LECTURE NOTES ON KOPPELMAN FORMULAS ON SINGULAR SPACES

MATS ANDERSSON

These notes are based on a series of five lectures given at the workshop "Differential forms on Singular Complex Spaces" in Bonn June 30 to July 04, 2014.

1. Lecture 1

I am grateful to the organizers for the opportunity to give this series of lectures on integral formulas and the $\bar\partial$-equation on singular varieties.

The $\bar\partial$-equation plays a fundamental role in SCV and the most powerful tool to study this equation is the L^2-methods introduced in the 60's by Hörmander, Kohn and others, and further developed by many authors since then. However, for various qualitative questions, for instance uniform estimates of solutions, integral formulas are are a necessary tool. Such formulas were introduced in the early 70's by Henkin and Ramirez. A few years later Henkin and Skoda independently proved the famous $L^1(\partial D)$-estimate bearing their name, by introducing the first weighted integral formula in a strictly pseudoconvex domain. More general weighted integral formlas were introduced together with by Berndtsson in the early 80's and there has been a further development since then by several authors.

Even in the singular case L^2-methods play a central role to understand possible obstruction for solvability on $X_{reg} := X \setminus X_{sing}$. By integral formulas however we can obtain solutions that have a reasonable meaning also across the singularity. For the moment we state the following result as a motivating aim for these lectures; later on we will discuss more elaborated formulations from [2]. Throughout these lectures X is a pure-dimensional reduced analytic space.

Theorem 1.1 (Theorem A). *If $\phi \in \mathcal{E}_{0,q}(X)$, X is Stein, and $\bar\partial\phi = 0$, then there is a current v that is smooth on X_{reg} such that $\bar\partial v = \phi$ on X.*

In particular, $\bar\partial v = \phi$ holds on X_{reg}. Such a solution on X_{reg} was found by Henkin and Polyakov -89 by an integral formula, [10], in the case when X is a complete intersection. In general, there is no solution that is smooth across X_{sing}.

Theorem A is basically a local statement, and it is proved by a local Koppelman formula. The main ingredients to construct and draw conclusions from such formulas are multilinear algebra and multivariable residue theory. In these lectures I will focus on some basic ideas and techniques and I will mainly consider a hypersurface X in a domain $\Omega \subset \mathbb{C}^n$. The general case and all details, as well as various references, can be found in [2] and in sketch of a monograph [1].

1.1. **The Cauchy-Green formula in the plane.**

Integral representation of a holomorphic functions f is often used to expressed f as a superposition of some class of

Date: September 12, 2014.

functions that are "simple" in some sense. For instance, by the Cauchy integral formula in one variable a functions in a bounded domain D with reasonable boundary is written as a superposition of simple rational functions $z \mapsto 1/(z-\zeta)$, for $\zeta \in \partial D$.

For fixed $z \in \mathbb{C}$,

$$k = \frac{1}{2\pi i}\frac{d\zeta}{\zeta - z}$$

is the Cauchy kernel with pole at z. It is holomorphic in $\mathbb{C}\setminus\{z\}$ and locally integrable in $\mathbb{C}$. It is well-known that

$$\bar\partial_\zeta \omega = [z], \tag{1.1}$$

where $[z]$ is the $(1,1)$-current "point evaluation at z". If $D \subset\subset \Omega$ we get the Cauchy integral formula

$$\phi(z) = \int_{\partial D} k\phi, \quad \phi \in \mathcal{O}(\Omega), z \in D. \tag{1.2}$$

It is often useful to have such a formula but where the integration is performed over a thickened boundary. Let χ be a smooth approximation of the characteristic function χ_D such that $\chi \equiv 1$ on D. From (1.1) we then get

$$\phi(z) = \int_{\partial D} -\bar\partial\chi \wedge k\phi, \quad \phi \in \mathcal{O}(\Omega), \quad z \in D. \tag{1.3}$$

Notice that for degree reasons we can replace $\bar\partial$ by d in (1.3) and that

$$-d\chi_D = [\partial D].$$

Thus we can think of the right hand side of (1.3) as an integral over "the boundary of χ".

We now turn our attention to the case $n > 1$. One can then solve (1.1) abstractly and all possible solutions form a cohomology class in a punctured ball around z. However, for qualitative questions a cohomology class is usually of no help as long as one has no access to concrete representatives to look at. We thus are to find locally integrable forms k that solves (1.1). To this end, however, it turns out that it is convenient to stick to a seemingly more involved equation. This detour involves some multilinear algebra that we first discuss.

1.2. **Functional calculus for forms of even degree.** Let E be an m-dimensional vector space and recall that k:th exterior product $\Lambda^k E$ consists of all alternating multilinear forms on the dual space E^*. If $v \in E^*$ we have contraction (interior multiplication) with v, $\delta_v\colon \Lambda^{k+1}E \to \Lambda^k E$, such that

$$\delta_v \xi = v \cdot \xi, \quad \xi \in E = \Lambda^1 E, \quad \delta_v(\alpha\wedge\beta) = \delta_v\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge\delta_v\beta. \tag{1.4}$$

One says that δ_v is an *anti-derivation.* From (1.4) follows that $\delta_v^2 = 0$.

