_\ell\phi=0$ on V. Thus the proposition is proved. □

Notice that if

$$\frac{\partial}{\partial w}=\frac{\partial}{\partial w_1}\wedge\ldots\wedge\frac{\partial}{\partial w_p},$$

$$Q=\sum_{0\le\alpha\le M} c_\alpha \frac{\partial^{M-\alpha}A}{\partial w^{M-\alpha}}\frac{\partial^\alpha}{\partial w^\alpha},$$

and ξ is any test form of bidegree $(n,n-p)$ with support in $\Omega\setminus W$, we have from (0.21) that

$$\langle\mu,\xi\rangle=\int_{w=0} Q((\partial/\partial w)\neg\xi).$$

It follows from the proof that

$$Q=\sum_{\alpha\le M} c_\alpha \frac{\partial^{M-\alpha}A}{\partial w^{M-\alpha}}\frac{\partial^\alpha}{\partial w^\alpha},$$

a priori defined in $\Omega\setminus W$ is equal to $Q=\mathcal{L}'/h^N$ for some N where $\mathcal{L}'$ is a holomorphic differential operator in Ω. Moreover,

$$\frac{\partial}{\partial w}=\frac{1}{h^p}\tau,$$

where τ is a holomorphic section of $\Lambda^p T_{1,0}(\Omega)$ in Ω. Notice that there is a holomorphic differential operator $\mathcal{L}$ such that $\mathcal{L}'(h^{-p}\eta)=h^{-M}\mathcal{L}\eta$. We now have

Proposition 0.20 (Björk). *Given $\mu\in\mathcal{CH}_{V,x}$ there is a neighborhood Ω of x, a holomorphic differential operator $\mathcal{L}$, a holomorphic function h not vanishing identically on any irreducible component of V, and a holomorphic $(p,0)$- vector field τ such that*

$$\langle\mu,\xi\rangle=\int_V \frac{1}{h^M}\mathcal{L}(\tau\neg\xi),\quad \xi\in\mathcal{D}_{n,n-p}(\Omega). \tag{0.24}$$

The right hand side here is defined as a principal value integral.

Proof. With the notation in the preceding proof and discussion we know that (0.24) holds for ξ with support in $\Omega\setminus W$. In view of the dimensional principle it is therefore enoughto see that the right hand side defines a pseudomeromorphic current in Ω.

Let $\mathcal{L}^*$ be the formal adjoint of $\mathcal{L}$ in Ω. Then the right hand side is $\pm$ the action of the current

$$\tau\neg\mathcal{L}^*\big(h^{-M}[V]\big)$$

on ξ, and this current is pseudomeromorphic according to Proposition 11.1. □

It follows that there is a holomorphic differential operator $\mathcal{N}$ such that

$$\mu=\tau\neg\big(h^{-M}\mathcal{N}[V]\big).$$

1. Vanishing of Coleff-Herrera currents

If $\mu \in \mathcal{CH}_V(X)$ and X Stein, then we can solve $\bar\partial v = \mu$ in X. Notice that such a solution v defines a Dolbeault cohomology class ω^μ in $X \setminus V$ that only depends on μ. We shall see now that $\mu = 0$ if and only if $\omega^\mu = 0$. In particular this implies that the annihilator $\operatorname{ann}\mu$ is the ideal of holomorphic functions ϕ in X such that $\phi\omega^\mu = 0$.

Theorem 1.1. *Assume that X is Stein and $V \subset X$ has pure codimension p. If $\mu \in \mathcal{CH}_V(X)$ and $\bar\partial v = \mu$ in X, then the following are equivalent:*
(i) $\mu = 0$.
(ii) For all $\psi \in \mathcal{D}_{n,n-p-1}(X \setminus V)$ such that $\bar\partial\psi = 0$ in some nbh of V we have that

$$\int v \wedge \bar\partial\psi = 0.$$

(iii) There is a solution to $\bar\partial w = v$ in $X \setminus V$.
(iv) For each neighborhood ω of V there is a solution to $\bar\partial u = v$ in $X \setminus \omega$.

Proof. It is readily checked that (i) implies all the other conditions. Assume that (ii) holds. We can mimick the proof of Lemma 0.7 above: Locally on $V_{reg} = \{w = 0\}$ we have (0.15), and by choosing $\xi(z, w) = \psi(z)\chi(w)dw^\beta \wedge sz \wedge d\bar z$ for a suitable cutoff function χ and test functions ψ, we can conclude successively from (ii) that all the coefficients a_α vanish, so that $\mu = 0$ there. It follows by the SEP that $\mu = 0$ globally.

Clearly (iii) implies (iv). Finally assume that (iv) holds. Given ψ in (ii) we can choose ω such that $\bar\partial\psi$ vanishes in a neighborhood of $\bar\omega$. Then

$$\int V \wedge \bar\partial\psi = \int d(w \wedge \bar\partial\psi) = 0$$

by Stokes' theorem, so (ii) holds. Alternatively, given $\omega \supset V$ choose $\omega' \subset\subset \omega$ and a solution to $\bar\partial w = V$ in $X \setminus \omega'$. If we extend w arbitrarily across ω' the form $U = V - \bar\partial w$ is a solution to $\bar\partial U = \mu$ with support in ω. In view of Lemma 0.7 thus $\mu = 0$. □

Corollary 1.2. *Assume that f defines a complete intersection at x and $V = V(f)$ and assume that $\bar\partial v = \mu^f$ in $X \setminus V$, where X is a small Sein neighborhood of x in X. Then $\phi \in \mathcal{O}_x$ is in $\mathcal{J}(f)_x$ if and only if*

$$\int \phi v \wedge \bar\partial\psi = 0$$

for all $\psi \in \mathcal{D}_{n,n-p-1}(X)$ such that $\bar\partial\psi = 0$ on V.

Chapter 4

Bochner-Martinelli type residues

Again let $f = f_1 e_1^* + \cdots + f_p e_p^*$ be a holomorphic section of the dual E^* of a trivial vector bundle $E \to X$. For the moment we also assume that f_j form a complete intersection at some given point x. We have seen that the $\Lambda^p E$-valued current $\mu^f \wedge e$ is invariantly defined. However, the current v in (0.11) such that $\nabla_f v = 1 - \mu^f \wedge e$ certainly depends on the choice of frame e_j. Moreover, although the residue current is only singular on the set $V(f)$, the current v is singular on the hypersurface $f_1 f_2 \cdots f_p = 0$. Let us try to find a somewhat more invariant such current by taking mean values in the following way. Let $\alpha = (\alpha^1, \ldots, \alpha^p)$ be a p-tuple of elements in $\mathbb{C}^p$ and consider the current

$$v_\alpha = \frac{\alpha^1 \cdot e}{\alpha^1 \cdot f} + \frac{\alpha^2 \cdot e}{\alpha^2 \cdot f} \wedge \bar\partial \frac{\alpha^1 \cdot e}{\alpha^1 \cdot f} + \cdots,$$

where $\alpha^\ell \cdot e = \alpha_1^\ell e_1 + \cdot + \alpha_p^\ell e_p$ and $\alpha^\ell \cdot f = \alpha_1^\ell e_1 + \cdot + \alpha_p^\ell f_p$. As long as α^j are linearly independent, this is just the current corresponding to the new frame $e_j' = \alpha^j \cdot e$, $j = 1, \ldots, p$, and hence $\nabla_f v_\alpha = 1 - \mu^f \wedge e$. Notice that v_α actually depends only on $[\alpha] = ([\alpha^1], \ldots, [\alpha^p]) \in (\mathcal{P}^{p-1})^p$.

Lemma 0.3. *If $a \in (\mathbb{C}^p)^*$ and $a \neq 0$, then*

$$\int_{[\beta]\in\mathcal{P}^{p-1}} \frac{\beta \cdot e}{\beta \cdot a} d\tau(\beta) = \frac{\bar a \cdot e}{|a|^2},$$

where $d\tau$ is the (normalized) Fubini-Study metric on $\mathcal{P}^{p-1}$.

Proof. It will follow from the argument below that the integrand in the lemma is integrable so the integral exists. It is clear that

$$\int_{[\beta]\in\mathcal{P}^{p-1}} \frac{\beta \cdot e}{\beta \cdot a} d\tau(\beta) = \int_{|\beta|=1} \frac{\beta \cdot e}{\beta \cdot a} dS(\beta),$$

where dS is normalized surface measure on the unit sphere in $\mathbb{C}^p$. By obvious homogeneity it is enough to assume that $|a| = 1$. First assume even that $a = (1, 0, \ldots, 0)$. Then the integral is

$$\int_{|\beta|=1} \frac{\beta_1 e_1 + \beta_2 e_2 + \cdots}{\beta_1} dS(\beta),$$

so the integrand is integrable, and the integral is in in fact equal to e_1 for symmetry reasons. Thus the lemma holds for this particular a. If $|a| = 1$, take a unitary mapping A such that $Aa = (1, 0, \ldots, 0)$. Then

$$\bar a \cdot e = \bar a \cdot A^* A e = \overline{Aa} \cdot Ae = \int_{|\beta|=1} \frac{\beta \cdot Ae}{\beta \cdot Aa} dS = \int_{|\beta|=1} \frac{A^t\beta \cdot e}{A^t\beta \cdot a} dS = \int_{|\beta|=1} \frac{\beta \cdot e}{\beta \cdot a} dS,$$

by the rotational invariance of dS. □

Outside $V(f)$ we thus have that

$$\int_{[\alpha]\in(\mathcal{P}^{p-1})^p} v_\alpha = \sigma + \sigma \wedge \bar\partial\sigma + \sigma \wedge (\bar\partial\sigma)^2 + \cdots =: u,$$

where

$$\sigma = \frac{\bar{f} \cdot e}{|f|^2}.$$

It is thus reasonable to guess that if we can extend this smooth form u across $V(f)$ to a current U, then $\nabla_f U = 1 - \mu^f \wedge e$. This is indeed the case as we shall see now. However, we shall consider slightly more general currents u corresponding to an arbitrary Hermitian metric on E.

0.1. **Bochner-Martinelli type residues.** Let now $E \to X$ be any Hermitian vector bundle of rank m and let f be a global holomorphic section if the dual bundle $E^* \to X$. Locally we can choose a holomorphic frame $e_1, \ldots, e_m$ so that $f = \sum f_j e_j^*$. To begin with we do not assume that f is a complete intersection. If E is trivial we can fix a global frame e_j and choose the metric on E so that e_j is orthonormal. Let

$$\sigma = \sum_j \sigma_j e_j$$

be the pointwise minimal solution to $f\sigma = 1$ in $X \setminus V$. If the metric on E^* is given by the Hermitian positively definite matrix h_{jk}, so that

$$|f|^2 = \sum_{jk} f_j \bar{f}_k h_{jk},$$

then it is easily checked that

$$\sigma_j = \sum_k \frac{\bar{f}_k h_{jk}}{|f|^2}.$$

In $X \setminus V$ we define

$$u = \frac{\sigma}{\nabla_f \sigma} = \sigma + \sigma \wedge \bar{\partial}\sigma + \ldots + \sigma \wedge (\bar{\partial}\sigma)^{m-1}.$$

It follows immediately that

$$\nabla_f u = 1$$

in $X \setminus V$.

Theorem 0.4. *The function $\lambda \mapsto |f|^{2\lambda} u$ has a current-valued analytic continuation to $\operatorname{Re}\lambda > -\epsilon$. The value at $\lambda = 0$,*

$$U = |f|^{2\lambda} u|_{\lambda=0},$$

is a $\mathcal{PM}$-current in X that coincides with u on $X \setminus V$, and

$$\nabla_f U = 1 - R,$$

where

$$R = \bar{\partial}|f|^{2\lambda} \wedge u|_{\lambda=0}$$

is a current with support on V.

Since R is pseudomeromorphic and has support on $V(f)$ it follows that it is annihilated by $\bar{h}$ and $d\bar{h}$ for $h \in I_V$. By the dimension principle we hav that

$$R = R_{\operatorname{codim} V} + \cdots + R_m.$$

Proof. If $f = f_0 f' = f_0(f'_1, \dots, f'_p)$, where $f' \neq 0$, then

$$\sigma = \frac{1}{f_0}\sigma'$$

where σ' is smooth across V. In fact, $\bar{f}_k = \bar{f}_0 \bar{f}'_k$ and $|f|^2 = |f_0|^2|f'|^2$ so that $\sigma_j = \sum_k \bar{f}'_k h_{jk}/f_0$. Thus

$$u_k = \sigma \wedge (\bar\partial\sigma)^{k-1} = \frac{\alpha}{f_0^k}, \tag{0.1}$$

where α is smooth.

Both the definition and the statement is clearly local and therefore we can assume that the bundle E is trivial in $\mathcal{U} \subset X$.

With a smooth principalization $\pi\colon \widetilde{X} \to X$ as in Section **??** above, we have

$$\int_{\mathcal{U}} |f|^{2\lambda} u_k \wedge \xi = \int_{\tilde{\mathcal{U}}} \sum_j |\pi^* f|^{2\lambda} (\pi^* u_k)\rho_j \wedge \pi^*\xi =$$

$$\sum_{jk} \int_{\tilde{\mathcal{U}}_j} |(\pi^j)^*\pi^* f|(\pi^j)^*\pi^* u_k \wedge (\pi^j)^*\pi^*\xi.$$

If we for each j choose a suitable partition of unity ρ_{jk} we have a local coordinate system t in a neighborhood of the support of In view of (0.1) each terms is like

$$\int |f_0|^{2\lambda}|f'|^{2\lambda}\frac{\alpha_j}{f_0^k} \wedge (\pi^j)^*\pi^*\xi,$$

and thus the proposed analytic continuation exists.

there, where f_0 is a monomial in t and f' is non-vanishing. In view of ??? it is now clear that the analytic continuation exists and moreover, that

$$\int_{\mathcal{U}} |f|^{2\lambda} u_k \wedge \phi|_{\lambda=0} = \sum_\ell \tau_\ell \wedge (\pi^j)^*\pi^*\phi,$$

where each τ_ℓ has the form

$$\tau = \frac{\alpha}{t_1^{a_1}\cdots t_r^{a_r}}$$

in suitable local coordinates t, where α has compact support. Thus we have that

$$U_k = \sum_\ell \pi_* \pi^\ell_* \tau_\ell. \tag{0.2}$$

Moreover,

$$\tau = \bar\partial \frac{1}{t_1^{a_1}} \wedge \frac{\alpha}{t_2^{a_2}\cdots t_r^{a_r}}$$

Since

$$\nabla(|f|^{2\lambda}u) = |f|^{2\lambda} - \bar\partial|f|^{2\lambda} \wedge u \tag{0.3}$$

and clearly $|f|^{2\lambda}$ has a continuation to $\operatorname{Re}\lambda > -\epsilon$ which is 1 for $\lambda = 0$, the desired continuation of the last term follows, and if we define the currents U and R^f as the values of the corresponding terms at $\lambda = 0$, then (1.4) follows from (0.3). In particular, it follows that R^f has support on Y.

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Thus we have that

$$R_k = \sum_\ell \pi_* \pi_*^\ell \tau_\ell. \tag{0.4}$$

$$\tau = \bar\partial \frac{1}{t_1^{a_1}} \wedge \frac{\alpha}{t_2^{a_2}\cdots t_r^{a_r}}$$

It now follows that

□

Corollary 0.5. *If $\phi \in \mathcal{O}$ and $\phi R = 0$, then $\phi \in (f)$.*

The algebraic meaning and generalizations will be discussed in ???? Let us just see that this leads to a simple proof of the

Theorem 0.6 (Briançon-Skoda)**.** *Suppose that $f = (f_1, \ldots, f_m)$ and ϕ are germs at 0 such that $|\phi| \le |f|^{\min(m,n)}$. Then $\phi \in (f)$.*

Notice that if f is a complete intersection, i.e., $V = V(f)$ has codimension p, then $R = R_p$ and $\bar\partial R = 0$. Thus $R = R_p = \mu \wedge e$ where $\mu \in \mathcal{CH}_V$. In view of ??? and Theorem **??** we thus have

Theorem 0.7. *If f is a complete intersection then*

$$R = R_p = \mu^f \wedge e_1 \ldots \wedge e_p,$$

*where μ^f is the Coleff-Herrera product (**??**).*

Let us point out a direct proof of this remarkable theorem.

A direct proof of Theorem 0.7. Let v be the current from ??? so that $\nabla_f v = 1 - \mu^f \wedge e$. Notice that $\bar\partial U_p^f = R^f$ and $\bar\partial v_p = \mu^f \wedge e$. If $\mathrm{Re}\,\lambda \gg 0$, then $|f|^{2\lambda} U^f$ is smooth and hence

$$\nabla_f(|f|^{2\lambda} U^f \wedge v) = |f|^{2\lambda} v - |f|^{2\lambda} U - \bar\partial |f|^{2\lambda} \wedge U \wedge v.$$

since $|f|^{2\lambda}\mu^f = 0$. Now

$$|f|^{2\lambda} V|_{\lambda=0} = v$$

since the difference is $\mathbf{1}_Z v$, that vanishes in view of the dimension principle, since v has degree at most $(0, p-1)$. Moreover,

$$\bar\partial |f|^{2\lambda} \wedge U \wedge v|_{\lambda=0} = 0$$

since again this is a pseudomeromorphic current of bidegree of a most $(0, p-1)$. Thus we have $\nabla A = v - U$ and in particular, $-\bar\partial A_p = U_p - v_p$ so that $0 = \bar\partial v_p - \bar\partial U_p = \mu^f \wedge e - R^f$. □

Corollary 0.8. *Assume that f defines a complete intersection at x and $V = V(f)$. Then $\phi \in \mathcal{O}_x$ is in (f) if and only if*

$$\int \phi u^f \wedge \bar\partial \psi = 0$$

for all $\psi \in \mathcal{D}_{n,n-p-1}(X)$ such that $\bar\partial \psi = 0$ on V.

We have already seen that if $\mu \in \mathcal{CH}_V$, then it can be factorized as $\mu = \psi\mu^f$, where ψ is holomorphic and f is a complete intersection (Corollary **??**). The disadvantage is that μ^f in general must have support on a larger set than V.

However, we shall now see (with basically the same argument) that if we choose a tuple f with common zero set V such that $\mathcal{J}(f)\mu = 0$, then we can write, $p = \operatorname{codim} V$,

$$\mu = \sum_{|I|=p}' \xi_I R_I^f,$$

where ξ_I are holomorphic and

$$R_p^f = \sum_{|I|=p}' R_I^f \wedge e_I.$$

Theorem 0.9. *Let $f = (f_1, \ldots, f_m)$ be an arbitrary tuple in $\mathcal{O}_x$. Assume that $V = V(f)$ has codimension p and let V' be the components of pure codimension p. Assume that $\mu \in \mathcal{CH}_{V'}$ and that $\mathcal{J}(f)\mu = 0$. Then there is $\xi \in \mathcal{O}_x(\Lambda^{m-p}E)$, with $\delta_f\xi = 0$, such that*

$$\mu \wedge e = R_p^f \wedge \xi. \tag{0.5}$$

Notice that if $\xi \in \mathcal{O}_x(\Lambda^{m-p}E)$, with $\delta_f\xi = 0$, then $R_p^f \wedge \xi$ is a $\bar\partial$-closed $(0,p)$-current with support on V' (by the dimension principle) , and hence an element in $\mathcal{CH}_{V'}(\Lambda^m E)$.

In fact,

$$\bar\partial\big(R_p^f \wedge \xi\big) = \delta_f R_{p+1}^f \wedge \xi = \delta_f\big(R_{p+1}^f \wedge \xi\big) = 0,$$

where the last equality holds since $R_{p+1}^f \wedge \xi = 0$ for degree reasons.

Proof. Since $\nabla_f(\mu \wedge e) = 0$, by (0.7) there is $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ (with $\delta_f\xi = 0$) such that $\nabla_f v = \xi - \mu \wedge e$ for some current v. On the other hand, $\nabla_f(U \wedge \xi) = \xi - R^f \wedge \xi = \xi - R_p^f \wedge \xi$ for degree reasons. It follows that there is a current w in $\mathcal{L}^{p-m-1}$ such that

$$\nabla_f\bar\partial w = R_p^f \wedge \xi - \mu \wedge e.$$

Now (0.5) follows from a slight modification of (the proof of) Theorem 0.8. ☐

0.2. **A more geometric point of view.** Again arbitrary f section of $E^* \to X$ and assume that $\pi\colon \widetilde{X} \to X$ is a smooth modification such that $\pi^* f = f^0 f'$, where f^0 is a section of a line bundle $L \to \widetilde{X}$ and f' is a section of $L^{-1} \otimes \pi^* E^*$.

Notice that the zero set $|D|$ of f^0 is precisely $\pi^{-1}V(f)$. here D denotes the divisor of F^0. Will ne imortant later on. Over $\widetilde{X}$ we thus have, suppressing π^* on vector bundles for simplicity in notation, that

$$E \xrightarrow{f} \mathbb{C} \tag{0.6}$$

factorizes as

$$E \xrightarrow{f'} L^{-1} \xrightarrow{f^0} \mathbb{C},$$

and we have a pointwise exact sequence

$$0 \to S \xrightarrow{i} E \xrightarrow{f'} L^{-1} \to 0, \tag{0.7}$$

Notice that the holomorphic vector bundle $S \to \widetilde{X}$ coincides with the kernel of (0.6) over $\widetilde{X} \setminus \pi^{-1}V(f)$.

Since E and S already have got a Hermitian metric, we equip $L^{-1} = E/S$ with the quotient metric. Let σ' be the minimal inverse of f', i.e., so that $\sigma'\xi$ is the element in E with minimal norm such that $f'\sigma\xi = \xi$. Then by definition of quotient metric $|\xi|_L = |\sigma'\xi|_E$. We claim that

$$\pi^*\sigma = \sigma'/f^0$$

on $\widetilde{X} \setminus |D|$. In fact, orthogonal to $\operatorname{Ker} f$ that is the same as $\operatorname{Ker} f'$ on this set.

Over $\widetilde{X} \setminus |D|$ we thus have that globally, cf., the proof of ??? in Section ????,

$$\pi^*u_k = \pi^*(\sigma \wedge (\bar\partial\sigma)^{k-1}) = \frac{1}{(f^0)^k}\sigma' \wedge (\bar\partial\sigma')^{k-1}.$$

It is clear now that u_k admits an obvious (unique) semi-meromorphic extension across $|D|$ in $\widetilde{X}$ as the principal value current $1/(f^0)^k$ times the smooth form $\sigma' \wedge (\bar\partial\sigma')^{k-1}$. Moreover, since this current is the value at $\lambda = 0$ of

$$|f^0 f'|^{2\lambda}/(f^0)^k\sigma' \wedge (\bar\partial\sigma')^{k-1} = \pi^*|f|^{2\lambda}u_k$$

we conclude, cf., ????, that U_k is the direct image under π_* of the (natural extension across $|D|$) of the current π^*u_k. It follows then that

$$R_k = \pi_*\Big(\bar\partial\frac{1}{(f^0)^k} \wedge \sigma' \wedge (\bar\partial\sigma')^{k-1}\Big), \quad k \geq 1,$$

and

$$R_0 = \pi_*\mathbf{1}_{|D|}.$$

Notice that $1/f^0$ is a meromorphic section of L^{-1}. Thus

$$|1/f^0|_{L^{-1}} = |\sigma'(1/f^0)|_E = |\sigma \cdot 1|_E = 1/|f|_{E^*},$$

since

$$|\sigma|_E = 1/|f|_{E^*}.$$

?????????????????

We conclude that

$$|f^0|_L = |f|_{E^*}.$$

0.3. **Another regularization of Bochner-Martinelli currents.** Although the approximands $U^{f,\lambda} = |f|^{2\lambda}U^f$ and $R^{f,\lambda} = \bar\partial|f|^{2\lambda} \wedge U^f$ of U^f and R^f are arbitrarily smooth if $\operatorname{Re}\lambda$ is large enough, if is sometimes desirable to have infinitley smooth approximations, just as for the case when f is one single function.

Recall that $\alpha := |f|^2\sigma$ is smooth in X. For $\epsilon > 0$ let

$$\sigma_\epsilon := \frac{\alpha}{|f|^2 + \epsilon}$$

and let us introduce the smooth forms

$$U^{f,\epsilon} := \frac{\sigma_\epsilon}{1 - \bar\partial\sigma_\epsilon} = \sigma_\epsilon + \sigma_\epsilon \wedge \bar\partial \wedge \sigma_\epsilon + \sigma_\epsilon \wedge (\bar\partial\sigma_\epsilon)^2 + \cdots, \tag{0.8}$$

and

$$R^{f,\epsilon} = \sum_{k=0}^{n} \frac{\epsilon}{|f|^2 + \epsilon}\Big(\frac{\bar\partial\alpha}{|f|^2 + \epsilon}\Big)^k. \tag{0.9}$$

Proposition 0.10. *We have that*

$$\nabla_f U^{f,\epsilon} = 1 - R^{f,\epsilon}, \tag{0.10}$$

$$\lim_{\epsilon\to 0} U^{f,\epsilon} = U^f \tag{0.11}$$

and

$$\lim_{\epsilon\to 0} R^{f,\epsilon} = R^f. \tag{0.12}$$

Proof. We use the notation from ???. Recall that

$$U_k = \pi_*\Big(\frac{1}{(f^0)^k}\wedge\sigma'\wedge(\bar\partial\sigma')^{k-1}\Big)$$

where $\pi^*\sigma = (1/f^0)\sigma'$. Since

$$\sigma_\epsilon = \frac{|f|^2}{|f|^2+\epsilon}\sigma$$

we have that

$$\pi^*\Big(\sigma_\epsilon\wedge(\bar\partial\sigma_\epsilon)^{k-1}\Big) = \Big(\frac{|f^0|^2|f'|^2}{|f^0|^2|f'|^2+\epsilon}\Big)^k\frac{1}{(f^0)^k}\sigma'\wedge(\bar\partial\sigma')^{k-1}$$

which tends to

$$\frac{1}{(f^0)^k}\wedge\sigma'\wedge(\bar\partial\sigma')^{k-1}$$

in view of Example **??**. It follows that

$$U^{f,\epsilon} = \pi_*(\pi^* U^{f,\epsilon}) \to U^f$$

so that (0.11) is settled.

Since $\delta_f\sigma = 1$ we have that $\delta_f\alpha = |f|^2$. Thus

$$U^{f,\epsilon} = \frac{\alpha}{|f|^2+\epsilon-\bar\partial\alpha} = \frac{\alpha}{\epsilon+\nabla\alpha}.$$

Thus

$$\nabla_f U^\epsilon = \frac{\nabla\alpha}{\epsilon+\nabla\alpha} = 1-\frac{\epsilon}{\epsilon+\nabla\alpha}.$$

Moreover,

$$\frac{\epsilon}{\epsilon+\nabla\alpha} = \frac{\epsilon}{\epsilon+|f|^2-\bar\partial\alpha} = \frac{\epsilon}{|f|^2+\epsilon}\frac{1}{1-\frac{\bar\partial\alpha}{|f|^2+\epsilon}},$$

and developing the right most factor we get (0.10) Now (0.12) follows from ???. □

Remark 0.11 (Resolutions and dimension of subvarieties). In a resolution $\pi\colon\tilde X\to X$, the inverse image $\tilde Y$ of a variety Y in X is (usually) a hypersurface in $\tilde X$ so any assumption about big codimension, e.g., an assumption about complete intersection, will necessarily be destroyed. However, it will be reflected on the pullback of a test form in the following way. Any smooth $(0,q)$-form ψ can locally be written $\psi = \sum_\nu \psi_\nu\bar\omega_\nu$, where ω_ν are holomorphic $(0,q)$-forms and ψ_ν are smooth. Now assume that the complex dimension of Y is smaller than q, so that (the pullback of) ψ vanishes of Y for degree reasons. Moreover, assune that s is a local coordinate function in $\tilde X$ such that $\{s=0\}\subset\tilde Y$. Then $\pi^*\omega_\nu$ is holomorphic and vanishes on the hyperplane $\{s=0\}$ and therefore it is a sum of terms, each of which is either divisible by s or by ds. It follows that $\tilde\psi$ is a sum of terms each of which is a smooth form times $\bar s$ or a smooth form times $d\bar s$. □

Example 0.12. Let $X = \mathbb{C}^2_{z,w}$ and $Y = \{0\}$ and let $\tilde{X}$ be the blow-up at 0, and assume that $z = s,\ w = st$, so that $\tilde{Y} = \{s = 0\}$. Then $\pi^* d\bar{w} = \bar{d\bar{t}} + \bar{s}d\bar{t}$, so both kind of terms may appear. □

1. Multivariable Poincare-Lelong

Chapter 5

Residue currents of generically exact complexes

1. Residue currents of generically exact complexes

Let E, Q be Hermitian holomorphic vector bundles over a connected manifold X and let $f\colon E \to Q$ be a holomorphic morphism. If f has optimal rank ρ then the rank is precisely ρ outside the analytic set $Z = \{F = 0\}$, where $F = \det^\rho f$ is a section of $\Lambda^\rho E^* \otimes \Lambda^\rho Q$. Let $\sigma\colon Q \to E$ be the minimal inverse in $X \setminus Z$, i.e., $\sigma\xi$ is the minimal solution to $f\eta = \xi$ if ξ is in the image of f and $\sigma\xi = 0$ if ξ is orthogonal to $\operatorname{Im} f$. Then clearly σ is smooth outside Z, and following the proof of Lemma 4.1 in [45] we get

Lemma 1.1. *If $F = F^0F'$ in X, where F^0 is a holomorphic function and F' is non-vanishing, then $F^0\sigma$ is smooth across Z.*

Let

$$0 \to E_N \xrightarrow{f_N} E_{N-1} \xrightarrow{f_{N-1}} \dots \xrightarrow{f_{-M+2}} E_{-M+1} \xrightarrow{f_{-M+1}} E_{-M} \to 0 \tag{1.1}$$

be a holomorphic complex of Hermitian vector bundles over the n-dimensional complex manifold X, and assume that it is pointwise exact outside the analytic set Z of positive codimension. Then for each k, rank f_k is constant in $X \setminus Z$ and equal to

$$\rho_k = \dim E_k - \dim E_{k+1} + \dots \pm \dim E_N. \tag{1.2}$$

The bundle $E = \oplus E_k$ has a natural superbundle structure, i.e., a $\mathbb{Z}_2$-grading, $E = E^+ \oplus E^-$, E^+ and E^- being the subspaces of even and odd elements, respectively, by letting $E^+ = \oplus_{2k} E_k$ and $E^- = \oplus_{2k+1} E_k$, see [75] and, e.g., [46], for details. The mappings $f = \sum f_j$ and $\bar\partial$ are then odd mappings on $\mathcal{D}'_\bullet(E)$ and they anticommute so that $\nabla^2 = 0$, where $\nabla = f - \bar\partial$ is (minus) the $(0,1)$-part of Quillen's superconnection $D - \bar\partial$. Moreover, ∇ extends to an odd mapping ∇_{End} on $\mathcal{D}'_\bullet(\mathrm{End}E)$ and $\nabla^2_{\mathrm{End}} = 0$. In $X \setminus Z$ let $\sigma_k\colon E_{k-1} \to E_k$ be the minimal inverses of f_k. If $\sigma = \sigma_{-M+1} + \dots + \sigma_N\colon E \to E$ and I denotes the identity endomorphism on E, then $f\sigma + \sigma f = I$. Moreover, $\sigma\sigma = 0$ and thus

$$\sigma(\bar\partial\sigma) = (\bar\partial\sigma)\sigma. \tag{1.3}$$

Since σ is odd, $\nabla_{\mathrm{End}}\sigma = \nabla \circ \sigma + \sigma \circ \nabla = f\sigma + \sigma f - (\bar\partial \circ \sigma + \sigma \circ \bar\partial)$, so we get

$$\nabla_{\mathrm{End}}\sigma = I - \bar\partial\sigma. \tag{1.4}$$

Notice that $\bar\partial\sigma$ has even degree. In $X \setminus Z$ we define the $\mathrm{End}E$-valued form, cf., (1.4),

$$u = \sigma(\nabla_{\mathrm{End}}\sigma)^{-1} = \sigma(I - \bar\partial\sigma)^{-1} = \sigma + \sigma(\bar\partial\sigma) + \sigma(\bar\partial\sigma)^2 + \dots. \tag{1.5}$$

Now, $\nabla_{\mathrm{End}}u = \nabla_{\mathrm{End}}\sigma(\nabla_{\mathrm{End}}\sigma)^{-1} - \sigma\nabla_{\mathrm{End}}(\nabla_{\mathrm{End}}\sigma)^{-1}$, and since $\nabla^2_{\mathrm{End}} = 0$ we thus have

$$\nabla_{\mathrm{End}}u = I. \tag{1.6}$$

Notice that

$$u = \sum_\ell \sum_{k \geq \ell+1} u_k^\ell$$

where

$$u_k^\ell = \sigma_k(\bar\partial\sigma_{k-1}) \cdots (\bar\partial\sigma_{\ell+1})$$

is in $\mathcal{E}_{0,k-\ell-1}(\mathrm{Hom}\,(E_\ell, E_k))$ over $X \setminus Z$. In view of (1.3) we also have

$$u_k^\ell = (\bar\partial\sigma_k)(\bar\partial\sigma_{k-1})\cdots(\bar\partial\sigma_{\ell+2})\sigma_{\ell+1}. \tag{1.7}$$

Let

$$u^\ell = \sum_{k\geq \ell+1} u_k^\ell,$$

be u composed with the projection $E \to E_\ell$. We can make a current extension of u across Z following [74] and the proof of Theorem 1.1 in [42]. In fact, after a sequence of suitable resolutions we may assume that the sections $F_j = \det^{\rho_j} f_j$ of $\Lambda^{\rho_j} E_j^* \otimes \Lambda^{\rho_j} E_{j-1}$ are of the form $F_j = F_j^0 F_j'$, where F_j^0 is a monomial and F_j' are non-vanishing. If F is a holomorphic function that vanishes on Z, in the same way we may assume that $F = F^0 F'$. By Lemma 1.1, $\sigma_j = \alpha_j / F_j^0$, where α_j is smooth across Z. Since $\alpha_{j+1}\alpha_j = 0$ outside the set $\{F_{j+1}^0 F_j^0 = 0\}$, thus $\alpha_{j+1}\alpha_j = 0$ everywhere. Therefore, cf., (1.7), it is easy to see that

$$u_{\ell+k}^\ell = \frac{(\bar\partial\alpha_{\ell+k})(\bar\partial\alpha_{\ell+k-1})\cdots(\bar\partial\alpha_{\ell+2})\alpha_{\ell+1}}{F_{\ell+k}^0\cdots F_{\ell+1}^0}. \tag{1.8}$$

Since F_j only vanish on Z and F vanishes there, F^0 must contain each coordinate factor that occurs in any F_j^0. It follows now that $\lambda \mapsto |F|^{2\lambda} u$ has a current-valued analytic continuation to $\mathrm{Re}\,\lambda > -\epsilon$, and that $U = |F|^{2\lambda} u|_{\lambda=0}$ is a current extension of u.

In the same way we can now define the residue current $R = R(E_\bullet)$ associated to (1.1) as

$$R = \bar\partial |F|^{2\lambda} \wedge u|_{\lambda=0}.$$

It clearly has its support on Z. If $R_k^\ell = \bar\partial|F|^{2\lambda} \wedge u_k^\ell|_{\lambda=0}$ and R^ℓ is defined analogously, then

$$R = \sum_\ell R^\ell = \sum_\ell \sum_{k\geq \ell+1} R_k^\ell.$$

Notice that R_k^ℓ is a $\mathrm{Hom}\,(E_\ell, E_k)$-valued $(0, k-\ell)$-current. The currents U^ℓ and U_k^ℓ are defined analogously. Notice that U has odd degree and R has even degree. In analogy with Theorems 1.1 and 1.2 in [42] we have:

Proposition 1.2. *If U and R are the currents associated to the complex* (1.1) *then*

$$\nabla_{\mathrm{End}} U = I - R, \qquad \nabla_{\mathrm{End}} R = 0. \tag{1.9}$$

Moreover, R_k^ℓ vanishes if $k - \ell < \mathrm{codim}\, Z$, and $\bar\xi R = d\bar\xi \wedge R = 0$ if ξ is holomorphic and vanishes on Z.

The residue current $R = R(E_\bullet)$ is related to the (lack of) exactness of the sheaf complex associated to (1.1) in the following way.

Proposition 1.3. *Let $R = R(E_\bullet)$ be the residue current associated with* (1.1) *and let ϕ be a holomorphic section of E_ℓ.*
(i) If $f_\ell\phi = 0$ and $R^\ell\phi = 0$, then locally there is a holomorphic section ψ of $E_{\ell+1}$ such that $f_{\ell+1}\psi = \phi$.
(ii) If moreover $R^{\ell+1} = 0$, then the existence of such a local solution ψ implies that $R^\ell\phi = 0$.

Proof. Let U be the associated current such that (1.9) holds. Then $\nabla(U\phi) = \phi - U(\nabla\phi) - R\phi$. Since $U\phi = U^\ell\phi$, $R\phi = R^\ell\phi$, and $\nabla\phi = f_\ell\phi - \bar\partial\phi$, it follows from the assumptions of ϕ that $\nabla(U^\ell\phi) = \phi$. Now (i) follows by solving a sequence of $\bar\partial$-equations locally. For the second part, assume that $f_{\ell+1}\psi = \phi$. Then by (1.9), $R^\ell\phi = R\phi = R(\nabla\psi) = \nabla(R\psi) = \nabla(R^{\ell+1}\psi) = 0$. □

If now (**??**) is a generically exact holomorphic complex of Hermitian bundles, since $\operatorname{rank} f_1$ is generically constant, we can define σ_1 in an unambiguous way in $X \setminus Z$, and therefore the currents R^ℓ for $\ell \geq 0$ can be defined as above, and we have:

Corollary 1.4. *If $R = R(E_\bullet)$ is the residue current associated to* (**??**)*, then Proposition 1.3 holds (for $\ell \geq 0$), provided that $f_0\phi = 0$ is interpreted as ϕ belonging generically (outside Z) to the image of f_1.*

If f_1 is generically surjective, in particular if $\operatorname{rank} E_0 = 1$ and f_1 is not identically 0, then this latter condition is of course automatically fulfilled.

Proof. The corollary actually follows just from a careful inspection of the arguments in the proof of Proposition 1.3. Another way is to extend (**??**) to a generically exact complex (1.1) and then refer directly to Proposition 1.3, noting that the definition of R^ℓ for $\ell \geq 0$ as well as the condition $f_0\phi = 0$ are independent of such an extension. □

2. Residue currents with prescribed annihilators

The exactness of (**??**) is characterized by the current R associated with (**??**).

Theorem 2.1. *Assume that* (**??**) *is generically exact, let R be the associated residue current, and let* (**??**) *be the associated complex of sheaves. Then $R^\ell = 0$ for all $\ell \geq 1$ if and only if* (**??**) *is exact.*

For the proof we will use the following characterization of exactness due to Buchsbaum-Eisenbud, see [62] Theorem 20.9: The complex (**??**) is exact if and only if

$$\operatorname{codim} Z_j \geq j \tag{2.1}$$

for all j, where, cf., (1.2),

$$Z_j = \{z;\ \operatorname{rank} f_j < \rho_j\}.$$

Remark 2.2. To be precise we will only use the "only if"-direction. The other direction is actually a consequence of Corollary 1.4 and (the proof of) Theorem 2.1. □

Proof. From Corollary 1.4 it follows that (**??**) is exact if $R^\ell = 0$ for $\ell \geq 1$. For the converse, let us now assume that (**??**) is exact; by the Buchsbaum-Eisenbud theorem then (2.1) holds. We will prove that $R^1 = 0$; the case when $\ell > 1$ is handled in the same way. The idea in the proof is based on the somewhat vague principle that a residue current of bidegree $(0, q)$ cannot be supported on a variety of codimension $q + 1$. Taking this for granted, we notice to begin with that $R^1_2 = \bar\partial|F|^{2\lambda} \wedge \sigma_2|_{\lambda=0}$ is a $(0, 1)$-current and has its support on Z_2, which has codimension at least 2. Hence R^1_2 must vanish according to the vague principle. Now, σ_3 is smooth outside Z_3, and hence $R^1_3 = \bar\partial\sigma_3 \wedge R^1_2 = 0$ outside Z_3; thus R^1_3 is supported on Z_3 and again, by the same principle, R^1_3 must vanish etc. To make this into a strict argument we will use the following simple lemma which follows from a Taylor expansion.

Lemma 2.3. *Suppose that $\gamma(s, \tau)$ is smooth in $\mathbb{C} \times \mathbb{C}^r$ and that moreover $\gamma(s, \tau)/\bar s$ is smooth where $\tau_1 \cdots \tau_k \neq 0$. Then $\gamma(s, \tau)/\bar s$ is smooth everywhere.*

After a sequence of resolutions of singularities the action of R^1_k on a test form ξ is a finite sum of integrals of the form

$$\int \bar\partial |F^0|^{2\lambda} \wedge \frac{(\bar\partial \alpha_k)(\bar\partial\alpha_{k-1})\cdots(\bar\partial\alpha_3)\alpha_2}{F^0_k F^0_{k-1}\cdots F^0_3 F^0_2} \wedge \left.\tilde\xi\right|_{\lambda=0}$$

where F^0, F^0_i and α_i are as (1.8) above, and where $\tilde\xi$ is the pullback of ξ. To be precise, there are also cutoff functions involved that we suppress for simplicity. Observe that $\bar\partial|F^0|^{2\lambda}$ is a finite sum of terms like $a\lambda|F^0|^{2\lambda}d\bar s/\bar s$, where a is a positive integer and s is just one of the coordinate functions that divide F^0. We need to show that all the corresponding integrals vanish when $\lambda = 0$, and to this end it is enough to show, see, e.g., Lemma 2.1 in [42], that

$$\eta = \frac{d\bar s}{\bar s} \wedge (\bar\partial\alpha_k)(\bar\partial\alpha_{k-1})\cdots(\bar\partial\alpha_3)\alpha_2 \wedge \tilde\xi$$

is smooth ($(d\bar s/\bar s)\wedge\beta$ being smooth for a smooth β, means that each term of β contains a factor $\bar s$ or $d\bar s$).

Let ℓ be the largest index among $2,\dots,k$ such that s is a factor in F^0_ℓ (possibly there is no such index at all; then ℓ below is to be interpreted as 1) and let $\tau_1,\dots,\tau_r$ denote the coordinates that divide $F^0_k\cdots F^0_{\ell+1}$. We claim that, outside $\tau_1\cdots\tau_r = 0$, the form

$$\frac{d\bar s}{\bar s} \wedge \frac{(\bar\partial\alpha_k)\cdots(\bar\partial\alpha_{\ell+1})}{F^0_k\cdots F^0_{\ell+1}} \wedge \tilde\xi$$

is smooth. This follows by standard arguments, see, e.g., the proof of Lemma 2.2 in [74] or the proof of Theorem 1.1 in [42]; in fact, outside $Z_k\cap\ldots\cap Z_{\ell+1}$ the $(n, n-\ell+1)$-form $(\bar\partial\sigma_k)\ldots(\bar\partial\sigma_{\ell+1})\wedge\xi$ is smooth and it must vanish on Z_ℓ for degree reasons, since Z_ℓ has codimension at least ℓ. Thus the form

$$\tilde\eta = \frac{d\bar s}{\bar s} \wedge (\bar\partial\alpha_k)\cdots(\bar\partial\alpha_{\ell+1}) \wedge \tilde\xi$$

is smooth outside $\tau_1\cdots\tau_r = 0$. By Lemma 2.3, applied to

$$\gamma = d\bar s \wedge (\bar\partial\alpha_k)\cdots(\bar\partial\alpha_{\ell+1}) \wedge \tilde\xi,$$

$\tilde\eta$ is smooth everywhere, and therefore η is smooth. □

If (**??**) is exact, then, with no ambiguity, we can write R_k rather than R^0_k.

Proof of Theorem **??**. Since a free resolution of a free sheaf is pointwise exact, it follows that $Z_N \subset \cdots \subset Z_1 = Z$. Therefore u^0 is smooth outside Z and thus the support of R must be contained in Z. By Theorem 2.1, $R^1 = 0$, and so the second assertion, the Noetherian property of $R = R^0$, follows from Corollary 1.4. □

Given any coherent sheaf $\mathcal{F}$ in a Stein manifold X and compact subset $K \subset X$, one can always find a resolution

$$\cdots \to \mathcal{O}^{\oplus r_2} \to \mathcal{O}^{\oplus r_1} \to \mathcal{O}^{\oplus r_0} \tag{2.2}$$

of $\mathcal{F}$ in a neighborhood of K, e.g., by iterated use of Theorem 7.2.1 in [66]. The key stone in the proof of Theorem 2.1, the Buchsbaum-Eisenbud theorem, in general requires that the resolution (2.2) starts with 0 somewhere on the left. However, by the Syzygy theorem and Oka's lemma, Ker $(\mathcal{O}^{\oplus r_\ell} \to \mathcal{O}^{\oplus r_{\ell-1}})$ is (locally) free for large ℓ, so we can replace such a module $\mathcal{O}^{\oplus r_\ell}$ with this kernel and 0 before that. Therefore Theorem 2.1 holds and we have

Proposition 2.4. *Let $\mathcal{J}$ be a coherent subsheaf of $\mathcal{O}^{\oplus r_0}$ in a Stein manifold X. For each compact subset $K \subset X$ there is a residue current R defined in a neighborhood of K such that* $\text{ann}\, R = \mathcal{J}$.

The degree of explicitness of the Noetherian residue current R in Theorem **??** is of course directly depending on the degree of explicitness of the resolution.

Example 2.5 (The Koszul complex). Let H be a Hermitian bundle over X of rank m and let h be a non-trivial holomorphic section of the dual bundle H^*. Then h can be considered as a morphism $H \to \mathbb{C} \times X$, and we get a generically exact complex (**??**) by taking $E_k = \Lambda^k H$ and let all the mappings f_k be interior multiplication with f. If η is the section of E over $X \setminus Z$ of minimal norm such that $f \cdot \eta = 1$, then $\sigma_k \xi = \eta \wedge \xi$ for sections ξ of E_{k-1}, and hence $u_k^\ell = \eta \wedge (\bar\partial \eta)^{k-\ell-1}$, acting on $\Lambda^\ell H$ via wedge multiplication. Thus $R_k^\ell = \bar\partial |h|^{2\lambda} \wedge \xi \wedge (\bar\partial \xi)^{k-\ell-1}|_{\lambda=0}$ are precisely the currents considered in [42]. If h is a complete intersection and $h = h_1 e_1^* + \cdots + h_m e_m^*$ in some local holomorphic frame e_j^* for H^*, then R is precisely the Coleff-Herrera product (**??**) times $e_1 \wedge \ldots \wedge e_m$, where e_j is the dual frame, see [42]. □

We now consider a simple example of a non-complete intersection ideal.

Example 2.6. Consider the ideal $J = (z_1^2, z_1 z_2)$ in $\mathbb{C}^2$ with zero variety $\{z_1 = 0\}$. It is easy to see that

$$0 \to \mathcal{O} \xrightarrow{f_2} \mathcal{O}^{\oplus 2} \xrightarrow{f_1} \mathcal{O}, \tag{2.3}$$

where

$$f_1 = \begin{bmatrix} z_1^2 & z_1 z_2 \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix},$$

is a (minimal) resolution of $\mathcal{O}/J$. We equip the corresponding vector bundles with the trivial Hermitian metrics. Since Z has codimension 1, R consists of the two parts $R_2 = \bar\partial |F|^{2\lambda} \wedge u_2^0|_{\lambda=0}$ and $R_1 = \bar\partial |F|^{2\lambda} \wedge u_1^0|_{\lambda=0}$, where $u_2^0 = \sigma_2 \bar\partial \sigma_1$ and $u_1^0 = \sigma_1$, respectively. To compute R it is enough to make a simple blow-up at the origin, and one gets, cf., [78] and [77], that

$$R_2 = \bar\partial\Big[\frac{1}{z_1^2}\Big] \wedge \bar\partial\Big[\frac{1}{z_2}\Big] \quad \text{and} \quad R_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big[\frac{1}{z_2}\Big] \bar\partial \Big[\frac{1}{z_1}\Big].$$

We see that $\text{ann}\, R_2 = (z_1^2, z_2)$ and $\text{ann}\, R_1 = (z_1)$, and hence $\text{ann}\, R = (z_1^2, z_2) \cap (z_1) = J$ as expected. Notice that the Koszul complex associated with the ideal J is like (2.3) but with an extra factor z_1 in the mapping f_2. Then the current R_1^0 is of course the same as before, but

$$R_2^0 = \frac{1}{2} \bar\partial \Big[\frac{1}{z_1^3}\Big] \wedge \bar\partial \Big[\frac{1}{z_2}\Big].$$

In this case $\text{ann}\, R^0 = \text{ann}\, R_2^0 \cap \text{ann}\, R_1^0 = (z_1^3, z_2) \cap (z_1)$ which is strictly smaller than J. Roughly speaking, the annihilator of R_2^0 is too small, since the singularity of σ_2 and hence of u_2^0 is too big, due to the extra factor z_1 in f_2. □

There has recently been a lot of work done on finding free resolutions of monomial ideals, see for example [69], [48] or [50]. For more involved explicit computations of residue currents for monomial ideals, see [78]. We conclude with a simple example where $\text{ann}\, (\mathcal{O}(E_0)/J) = 0$.

Example 2.7. Consider the submodule J of $\mathcal{O}^{\oplus 2}$ generated by $f_1 = [z_1 z_2 \quad -z_1^2]^T$ and the resolution $0 \to \mathcal{O} \xrightarrow{f_1} \mathcal{O}^{\oplus 2}$, which is easily seen to be minimal. Notice that $Z = \{z_1 = 0\}$ is the associated set where $\mathcal{O}^{\oplus 2}/J$ is not locally free, or equivalently where f_1 is not locally constant. Moreover, notice that $\operatorname{ann}(\mathcal{O}^{\oplus 2}/J) = 0$. The associated residue current is

$$R = R_1 = \Big[\frac{1}{z_2}\Big]\bar\partial\Big[\frac{1}{z_1}\Big]\,[\,0 \quad 1\,]\,.$$

If we extend the complex with the mapping $f_0 = [z_1\ z_1]$ the new complex is still exact outside Z. Observe that $\operatorname{ann} R$ is generated by $z_1[1\ 1]^T$ and moreover that $\operatorname{Ker} f_0$ is generated by $[z_2 \quad -z_1]^T$. Thus $\operatorname{Ker} f_0 \cap \operatorname{ann} R = J$ as expected. □

3. Division and interpolation formulas

To obtain formulas for division and interpolation that involve our currents R and U we will use the general scheme developed in [46]. Let z be a fixed point in $\mathbb{C}^n$, let $\delta_{\zeta - z}$ denote interior multiplication by the vector field $2\pi i\sum_1^n(\zeta_j - z_j)(\partial/\partial\zeta_j)$, and let $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar\partial$. Let $g = g_{0,0} + \cdots + g_{n,n}$ be a smooth form such that $\nabla_{\zeta-z} g = 0$ and $g_{0,0}(z) = 1$ (here lower indices denote bidegree); such a form will be called a weight with respect to the point z. If g has compact support then

$$\phi(z) = \int g\phi \tag{3.1}$$

for ϕ that are holomorphic in a neighborhood of the support of g, [46].

Let D be a ball with center at the origin in $\mathbb{C}^n$ and let χ be a cutoff function that is 1 in a neighborhood of $\overline{D}$. Then for each $z \in \overline{D}$,

$$g = \chi - \bar\partial\chi \wedge \frac{s}{\nabla_{\zeta-z} s} = \chi - \bar\partial\chi \wedge [s + s\wedge\bar\partial s + \cdots + s\wedge(\bar\partial s)^{n-1}] \tag{3.2}$$

is a weight, and it depends holomorphically on z. Assume that (1.1) is a complex of (trivial) bundles over a neighborhood of $\overline{D}$ and let $\mathcal{J} = \operatorname{Im} f_1$. Let us also fix global frames for the bundles E_k. Then $E_k \simeq \mathbb{C}^{\operatorname{rank} E_k}$ and the morphisms f_k are just matrices of holomorphic functions. One can find (see [46] for explicit choices) $(k-\ell, 0)$-form-valued holomorphic Hefer morphisms, i.e., matrices, $H_k^\ell\colon E_k \to E_\ell$ depending holomorphically on z and ζ, such that $H_k^\ell = 0$ for $k < \ell$, $H_\ell^\ell = I_{E_\ell}$, and in general,

$$\delta_{\zeta-z} H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z) H_k^{\ell+1}; \tag{3.3}$$

here f stands for $f(\zeta)$. Let

$$HU = \sum_\ell H^{\ell+1}U = \sum_{\ell k} H_k^{\ell+1}U_k^\ell, \quad HR = \sum_\ell H^\ell R = \sum_{\ell k} H_k^\ell R_k^\ell.$$

Then $g' = f(z)HU + HUf + HR$ maps a section of E_ℓ depending on ζ into a (current-valued) section of E_ℓ depending on both ζ and z. Moreover, $\nabla_{\zeta-z} g' = 0$ and $g'_{0,0} = I_E$. If g is weight with compact support, cf., Proposition 5.4 in [46], we therefore have the representation

$$\phi(z) = f_{k+1}(z)\int_\zeta H^{k+1}U\phi\wedge g + \int_\zeta H^k U f_k\phi \wedge g + \int_\zeta H^k R\phi\wedge g, \tag{3.4}$$

$z \in \overline{D}$, for $\phi \in \mathcal{O}(\overline{D}, E_k)$. Thus we get an explicit realization (in terms of U) of to $f_{k+1}\psi = \phi$, if $f_k\phi = 0$ and $R\phi = 0$, and thus an explicit proof of Proposition 1.3 (i).

If we have a complex (**??**) over a neighborhood of $\overline{D}$, and either f_1 is generically surjective or we have an extension to a generically exact complex ending at E_{-1}, then (4.6) still holds for $k=0$. If R is Noetherian, then the last two terms vanish if and only if ϕ is in $\mathcal{J}$. We thus obtain an explicit realization of the membership of $\mathcal{J}$.

In the same way as in [43] one can extend these formulas slightly, to obtain a characterization of the module $\mathcal{E}J$ of smooth tuples of functions generated by J, i.e., the set of all $\phi = f_1\psi$ for smooth ψ. For simplicity we assume that $\mathcal{O}(E_0)/J$ has positive codimension so that $f_0 = 0$. Let R be a Noetherian current for J. First notice that if $\phi = f_1\psi$, then, cf., Proposition 1.2, $R\phi = R^0\phi = R^0 f_1\psi - R^1\bar\partial\psi = R\nabla\psi = \nabla R^1\psi = 0$, so that $R\phi = 0$. Since each partial derivative $\partial/\partial\bar z_j$ commutes with f_1, we get that

$$R(\partial^\alpha\phi/\partial\bar z^\alpha) = 0 \tag{3.5}$$

for all multiindices α. The converse can be proved by integral formulas precisely as in [43], and thus we have

Theorem 3.1. *Assume that $J \subset \mathcal{O}^{\oplus r_0}$ is a coherent subsheaf such that $\mathcal{O}^{\oplus r_0}/J$ has positive codimension, and let R be a Noetherian residue current for J. Then an r_0-tuple $\phi \in \mathcal{E}^{\oplus r_0}$ of smooth functions is in $\mathcal{E}J$ if and only if* (3.5) *holds for all α.*

Let J be a coherent Cohen-Macaulay ideal sheaf of codimension p over some pseudoconvex set X and let μ be an analytic functional that annihilates J. In [60] was proved (Theorem 4.4) that μ can be represented by an (n,n)-current $\tilde\mu$ with compact support of the form $\tilde\mu = \alpha\wedge R$, where α is a smooth $(n, n-p)$-form with compact support and R is the Coleff-Herrera product of a complete intersection ideal contained in J. In particular, $\tilde\mu$ vanishes on $\mathcal{E}J$. As another application of our integral formulas we prove the following more general result.

Theorem 3.2. *Let X be a pseudoconvex set in $\mathbb{C}^n$ and let J be a coherent subsheaf of $\mathcal{O}(E_0) \simeq \mathcal{O}^{\oplus r_0}$ such that $\mathcal{O}(E_0)/J$ has positive codimension. If $\mu \in \mathcal{O}'(X, E_0^*)$ is an analytic functional that vanishes on J, then there is an (n,n)-current $\tilde\mu$ with compact support that represents μ, i.e.,*

$$\mu.\xi = \tilde\mu.\xi, \quad \xi \in \mathcal{O}(X, E_0), \tag{3.6}$$

and such that $\tilde\mu$ vanishes on $\mathcal{E}J$. More precisely we can choose $\tilde\mu$ of the form

$$\tilde\mu = \sum_k \alpha_k R_k,$$

where R is a Noetherian residue current for J and $\alpha_k \in \mathcal{D}_{n,n-k}(X, E_k^*)$.

Here E_k refers to the trivial vector bundles associated to a free resolution of $\mathcal{O}(E_0)/J$.

Proof. Assume that μ is carried by the $\mathcal{O}(X)$-convex compact subset $K \subset X$ and let V be an open neighborhood of K. For each $z \in V$ we can choose a weight g^z with respect to z, such that $z \mapsto g^z$ is holomorphic in V and all g^z have support in some compact $\widetilde K \subset X$, see Example 10 in [42]. Let R be a residue current for J, associated to a free resolution of $\mathcal{O}(E_0)/J$ in a neighborhood of $\widetilde K$, cf, Proposition 2.4. Now consider the corresponding decomposition (4.6) (with $k=0$) that holds for $z \in V$, with $g = g^z$; notice that $f_0 = 0$ by the assumption on J. The analytic functional

μ has a continuous extension to $\mathcal{O}(K, E_0)$ and since $\mathcal{O}(X)$ is dense in $\mathcal{O}(K)$ μ will vanish on the first term on the right hand side in (4.6). If we define the (n, n)-current

$$\tilde{\mu} = \mu_z(g^z \wedge H^0)R = \sum_k \mu_z(g^z_{n-k,n-k} \wedge H^0_k)R_k = \sum_k \alpha_k R_k,$$

then α_k have compact support and (3.6) holds. Since R is Noetherian, $\tilde{\mu}$ annihilates $\mathcal{E}J$. $\square$

4. NALLE

Recall that $\mathcal{F}$ has *pure* codimension p if the associated prime ideals (of each stalk) all have codimension p. The starting point in this paper is the following result that follows from [13] (see also Section **??** below); as we will see later on it is in a way equivalent to Roos' characterization of purity.

Theorem 4.1. *The sheaf $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{I}$ has pure codimension p if and only if $\mathcal{I}$ is equal to the annihilator of R_p, i.e.,*

$$\mathcal{I} = \{\phi \in \mathcal{O}(E_0);\ R_p\phi = 0\}.$$

If $\mathcal{F}$ is Cohen-Macaulay we can choose a resolution (**??**) with $N = p$, and then $R = R_p$ is a matrix of $\mathcal{CH}_Z$-currents which thus solves our problem. However, in general R_p is not $\bar{\partial}$-closed even if $\mathcal{F}$ has pure codimension. Let

$$0 \to \mathcal{O}(E_0^*) \xrightarrow{f_1^*} \mathcal{O}(E_1^*) \xrightarrow{f_2^*} \dots \xrightarrow{f_{p-1}^*} \mathcal{O}(E_{p-1}^*) \xrightarrow{f_p^*} \mathcal{O}(E_p^*) \xrightarrow{f_{p+1}^*} \tag{4.1}$$

be the dual complex of (**??**) and let

$$\mathcal{H}^k(\mathcal{O}(E_\bullet^*)) = \frac{\text{Ker}\ _{f_{k+1}^*}\mathcal{O}(E_k^*)}{f_k^*\mathcal{O}(E_{k-1}^*)} \tag{4.2}$$

be the associated cohomology sheaves. It turns out that for each choice of $\xi \in \mathcal{O}(E_p^*)$ such that $f_{p+1}^*\xi = 0$, the current ξR_p is in $\mathcal{CH}_Z(E_0^*)$, and we have in fact a bilinear (over $\mathcal{O}$) pairing

$$\mathcal{H}^p(\mathcal{O}(E_\bullet^*)) \times \mathcal{F} \to \mathcal{CH}_Z, \quad (\xi, \phi) \mapsto \xi R_p \phi. \tag{4.3}$$

Moreover, (4.3) is independent of the choice of Hermitian metrics on E_k. It is well-known that the sheaves in (4.2) represent the intrinsic sheaves $\mathcal{E}xt^k_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$. (If Z does not have pure codimension p then we define $\mathcal{CH}_Z$ as $\mathcal{CH}_{Z'}$, where Z' is the union of irreducible components of codimension p; this is reasonable, in view of the SEP.)

Theorem 4.2. *Assume that $\mathcal{F}$ has codimension $p > 0$. The pairing* (4.3) *induces an intrinsic pairing*

$$\mathcal{E}xt^p_{\mathcal{O}}(\mathcal{F}, \mathcal{O}) \times \mathcal{F} \to \mathcal{CH}_Z. \tag{4.4}$$

If $\mathcal{F}$ has pure codimension, then the pairing is non-degenerate.

Notice that $\mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)$ is the subsheaf of $\mathcal{H}om\,(\mathcal{O}(E_0), \mathcal{CH}_Z) = \mathcal{CH}_Z(E_0^*)$ consisting of all Coleff-Herrera currents μ with values in E_0^* such that $\mu\phi = 0$ for all $\phi \in \mathcal{I}$. It follows that we have the equality

$$\mathcal{I} = \{\phi \in \mathcal{O}(E_0);\ \mu\phi = 0 \text{ for all } \mu \in \mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)\} \tag{4.5}$$

if $\mathcal{F}$ is pure. The sheaf $\mathcal{H}^p(\mathcal{O}(E_\bullet^*))$ is coherent and thus locally finitely generated. Therefore we have now a solution to our problem:

Corollary 4.3. *Assume that $\mathcal{F}$ has pure codimension. If $\xi_1, \ldots, \xi_\nu \in \mathcal{O}(E_p^*)$ generate $\mathcal{H}^p(\mathcal{O}(E_\bullet^*))$, then $\mu_j = \xi_j R_p$ are in $\mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)$ and*

$$\mathcal{I} = \cap_{j=1}^{\nu} ann\, \mu_j. \tag{4.6}$$

Remark 4.4. If $\mathcal{I}$ is not pure, one obtains a decomposition (4.6) after a preliminary decomposition $\mathcal{I} = \cap \mathcal{I}_\nu$, where each $\mathcal{I}_\nu$ has pure codimension. □

In case of a complete intersection, $\mathcal{E}xt^p(\mathcal{F}, \mathcal{O})$ is isomorphic to $\mathcal{F}$ itself. If $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{I}$ is a sheaf of Cohen-Macaulay modules there is also a certain symmetry: If (**??**) is a resolution with $N = p$, then it is well-known, cf., also Example 4.12 below, that the dual complex (4.1) is a resolution of $\mathcal{O}(E_p^*)/\mathcal{I}^*$, where $\mathcal{I}^* = f_p^* \mathcal{O}(E_{p-1}^*) \subset \mathcal{O}(E_p^*)$, and we have

Corollary 4.5. *If $\mathcal{O}(E_0)/\mathcal{I}$ is Cohen-Macaulay, then $\mathcal{O}(E_p^*)/\mathcal{I}^*$ is Cohen-Macaulay as well and we have a non-degenerate pairing*

$$\mathcal{O}(E_0)/\mathcal{I} \times \mathcal{O}(E_p^*)/\mathcal{I}^* \to \mathcal{CH}_Z, \quad (\xi, \phi) \mapsto \xi R_p \phi.$$

Remark 4.6. Assume that $\mathcal{F}$ has codimension $p = 0$, or equivalently, $\operatorname{ann} \mathcal{F} = 0$. If it is pure, i.e., (0) is the only associated prime ideal, then there is a homomorphism $f_0 \colon \mathcal{O}(E_0) \to \mathcal{O}(E_{-1})$ such that $\mathcal{I} = \operatorname{Ker} f_0$. It is natural to consider f_0 as a Coleff-Herrera current μ associated with the zero-codimensional "variety" X. Then $\mathcal{I} = \operatorname{ann} \mu$ and thus analogues of Theorem 4.1 and Corollary 4.3 still hold. □

The duality discussed here leads to a generalization of the Dickenstein-Sessa decomposition that we now will describe. It was proved by Malgrange, see, e.g., [56], that the analytic sheaf of distributions $\mathcal{C}$ is stalkwise injective. Thus the double complex

$$\mathcal{H}om_{\,\mathcal{O}}(\mathcal{O}(E_\ell), \mathcal{C}^{0,k}) = \mathcal{C}^{0,k}(E_\ell^*), \tag{4.7}$$

with differentials $\bar\partial$ and f^*, is exact except at $k = 0$ and $\ell = 0$, where we have the cohomology sheaves $\mathcal{O}(E_\ell^*)$ and $\mathcal{H}om\,(\mathcal{F}, \mathcal{C}^{0,\bullet})$, respectively. By standard homological algebra, we therefore have natural isomorphisms

$$\mathcal{H}^k(\mathcal{O}(E_\bullet^*), \mathcal{O}) \simeq \mathcal{H}^k(\mathcal{H}om\,(\mathcal{F}, \mathcal{C}^{0,\bullet})). \tag{4.8}$$

The residue calculus also gives

Theorem 4.7. *Assume that $\operatorname{codim} \mathcal{F} = p > 0$. Both mappings*

$$\mathcal{H}^p(\mathcal{O}(E_\bullet^*)) \overset{\Psi}{\simeq} \mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z) \simeq \mathcal{H}^p(\mathcal{H}om\,(\mathcal{F}, \mathcal{C}^{0,\bullet})) \tag{4.9}$$

are isomorphisms, and the composed mapping coincides with the isomorphism (4.8).

These isomorphisms seem to be known as "folklore" since long ago, cf., Section **??** below. Our contribution should be the proof by residue calculus, and especially, the realization of the mapping Ψ as $\xi \mapsto \xi R_p$.

Example 4.8. If $\mu \in \mathcal{CH}_Z$ is annihilated by $\mathcal{I}$ it follows that we have the factorization $\mu = \xi R_p$. There are analogous isomorphisms where $\mathcal{O}$ is replaced by Ω^r, the sheaf of holomorphic $(r, 0)$-forms, and Coleff-Herrera currents of bidegree (r, p), $\mathcal{CH}_Z^r = \mathcal{CH}_Z \otimes_{\mathcal{O}} \Omega^r$. For instance it follows that there is a factorization

$$[Z] = \xi R_p,$$

where $[Z]$ is the Lelong current, and ξ is in $\Omega^p(E_p^*)$ with $f_{p+1}^* \xi = 0$. □

Example 4.9. We can rephrase the second isomorphism in (4.9) as the decomposition

$$\text{Ker}\left(\mathcal{H}om\,(\mathcal{F},\mathcal{C}^{0,p})\xrightarrow{\bar\partial}(\mathcal{H}om\,(\mathcal{F},\mathcal{C}^{0,p+1})\right)= \\ =\mathcal{H}om\,(\mathcal{F},\mathcal{CH}_Z)\oplus\bar\partial\mathcal{H}om\,(\mathcal{F},\mathcal{C}^{0,p-1}). \tag{4.10}$$

For a given $\bar\partial$-closed $(0,p)$-current μ (with values in E_0^* and annihilated by $\mathcal{I}$), its canonical projection in $\mathcal{H}om\,(\mathcal{F},\mathcal{CH}_Z)$ is given by ξR_p, where ξ is obtained from μ via the isomorphism (4.8). □

Example 4.10. Assume that Z has pure codimension p and let $\mathcal{C}_Z^{0,k}$ denote the sheaf of $(0,k)$-currents with support on Z. If $\mathcal{F}$ has support on Z, then $\mathcal{H}om\,(\mathcal{F},\mathcal{C}^{0,k})=\mathcal{H}om\,(\mathcal{F},\mathcal{C}_Z^{0,k})$. Since any current with support on Z must be annihilated by some power of $\mathcal{I}_Z$, (4.10) implies the decomposition

$$\text{Ker}\left(\mathcal{C}_Z^{0,p}\xrightarrow{\bar\partial}\mathcal{C}_Z^{0,p+1}\right)=\mathcal{CH}_Z\oplus\bar\partial\mathcal{C}_Z^{0,p-1} \tag{4.11}$$

that was first proved in [59] by Dickenstein and Sessa (in the case of a complete intersection; see [56] for the general case). □

Since (**??**) is generically exact, so is its dual complex

$$0\to E_{-M}^*\xrightarrow{f_{-M+1}^*}\cdots\xrightarrow{f_N^*}E_N^*\to 0 \tag{4.12}$$

of Hermitian vector bundles, and we have the corresponding dual complex of locally free sheaves

$$0\to\mathcal{O}(E_{-M}^*)\xrightarrow{f_{-M+1}^*}\cdots\xrightarrow{f_N^*}\mathcal{O}(E_N^*)\to 0. \tag{4.13}$$

Using the induced metrics, we get a residue current

$$R^*=\sum_k (R^*)^k=\sum_{k,\ell}(R^*)^k_\ell,$$

where $(R^*)^k_\ell$ takes values in $\text{Hom}\,(E_k^*,E_\ell^*)$.

Proposition 4.11. *Using the natural isomorphisms* $\text{Hom}\,(E_k^*,E_\ell^*)=\text{Hom}\,(E_\ell,E_k)$ *we have that* $(R^*)^k_\ell=R^\ell_k$.

Proof. It is readily verified that the adjoint $\sigma^*\colon E^*\to E^*$ of $\sigma\colon E\to E$ over $X\setminus Z$ is the minimal inverse of f^*. Therefore,

$$u^*=(\sigma+\sigma(\bar\partial\sigma)+\sigma(\bar\partial\sigma)^2+\cdots)^*=\sigma^*+\sigma^*(\bar\partial\sigma^*)+\sigma^*(\bar\partial\sigma^*)^2+\cdots,$$

since, see [12], $\sigma^*\bar\partial\sigma^*=(\bar\partial\sigma^*)\sigma^*$. Now the proposition follows. □

If $\xi\in\mathcal{O}(E_k^*)$ and $\phi\in\mathcal{O}(E_\ell)$ we write

$$\xi R^\ell_k\phi=\phi(R^*)^k_\ell\xi.$$

Assume that $\mathcal{F}$ is a coherent sheaf of positive codimension p, and let (**??**) be a (locally) free resolution of $\mathcal{F}=\mathcal{O}(E_0)/\mathcal{I}$. Moreover, assume that f_1 is generically surjective so that the corresponding vector bundle complex

$$0\to E_N\xrightarrow{f_N}\cdots\xrightarrow{f_3}E_2\xrightarrow{f_2}E_1\xrightarrow{f_1}E_0\to 0 \tag{4.14}$$

is generically exact. It follows from Proposition **??** that

$$R^0=R^0_p+R^0_{p+1}+\cdots.$$

By Theorem 3.1 in [12], $R_k^\ell = 0$ for each $\ell \geq 1$, i.e., $R = R^0$, and combining with Proposition **??** above we find that a $\phi \in \mathcal{O}(E_0)$ is in $\mathcal{I}$ if and only if $R\phi = 0$. It is proved in Section 5 of [13] that $\mathcal{F}$ has pure codimension p if and only if $\operatorname{ann} R = \operatorname{ann} R_p$, i.e., Theorem 4.1 holds.

Proof of Theorem 4.2. It follows from (**??**) that

$$\bar\partial R_k = f_{k+1} R_{k+1} \tag{4.15}$$

for each k. If $\xi \in \mathcal{O}(E_k^*)$ and $f_{k+1}^* \xi = 0$ we therefore have

$$\bar\partial(\xi R_k) = \pm \xi \bar\partial R_k = \pm \xi f_{k+1} R_{k+1} = \pm (f_{k+1}^* \xi) R_{k+1} = 0.$$

Thus ξR_p is $\bar\partial$-closed and since it is also pseudomeromorphic, cf., Proposition **??**, it is in $\mathcal{CH}_Z$. Moreover, if $\xi = f_p^* \eta$, then

$$\xi R_p = (f_p^* \eta) R_p = \eta f_p R_p = \eta \bar\partial R_{p-1} = 0$$

since $R_k = 0$ for $k < p$. Thus ξR_p only depends on the cohomology class of ξ in $H^p(\mathcal{O}(E_\bullet^*))$. We now choose another Hermitian metric on E and let $\tilde R$ denote the current associated with the new metric. It is showed in [12] (see the proof of Theorem 4.4) that then

$$R_p - \tilde R_p = f_{p+1} M_{p+1}^0$$

for a certain residue current M. Thus $\xi R_p = \xi \tilde R_p$. It follows that the mapping (4.3) is well-defined and independent of the Hermitian metric on E.

It is enough to prove the invariance at a fixed point x, so we consider stalks of the sheaves at x. It is well-known that then our resolution $\mathcal{O}_x(E_\bullet), f_\bullet$ can be written

$$\mathcal{O}_x(E'_\bullet \oplus E''_\bullet) \simeq \mathcal{O}_x(E'_\bullet) \oplus \mathcal{O}_x(E''_\bullet), \ f_\bullet = f'_\bullet \oplus f''_\bullet,$$

where $\mathcal{O}_x(E'_\bullet)$ is a resolution of $\mathcal{F}_x$ and (since we assume that E_0 has minimal rank) $\mathcal{O}_x(E''_k), k \geq 1$, is a resolution of $\mathcal{O}_x(E''_0) = 0$. It follows that the natural mapping $H^p(\mathcal{O}_x((E'_\bullet)^*) \to H^p(\mathcal{O}_x((E_\bullet)^*)),\ \xi' \mapsto (\xi', 0)$, is an isomorphism. Moreover, if we choose a metric on $E_k = E'_k \oplus E''_k$ that respects the direct sum, then the resulting current R is $R' \oplus 0$, where R' is the current associated with $\mathcal{O}_x(E'_\bullet)$. Since all minimal resolutions are isomorphic, the mapping (4.4) is therefore well-defined.

It remains to check that (4.4) is non-degenerate. If $\xi \in \mathcal{O}(E_p^*)$ with $f_{p+1}^* \xi = 0$ and $\xi R_p \phi = 0$ for all $\phi \in \mathcal{O}(E_0)$, then clearly $\xi R_p = 0$. Since $R = R_p^0$, by Proposition 4.11 therefore $(R^*)_\ell^p \xi = 0$ for all ℓ, and now it follows from Proposition **??** that $\xi = f_p^* \eta$ for some η. Thus (the class of) ξ is zero in $\mathcal{H}^p(\mathcal{O}(E_\bullet^*))$.

Now, assume that $\xi R_p \phi = 0$ for all ξ such that $f_{p+1}^* \xi = 0$. If $\mathcal{F}$ is Cohen-Macaulay and $N = p$, then $f_{p+1}^* = 0$ so the assumption implies that $R_p \phi = 0$, and thus $\phi \in \mathcal{I}$. However, generically on Z, $\mathcal{F}$ is Cohen-Macaulay, and hence for an arbitrary resolution we must have that $R_p \phi = 0$ outside a variety of codimension $\geq p + 1$. Since $R_p \phi$ is pseudomeromorphic with bidegree $(0, p)$ it follows from Proposition **??** that $R_p \phi$ vanishes identically. If we in addition assume that $\mathcal{F}$ has pure codimension it follows from Theorem 4.1 that $\phi \in \mathcal{I}$. Thus the pairing is non-degenerate. □

Example 4.12 (The Cohen-Macaulay case). It is well-known, see, e.g., [62], that $\mathcal{F}$ is Cohen-Macaulay if and only if it admits resolutions of length $p = \operatorname{codim} Z$. If (**??**) is a resolution with $N = p$, then $R = R_p^0$, and hence $R^* = (R^*)_p^0$. It follows from Proposition **??**, applied to R^*, that the dual complex (4.1) is a resolution of $\mathcal{O}(E_0^*)/\mathcal{I}^*$ and, in particular, that $\mathcal{O}(E_0^*)/\mathcal{I}^*$ is Cohen-Macaulay. □

Proof of Theorem 4.7. Let
$$\mathcal{L}^\nu = \sum_{\ell+k=\nu} \mathcal{C}^{0,k}(E^*_\ell)$$
be the total complex with differential $\nabla^* = f^* - \bar\partial$, associated with the double complex (4.7). We thus have natural isomorphisms

$$\mathcal{H}^k(\mathcal{O}(E^*_\bullet)) \simeq \mathcal{H}^k(\mathcal{L}) =_{def} \frac{\text{Ker}_{\nabla^*}\mathcal{L}^k}{\nabla^*\mathcal{L}^{k-1}} \simeq \mathcal{H}^k(\mathcal{H}om\,(\mathcal{F}, \mathcal{C}^{0,\bullet})). \tag{4.16}$$

The naturality means that the ismorphisms are induced by the natural mappings $\mathcal{O}(E^*_k) \to \mathcal{L}^k$ and $\mathcal{H}om\,(\mathcal{F}, \mathcal{C}^{0,\ell}) \to \mathcal{L}^k$, respectively, and that $\xi \in \mathcal{O}(E^*_k)$ such that $f^*_{k+1}\xi = 0$ defines the same class as $\mu \in \text{Hom}\,(\mathcal{F}, \mathcal{C}^{0,k})$ with $\bar\partial\mu = 0$ if and only if there is $W \in \mathcal{L}^{k-1}$ such that $\nabla^* W = \xi - \mu$.

If now $\xi \in \mathcal{O}(E^*_k)$ and $f^*_{k+1}\xi = 0$, then $\nabla^*\xi = 0$, and hence

$$\nabla^*(U^*)^k\xi = \xi - (R^*)^k\xi = \xi - \xi R_k, \tag{4.17}$$

cf., (**??**) and Proposition 4.11 above. Therefore the composed mapping in (4.9) coincides with the isomorphisms in (4.8). It is readily verified that the second mapping in (4.9) is injective, see, e.g., Lemma 3.3 in [**?**], and hence both mappings must be isomorphisms. Thus Theorem 4.7 is proved. □

We think it may be enlightening with a proof of the first isomorphism in (4.9) that does not rely on Malgrange's theorem. We already know from Theorem 4.2 that this mapping is injective, so we have to prove the surjectivity. The proof is based on the following lemma.

Lemma 4.13. *If there is a current $W \in \mathcal{L}^{p-1}$ such that $\nabla^* W = \mu \in \mathcal{CH}_Z(E^*_0)$, then $\mu = 0$.*

Proof. Let u be a smooth form u such that $\nabla^*_{\text{End}} u = I_{E^*}$ in $X \setminus Z$. For a given neighborhood ω of Z, take a cutoff function χ with support in ω and equal to 1 in some neighborhood of Z. Then $g = \chi I_{E^*} - \bar\partial\chi \wedge u$ is smooth with compact support in ω, equal to I_{E^*} in a neighborhood of Z, and moreover $\nabla^* g = 0$. Therefore, $\nabla^*(gW) = g\mu = \mu$ and hence, for degree reasons, we have a solution $\bar\partial w = \mu$ with support in ω. Since $\omega \supset Z$ is arbitrary it follows, cf., Lemma 3.3 in [**?**], that $\mu = 0$. □

Since (4.7) is exact in k except at $k = 0$, the first equivalence in (4.16) holds. Take $\mu \in \mathcal{H}om\,(\mathcal{F}, \mathcal{CH}_Z)$. Then $\nabla^*\mu = (f^*_1 - \bar\partial)\mu = 0$ so by (4.16) (with $k = p$) there is $\xi \in \mathcal{O}(E^*_p)$ such that $\nabla^* W = \xi - \mu$ has a current solution $W \in \mathcal{L}^{p-1}$. In view of (4.17) it now follows from Lemma 4.13 that $\mu = \xi R^0_p$.

5. Cohen-Macaulay ideals and modules

Let $\mathcal{F}_x$ be a $\mathcal{O}^r_x$-module. The minimal length ν_x of a resolution of $\mathcal{F}_x$ is precisely $n - \text{depth}\,\mathcal{F}_x$, and $\text{depth}\,\mathcal{F}_x \le \dim \mathcal{F}_x$, so the length of the resolution is at least equal to $\text{codim}\,\mathcal{F}_x$. Recall that the $\mathcal{F}_x$ is Cohen-Macaulay if $\text{depth}\,\mathcal{F}_x = \dim \mathcal{F}_x$, or equivalently, $\nu_x = \text{codim}\,\mathcal{F}_x$, see [62]. As usual we say that an ideal $J_x \subset \mathcal{O}_x$ is Cohen-Macaulay if $\mathcal{F}_x = \mathcal{O}_x/J_x$ is a Cohen-Macaulay module.

A coherent analytic sheaf $\mathcal{F}$ is Cohen-Macaulay if $\mathcal{F}_x$ is Cohen-Macaulay for each x. If we have any locally free resolution of $\mathcal{F}$ and $\text{codim}\,\mathcal{F} = p$, then at each point $\text{Ker}\,(\mathcal{O}(E_{p-1}) \to \mathcal{O}(E_{p-2}))$ is free by the uniqueness theorem, see below, so by Oka's lemma the kernel is locally free; hence we can modify the given resolution to a locally free resolution of minimal length p. Notice that the residue current associated with a resolution of minimal length p just consists of the single term $R = R_p^0$, which locally is a $r_p \times r_0$-matrix of currents.

Theorem 5.1. *Suppose that $\mathcal{F}$ is a coherent analytic sheaf with codimension $p > 0$ that is Cohen-Macaulay, and assume that*

$$0 \to \mathcal{O}(E_p) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0) \tag{5.1}$$

is a locally free resolution of $\mathcal{F}$ of minimal length p. Then the associated Noetherian current is independent of the Hermitian metric.

Proof. Assume that u and u' are the forms in $X \setminus Z$ constructed by means of two different choices of metrics on E. Then $\nabla_{\text{End}} u = I$ and $\nabla_{\text{End}} u' = I$ in $X \setminus Z$, and hence

$$\nabla_{\text{End}}(uu') = (\nabla_{\text{End}} u)u' - u\nabla_{\text{End}} u' = u' - u,$$

where the minus sign occurs since u has odd order. For large $\text{Re}\,\lambda$ we thus have, cf., the proof of Proposition 1.2,

$$\nabla_{\text{End}}\big(|F|^{2\lambda} uu'\big) = |F|^{2\lambda} u' - |F|^{2\lambda} u - \bar\partial |F|^{2\lambda} \wedge uu'.$$

As before one can verify that each term admits an analytic continuation to $\text{Re}\,\lambda > -\epsilon$, and evaluating at $\lambda = 0$ we get $\nabla_{\text{End}} W = U' - U - M$, where $W = |F|^{2\lambda} uu'|_{\lambda=0}$, and M is the residue current

$$M = \bar\partial |F|^{2\lambda} \wedge uu'|_{\lambda=0}. \tag{5.2}$$

Since $\nabla_{\text{End}}^2 = 0$, by Proposition 1.2 we therefore get

$$R - R' = \nabla_{\text{End}} M. \tag{5.3}$$

However, since the complex ends up at p, each term in uu' has at most bidegree $(0, p-2)$ and hence the current M has at most bidegree $(0, p-1)$. Since it is supported on Z with codimension p, it must vanish, cf., the proof of Proposition 1.2. $\square$

When $\mathcal{F} = \mathcal{O}(E_0)/\mathcal{J}$ is Cohen-Macaulay we can also define a cohomological residue that characterizes the module sheaf $\mathcal{J} = \text{Im}\,(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$ locally. Suppose that we have a fixed resolution (5.1) of minimal length and let us assume that $p > 1$. If u is any solution to $\nabla_{\text{End}} u = I$ in $X \setminus Z$, then u_p^0 is a $\bar\partial$-closed $\text{Hom}\,(E_0, E_p)$-valued $(0, p-1)$-form. Moreover if u' is another solution, then it follows from the preceding proof that $\bar\partial(uu')_p^0 = u_p^0 - u_p'^0$. Therefore u_p^0 defines a Dolbeault cohomology class $\omega \in H^{0,p-1}(X \setminus Z, \text{Hom}\,(E_0, E_p))$. If ϕ is a holomorphic section of E_0 then $\omega\phi = [u_p^0 \phi]$ is an element in $H^{0,p-1}(X \setminus Z, E_p)$. Moreover, if v is any solution in $X \setminus Z$ to $\nabla v = \phi$, then v_p defines the class $\omega\phi$. In fact, $\nabla(uv) = v - u\phi = v - u^0\phi$ so that $\bar\partial(uv)_p = u_p^0 \phi - v_p$. Precisely as for a complete intersection, [59] and [72], we have the following cohomological duality principle.

Theorem 5.2. *Let X be a Stein manifold and let* (5.1) *be a resolution of minimal length p of the Cohen-Macaulay sheaf $\mathcal{O}(E_0)/\mathcal{J}$ over X, and assume that $p > 1$. Moreover, let ω be the associated class in $H^{0,p-1}(X \setminus Z, \text{Hom}\,(E_0, E_p))$. For a holomorphic section ϕ of E_0 the following conditions are equivalent:*

(i) ϕ is a global section of $\mathcal{J}$.

(ii) The class $\omega\phi$ in $X \setminus Z$ vanishes.

(iii) $\int \omega\phi \wedge \bar\partial\xi = 0$ *for all* $\xi \in \mathcal{D}_{n,n-p}(X, E_p^*)$ *such that* $\bar\partial\xi = 0$ *in a neighborhood of* Z.

Notice that if R is the associated Noetherian current, then $\bar\partial U_p^0 = R_p$, so by Stokes' theorem, (iii) is equivalent to that $\int R_p\phi \wedge \xi = 0$ for all $\xi \in \mathcal{D}_{n,n-p}(X, E_p^*)$ such that $\bar\partial\xi = 0$ in a neighborhood of Z.

If $p = 1$, then f_1 is an isomorphism outside Z, so its inverse $\omega = \sigma_1$ is a holomorphic $(0,0)$-form in $X \setminus Z$. Thus a holomorphic section ϕ of E_0 belongs to $\mathcal{J}$ if and only if $\omega\phi$ has a holomorphic extension across Z.

Proof. If (i) holds, then $\phi = f_1\psi$ for some holomorphic ψ; thus $\nabla\psi = \phi$. However, since $p > 1$, ψ has no component in E_p, and hence by definition the class $\omega\phi$ vanishes. The implication $(ii) \to (iii)$ follows from Stokes' theorem.

Let us now assume that (iii) holds, and choose a point x on Z. Let $v_k = u_k^0\phi$. If X' is an appropriate small neighborhood of x, then, since Z has codimension p and v_p is a $\bar\partial$-closed $(0,p)$-current, one can verify that the condition (iii) ensures that $\bar\partial w_p = v_p$ has a solution in $X' \setminus \overline{W}$, where W is a small neighborhood of Z in X'. Then, successively, all the lower degree equations $\bar\partial w_k = v_k + f_{k+1}w_{k+1}$, $k \geq 2$, can be solved in similar domains. Finally, we get a holomorphic solution $\psi = v_1 + f_2w_2$ to $f_1\psi = \phi$, in such a domain. By Hartogs' theorem ψ extends across Z in X'. Alternatively, one can obtain such a local holomorphic solution ψ, using the decomposition formula (4.6) below and mimicking the proof of the corresponding statement for a complete intersection in [72]; cf., also the proof of Proposition 7.1 in [46]. Since X is Stein, one can piece together to a global holomorphic solution to $f_1\psi = \phi$, and hence ϕ is a section of $\mathcal{J}$. □

Example 5.3. Let J be an ideal in $\mathcal{O}_0$ of dimension zero. Then it is Cohen-Macaulay and for each germ ϕ in $\mathcal{O}_0$, $\omega\phi$ defines a functional on $\mathcal{O}_0(E_n^*) \simeq \mathcal{O}_0^{r_n}$. If J is defined by a complete intersection, then we may assume that (5.1) is the Koszul complex. Then $r_n = 1$, and in view of the Dolbeault isomorphism, see, e.g., Proposition 3.2.1 in [72], $\omega\phi$ is just the classical Grothendieck residue. □

For the rest of this section we will restrict our attention to modules over the local ring $\mathcal{O}_0$, and we let $\mathcal{O}(E_k)$ denote the free $\mathcal{O}_0$-module of germs of holomorphic sections at 0 of the vector bundle E_k. Given a free resolution (**??**) of a module $\mathcal{F}_0$ over $\mathcal{O}_0$ and given metrics on E_k we thus get a germ R of a Noetherian residue current at 0. Recall that the resolution (**??**) is *minimal* if for each k, f_k maps a basis of $\mathcal{O}(E_k)$ to a minimal set of generators of $\operatorname{Im} f_k$. The uniqueness theorem, see, e.g., Theorem 20.2 in [62], states that any two minimal (free) resolutions are equivalent, and moreover, that any (free) resolution has a minimal resolution as a direct summand.

For a Cohen-Macaulay module $\mathcal{F}_0$ over $\mathcal{O}_0$ we have the following uniqueness.

Proposition 5.4. *Let* $\mathcal{F}_0$ *be a Cohen-Macaulay module over* $\mathcal{O}_0$ *of codimension* p. *If we have two minimal free resolutions* $\mathcal{O}(E_\bullet)$ *and* $\mathcal{O}(E'_\bullet)$ *of* $\mathcal{F}_0$, *then there are holomorphic invertible matrices* g_p *and* g_0 *(local holomorphic isomorphism* $g_p\colon E'_p \simeq E_p$ *and* $g_0\colon E'_0 \simeq E_0$*) such that* $R = g_pR'g_0^{-1}$.

Since minimal resolutions have minimal length p, the currents are independent of the metrics, in view of Proposition 5.1.

Proof. By the uniqueness theorem there are holomorphic local isomorphisms $g_k\colon E'_k \to E_k$ such that

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(E'_p) & \xrightarrow{f'_p} & \cdots & \xrightarrow{f'_2} & \mathcal{O}(E'_1) & \xrightarrow{f'_1} & \mathcal{O}(E'_0) \\ & & g_p \downarrow & & & & g_1 \downarrow & & g_0 \downarrow \\ 0 & \to & \mathcal{O}(E_p) & \xrightarrow{f_p} & \cdots & \xrightarrow{f_2} & \mathcal{O}(E_1) & \xrightarrow{f_1} & \mathcal{O}(E_0) \end{array}$$

commutes. Let g denote the induced isomorphism $E \to E'$. Choose any metric on E and equip E' with the induced metric, i.e., such that $|\xi| = |g^{-1}\xi|$ for a section ξ of E'. If $\sigma\colon E \to E$ and $\sigma'\colon E' \to E'$ are the associated endomorphisms over $X \setminus Z$, cf., Section 1, then $\sigma' = g\sigma g^{-1}$ in $X \setminus Z$, and therefore

$$u' = \sigma' + (\bar\partial\sigma')\sigma' + \cdots = g(\sigma + (\bar\partial\sigma)\sigma + \cdots)g^{-1} = gug^{-1}.$$

Therefore, $(u')^0_p = g_p u^0_p g_0^{-1}$, and hence the statement follows since $R = R_p = R^0_p$. □

We shall now consider the residue current associated to a general free resolution.

Theorem 5.5. *Let $\mathcal{F}_0$ be a Cohen-Macaulay module over $\mathcal{O}_0$ of codimension p. If R is the residue current associated to an arbitrary free resolution* (**??**) *(and given metrics on E_k) and $R' = R'_p$ is associated to a minimal resolution $0 \to \mathcal{O}(E'_p) \xrightarrow{f'_p} \cdots \xrightarrow{f'_2} \mathcal{O}(E'_1) \xrightarrow{f'_1} \mathcal{O}(E'_0)$, then*

$$R_p = h_p R'_p \beta_0, \tag{5.4}$$

where $\beta_0\colon E_0 \to E'_0$ is a local holomorphic pointwise surjective morphism and h_p is a local smooth pointwise injective morphism $h_p\colon E'_p \to E_p$. Moreover, for each $\ell > 0$,

$$R_{p+\ell} = \alpha_\ell R_p,$$

where α_ℓ is a smooth Hom $(E_p, E_{p+\ell})$*-valued $(0,\ell)$-form.*

Proof. By the uniqueness theorem for resolutions, the resolution $E'_\bullet$ is isomorphic to a direct summand in $E_\bullet$, and in view of the preceding proposition, we may assume that

$$\mathcal{O}(E_k) = \mathcal{O}(E'_k \oplus E''_k) = \mathcal{O}(E'_k) \oplus \mathcal{O}(E''_k)$$

and $f_k = f'_k \oplus f''_k$, so that

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(E'_p) & \xrightarrow{f'_p} & \cdots & \xrightarrow{f'_2} & \mathcal{O}(E'_1) & \xrightarrow{f'_1} & \mathcal{O}(E'_0) \\ i_{p+1} \downarrow & & i_p \downarrow & & & & i_1 \downarrow & & i_0 \downarrow \\ \to \mathcal{O}(E_{p+1}) & \xrightarrow{f_{p+1}} & \mathcal{O}(E_p) & \xrightarrow{f_p} & \cdots & \xrightarrow{f_2} & \mathcal{O}(E_1) & \xrightarrow{f_1} & \mathcal{O}(E_0), \end{array}$$

where $i_k\colon E'_k \to E'_k \oplus E''_k$ are the natural injections, and

$$\to \mathcal{O}(E''_{p+1}) \xrightarrow{f''_{p+1}} \mathcal{O}(E''_p) \xrightarrow{f''_p} \cdots \xrightarrow{f''_2} \mathcal{O}(E''_1) \xrightarrow{f''_1} \mathcal{O}(E''_0)$$

is a resolution of 0. In particular,

$$\to E_{p+1} \xrightarrow{f''_{p+1}} E''_p \xrightarrow{f''_p} \cdots \xrightarrow{f''_2} E''_1 \xrightarrow{f''_1} E''_0 \to 0$$

is a pointwise exact sequence of vector bundles, and therefore the set Z_k where rank f_k is not optimal coincides with the set Z'_k where rank f'_k is not optimal. In particular, $Z_k = \emptyset$ for $k > p$. If we choose, to begin with, Hermitian metrics on E_k that respect this direct sum, and let σ_k, σ'_k, and σ''_k be the corresponding minimal inverses, then $\sigma_k = \sigma'_k \oplus \sigma''_k$ and hence

$$u_k^0 = (\bar\partial\sigma'_k \oplus \bar\partial\sigma''_k)(\bar\partial\sigma'_{k-1} \oplus \bar\partial\sigma''_{k-1}) \cdots (\bar\partial\sigma'_2 \oplus \bar\partial\sigma''_2)(\sigma'_1 \oplus \sigma''_1) = (u')^0_k \oplus (u'')^0_k$$

for all k. However, $(u'')^0_k$ is smooth, and hence

$$R_p = R'_p \oplus 0, \quad R_k = 0 \text{ for } k \neq p.$$

For this particular choice of metric thus (5.4) holds with h_p as the natural injection $i_p\colon E'_p \to E_p$ and β_0 as the natural projection.

Without any risk of confusion we can therefore from now on let R'_p denote the residue current with respect to this particular metric on E, and moreover let σ' denote the minimal inverse of f with respect to this metric etc. We now choose other metrics on E_k and let R_k from now on denote the residue current associated with this new metric. Following the notation in the proof of Proposition 5.1 we again have (5.3), and for degree reasons still $M_p^0 = 0$; here M_k^ℓ denotes the component of M that takes values in $\text{Hom}\,(E_\ell, E_k)$. Thus

$$R_p - R'_p = f_{p+1} M^0_{p+1}.$$

Moreover, if we expand uu', we get

$$M^0_{p+1} = \bar\partial|F|^{2\lambda} \wedge \big[\sigma_{p+1}\sigma'_p(\bar\partial\sigma'_{p-1}) \cdots (\bar\partial\sigma'_1) + \sigma_{p+1}(\bar\partial\sigma_p)\sigma'_{p-1}(\bar\partial\sigma'_{p-2}) \cdots (\bar\partial\sigma'_1) + \cdots\big]|_{\lambda=0}.$$

However, $\sigma_{p+1}(\bar\partial\sigma_p) = (\bar\partial\sigma_{p+1})\sigma_p$ and σ_{p+1} is smooth since Z_{p+1} is empty, so

$$M^0_{p+1} = -\sigma_{p+1}R'_p + (\bar\partial\sigma_{p+1})M^0_p = -\sigma_{p+1}R'_p.$$

Thus,

$$R_p = R'_p - f_{p+1}\sigma_{p+1}R'_p = (I_{E_p} - f_{p+1}\sigma_{p+1})R'_p.$$

Since f_{p+1} has constant rank, $H = \text{Im}\, f_{p+1}$ is a smooth subbundle of E_p. Notice that $\Pi = I_{E_p} - f_{p+1}\sigma_{p+1}$ is the orthogonal projection of E_p onto the orthogonal complement of H with respect to the new metric. In this case therefore h in (5.4) becomes the natural injection $i_p\colon E'_p \to E_p$ composed by Π, and since $E'_p \cap H = 0$, h is pointwise injective.

Since Z_k is empty for $k > p$, σ_k is smooth for $k > p$ and hence for $\ell > p$,

$$R_\ell = \bar\partial|F|^{2\lambda} \wedge (\bar\partial\sigma_\ell) \cdots (\bar\partial\sigma_{p+1})u^0_p = (\bar\partial\sigma_\ell) \cdots (\bar\partial\sigma_{p+1})\bar\partial|F|^{2\lambda} \wedge u^0_p = \alpha_\ell R_p$$

where $\alpha_\ell = (\bar\partial\sigma_\ell) \cdots (\bar\partial\sigma_{p+1})$. □

6. The notion of structure form ω on X

To begin with, let $i\colon X \hookrightarrow \Omega$ be a (reduced) hypersurface in a pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$, i.e., $X = \{f = 0\}$ where f is holomorphic in Ω and $df \neq 0$ on X_{reg}. If ω' is a meromorphic $(n, 0)$-form in Ω (or in a small neighborhood of X in Ω) such that

$$(df/2\pi i) \wedge \omega' = d\zeta_1 \wedge \ldots \wedge d\zeta_{n+1} \tag{6.1}$$

on X, then $\omega := i^*\omega'$ is a meromorphic form on X that is independent of the choice of ω', and the classical Leray residue formula states that for test forms ψ of bidegree $(0, n-1)$, the principle value integral

$$\int_X \omega \wedge i^*\psi$$

is equal to the action of the residue current $\bar\partial(1/f)$ on the test form $\psi d\zeta_1\wedge\ldots\wedge d\zeta_{n+1}$. This equality can be rephrased as

$$(6.2)\qquad i_*\omega = \bar\partial\frac{1}{f}\wedge d\zeta_1\wedge\ldots\wedge d\zeta_{n+1}.$$

If $\partial f/\partial\zeta_{n+1}$ is not vanishing identically on (any irreducible component of) X, one can take, e.g., $\omega' = 1/(\partial f/\partial\zeta_{n+1})d\zeta_1\wedge\ldots\wedge d\zeta_n$. Notice that under this assumption on f, any meromorphic form on X can be written $hd\zeta_1\wedge\ldots\wedge d\zeta_n$ for a unique meromorphic function h. It follows from (6.2) that $\bar\partial\omega = 0$ so ω is in $\mathcal{B}_n(X)$, cf., Example 10.8. The form ω also has the following two properties:

(i) If ϕ is a meromorphic function on X, then ϕ is in $\mathcal{O}^X$ if (and only if) $\bar\partial(\phi\omega) = 0$.

(ii) If α is in $\mathcal{B}_n^X$ then $\alpha = h\omega$ for some h in $\mathcal{O}^X$.

Since any meromorphic $(n,0)$-form α is $h\omega$ for a unique meromorphic function h, (i) and (ii) are in fact equivalent; for a proof of (i), see, e.g., [23, Remark 3] or below.

For the rest of this section let $i\colon X\hookrightarrow\Omega\subset\mathbb{C}^N$ be a pure n-dimensional subvariety of the pseudoconvex domain Ω, and let $p := N-n$ be its codimension. We will introduce an almost semimeromorphic form ω on X, that satisfies an analogue of (6.2). In a reasonable sense it will also fulfill (i) and (ii). It can be noted that ω plays a central role in [11]. To begin with we look for an adequate generalization of the residue current $\bar\partial(1/f)$.

If f is any holomorphic function, then a holomorphic function ϕ is in the ideal (f) generated by f if and only if $\phi\bar\partial(1/f) = 0$. Given any ideal sheaf $\mathcal{J}$ in Ω, in [12] was constructed a residue current R such that

$$(6.3)\qquad \phi\in\mathcal{J} \text{ if and only if } \phi R = 0$$

if $\phi\in\mathcal{O}^\Omega$. It is thus reasonable to consider R when $\mathcal{J} = \mathcal{J}_X$ is the radical ideal sheaf $\mathcal{J} = \mathcal{J}_X$ associated with X, so let us first recall its definition. In a slightly smaller set, still denoted Ω, there is a free resolution

$$(6.4)\qquad 0\to\mathcal{O}(E_m)\xrightarrow{f_m}\cdots\xrightarrow{f_3}\mathcal{O}(E_2)\xrightarrow{f_2}\mathcal{O}(E_1)\xrightarrow{f_1}\mathcal{O}(E_0)$$

of the sheaf $\mathcal{O}^\Omega/\mathcal{J}_X$; here E_k are trivial vector bundles over Ω and $E_0\simeq\mathbb{C}\times\Omega$ is the trivial line bundle. This resolution induces a complex of vector bundles

$$(6.5)\qquad 0\to E_m\xrightarrow{f_m}\cdots\xrightarrow{f_3}E_2\xrightarrow{f_2}E_1\xrightarrow{f_1}E_0$$

that is pointwise exact outside X. Let X_k be the set where f_k does not have optimal rank. Then

$$\cdots\subset X_{k+1}\subset X_k\subset\cdots\subset X_{p+1}\subset X_{sing}\subset X_p=\cdots=X_1=X;$$

these sets are independent of the choice of resolution and thus invariants of the sheaf $\mathcal{F} := \mathcal{O}^\Omega/\mathcal{J}_X$. Since $\mathcal{F}$ has *pure* codimension p (i.e., no embedded prime ideals),

$$(6.6)\qquad \operatorname{codim} X_k\ge k+1,\quad \text{for}\quad k\ge p+1,$$

see Corollary 20.14 in [19]. There is a free resolution (6.4) if and only if $X_k=\emptyset$ for $k>m$. Thus we can always have $m\le N-1$. The variety is Cohen-Macaulay, i.e., the sheaf $\mathcal{F}$ is Cohen-Macaulay, if and only if $X_k=\emptyset$ for $k\ge p+1$. In this case we can thus choose a resolution (6.4) with $m=p$. If we define

$$(6.7)\qquad X^0 = X_{sing},\quad X^r = X_{p+r},\ r\ge 1,$$

then

$$X^{n-1} \subset \cdots \subset X^1 \subset X^0 \subset X, \qquad \operatorname{codim} X^k \geq k+1.$$

The sets X^k are independent of the choice of embedding, see [14, Lemma 4.2], and are thus intrinsic subvarieties of the complex space X and reflect the complexity of the singularities of X.

Let us now choose Hermitian metrics on the bundles E_k. We then refer to (6.4) as a *Hermitian free resolution* of $\mathcal{O}^X/\mathcal{J}_X$ in Ω. In $\Omega \setminus X_k$ we have a well-defined vector bundle morphism $\sigma_{k+1}\colon E_k \to E_{k+1}$, if we require that σ_{k+1} vanishes on $(\operatorname{Im} f_{k+1})^\perp$, takes values in $(\operatorname{Ker} f_{k+1})^\perp$ and that $f_{k+1}\sigma_{k+1}$ is the identity on $\operatorname{Im} f_{k+1}$. Following [12] we define the smooth E_k-valued forms

$$u_k = (\bar\partial\sigma_k)\cdots(\bar\partial\sigma_2)\sigma_1 = \sigma_k(\bar\partial\sigma_{k-1})\cdots(\bar\partial\sigma_1) \tag{6.8}$$

in $\Omega \setminus X$; for the second equality, see [12, (2.3)]. We have that

$$f_1 u_1 = 1, \quad f_{k+1}u_{k+1} - \bar\partial u_k = 0, \quad k \geq 1,$$

in $\Omega\setminus X$. If $f := \oplus f_k$ and $u := \sum u_k$, then these relations can be written economically as $\nabla_f u = 1$ where $\nabla_f := f - \bar\partial$. To make the algebraic machinery work properly one has to introduce a superstructure on the bundle $E =: \oplus E_k$ so that vectors in E_{2k} are even and vectors in E_{2k+1} are odd, and hence f, $\sigma := \oplus\sigma_k$, and $u := \sum u_k$ are odd. For details, see [12]. It turns out that u has a (necessarily unique) almost semi-meromorphic extension U to Ω, and the current R is defined by the relation

$$\nabla_f U = 1 - R.$$

If F is any holomorphic tuple that vanishes on X, then

$$U = |F|^{2\lambda}u|_{\lambda=0}, \quad R = \bar\partial|F|^{2\lambda}\wedge u|_{\lambda=0}. \tag{6.9}$$

Thus R has support on X and is a sum $\sum R_k$, where R_k is a pseudomeromorphic E_k-valued current of bidegree $(0,k)$. It follows from the dimension principle that $R = R_p + R_{p+1} + \cdots + R_N$. Since we can always choose a resolution that ends at level $N-1$, cf., (6.6), we may assume that $R_N = 0$. If X is Cohen-Macaulay and $m = p$ in (6.4), then $R = R_p$ is $\bar\partial$-closed; in general, R is ∇_f-closed.

Remark 6.1. If $\mathcal{J}$ is an arbitrary ideal sheaf and R is defined in the same way as above, then (6.3) holds, [12]. In case $\mathcal{J}$ is Cohen-Macaulay, one can express this duality in a way that only involves the smooth form u in $\Omega \setminus X$, where X is the zero set of $\mathcal{J}$, see [12, Theorem 4.2]. This result was recently proved algebraically in [28] with no reference to residue calculus and resolution of singularities. □

Remark 6.2. In case $\mathcal{J}$ is generated by the single function f, then we have the free resolution $0 \to \mathcal{O} \stackrel{f}{\to} \mathcal{O} \to \mathcal{O}/(f) \to 0$; thus U is just the principal value current $1/f$ and $R = \bar\partial(1/f)$. □

Notice that (6.4) gives rise to the dual Hermitian complex

$$0 \to \mathcal{O}(E_0^*) \stackrel{f_1^*}{\to} \cdots \to \mathcal{O}(E_{p-1}^*) \stackrel{f_p^*}{\to} \mathcal{O}(E_p^*) \stackrel{f_{p+1}^*}{\longrightarrow} \cdots . \tag{6.10}$$

Since the sheaf $\mathcal{K}er\,(\mathcal{O}(E_p^*) \stackrel{f_{p+1}^*}{\to} \mathcal{O}(E_{p+1}^*))$ is coherent, there is a (trivial) Hermitian vector bundle F in Ω and a holomorphic morphism $g\colon E_p \to F$ such that

$$\mathcal{O}(F^*) \stackrel{g^*}{\to} \mathcal{O}(E_p^*) \stackrel{f_{p+1}^*}{\to} \mathcal{O}(E_{p+1}^*) \tag{6.11}$$

is exact. Since f_{p+1} has constant rank outside X_{p+1}, also f^*_{p+1} has, and it follows that g has as well. Outside X_{p+1} we can thus define the mapping $\sigma_F\colon F\to E_p$ such that $\sigma_F=0$ on $(\operatorname{Im} g)^\perp\subset F$, $\sigma_F g=Id$ on $(\operatorname{Ker} g)^\perp=(\operatorname{Im} f_{p+1})^\perp$ and $\operatorname{Im}\sigma_F$ is orthogonal to $\operatorname{Ker} g$. If $m=p$, then we can take $F=E_p$ and $g=Id$.

Let $d\zeta:=d\zeta_1\wedge\ldots\wedge d\zeta_N$. We also introduce the notation

$$E^r:=E_{p+r}|_X,\quad f^r:=f_{p+r}|_X$$

so that f^r becomes a holomorphic section of $\operatorname{Hom}(E^r,E^{r-1})$. Notice that for $k\geq 1$, $\alpha^k:=i^*\bar\partial\sigma_{p+k}$ are smooth in $X\setminus X^k$.

Proposition 6.3. *Let* (6.4) *be a Hermitian free resolution of* $\mathcal{O}^\Omega/\mathcal{J}_X$ *in* Ω *and let* R *be the associated residue current. Then there is a unique almost semi-meromorphic current*

$$\omega=\omega_0+\omega_1+\cdots+\omega_{n-1}$$

on X*, where* ω_r *has bidegree* (n,r) *and takes values in* E^r*, such that*

$$i_*\omega=R\wedge d\zeta. \tag{6.12}$$

Moreover,

$$f^0\omega_0=0,\quad f^r\omega_r=\bar\partial\omega_{r-1},\ r\geq 1,\quad \text{on } X, \tag{6.13}$$

and

$$|\omega|=\mathcal{O}(\delta^{-M}) \tag{6.14}$$

for some $M>0$*, where* δ *is the distance to* X_{sing}*.*

Assume that (6.11) *is exact. The forms* α^k*,* $1\leq k\leq n-1$*, defined and smooth outside* X^k*, and* σ_F*, defined and smooth outside* X^1*, extend to almost semimermorphic currents on* X*. There is an* F*-valued section* ϑ *of* $\mathcal{B}^X_n$ *such that*

$$\omega_0=\sigma_F\vartheta. \tag{6.15}$$

Moreover,

$$\omega_r=\alpha_r\omega_{r-1},\quad r\geq 1,\quad \text{on } X. \tag{6.16}$$

We say that ω so obtained is a *structure form* on X. The products in (6.15) and (6.16) are well-defined by Proposition **??**. Notice that if X is Cohen-Macaulay and $m=p$, then ω_0 is an E^0-valued section of $\mathcal{B}^X_n$.

Proof. Let x be an arbitrary point on X_{reg}. Since the ideal sheaf $\mathcal{J}_X$ is generated by the functions f_1^j that constitute the map f_1, cf. (6.4), we can extract holomorphic functions $a_1\ldots,a_p$ from the f_1^j's such that $da_1\wedge\cdots\wedge da_p\neq 0$ at x. Possibly after a re-ordering of the variables ζ in the ambient space, we may assume that $\zeta=(\zeta',\zeta'')=(\zeta',\zeta''_1,\ldots,\zeta''_p)$ and that $A:=\det(\partial a/\partial\zeta'')\neq 0$ at x. We also note that $d\zeta'\wedge da_1\wedge\cdots\wedge da_p=Ad\zeta'\wedge d\zeta''=Ad\zeta$ close to x.

Now, $\mathcal{J}_X$ is generated by $a=(a_1,\ldots,a_p)$ at x and so the Koszul complex generated by the a_j provides a minimal resolution of $\mathcal{O}^\Omega/\mathcal{J}_X$ there. The associated residue current $R^a=R^a_p$ is just the Coleff-Herrera product formed from the tuple a, cf., Section 12. The original resolution (6.4) contains the Koszul complex as a direct summand in a neighborhood of x and so it follows from Theorem 4.4 in [12] that

$$R_p=\alpha\,\bar\partial\frac{1}{a_p}\wedge\cdots\wedge\bar\partial\frac{1}{a_1}, \tag{6.17}$$

where α is a smooth section of E_p close to x. By the Poincaré-Lelong formula thus

$$R_p \wedge d\zeta = \pm\alpha\,\bar\partial\frac{1}{a_p}\wedge\cdots\wedge\bar\partial\frac{1}{a_1}\wedge da_1\wedge\cdots\wedge da_p\wedge\frac{d\zeta'}{A} = \pm(2\pi i)^p\alpha\frac{d\zeta'}{A}\wedge[X] \tag{6.18}$$

close to x. If ω_0 is the pullback of $\pm(2\pi i)^p\alpha\, d\zeta'/A$ to X_{reg}, then the preceding equation means that

$$R_p\wedge d\zeta.\psi = \int_X \omega_0\wedge i^*\psi, \tag{6.19}$$

where ψ is a test form with support close to x. Thus ω_0 is determined by R_p and so it extends to a global E_p-valued $(n,0)$-form on X_{reg}, still denoted ω_0. Since $\sigma_{p+1}u_p = 0$ outside X_{p+1}, cf., (6.8), we find that R_p and hence ω_0 takes values in $(\operatorname{Im} f_{p+1})^\perp \subset E_p$, cf., (6.9) and (6.19). Thus $\omega_0 = \sigma_F g\omega_0 = \sigma_F\vartheta$ where $\vartheta := g\omega_0$. On X_{reg} we have

$$i_*\bar\partial\vartheta = -i_*g\bar\partial\omega_0 = -g\bar\partial i_*\omega_0 = -g\bar\partial R_p\wedge d\zeta = -gf_{p+1}R_{p+1}\wedge d\zeta = 0$$

since $gf_{p+1}=0$. Thus $\bar\partial\vartheta = 0$ and from Example 10.8 we conclude that ϑ is a section of $\mathcal{B}_n^X$.

Let $\mathfrak{a}_F$ be the Fitting ideal of g, restricted to X, i.e., the ideal (on X) generated by the $r\times r$-minors of g, where r is the generic rank of g; notice that g has rank r on $X\setminus X^1$. Let $\mathfrak{a}_k$ be the Fitting ideals of f^k, $k=1,\ldots,n-1$. By Hironaka's theorem there is a smooth modification $\tau\colon \tilde X\to X$ such that all the ideals $\tau^*\mathfrak{a}_F, \tau^*\mathfrak{a}_1,\ldots,\tau^*\mathfrak{a}_{n-1}$ are principal on $\tilde X$. This means that there are holomorphic sections $s_F, s_1,\ldots,s_{n-1}$ of line bundles on $\tilde X$ that generate these ideals. It follows from [12, Lemma 2.1] that $\tau^*\sigma_F = \beta_F/s_F$ and $\tau^*\sigma^k = \beta_k/s_k$, $k\ge 1$, where β_F and β_k are smooth. Hence, $\tau^*\alpha^k = \bar\partial\beta_k/s_k$. We conclude that σ_F as well as α^k are almost semi-meromorphic on X.

Let us now define ω_r inductively by (6.16). We claim that

$$i_*\omega_k = R_{p+k}\wedge d\zeta, \quad k\ge 0. \tag{6.20}$$

If $k=0$ it is just (6.19). Assume (6.20) is proved for $k-1$. It follows from (6.8) and (6.9) that $R_{p+k} = \alpha_{p+k}R_{p+k-1}$ in $\Omega\setminus X_{p+k+1}$. In this set we thus have that

$$i_*\omega_k = i_*\alpha^k\omega_{k-1} = \alpha_{p+k}i_*\omega_{k-1} = \alpha_{p+k}R_{k-1}\wedge d\zeta.$$

Let $\chi_\delta = \chi(|h|/\delta)$, where h is a holomorphic tuple that cuts out X_{p+k}, cf., (6.1). Then $i_*(i^*\chi_\delta\omega_k) = \chi_\delta R_k\wedge d\zeta$. Now $\chi_\delta R_{p+k}\to R_{p+k}$ in view of (6.6) and the dimension principle, and $i^*\chi_\delta\omega_k\to\omega_k$, and hence (6.20) holds in Ω.

The estimate (6.14) follows since it holds for Θ, being a tuple of meromorphic forms on X that are holomorphic on X_{reg}, and for each of $i^*\sigma_F, \alpha^1,\ldots,\alpha^{n-1}$. Finally, (6.13) follows since $(f-\bar\partial)R = \nabla_f R = 0$. □

Let Θ be an F-valued meromorphic form in Ω such that $i^*\Theta = \vartheta$. Notice that

$$\Theta = \gamma_\Theta \lrcorner d\zeta_1\wedge\ldots\wedge d\zeta_N$$

for a (unique) meromorphic section of $F\otimes\Lambda^p T^{1,0}(\Omega)$. If $\gamma := \sigma_F\gamma_\Theta + \alpha^1\sigma_F\gamma_\Theta + \alpha^2\alpha^1\sigma_F\gamma_\Theta + \cdots$ and $\omega' := \gamma\lrcorner d\zeta$, thus $\omega = i^*\omega'$. Since $[X]\wedge\gamma\lrcorner d\zeta = [X]\wedge\omega' = i_*\omega = R\wedge d\zeta$, and $[X]\wedge\gamma\lrcorner d\zeta = (-1)^p\gamma\lrcorner[X]\wedge d\zeta$ we have

$$R = (-1)^p\gamma\lrcorner[X], \quad i_*\omega = [X]\wedge\omega' =: [X]\wedge\omega. \tag{6.21}$$

—

We will now discuss generalizations of (i) and (ii) above. It is proved in [7] that if Φ is meromorphic in Ω, then $\phi := i^*\Phi$ is in $\mathcal{O}^X$ if and only if $\nabla_f(\Phi R) = 0$ in Ω. Combining with Proposition 6.3 we get:

$(i)'$ If ϕ is a meromorphic function on X, then ϕ is in $\mathcal{O}^X$ if and only if $\nabla_f(\phi\omega) = 0$ on X.

Let Ω^k denote the sheaf $\mathcal{O}(\Lambda^k T^*_{1,0}(\Omega))$. Let $\xi d\zeta$ be a section of the sheaf

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(E_p), \Omega^N) \simeq \mathcal{O}(E_p^*) \otimes_{\mathcal{O}} \Omega^N$$

such that $f^*_{p+1}\xi = 0$. Then $\bar\partial(\xi \cdot \omega_0) = -\xi \cdot \bar\partial\omega_0 = -\xi \cdot f_{p+1}\omega_1 = f^*_{p+1}\xi \cdot \omega_1 = 0$, so that $\xi \cdot \omega_0$ is in $\mathcal{B}^X_n$. The minus signs appear since f is an odd mapping with respect to the superstrucure. Moreover, if $\xi = f^*_p\eta$ for $\eta \in \mathcal{O}(E^*_{p-1})$, then $\xi \cdot \omega_0 = f^*_p\eta \cdot \omega_0 = -\eta \cdot f_p\omega_0 = 0$. We thus have a sheaf mapping

$$\mathcal{H}^p(\mathcal{H}om\,(\mathcal{O}(E_\bullet), \Omega^N)) \to \mathcal{B}^X_n, \quad \xi d\zeta \mapsto \xi \cdot \omega_0. \tag{6.22}$$

Proposition 6.4. *The mapping* (6.22) *is an isomorphism, and it is independent of the specific choice of resolution, hence establishing an isomorphism*

$$\mathcal{E}xt^{\,p}(\mathcal{O}^\Omega/\mathcal{J}_X, \Omega^N) \simeq \mathcal{B}^X_n.$$

This isomorphism is well-known, cf., [23, Remark 5]. Our contribution is the realization (6.22). Thus $\mathcal{B}^X_n$ is coherent and we have:

$(ii)'$ If $\xi_1, \dots, \xi_\nu$ are generators of $\mathcal{H}^p(\mathcal{H}om\,(\mathcal{O}(E^*_\bullet)))$, then $\eta_\ell := \xi_\ell \cdot \omega_0$, $\ell = 1, \dots, \nu$, generate the $\mathcal{O}^X$-module $\mathcal{B}^X_n$.

Proof of Proposition 6.4. If $h \in \mathcal{B}^X_n$, then $i_*h = h \wedge [X]$ is a so-called Coleff-Herrera current with respect to X (taking values in the holomorphic vector bundle $\Lambda^N T^*_{1,0}(\Omega)$) that is annihilated by $\mathcal{J}_X$, cf., [8]. Thus we have mappings

$$\mathcal{H}^p(\mathcal{H}om\,(\mathcal{O}(E_\bullet), \Omega^N)) \to \mathcal{B}^X_n \to \mathcal{H}om\,(\mathcal{O}^\Omega/\mathcal{J}_X, \mathcal{CH}_X) \otimes_{\mathcal{O}} \Omega^N, \tag{6.23}$$

defined by $\xi d\zeta \mapsto \xi \cdot \omega_0$ and $h \mapsto i_*h$. The latter mapping is certainly injective. The composed mapping is an isomorphism according to [8, Theorem 1.5]. It follows that both mappings are isomorphisms. From [8, Theorem 1.5] we also know that the composed mapping is independent of the particular Hermitian resolution, and choice of $d\zeta$, and thus induces an isomorphism $\mathcal{E}xt^{\,p}(\mathcal{O}^\Omega/\mathcal{J}_X, \Omega^N) \simeq \mathcal{H}om\,(\mathcal{O}^\Omega/\mathcal{J}_X, \mathcal{CH}_X) \otimes_{\mathcal{O}} \Omega^N$. Hence the proposition follows. □

We conclude with a lemma that roughly speaking says that one can "divide" by ω.

Lemma 6.5. *If ϕ is a smooth (n,q)-form on X, then there is a smooth $(0,q)$-form ϕ' on X with values in $(E^0)^*$ such that $\phi = \omega_0 \wedge \phi'$.*

Proof. Let Φ be a smooth extension of ϕ to Ω. Since $[X]$ is a Coleff-Herrera current (with values in $\Lambda^p T^*_{1,0}(\Omega)$), it follows from [8, Theorem 1.5 and Example 1] that locally there is a holomorphic E^*_p-valued $(p,0)$-form a such that $R_p \wedge a = [X]$.

By a partition of unity we can find a global smooth $\tilde a$ such that $R_p \wedge \tilde a = [X]$ in Ω. Since $\tilde a \wedge \Phi$ has bidegree (N,q), there is an E^*_p-valued smooth $(0,q)$-form Φ' in Ω

such that $\tilde{a}\wedge\Phi = d\zeta\wedge\Phi'$. For every test form Ψ in Ω we now get

$$\int_X \phi\wedge i^*\Psi \;=\; [X].(\Phi\wedge\Psi) = R_p\wedge\tilde{a}.\,(\Phi\wedge\Psi) = R_p\wedge d\zeta.\,(\Phi'\wedge\Psi)$$
$$= \int_X \omega_0\wedge\phi'\wedge i^*\Psi,$$

where $\phi' = i^*\Phi'$. Hence, $\phi = \omega_0\wedge\phi'$ on X. □

An algebraic counterpart of the factorization $R_p\wedge a = [X]$ appeared in [27] in case X is Cohen-Macaulay; then one can take $a = df_1 df_2\cdots df_p$.

7. The strong $\bar\partial$-operator on X

Let ω be a structure form on X, and let $\chi_\delta := \chi(|h|/\delta)$, where χ is a smooth approximand of the characteristic function of $[1,\infty)$, and h is a holomorphic tuple such that $X_{sing} = \{h = 0\}$. Notice that if $\alpha\in\mathcal{W}(X)$, then

$$\mathbf{1}_{X_{sing}}\nabla_f\alpha = 0 \iff \mathbf{1}_{X_{sing}}\bar\partial\alpha = 0 \iff \bar\partial\chi_\delta\wedge\alpha\to 0,\ \delta\to 0. \tag{7.1}$$

In fact, since $\mathbf{1}_{X_{sing}}\alpha = 0$ and f is smooth we have that $\mathbf{1}_{X_{sing}}f\alpha = 0$; hence the first equivalence follows. For the second one, consider the equality

$$\bar\partial(\chi_\delta\alpha) = \chi_\delta\bar\partial\alpha + \bar\partial\chi_\delta\wedge\alpha.$$

Since $\chi_\delta\alpha\to\alpha$ it follows that $\mathbf{1}_{X_{sing}}\bar\partial\alpha = \lim(1-\chi_\delta)\bar\partial\alpha = 0$ if and only if $\bar\partial\chi_\delta\wedge\alpha\to 0$.

Lemma 7.1. *Assume that $\mu\in\mathcal{W}(X)$.*

(i) *If there is $\tau\in\mathcal{W}(X)$ such that*

$$-\nabla_f(\mu\wedge\omega) = \tau\wedge\omega, \tag{7.2}$$

then $\bar\partial\mu = \tau$ and

$$\bar\partial\chi_\delta\wedge\mu\wedge\omega\to 0,\ \delta\to 0. \tag{7.3}$$

(ii) *If $\bar\partial\mu\in\mathcal{W}(X_{reg})$ and (7.3) holds, then there is $\tau\in\mathcal{W}(X)$ such that (7.2) holds.*

From Proposition **??** we know that $\mu\wedge\omega$ is a well-defined current in $\mathcal{W}(X)$.

Proof. Assume that (7.2) holds. Then $-\mathbf{1}_{X_{sing}}\nabla_f(\mu\wedge\omega) = \mathbf{1}_{X_{sing}}\tau\wedge\omega = 0$, since $\tau\wedge\omega$ is in $\mathcal{W}(X)$. Thus (7.3) holds, in view of (7.1). Since ω is smooth on X_{reg} and $\nabla_f\omega = 0$, (7.2) implies that $\bar\partial\mu\wedge\omega_0 = \tau\wedge\omega_0$ on X_{reg}. It follows from Lemma 6.5 that $\bar\partial\mu = \tau$ on X_{reg}. Moreover, from (7.3) and Lemma 6.5 we find that $\bar\partial\chi_\delta\wedge\mu\to 0$, so that, cf., (7.1), $\mathbf{1}_{X_{sing}}\bar\partial\mu = 0$. It follows that $\bar\partial\mu = \mathbf{1}_{Xreg}\bar\partial\mu = \tau$. Thus (i) is proved.

If $\bar\partial\mu$ has the SEP on X_{reg}, then $\tau := \mathbf{1}_{X_{reg}}\bar\partial\mu$ has the SEP on X and hence is in $\mathcal{W}(X)$. Since ω is smooth on X_{reg}, $-\nabla_f(\mu\wedge\omega) = \tau\wedge\omega$ there. In view of (7.3) and (7.1), $\mathbf{1}_{X_{sing}}\nabla_f(\mu\wedge\omega) = 0$, and since $\nabla_f(\mu\wedge\omega)$ has the SEP on X_{reg} it follows that it has the SEP on X, i.e., is in $\mathcal{W}(X)$. Since ω is smooth on X_{reg}, (7.2) holds on X_{reg}. Since both sides have the SEP, the equality must hold on X. □

Let x be a point in an arbitrary complex space X. By choosing local embeddings $X\hookrightarrow\Omega\subset\mathbb{C}^N$ at x and Hermitian free resolutions of $\mathcal{O}^\Omega/\mathcal{J}_X$ (and choice of coordinates on Ω, cf., (6.12)) we get the collection $\mathfrak{S}_x$ of all structure forms ω at x.

Given $\mu,\tau\in\mathcal{W}^X_{0,*,x}$ we say that $\bar\partial_X\mu = \tau$ at x if (7.2) holds at x for all $\omega\in\mathfrak{S}_x$. It follows from Lemma 7.1 that $\bar\partial_X\mu = \tau$ if and only if $\bar\partial\mu = \tau$ and the "boundary condition" (7.3) holds for every $\omega\in\mathfrak{S}_x$.

Definition 4 (The sheaves $\mathrm{Dom}_q\,\bar\partial_X$)**.** We say that a $(0,q)$-current μ is a section of $\mathrm{Dom}_q\,\bar\partial_X$ in the open set $\mathcal{U}\subset X$ if $\mu\in\mathcal{W}_{0,q}(\mathcal{U})$ and there is $\tau\in\mathcal{W}_{0,q+1}(\mathcal{U})$ such that $\bar\partial_X\mu=\tau$ in $\mathcal{U}$, i.e., $\bar\partial_X\mu=\tau$ at each point $x\in\mathcal{U}$.

If $\mu\in\mathcal{W}(X)$ is smooth on X_{reg}, then it follows from Lemma 7.1 that $\mu\in\mathrm{Dom}_q\,\bar\partial_X$ if and only if (7.3) holds at each $x\in X$ for each $\omega\in\mathfrak{S}_x$. If μ is smooth on X, then $\bar\partial(\mu\wedge\omega)$ has the SEP, and so (7.3) holds for each ω. Thus $\mathcal{E}^X_{0,q}$ is a subsheaf of $\mathrm{Dom}_q\,\bar\partial_X$.

Proposition 7.2. *The sheaves $Dom_*\,\bar\partial_X$ are $\mathcal{E}^X_{0,*}$-modules and*

$$0\to\mathcal{O}^X\hookrightarrow Dom_0\,\bar\partial_X\xrightarrow{\bar\partial}Dom_1\,\bar\partial_X\xrightarrow{\bar\partial}\cdots\tag{7.4}$$

is a complex. Moreover, the kernel of $\bar\partial$ in $Dom_0\,\bar\partial_X$ is $\mathcal{O}^X$.

When $\dim X=1$ the complex (7.4) is exact, i.e., a fine resolution of $\mathcal{O}^X$, see Section 13 below. We do not know whether this is true if $\dim X>1$.

Proof. Assume that μ is in $\mathrm{Dom}\,\bar\partial_X$ and that $\omega\in\mathfrak{S}_x$. In view of Lemma 7.1, μ and $\bar\partial\mu$ are both in $\mathcal{W}^X$ and (7.3) holds. Since ω is smooth on X_{reg} and $\nabla_f\omega=0$, $\nabla_f(\bar\partial\chi_\delta\wedge\mu\wedge\omega)=-\bar\partial\chi_\delta\wedge\bar\partial\mu\wedge\omega$. Therefore (7.3), with μ replaced by $\bar\partial\mu$, holds as well and it follows from Lemma 7.1 that $\bar\partial\mu\in\mathrm{Dom}\,\bar\partial_X$. Moreover, if ξ is smooth it is clear that (7.3) holds with μ replaced by $\xi\wedge\mu$. Since $\bar\partial(\bar\partial\mu)=0$ and $\bar\partial(\xi\wedge\mu)$ is in $\mathcal{W}^X$ we conclude that $\xi\wedge\mu\in\mathrm{Dom}\,\bar\partial_X$.

Now assume $\mu\in\mathcal{W}^X_{0,0}$ and (7.2) holds with $\tau=0$. Then $\bar\partial\mu=0$ by Lemma 7.1 and hence μ is holomorphic on X_{reg}, and has a meromorphic extension to X, cf., Example 10.8. Thus $\mu\in\mathcal{O}^X$ in view of $(i)'$ above. □

If (7.2) holds at x for a given $\omega\in\mathfrak{S}_x$, then in particular, $\bar\partial(\mu\wedge\omega_0)\pm\mu f_{p+1}\omega_1=\tau\wedge\omega_0$. Applying various $\xi\in\mathcal{O}(E_p^*)$ with $f^*_{p+1}\xi=0$ to this equality we conclude, by Proposition 6.4, that

$$\bar\partial(\mu\wedge\theta)=\tau\wedge\theta,\quad \theta\in\mathcal{B}^X_x.\tag{7.5}$$

If X is Cohen-Macaulay, and (7.2) holds for one ω, then it holds for all $\omega\in\mathfrak{S}_x$. In fact we have:

Proposition 7.3. *If X is Cohen-Macaulay, then $\mu\in\mathcal{W}(X)$ is in $Dom\,\bar\partial_X$ and $\bar\partial_X\mu=\tau$ if and only if (locally)* (7.5) *holds.*

Proof. It follows from Proposition 6.3 that if X is Cohen-Macaulay at $x\in X$, and thus $X^1=\emptyset$, any $\omega\in\mathfrak{S}_x$ has the form $a\vartheta$ where ϑ is (a vector-valued) section of $\mathcal{B}^X$ and a is smooth. If $\tau\in\mathcal{W}(X)$ and (7.5) holds, then

$$\bar\partial(\mu\wedge\omega)=\pm\bar\partial(a\mu\wedge\vartheta)=\pm\bar\partial a\wedge\mu\wedge\vartheta\mp a\bar\partial(\mu\wedge\vartheta)=\pm\bar\partial a\wedge\mu\wedge\vartheta\mp a\tau\wedge\vartheta.$$

It follows that $\mathbf{1}_{X_{sing}}\bar\partial(\mu\wedge\omega)=0$ and hence μ is in $\mathrm{Dom}\,\bar\partial_X$. □

Notice that $\mathcal{W}^X_{0,n}=\mathrm{Dom}_n\,\bar\partial_X$. Assume now that

$$\mathrm{codim}\,X^r\ge r+\ell,\quad r\ge 0.\tag{7.6}$$

We claim that if $q\le\ell-2$, $\mu\in\mathcal{W}^X_{0,q}$ and $\bar\partial\mu\in\mathcal{W}^X_{0,q+1}$, then $\mu\in\mathrm{Dom}_q\,\bar\partial_X$. To see this, we have to verify that $\mathbf{1}_{X_{sing}}\bar\partial(\mu\wedge\omega_k)=0$ for each $k\ge 0$. For $k=0$ it follows directly by the dimension principle since $\mathbf{1}_{X_{sing}}\bar\partial(\mu\wedge\omega_0)$ has bidegree (at most) $(n,\ell-1)$ and support on X^0 that has codimension ℓ. Now, $\omega_1=\alpha^1\omega_0$ and

α^1 is smooth outside X^1, so $\mathbf{1}_{X_{sing}}\bar\partial(\mu\wedge\omega_1) = \pm\alpha^1\mathbf{1}_{X_{sing}}\bar\partial(\mu\wedge\omega_0) = 0$ outside X^1. Thus $\mathbf{1}_{X_{sing}}\bar\partial(\mu\wedge\omega_1)$ has support on X^1 and hence must vanish by (7.6) and the dimension principle. The claim follows in this way by induction. It follows in particular that X is normal if (7.6) holds for $\ell = 2$. One can verify that (7.6) with $\ell = 2$ is a way to formulate Serre's conditions $R1$ and $S2$ for normality.

8. The injectivity of the analytic sheaf $\mathcal{C}$

Here is a proof of Malgrange's theorem by residue calculus. Let $\mathcal{F}$ be any module over the local ring $\mathcal{O}_0$ and let (**??**) be a resolution of $\mathcal{F}$. We have to prove that then the complex

$$(8.1) \qquad 0 \to \mathrm{Hom}\,(\mathcal{O}_0(E_0),\mathcal{C}) \xrightarrow{f_1^*} \mathrm{Hom}\,(\mathcal{O}_0(E_1),\mathcal{C}) \xrightarrow{f_2^*}$$

is exact except at $k = 0$. Fix a natural number N. Given a smooth function ϕ in $X\subset\mathbb{C}^n$, let $\tilde\phi$ be the function

$$\tilde\phi(\zeta,\omega) = \sum_{|\alpha|<N} \partial^\alpha_{\bar\zeta}\phi(\zeta)(\omega-\bar\zeta)^\alpha/\alpha!,$$

in $\tilde X = \{(\zeta,\bar\zeta)\in\mathbb{C}^{2n};\ \zeta\in X\}$. Then

$$\tilde\phi(\zeta,\bar\zeta) = \phi(\zeta), \quad \bar\partial\tilde\phi = O(|\omega-\bar\zeta|^N).$$

Moreover, if f is holomorphic then $\widetilde{f\phi} = f\tilde\phi$. Combining the formulas in [46] with the construction in [43], we get

$$\tilde\phi(z,\bar z) = \int_{\zeta,\omega} (f_{k+1}(z)H^kU^k + H^kR^k + H^kU^{k-1}f_k)\wedge(\tilde\phi + \bar\partial\tilde\phi\wedge v^z)\wedge g,$$

where g is a suitable form in $\mathbb{C}^{2n}$ with compact support and v^z is the Bochner-Martinelli form in $\mathbb{C}^{2n}$ with pole at $(z,\bar z)$, and H^ℓ are holomorphic forms. Since $R^k = 0$ for $k\ge 1$ when (**??**) is a resolution, we have the homotopy formula

$$\phi = f_{k+1}T_{k+1}\phi + T_k(f_k\phi), \quad k\ge 1,$$

where

$$T_k\phi(z) = \int_{\zeta,\omega} H^kU(\tilde\phi + \bar\partial\tilde\phi\wedge v^z)\wedge g^z.$$

Moreover, as in [43] one can verify that $T_k\phi$ is of class C^M if N is large enough. If now μ has order at most M, then we have

$$\mu = T^*_{k+1}f^*_{k+1}\mu + f^*_kT^*_k\mu,$$

so if $f^*_{k+1}\mu = 0$, then $\mu = f^*_k\gamma$ if $\gamma = T^*_k\mu$. Thus (8.1) is exact at k.

9. Koppelman formulas on $X\subset\Omega$ (the embedded context)

We first recall the construction of integral formulas in [5] on an open set $\Omega\subset\mathbb{C}^N$. Let $\eta = (\eta_1,\ldots,\eta_N)$ be a holomorphic tuple in $\Omega_\zeta\times\Omega_z$ that generates the ideal associated with the diagonal $\Delta\subset\Omega_\zeta\times\Omega_z$. For instance one can take $\eta = \zeta - z$. Following the last section in [5] we consider forms in $\Omega_\zeta\times\Omega_z$ with values in the exterior algebra Λ_η spanned by $T^*_{0,1}(\Omega\times\Omega)$ and the $(1,0)$-forms $d\eta_1,\dots,d\eta_n$. On such forms interior multiplication δ_η with

$$\eta = 2\pi i\sum_1^N \eta_j\frac{\partial}{\partial\eta_j}$$

is defined. Let $\nabla_\eta=\delta_\eta-\bar\partial$. A smooth section $g=g_0+\cdots+g_N$ of Λ_η, defined for $z\in\Omega'\Subset\Omega$ and $\zeta\in\Omega$, such that $\nabla_\eta g=0$ and $g_0|_\Delta=1$, lower indices denote degree in $d\eta$, will be called a *weight with respect to* $z\in\Omega'$. Notice that if g and g' are weights, then $g\wedge g'$ is again a weight. We will use one weight that has compact support in Ω and one weight which gives a division-interpolation type formula (for $z\in\Omega'$) for the ideal sheaf $\mathcal{J}_X$ associated with a subvariety $X\hookrightarrow\Omega$. We first discuss weights with compact support.

Example 9.1 (Weights with compact support). If Ω is pseudoconvex and K is a holomorphically convex compact subset, then one can find a weight with respect to z in some neighborhood $\Omega'\Subset\Omega$ of K, depending holomorphically on $z\in\Omega'$, that has compact support in Ω, see, e.g., Example 2 in [6]. Here is an explicit choice when Ω is a neighborhood of the closed unit ball $\overline{\mathbb{B}}$, $K=\overline{\mathbb{B}}$, and $\eta=\zeta-z$: Let $\sigma=\bar\zeta\cdot d\eta/(2\pi i(|\zeta|^2-\bar\zeta\cdot z))$. Then $\delta_\eta\sigma=1$ for $\zeta\neq z$ and

$$\sigma\wedge(\bar\partial\sigma)^{k-1}=\frac{1}{(2\pi i)^k}\frac{(\bar\zeta\cdot d\eta)\wedge(d\bar\zeta\cdot d\eta)^{k-1}}{(|\zeta|^2-\bar\zeta\cdot z)^k}.$$

If $\chi=\chi(\zeta)$ is a cutoff function that is 1 in a slightly larger ball Ω', then

$$g=\chi-\bar\partial\chi\wedge\frac{\sigma}{\nabla_\eta\sigma}=\chi-\bar\partial\chi\wedge\Big(\sum_{k=1}^N\sigma\wedge(\bar\partial\sigma)^{k-1}\Big).$$

is a weight with respect to $z\in\Omega'$ with compact support in Ω. □ □

Let s be a smooth $(1,0)$-form in Λ_η such that $|s|\lesssim|\eta|$ and $|\eta|^2\lesssim|\delta_\eta s|$; such an s is called admissible. Then $B:=s/\nabla_\eta s=\sum_k s\wedge(\bar\partial s)^{k-1}$ satisfies $\nabla_\eta B=1-[\Delta]$, where $[\Delta]$ is the (N,N)-current of integration over Δ. If $\eta=\zeta-z$, then $s=\partial|\eta|^2$ will do and we refer to the resulting B as the Bochner-Martinelli form. If g is any weight, we have $\nabla_\eta(g\wedge B)=g-[\Delta]$, and identifying terms of bidegree $(N,N-1)$ we see that

$$\bar\partial(g\wedge B)_N=[\Delta]-g_N, \tag{9.1}$$

which is equivalent to a weighted Koppelman formula in Ω.

—

We now turn our attention to construction of weights for division-interpolation with respect to the ideal $\mathcal{J}_X$. For the rest of this section we assume that $\Omega\subset\mathbb{C}^N$ is pseudoconvex and that $X\hookrightarrow\Omega$ is a subvariety. Let us fix global holomorphic frames for the bundles E_k in (6.5) over Ω. Then $E_k\simeq\mathbb{C}^{\operatorname{rank}E_k}\times\Omega$, and the morphisms f_k are just matrices of holomorphic functions. One can find, see [6] for explicit choices, $(k-\ell,0)$-form-valued Hefer morphisms, i.e., matrices $H^\ell_k\colon E_k\to E_\ell$, depending holomorphically on z and ζ, such that $H^k_k=I_{E_k}$ and

$$\delta_\eta H^\ell_k=H^\ell_{k-1}f_k-f_{\ell+1}(z)H^{\ell+1}_k,\quad k>\ell,$$

where I_{E_k} is the identity operator on E_k and f stands for $f(\zeta)$. For Re $\lambda \gg 0$ we put $U^\lambda = |F|^{2\lambda}u$, see Section 6 for the notation, and

$$R^\lambda = \sum_{k=0}^N R_k^\lambda = 1 - \nabla_f U^\lambda = 1 - |F|^{2\lambda} + \bar\partial |F|^{2\lambda} \wedge u.$$

Then

$$\begin{aligned} g^\lambda &:= 1 - \nabla_\eta \sum_{k=1}^N H_k^0 U_k^\lambda = \sum_{k=0}^N H_k^0 R_k^\lambda + f_1(z) \sum_{k=1}^N H_k^1 U_k^\lambda \\ &= HR^\lambda + f_1(z) H U^\lambda \end{aligned}$$

is a weight that is as smooth as we want if Re λ is large enough. Let g be any smooth weight with respect to $\Omega' \Subset \Omega$ (but not necessarily holomorphic in z) with compact support in Ω_ζ. Then (9.1) holds with g replaced by $g^\lambda \wedge g$. Since $R(z)$ is $\nabla_{f(z)}$-closed we thus get

$$\begin{aligned} -\nabla_{f(z)}\big(R(z) \wedge dz \wedge (g^\lambda \wedge g \wedge B)_N\big) &= R(z) \wedge dz \wedge [\Delta] - \\ &- R(z) \wedge dz \wedge (g^\lambda \wedge g)_N. \end{aligned}$$

Notice that the products of currents are well-defined; they are just tensor products since z and η are independent variables in $\Omega \times \Omega$. Moreover, since $R(z) f_1(z) = 0$ we have

$$\begin{aligned} -\nabla_{f(z)}\big(R(z) \wedge dz \wedge (HR^\lambda \wedge g \wedge B)_N\big) &= R(z) \wedge dz \wedge [\Delta] - \\ &- R(z) \wedge dz \wedge (HR^\lambda \wedge g)_N. \end{aligned} \tag{9.2}$$

It follows from (6.12) that (recall that $\Delta \subset \Omega \times \Omega$ is the diagonal)

$$R(z) \wedge dz \wedge [\Delta] = \iota_* \omega, \tag{9.3}$$

where $\iota \colon \Delta^X \hookrightarrow \Omega \times \Omega$ is the inclusion of the diagonal $\Delta^X \subset X \times X \subset \Omega \times \Omega$. We notice that the analytic continuation to $\lambda = 0$ of the last term on the right hand side of (9.2) exists and yields the well-defined current $R(z) \wedge dz \wedge (HR \wedge g)_N$ in $\Omega_\zeta \times \Omega'_z$. The existence of the analytic continuation to $\lambda = 0$ of the left hand side of (9.2) follows from Proposition 2.1 in [13] since $R(z) \wedge B$ is pseudomoromorphic in $\Omega \times \Omega$. Our Koppelman formulas will follow by letting $\lambda = 0$ in (9.2).

To begin with, let us consider (9.2) for $\lambda = 0$ in $(\Omega \setminus X_{sing}) \times (\Omega' \setminus X_{sing})$. In this set we have, by (6.12) and (6.21), that

$$\begin{aligned} R(z) \wedge dz \wedge (HR \wedge g)_N &= \pm \omega(z) \wedge [X_z] \wedge \big(H(\gamma(\zeta) \lrcorner [X_\zeta]) \wedge g\big)_N \\ &= \pm \omega(z) \wedge [X_z] \wedge [X_\zeta] \wedge \gamma(\zeta) \lrcorner (H \wedge g)_N = \omega(z) \wedge [X_z \times X_\zeta] \wedge p(\zeta, z), \end{aligned} \tag{9.4}$$

where

$$p(\zeta, z) := \pm\big(\gamma(\zeta) \lrcorner (H \wedge g)_N\big)_{(n)} \tag{9.5}$$

is the term of $\pm \gamma(\zeta) \lrcorner (H \wedge g)_N$ of degree n in $d\zeta$; this is the only term of $\pm \gamma(\zeta) \lrcorner (H \wedge g)_N$ that can contribute in (9.4) since $\omega(z) \wedge [X_z]$ has full degree in the dz_j. Notice that $p(\zeta, z)$ is almost semi-meromorphic on $X \times X'$ ($X' = X \cap \Omega'$) and smooth on $X_{reg} \times \Omega'_z$; if g is holomorphic in z then $z \mapsto p(\zeta, z)$ is holomorphic in Ω'.

Lemma 9.2. *In* $(\Omega_\zeta \setminus X_{sing}) \times (\Omega'_z \setminus X_{sing})$ *we have*

$$R(z) \wedge dz \wedge (HR^\lambda \wedge g \wedge B)_N|_{\lambda=0} = R(z) \wedge dz \wedge (HR \wedge g \wedge |\eta|^{2\lambda} B)_N|_{\lambda=0}. \tag{9.6}$$

Proof. Recall from Section 6 that in $\Omega \setminus X_{sing}$, R is a smooth form times R_p. Notice that

$$T_{jk} := R_p(z) \wedge R_k^\lambda \wedge B_{j-k}|_{\lambda=0} - R_p(z) \wedge R_k \wedge |\eta|^{2\lambda} B_{j-k}|_{\lambda=0}, \ \ j \le N.$$

is a pseudomeromorphic current in $\Omega \times \Omega$ of bidegree $(j-k, p+k+j-k-1) = (j-k, p+j-1)$ that clearly vanishes outside X_z. It also vanishes outside Δ since B is smooth there. Thus T_{jk} has support contained in $\Delta^X \simeq X$, which has codimension $2N - n = p + N$ in $\Omega \times \Omega$. Since $p + N > p + j - 1$ for $j \le N$, it follows from the dimension principle that T_{jk} must vanish; in particular, $R_p(z) \wedge R_k^\lambda \wedge B_{j-k}|_{\lambda=0} = 0$ for $k < p$ since $R_k = 0$ for $k < p$. We conclude that (9.6) holds in $(\Omega \setminus X_{sing}) \times (\Omega' \setminus X_{sing})$ since $T_{jk} = 0$ there. □

Notice that the right hand side of (9.6) only involves B_j with $j \le n$ since all terms in HR have degree at least p in $d\eta$. If $\mathfrak{Re}\, \lambda \gg 0$ we may replace g by $g \wedge |\eta|^{2\lambda} B$ in (9.4) and combining with Lemma 9.2 we get

$$(9.7) \quad R(z) \wedge dz \wedge (HR^\lambda \wedge g \wedge B)_N\big|_{\lambda=0} = R(z) \wedge dz \wedge (HR \wedge g \wedge |\eta|^{2\lambda} B)_N\big|_{\lambda=0}$$

$$= \ \omega(z) \wedge [X \times X] \wedge \big(\gamma(\zeta) \lrcorner (H \wedge g \wedge |\eta|^{2\lambda} B)\big)_N\big|_{\lambda=0}$$

$$= \ \omega(z) \wedge [X \times X] \wedge \big(\gamma(\zeta) \lrcorner \sum_{j=1}^{n} (H \wedge g)_{N-j} \wedge |\eta|^{2\lambda} B_j\big)\big|_{\lambda=0}$$

in $(\Omega \setminus X_{sing}) \times (\Omega' \setminus X_{sing})$. Since $B_j = \mathcal{O}(|\eta|^{-2j+1})$, we see that B_j is locally integrable on $X_{reg} \times X_{reg}$ for $j \le n$. It is thus innocuous to put $\lambda = 0$ in the right hand side of (9.7) as long as we restrict our attention to $X_{reg} \times X'_{reg}$. Notice that the integral kernel

$$(9.8) \qquad k(\zeta, z) := \pm \big(\gamma(\zeta) \lrcorner \sum_{j=1}^{n} (H \wedge g)_{N-j} \wedge B_j\big)_{(n)}$$

is almost semi-meromorphic on $X \times X'$ and locally integrable on $X_{reg} \times X'_{reg}$.

In view of (9.2), (9.3), (9.4), (9.7), and (9.8) we have that

$$(9.9) \qquad -\nabla_{f(z)}\big(\omega(z) \wedge k(\zeta, z)\big) = \omega \wedge [\Delta^X] - \omega(z) \wedge p(\zeta, z)$$

in the current sense on $X_{reg} \times X'_{reg}$. Combined with Lemma 6.5 this gives

Lemma 9.3. *With $k(\zeta, z)$ and $p(\zeta, z)$ defined by* (9.8) *and* (9.5) *respectively, we have*

$$\bar\partial k(\zeta, z) = [\Delta^X] - p(\zeta, z)$$

in the current sense on $X_{reg} \times X'_{reg}$.

We can write our integral kernels $p(\zeta, z)$ and $k(\zeta, z)$ in terms of the structure form ω as follows: Let F be a trivial vector bundle over $\Omega \times \Omega$ with basis elements $\epsilon_1, \ldots, \epsilon_N$. Now replace each occurrence of $d\eta_j$ in H and g by ϵ_j and let $\hat H$ and $\hat g$ be the forms so obtained. Then

$$(H \wedge g)_N = \epsilon_N^* \wedge \cdots \wedge \epsilon_1^* \lrcorner \big(d\eta_1 \wedge \cdots \wedge d\eta_N \wedge (\hat H \wedge \hat g)_N\big),$$

where $\{\epsilon_j^*\}$ is the dual basis and the lower index N on the right hand side means the term with degree N in the ϵ_j. If $C = C(\zeta, z)$ is the invertible holomorphic function defined by $d\eta = C d\zeta + \cdots$, we thus have, cf., (6.21),

$$(9.10) \qquad p(\zeta, z) \pm C \epsilon_N^* \wedge \cdots \wedge \epsilon_1^* \lrcorner (\hat H \wedge \hat g)_N \wedge \omega(\zeta).$$

Similarly, we get that

$$k(\zeta,z)=\pm C\epsilon_N^*\wedge\cdots\wedge\epsilon_1^*\lrcorner\sum_{j=1}^{n}(\hat{H}\wedge\hat{g})_{N-j}\wedge\hat{B}_j\wedge\omega(\zeta). \tag{9.11}$$

10. Koppelman formulas on X (in the intrinsic context)

Let X be a reduced complex space of pure dimension n. Locally, X can be embedded as a subvariety of a pseudoconvex domain in some $\mathbb{C}^N$, so let us, for notational convenience, assume that X can be embedded, $X\hookrightarrow\Omega$, in a pseudoconvex domain $\Omega\subset\mathbb{C}^N$. Then, following the previous section, for any $\Omega'\Subset\Omega$ we can construct integral kernels $k(\zeta,z)$ and $p(\zeta,z)$ which are almost semi-meromorphic on $X\times X'$, where $X'=X\cap\Omega'$, such that (9.9) and Lemma 9.3 hold. Moreover, $k(\zeta,z)$ and $p(\zeta,z)$ are locally integrable on $X_{reg}\times X'_{reg}$ and smooth on $X_{reg}\times\Omega'$ respectively.

Now assume that $\mu(\zeta)\in\mathcal{W}_{0,q}(X)$. Since $k(\zeta,z)$ and $p(\zeta,z)$ are almost semi-meromorphic, the products $k(\zeta,z)\wedge\mu(\zeta)$ and $p(\zeta,z)\wedge\mu(\zeta)$ are well-defined currents in $\mathcal{W}(X_\zeta\times X'_z)$ in view of Proposition **??**. Let $\pi\colon X_\zeta\times X_z\to X_z$ be the projection and put $\mathcal{K}\mu(z)=\pi_*(k(\zeta,z)\wedge\mu(\zeta))$ and $\mathrm{P}\mu(z)=\pi_*(p(\zeta,z)\wedge\mu(\zeta))$. Since k and p have compact support in $\zeta\in\Omega$, $\mathcal{K}\mu$ and $\mathrm{P}\mu$ are well-defined currents in $\mathcal{W}(X'_z)$, and in fact, $\mathrm{P}\mu(z)$ is a smooth function in Ω' since $p(\zeta,z)$ is smooth in $z\in\Omega'$; if we choose the weight g to be holomorphic in z, then $\mathrm{P}\mu(z)$ is holomorphic in Ω'. It is of course natural to write

$$\mathcal{K}\mu(z)=\int_{X_\zeta}k(\zeta,z)\wedge\mu(\zeta),\quad \mathrm{P}\mu(z)=\int_{X_\zeta}p(\zeta,z)\wedge\mu(\zeta). \tag{10.1}$$

Lemma 10.1. *Let $\mu\in\mathcal{W}_{0,q}(X)$.*

(i) *If μ is smooth in a neighborhood of a given point $x\in X'_{reg}$, then $\mathcal{K}\mu(z)$ is smooth in a neighborhood of x.*

(ii) *If μ vanishes in a neighborhood of $x\in X'$, then $\mathcal{K}\mu(z)$ is smooth close to x.*

Proof. Since $k(\zeta,z)$ is smooth in z close to x if ζ avoids a neighborhood of x, (ii) follows. To see (i) it is enough to assume that μ is smooth and has compact support close to $x\in X_{reg}$. Close to the point (x,x) $X\times X$ is a smooth manifold, $\omega(\zeta)$ is smooth, and $\hat{B}_j\sim|\zeta-z|^{-2j+1}$. Thus, (i) follows from the following lemma, cf., the definition (9.8) and (9.11) of $k(\zeta,z)$. □

Lemma 10.2. *Let Φ be a non-negative function on $\mathbb{R}^d_x\times\mathbb{R}^d_y$ such that Φ^2 is smooth and $\Phi\sim|x-y|$. For each integer $m\geq0$, let φ_m denote an arbitrary smooth function that is $\mathcal{O}(|x-y|^m)$, and let $\mathcal{E}_\nu$ denote a finite sum $\sum_{m\geq0}\varphi_m/\Phi^{\nu+m}$. If $\nu\leq d-1$ and $\xi\in C^k_c(\mathbb{R}^d)$, then*

$$T\xi(x)=\int_{\mathbb{R}^d_y}\mathcal{E}_\nu(x,y)\xi(y)dy$$

is in $C^k(\mathbb{R}^d)$.

This lemma should be well-known, but for the reader's convenience we sketch a proof.

Sketch of proof. Let $L_j=\partial/\partial x_j+\partial/\partial y_j$. It is readily checked (e.g., by Taylor expanding) that $L_j\varphi_m=\varphi_m$ from which we conclude that $L_j\mathcal{E}_\nu=\mathcal{E}_\nu$. Let

$$T^\lambda\xi(x)=\int_{\mathbb{R}^d_y}|x-y|^{2\lambda}\mathcal{E}_\nu(x,y)\xi(y)dy.$$

For Re $\lambda > -1/2$, it is clear that $T^\lambda \xi$ is an analytic $C^0(\mathbb{R}^d)$-valued function. Moreover, for Re $\lambda > 0$, one easily checks by using $L_j \mathcal{E}_\nu = \mathcal{E}_\nu$ that all distributional derivatives of order $\leq k$ of $T^\lambda \xi$ are continuous and analytic in λ for Re $\lambda > -1/2$. It follows that $T\xi = T^0 \xi \in C^k(\mathbb{R}^d)$. □

Proposition 10.3. *If μ is a section of $Dom_q \bar{\partial}_X$ over X, then $\bar{\partial}\mu \in \mathcal{W}_{0,q+1}(X)$ and the current equation*

$$\mu = \bar{\partial}\mathcal{K}\mu + \mathcal{K}(\bar{\partial}\mu) + \mathrm{P}\mu \tag{10.2}$$

holds on $X'_{reg} = X_{reg} \cap \Omega'$.

Proof. From Proposition 7.2 it follows that $\bar{\partial}\mu \in \mathcal{W}_{0,q+1}(X)$ and so $\mathcal{K}(\bar{\partial}\mu)$ is a well-defined current in $\mathcal{W}(X')$. Moreover, from Lemma 9.3 it follows that if $\phi(z)$ is a test form on X'_{reg}, then

$$\phi(\zeta) = \int_{X_z} k(\zeta,z) \wedge \bar{\partial}\phi(z) + \bar{\partial}_\zeta \int_{X_z} k(\zeta,z) \wedge \phi(z) + \int_{X_z} p(\zeta,z) \wedge \phi(z) \tag{10.3}$$

for $\zeta \in X_{reg}$. We also see from Lemma 10.1 that all terms in (10.3) are smooth on X. If μ has compact support in X_{reg}, then the proposition follows by duality.

For the general case, let $\chi_\delta = \chi(|h|/\delta)$, where $h = h(\zeta)$ is a holomorphic tuple cutting out X_{sing}. Then the proposition holds for $\chi_\delta \mu$. Since $k(\zeta,z) \wedge \mu(\zeta)$ and $p(\zeta,z) \wedge \mu(\zeta)$ has the SEP on $X \times X'$, we have that $\mathcal{K}(\chi_\delta \mu) \to \mathcal{K}\mu$ and $\mathrm{P}(\chi_\delta \mu) \to \mathrm{P}\mu$. Moreover, $\bar{\partial}\mu \in \mathcal{W}_{0,q+1}(X)$ so $k(\zeta,z) \wedge \bar{\partial}\mu(\zeta)$ has the SEP, which implies that $\mathcal{K}(\chi_\delta \bar{\partial}\mu) \to \mathcal{K}(\bar{\partial}\mu)$. Hence,

$$\lim_{\delta \to 0^+} \mathcal{K}(\bar{\partial}(\chi_\delta \mu)) = \mathcal{K}(\bar{\partial}\mu) + \lim_{\delta \to 0^+} \mathcal{K}(\bar{\partial}\chi_\delta \wedge \mu).$$

The singularities of $k(\zeta,z)$ only come from the structure form $\omega(\zeta)$ when z and ζ "far apart", e.g., for z in a compact subset of X'_{reg} and ζ close to X_{sing}. From Lemma 7.1 we have that $\bar{\partial}\chi_\delta \wedge \mu \wedge \omega \to 0$ and so $\lim_{\delta \to 0^+} \mathcal{K}(\bar{\partial}\chi_\delta \wedge \mu) = 0$ for z in X'_{reg}; thus the proposition follows. □

Notice that $\mathrm{P}\mu$ in general is smooth. If the weight g is holomorphic in z, then $\mathrm{P}\mu$ is holomorphic in Ω' for $q = 0$ and 0 for $q \geq 1$. In this case, Proposition 10.3 thus is a homotopy formula for $\bar{\partial}$ on X'_{reg} in the sense that if μ is in $\mathrm{Dom}_q \bar{\partial}_X$ on X and $\bar{\partial}\mu = 0$, then μ is holomorphic in Ω' for $q = 0$ and $\mu = \bar{\partial}\mathcal{K}\mu$ on X'_{reg} for $q \geq 1$.

Proof of Proposition **??**. We know that $\mathcal{K}\phi$ is defined and in $\mathcal{W}^X$ if $\phi \in \mathcal{W}^X$. By choosing the weight g to be holomorphic in z, we get that $\mathrm{P}\phi$ is in $\mathcal{O}^X$. Moreover, from Proposition 10.3 we have that the Koppelman formulas (**??**) and (**??**) hold on X'_{reg} if, in addition, ϕ is in $\mathrm{Dom}\, \bar{\partial}_X$.

□

We do not know whether $\bar{\partial}\mathcal{K}\mu$ is in $\mathcal{W}^X$ or not, still less whether $\mathcal{K}\mu$ is in $\mathrm{Dom}\, \bar{\partial}_X$ or not in general. However, we shall now see that this is true if μ is smooth, and more generally if μ is obtained by a finite number of applications of $\mathcal{K}$'s. Notice that $\mathcal{K}\mu$ is only defined in the slightly smaller set X'. Therefore, when we in the following lemma consider products of kernels $\wedge_j k_j(z^j, z^{j+1})$, where $(z^1, \ldots, z^m)$ are coordinates on $X \times \cdots \times X$, we will assume that $z^{j+1} \mapsto k_{j+1}(z^{j+1}, z^{j+2})$ has compact support where $z^{j+1} \mapsto k_j(z^j, z^{j+1})$ is defined.

Lemma 10.4 (Main lemma)**.** *Let k_j denote kernels* (9.8) *obtained via local embeddings and arbitrary Hermitian free resolutions of $\mathcal{O}^\Omega/\mathcal{J}_X$. Let $(z^1,\dots,z^m)$ be coordinates on $X\times\cdots\times X$ and assume that $z^{j+1}\mapsto k_{j+1}(z^{j+1},z^{j+2})$ has compact support where $z^{j+1}\mapsto k_j(z^j,z^{j+1})$ is defined. Then, for any $x^m\in X$ and any $\omega\in\mathfrak{S}_{x^m}$, we have*

$$\lim_{\delta\to 0^+}\bar\partial\chi(|h(z^m)|/\delta)\wedge\omega(z^m)\wedge\bigwedge_{j=1}^{m-1}k_j(z^j,z^{j+1})=0 \tag{10.4}$$

in the current sense in a neighborhood of $\{x^m\}\times X\times\cdots\times X$.

Proof. We proceed by induction over m. Every $\omega\in\mathfrak{S}_{x^m}$ is in $\mathcal{W}^X$, so $\chi_\delta\omega\to\omega$ and hence

$$-\bar\partial\chi_\delta\wedge\omega=\nabla_f(\chi_\delta\omega)\to\nabla_f\omega=0.$$

Thus the lemma holds for $m=1$ (i.e., when there are no k-kernels). Now consider the case $m+1$. Recall that the limit in (10.4) is a pseudomeromorphic current T in a neighborhood of $\{x^{m+1}\}\times X\times\cdots\times X$. When $z^1\neq z^2$, then $k_1(z^1,z^2)$ is a smooth form times $\omega(z^1)$, cf., (9.11). Thus, outside $z^1=z^2$, T is a smooth form times the tensor product of $\omega(z^1)$ and a current of the form (10.4) in the variables $z^{m+1},\cdots,z^2$; the support of T is thus contained in $\{z^1=z^2\}$ by the induction hypothesis. For a similar reason the support of T must be contained in $\{z^k=z^{k+1}\}$ and we see that T must have support contained in the diagonal $\Delta=\{z^{m+1}=\cdots=z^1=0\}$. Moreover, the support of T is clearly also contained in $X_{sing}\times X\times\cdots\times X$. Thus, the support of T is contained in $(\Delta^X)_{sing}\subset\Delta$, which has dimension (at most) $n-1$ and hence codimension (at least) $(m+1)n-(n-1)=mn+1$.

Now let T^0 be the component of T obtained from the component $\omega_0(z^{m+1})$. Then T^0 has bidegree $(mn,m(n-1)+1)$ since each k_j has bidegree $(n,n-1)$. However, since $m\geq 1$, we have $m(n-1)+1<mn+1$ and so $T^0=0$ by the dimension principle. Let T^1 be the component of T obtained from $\omega_1(z^{m+1})$. Since $\omega_1=\alpha^1\omega_0$ and α^1 is smooth outside X^1, it follows from what we have just proved that T^1 has support contained in $(X^1\times X\times\cdots\times X)\cap\Delta\simeq X^1$. This set has codimension at least $mn+1+1$ and T^1 has bidegree $(*,m(n-1)+1+1)$ so also $T^1=0$ by the dimension principle. Proceeding in this way we conclude that $T=0$. □

We can now show that Lemma 9.3 holds on $X\times X'$.

Proposition 10.5. *We have that*

$$-\nabla_{f(z)}\big(\omega(z)\wedge k(\zeta,z)\big)=\omega\wedge[\Delta^X]-\omega(z)\wedge p(\zeta,z)$$

in the current sense on $X\times X'$.

Proof. Let $\chi_\delta=\chi(|h(\zeta)|/\delta)$ and $\chi_\epsilon=\chi(|h(z)|/\epsilon)$, where h as before cuts out X_{sing}. From Lemma 9.3 we have that

$$-\nabla_{f(z)}\big(\chi_\delta\chi_\epsilon\omega(z)\wedge k(\zeta,z)\big)=\chi_\delta\chi_\epsilon\omega\wedge[\Delta^X]-\chi_\delta\chi_\epsilon\omega(z)\wedge p(\zeta,z)+V(\delta,\epsilon),$$

where

$$V(\delta,\epsilon)=\bar\partial\chi_\delta\wedge\chi_\epsilon\omega(z)\wedge k(\zeta,z)+\chi_\delta\bar\partial\chi_\epsilon\wedge\omega(z)\wedge k(\zeta,z).$$

Since ω, k, p, as well as the products $\omega(z)\wedge k(\zeta,z)$ and $\omega(z)\wedge p(\zeta,z)$ all are in $\mathcal{W}(X\times X)$, it is enough to see that $\lim_{\epsilon\to 0}\lim_{\delta\to 0}V(\delta,\epsilon)=0$. We have

$$\lim_{\delta\to 0}V(\delta,\epsilon)=\lim_{\delta\to 0}\bar\partial\chi_\delta\wedge\chi_\epsilon\omega(z)\wedge k(\zeta,z)+\bar\partial\chi_\epsilon\wedge\omega(z)\wedge k(\zeta,z). \tag{10.5}$$

Since $\chi_\epsilon \omega(z)$ is smooth and vanishing in a neighborhood of X_{sing}, $k(\zeta, z)$ is a smooth form times $\omega(\zeta)$, cf., (9.11), on the support of $\bar\partial\chi_\delta$ if δ is small enough. Therefore, the first term on the right hand side of (10.5) is 0 by Lemma 10.4 with $m = 1$. The second term on the right hand side of (10.5) tends to 0 as $\epsilon \to 0$, again by Lemma 10.4. □

11. The ad hoc sheaf $\mathcal{A}^X$

We are now ready to define the sheaf $\mathcal{A}^X$; it is indeed an ad hoc definition with respect to the Koppelman formulas in the intrinsic context. From the previous two sections we know that we locally (and semi-globally) on X can construct integral kernels $k(\zeta, z)$ and $p(\zeta, z)$, cf., (9.11) and (9.10), and corresponding integral operators $\mathcal{K}$ and P such that Proposition 10.3 holds.

Definition 5. We say that a $(0, q)$-current ϕ on an open set $\mathcal{U} \subset X$ is a section of $\mathcal{A}^X$ over $\mathcal{U}$, $\phi \in \mathcal{A}_q(\mathcal{U})$, if, for every $x \in \mathcal{U}$, the germ ϕ_x can be written as a finite sum of terms

$$\xi_\nu \wedge \mathcal{K}_\nu(\cdots \xi_2 \wedge \mathcal{K}_2(\xi_1 \wedge \mathcal{K}_1(\xi_0)) \cdots),$$

where $\mathcal{K}_j$ are integral operators with kernels $k_j(\zeta, z)$ at x of the form defined in Section 9 and ξ_j are smooth $(0, *)$-forms at x such that ξ_j has compact support in the set where $z \mapsto k_j(\zeta, z)$ is defined.

Recall from Section 10 that if $\phi \in \mathcal{W}(\mathcal{U})$ and $\mathcal{K}$ is an integral operator, as defined above, with kernel $k(\zeta, z)$, where $z \mapsto k(\zeta, z)$ is defined in $\mathcal{U}' \Subset \mathcal{U}$, then $\mathcal{K}\phi \in \mathcal{W}(\mathcal{U}')$. Therefore, $\mathcal{A}^X$ is a subsheaf of $\mathcal{W}^X$ and from Lemma 10.1 it follows that the currents in $\mathcal{A}^X$ are smooth on X_{reg}. In view of Lemmas 10.4 and 7.1 we see that $\mathcal{A}^X$ is in fact a subsheaf of $\text{Dom}\,\bar\partial_X$. We also note that if $\phi \in \mathcal{A}_q(\mathcal{U})$, then $\mathcal{K}\phi \in \mathcal{A}_{q-1}(\mathcal{U}')$.

Proof of Theorem **??**. It is clear that $\mathcal{A}_q^X \supset \mathcal{E}_{0,q}^X$ are fine sheaves satisfying (i) of Theorem **??** and we have just noted that also (ii) holds.

We must check condition (iii). We have already seen in Proposition 7.2 that the kernel of $\bar\partial$ in $\text{Dom}_0\,\bar\partial_X$ is $\mathcal{O}^X$. Let ϕ be a section of $\mathcal{A}_q^X$, $q \geq 1$, in a neighborhood of an arbitrary point $x \in X$, and assume that $\bar\partial\phi = 0$. Since $\mathcal{A}^X \subset \text{Dom}\,\bar\partial_X$ we also have $\bar\partial_X \phi = 0$. For some neighborhood $\mathcal{U}$ of x, by Proposition 10.3, we can find an operator $\mathcal{K}$ such that

$$\bar\partial \mathcal{K}\phi = \phi \tag{11.1}$$

in $\mathcal{U}_{reg}$; here $\mathcal{K}$ corresponds to a weight that is holomorphic in z. Since ϕ is a section of $\mathcal{A}_q^X$ we know that $\mathcal{K}\phi$ is a section of $\mathcal{A}_{q-1}^X$ and since $\mathcal{A}^X \subset \text{Dom}\,\bar\partial_X$ it follows from Proposition 7.2 that $\bar\partial\mathcal{K}\phi$ is in $\mathcal{W}^X$. Both sides of (11.1) thus have the SEP and we conclude that (11.1) in fact holds on $\mathcal{U}$.

It remains to prove that $\bar\partial$ is a map from $\mathcal{A}^X$ to $\mathcal{A}^X$. It is sufficient to show that

$$\bar\partial\big(\xi_\nu \wedge \mathcal{K}_\nu(\cdots \xi_2 \wedge \mathcal{K}_2(\xi_1 \wedge \mathcal{K}_1(\xi_0)) \cdots)\big) \in \mathcal{A}^X, \tag{11.2}$$

for any operators $\mathcal{K}_j$ (not necessarily corresponding to weights that are holomorphic in z) and smooth $(0, *)$-forms ξ_j with compact support where $\mathcal{K}_j(\xi_{j-1})$ is defined. We prove (11.2) by induction over ν. The case $\nu = 0$ is clear. Assume that (11.2) holds for $\nu = \ell - 1$. Let $\mathcal{K}_j$, $j = 1, \ldots, \ell$ be any integral operators and ξ_j, $j = 0, \ldots, \ell$, smooth forms with compact support where $\mathcal{K}_j(\xi_{j-1})$ are defined. Put $\phi_{\ell-1} = \xi_{\ell-1} \wedge$

$\mathcal{K}_{\ell-1}(\cdots\xi_1\wedge\mathcal{K}_1(\xi_0)\cdots)$ and let $\mathcal{U}$ be a sufficiently small neighborhood of $\operatorname{supp}\xi_\ell$. By Proposition 10.3 we have that

$$\phi_{\ell-1}=\mathcal{K}_\ell(\bar\partial\phi_{\ell-1})+\bar\partial\mathcal{K}_\ell\phi_{\ell-1}+\mathrm{P}_\ell\phi_{\ell-1} \tag{11.3}$$

in $\mathcal{U}_{reg}$; notice that $\mathrm{P}_\ell\phi_{\ell-1}$ is smooth. From the induction hypothesis we have that $\bar\partial\phi_{\ell-1}$ is in $\mathcal{A}^X$. Moreover, any $\mathcal{K}$ maps $\mathcal{A}^X$ to $\mathcal{A}^X$ and since $\mathcal{A}^X\subset\operatorname{Dom}\bar\partial_X$, all terms in (11.3) have the SEP. Hence, (11.3) holds on $\mathcal{U}$ and it follows that $\bar\partial\mathcal{K}_\ell\phi_{\ell-1}$ is in $\mathcal{A}(\mathcal{U})$. Thus, (11.2) holds for $\nu=\ell$ and the proof is complete.

□

Proof of Theorem 7. From Section 10 we have integral operators $\mathcal{K}$ and P such that $\mathrm{P}\varphi$ is holomorphic in Ω' if $\varphi\in\mathcal{W}_{0,0}(X)$ and 0 if $\varphi\in\mathcal{W}_{0,q}(X)$, $q\geq 1$. Moreover, we noted above that $\mathcal{K}\colon\mathcal{A}_{q+1}(X)\to\mathcal{A}_q(X')$ and that $\mathcal{A}^X$ is a subsheaf of $\operatorname{Dom}\bar\partial_X$. Let $\phi\in\mathcal{A}_q(X)$, $q\geq 1$. By Proposition 10.3 we have that

$$\phi=\bar\partial\mathcal{K}\phi+\mathcal{K}(\bar\partial\phi)+\mathrm{P}\phi \tag{11.4}$$

on X'_{reg}. Since ϕ and $\bar\partial\phi$ are in $\mathcal{A}^X$, all terms in (11.4) have the SEP, cf., the previous proof. Hence (11.4) holds on X' and so Theorem 7 follows. □

12. Example with a reduced complete intersection

Let $a_1,\dots,a_p\in\mathcal{O}(\overline{\mathbb{B}})$, where $\mathbb{B}\subset\mathbb{C}^n$ is the unit ball, and assume that $X=\{a_1=\cdots=a_p=0\}\cap\mathbb{B}$ is a reduced complete intersection, i.e., that X has pure codimension p and $da_1\wedge\cdots\wedge da_p\neq 0$ on X_{reg}. Let $e_1,\dots,e_p$ be a holomorphic frame for the trivial bundle A and let a be the section $a=a_1e_1^*+\cdots+a_pe_p^*$ of the dual bundle A^*, where $\{e_j^*\}$ is the dual frame. Put $E_k=\Lambda^kA$ and let $\delta_a\colon\mathcal{O}(E_{\bullet+1})\to\mathcal{O}(E_\bullet)$ be interior multiplication with a. The Koszul complex $(\mathcal{O}(E_\bullet),\delta_a)$ is then a free resolution of $\mathcal{O}^\Omega/\mathcal{J}_X$, cf., (6.4). It is clear that $s_a:=\sum_j\bar a_je_j/|a|^2$ is the solution to $\delta_as_a=1$, outside X, with pointwise minimal norm (with respect to the trivial metric on A). If we consider all forms as sections of the bundle $\Lambda(T^*(\Omega)\oplus A)$, then we can write (6.8) as $u_k=s_a\wedge(\bar\partial s_a)^{k-1}$. Following [12], cf., (6.9), we get that

$$R=R_p=\bar\partial|a|^{2\lambda}\wedge u_p\big|_{\lambda=0}=\bar\partial\frac{1}{a_p}\wedge\cdots\wedge\bar\partial\frac{1}{a_1}\wedge e_1\wedge\cdots\wedge e_p, \tag{12.1}$$

i.e., R is the classical Coleff-Herrera product (times $e_1\wedge\cdots\wedge e_p$). Let ω' be a smooth E_p-valued form in $\Omega\setminus X_{sing}$ such that $da_1\wedge\cdots\wedge da_p\wedge\omega'/(2\pi i)^p=e\wedge d\zeta$ where $e=e_1\wedge\cdots\wedge e_p$ and $d\zeta=d\zeta_1\wedge\cdots\wedge d\zeta_N$. Then the pullback $i^*\omega'$, where $i\colon X\hookrightarrow\mathbb{B}$, is unique and meromorphic on X. By the Leray residue formula we get that

$$R\wedge d\zeta=\bar\partial\frac{1}{a_p}\wedge\cdots\wedge\bar\partial\frac{1}{a_1}\wedge e\wedge d\zeta=\omega'\wedge[X],$$

and so, cf., (6.12) and (6.21), the structure form associated to R is $\omega:=i^*\omega'$. If we choose coordinates $\zeta=(\zeta',\zeta'')$ so that $\det(\partial a/\partial\zeta')$ is generically non-vanishing on X_{reg}, then we can take $\omega'=(-2\pi i)^p\,e\wedge d\zeta''/\det(\partial a/\partial\zeta')$ and the structure form is explicitly given as

$$\omega=i^*\big((-2\pi i)^p\,e\wedge d\zeta''/\det(\partial a/\partial\zeta')\big).$$

If we let

$$\gamma=\frac{(-2\pi i)^p}{\det(\partial a/\partial\zeta')}\,e\wedge\frac{\partial}{\partial\zeta_p}\wedge\cdots\wedge\frac{\partial}{\partial\zeta_1},$$

then we have $R=(-1)^p\gamma\lrcorner[X]$, cf., (6.21).

Let $\mu \in \mathcal{W}_{0,q}(X)$ and assume that μ is smooth on X_{reg}. Then, cf., Section 7, μ is a section of $\operatorname{Dom}\bar\partial_X$ if and only if $\bar\partial\chi_\delta \wedge \mu \wedge i^*(d\zeta''/\det(\partial a/\partial\zeta')) \to 0$ in the current sense as $\delta \to 0$; here $\chi_\delta = \chi(|h|/\delta)$, χ is a smooth approximand of the characteristic function of $[1,\infty)$ and h cuts out X_{sing}.

To construct integral kernels, cf., Section 9, let h_j be $(1,0)$-forms so that $\delta_\eta h_j = a_j(\zeta) - a_j(z)$, where $\eta = \zeta - z$. We then have Hefer morphisms H_k^ℓ given as interior multiplication with $(\sum h_j \wedge e_j^*)^{k-\ell}/(k-\ell)!$. Let g be the weight from Example 9.1 and let B be the Bochner-Martinelli form. Then $(HR\wedge g\wedge B)_N = H_p^0 R_p \wedge (g\wedge B)_n$ since $R = R_p$ and a straight forward computation shows that

$$HR = H_p^0 R_p = \bar\partial\frac{1}{a_p}\wedge\cdots\wedge\bar\partial\frac{1}{a_1}\wedge h_1\wedge\cdots\wedge h_p.$$

There is a $\tilde k(\zeta,z) = \mathcal{O}(|\eta|^{-2n+1})$ such that

$$h_1\wedge\cdots\wedge h_p\wedge(g\wedge B)_n = (2\pi i)^{-p} d\eta\wedge\tilde k(\zeta,z)$$

and so from (9.8) we see that our solution kernel for $\bar\partial$ on X is

$$k(\zeta,z) = \pm\big(\gamma\lrcorner H_p^0\wedge(g\wedge B)_n\big)_{(n)} = \pm\tilde k(\zeta,z)\wedge\frac{d\zeta''}{\det(\partial a/\partial\zeta')}.$$

Similarly, there is a smooth form $\tilde p(\zeta,z)$, depending holomorphically on z if g does, such that

$$h_1\wedge\cdots\wedge h_p\wedge(\bar\zeta\cdot d\eta)\wedge(d\bar\zeta\cdot d\eta)^{n-1} = (2\pi i)^{-p} d\eta\wedge\tilde p(\zeta,z)$$

and we can compute $p(\zeta,z)$ from (9.5) using $\tilde p(\zeta,z)$. We get the representation formula

$$\phi(z) = \int_X \bar\partial\chi(\zeta)\wedge\frac{d\zeta''}{\det(\partial a/\partial\zeta')}\wedge\frac{\tilde p(\zeta,z)}{(|\zeta|^2 - z\cdot\bar\zeta)^n}\,\phi(\zeta) \tag{12.2}$$

for (strongly) holomorphic functions ϕ on X. If X intersects $\partial\mathbb{B}$ properly and X_{sing} avoids $\partial\mathbb{B}$ then we may let χ tend to the characteristic function for $\mathbb{B}$. The integral (12.2) then becomes an integral over $X\cap\partial\mathbb{B}$ and the resulting representation formula coincides with a formula of Stout [41] and Hatziafratis [22].

Let us consider the cusp $X = \{a(z) = z_1^r - z_2^s = 0\}\subset\mathbb{B}\subset\mathbb{C}^2$, where $2\le r<s$ are relatively prime integers, in more detail. In this case the structure form is the pullback of $-2\pi i\, e_1\wedge d\zeta_2/(r\zeta_1^{r-1})$ to X and we can take $\gamma(\zeta) = (-2\pi i/r\zeta_1^{r-1})\cdot e_1\wedge(\partial/\partial\zeta_1)$. The Hefer form is given by

$$h = h_1 d\eta_1 + h_2 d\eta_2 = \frac{1}{2\pi i}\Big(\frac{\zeta_1^r - z_1^r}{\zeta_1 - z_1}d\eta_1 + \frac{\zeta_2^s - z_2^s}{\zeta_2 - z_2}d\eta_2\Big)$$

and we get

$$h\wedge(g\wedge B)_1 = h\wedge\chi(\zeta)\frac{\partial|\eta|^2}{2\pi i|\eta|^2} = (2\pi i)^{-1} d\eta_1\wedge d\eta_2\,\tilde k(\zeta,z) \tag{12.3}$$

for a certain function $\tilde k(\zeta,z)$. The restriction of this function to $X\times X$ can be computed by applying δ_η to (12.3) and noting that $\delta_\eta h = a(\zeta) - a(z) = 0$ on $X\times X$. One gets that $\tilde k(\zeta,z) = \chi(\zeta)\,h_1/\eta_2$ on $X\times X$ and so our solution kernel for $\bar\partial$ on the cusp is

$$k(\zeta,z) = \frac{d\zeta_2}{r\zeta_1^{r-1}}\,\tilde k(\zeta,z) = \frac{\chi(\zeta)}{2\pi i}\frac{\zeta_1^r - z_1^r}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\frac{d\zeta_2}{r\zeta_1^{r-1}}.$$

Expressed in the parametrization $\tau \mapsto (\tau^s, \tau^r) = (\zeta_1, \zeta_2)$ our solution operator for $(0,1)$-forms thus becomes

$$\mathcal{K}\phi\,(t) = \frac{1}{2\pi i}\int_\tau \chi(\tau)\frac{\tau^{rs}-t^{rs}}{(\tau^s-t^s)(\tau^r-t^r)}\frac{d\tau}{\tau^{(s-1)(r-1)}} \wedge \phi(\tau).$$

One similarly shows that the projection operator P looks the same but with χ replaced by $\bar\partial\chi$, i.e., the kernel is the same but the solid integral is replaced by a boundary integral.

13. The one-dimensional case

In the case when X is a complex curve we have some further results. In particular, we have a stronger version of Proposition 7.2.

Proposition 13.1. *Let X be a reduced complex curve.*

(i) *If the complex* $0 \to \mathcal{O}^X \hookrightarrow \mathcal{E}^X_{0,0} \xrightarrow{\bar\partial} \mathcal{E}^X_{0,1} \to 0$ *is exact, then* $\mathcal{A}^X_* = \mathcal{E}^X_{0,*}$.

(ii) *The complex* $0 \to \mathcal{O}^X \hookrightarrow Dom_0\,\bar\partial_X \xrightarrow{\bar\partial} Dom_1\,\bar\partial_X \to 0$ *is exact.*

Proof. To prove (i), according to Definition 5, it is enough to show that $\mathcal{K}\xi$ is smooth for every $\mathcal{K}$ if ξ is a smooth $(0,1)$-form. If ξ is a smooth $(0,1)$-form, there is (locally) a smooth function ψ such that $\bar\partial\psi = \xi$. Smooth forms are in $\mathcal{A}^X$ and so, cf., the proof of Theorem 7, we get that

$$\mathcal{K}\xi = \mathcal{K}(\bar\partial\psi) = \psi - \mathrm{P}\psi$$

on X. Since $\mathrm{P}\psi$ is smooth, $\mathcal{K}\xi$ is indeed smooth on X.

From Proposition 7.2 we have that the kernel of $\bar\partial$ in $\mathrm{Dom}_0\,\bar\partial_X$ is $\mathcal{O}^X$ so to prove (ii) it remains to see that $\bar\partial\colon \mathrm{Dom}_0\,\bar\partial_X \to \mathrm{Dom}_1\,\bar\partial_X = \mathcal{W}^X_{0,1}$ is surjective. We take a minimal local embedding $X \hookrightarrow \mathbb{C}^N$ so that $X_{sing} = \{0\}$ and we let μ be a section of $\mathcal{W}^X_{0,1}$ in a neighborhood of 0. We choose a Hermitian minimal free resolution of $\mathcal{O}^X$ and we get the structure form $\omega = \omega_0$; notice that $\mathcal{O}^X$ is Cohen-Macaulay since $\dim X = 1$. Let $\mathcal{K}$ and P be integral operators as in Section 10 associated with a weight g which is holomorphic in z. From Proposition 10.3 we have that $u_1 := \mathcal{K}\mu$ is in $\mathcal{W}^X_{0,0}$ and solves $\bar\partial u_1 = \mu$ outside 0; we will modify this solution to a solution in $\mathrm{Dom}\,\bar\partial_X$.

Let $\pi\colon \tilde X \to X$ be the normalization of X. Then $\tilde\omega := \pi^*\omega$ is a meromorphic $(1,0)$-form and from (5.5) we see that there is $\tilde u_1$ in $\mathcal{W}^{\tilde X}_{0,0}$ such that $\pi_*\tilde u_1 = u_1$. Let h be a holomorphic tuple such that $\{h=0\} = \{0\}$ and put $\chi_\delta = \chi(|h|/\delta)$. Then $\nu := \lim_{\delta\to 0^+} \bar\partial\chi_\delta \wedge \tilde u_1\tilde\omega$ is a pseudomeromorphic $(1,1)$-current on $\tilde X$ with support in the finite set of points $\pi^{-1}(0)$. Let us for simplicity assume that X is irreducible at 0 so that $\tilde X$ is connected and $\pi^{-1}(0)$ is just one point $t=0$ for some holomorphic coordinate t on $\tilde X$. Then ν has support at $t=0$ and hence equals a finite linear combination of derivatives of the Dirac mass, δ_0, at $t=0$. Moreover, since ν is pseudomeromorphic, only holomorphic derivatives occur, cf., the first part of the proof of Proposition **??**, and so we have

$$\nu = \sum_0^\ell c'_j \frac{\partial^j}{\partial t^j}\delta_0 = \sum_0^\ell c_j\,\bar\partial\Big(\frac{1}{t^{j+1}}\Big) \wedge dt, \quad c'_j, c_j \in \mathbb{C}.$$

Also, since $\tilde\omega$ is meromorphic, $\tilde\omega = f(t)dt/t^k$ for some $k \geq 0$ and some holomorphic function f with $f(0) \neq 0$. The current

$$\tilde u_2 := \sum_{j=0}^{\ell} c_j \frac{t^{k-j-1}}{f(t)}.$$

is holomorphic for $t \neq 0$ and by construction, $\nu = \bar\partial(\tilde u_2 \tilde\omega)$. If $\tilde u := \tilde u_1 - \tilde u_2$, it is then straightforward to verify that $\bar\partial\chi_\delta \wedge \tilde u \tilde\omega \to 0$ on $\tilde X$. Hence, $u := \pi_* \tilde u = u_1 - \pi_* \tilde u_2$ is in $\mathrm{Dom}_0\, \bar\partial_X$ and solves $\bar\partial u = \mu$. □

Notice that once we know that $\bar\partial\colon \mathrm{Dom}_0\, \bar\partial_X \to \mathrm{Dom}_1\, \bar\partial_X = \mathcal{W}^X_{0,1}$ is surjective, it is easy to show, using an argument similar to the proof of statement (i) above, that our solution operators for $\bar\partial$ indeed produce solutions in $\mathrm{Dom}_0\, \bar\partial_X$.

Also notice that, in view of Proposition 7.2, it follows from (ii) of Proposition 13.1 that if $H^1(X, \mathcal{O}^X) = 0$ and $\phi \in \mathcal{W}_{0,1}(X) = \mathrm{Dom}_1\, \bar\partial_X$, then there is a $\psi \in \mathcal{W}_{0,0}(X)$ such that $\bar\partial_X \psi = \phi$ on X.

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