Now let $\omega_1, \ldots, \omega_m$ be even forms, i.e., in $\oplus_\ell \Lambda^{2\ell}E$, and let $\omega_j = \omega_j' + \omega_j''$ be the decomposition in components of degree zero and positive degree, respectively. Notice that $\wedge$ is commutative for even forms. If $p(z) = \sum_\alpha c_\alpha z^\alpha = \sum_\alpha c_\alpha z_1^{\alpha_1}\cdots z_m^{\alpha_m}$ is a polynomial, we therefore have a natural definition of $p(\omega)$ as the form $\sum_\alpha c_\alpha \omega_1^{\alpha_1}\wedge\ldots\wedge\omega_m^{\alpha_m}$. However, it is often convenient to be able to apply more general holomorphic functions.

If f is holomorphic in a neighborhood of the point $\omega' = (\omega'_1, \dots, \omega'_m) \in \mathbb{C}^m$, then we define

$$f(\omega) = \sum_\alpha f^{(\alpha)}(\omega')(\omega'')_\alpha, \tag{1.5}$$

where we use the convention that

$$w_\alpha = \frac{w_1^{\alpha_1} \wedge \ldots \wedge w_m^{\alpha_m}}{\alpha_1! \cdots \alpha_m!}.$$

Thus $f(\omega) = f(\omega' + \omega'')$ is defined as the formal power series expansion at the point ω'. Since the sum is finite, $f(\omega)$ is a well-defined form, and if ω depends continuously (smoothly, holomorphically) on some parameters, $f(\omega)$ will do as well.

One can check that this definition coincides with the natural one in case f is a polynomial. Notice that if $f(z) - g(z) = \mathcal{O}((z - \omega')^M)$ for a large enough M, then $f(\omega) = g(\omega)$. By the Cauchy estimates one can prove

Lemma 1.2. *Suppose that $f_k \to f$ in a neighborhood of $\omega' \in \mathbb{C}^m$ and that $\omega_k \to \omega$. Then $f_k(\omega_k) \to f(\omega)$.*

Clearly

$$(af + bg)(\omega) = af(\omega) + bg(\omega), \quad a, b \in \mathbb{C}.$$

From Lemma 1.2 one can prove

Proposition 1.3. *If f, g are holomorphic in a neighborhood of ω', then*

$$(fg)(\omega) = f(\omega) \wedge g(\omega). \tag{1.6}$$

If f is holomorphic in a neighborhood of ω' (possibly $\mathbb{C}^r$-valued) and h is holomorphic in a neighborhood of $f(\omega')$, then

$$(h \circ f)(\omega) = h(f(\omega)). \tag{1.7}$$

If v is in E^*, then

$$\delta_v f(\omega) = \sum_1^m \frac{\partial f}{\partial z_j}(\omega) \wedge \delta_v \omega_j, \tag{1.8}$$

and if ω depends on a parameter, then

$$df(\omega) = \sum_1^m \frac{\partial f}{\partial z_j}(\omega) \wedge d\omega_j. \tag{1.9}$$

For instance, if ω_1 and ω_2 are even forms, then

$$e^{\omega_1 + \omega_2} = e^{\omega_1} \wedge e^{\omega_2}.$$

We will frequently use the formula

$$\frac{1}{1 - \omega''} = 1 + \omega'' + (\omega'')^2 + \cdots (\omega'')^m.$$

1.3. **Integral representation of holomorphic functions.** For $U \subset \mathbb{C}^n$ and any integer m, let $\mathcal{L}^m(U) = \oplus_{k=0}^n \mathcal{C}_{k,k+m}(U)$. For instance, $u \in \mathcal{L}^{-1}(U)$ can be written $u = u_{1,0} + \ldots + u_{n,n-1}$, where the indices denote bidegree in $d\zeta$. We let $\mathcal{L}^m_{\mathcal{E}}(U)$ denote the subspace of $\mathcal{L}^m(U)$ of smooth forms.

Fix a point $z \in \mathbb{C}^n$, let $\delta = \delta_{\zeta - z} \colon \mathcal{E}_{p,q}(U) \to \mathcal{E}_{p-1,q}(U)$ be contraction with the vector field

$$2\pi i \sum_1^n (\zeta_k - z_k) \frac{\partial}{\partial \zeta_k}.$$

Since $\delta_{\zeta-z}\bar\partial f = -\bar\partial \delta_{\zeta-z} f$ (this is verified, e.g., by induction over the degree of f) $\mathcal{E}_{p,q}$ is a double complex with mappings $\bar\partial$ and δ. If $\nabla = \nabla_{\zeta-z} := \delta_{\zeta-z} - \bar\partial$, then $\nabla^2 = 0$, and we have the associated total complex

$$\cdots \xrightarrow{\nabla} \mathcal{L}^m \xrightarrow{\nabla} \mathcal{L}^{m+1} \xrightarrow{\nabla} \cdots .$$

The usual wedge product extends to a mapping $\mathcal{L}^m(U) \times \mathcal{L}^{m'}(U) \to \mathcal{L}^{m+m'}_{\mathcal{E}}(U)$, such that $g \wedge f = (-1)^{mm'} f \wedge g$. Since δ and $\bar\partial$ are both anti-derivations, so is ∇, i.e.,

$$\nabla(f \wedge g) = \nabla f \wedge g + (-1)^m f \wedge \nabla g, \quad f \in \mathcal{L}^m(U). \tag{1.10}$$

In order to generalize Cauchy's formula to higher dimensions we will look for $u \in \mathcal{L}^{-1}_{\mathcal{E}}(U \setminus \{z\})$ with integrable singularity at $z \in U$ such that

$$\nabla u(\zeta) = 1 - [z]. \tag{1.11}$$

If $n = 1$ the Cauchy kernel with pole at z, $u(\zeta) = d\zeta/2\pi i(\zeta - z)$, is the only possible solution. If $n > 1$, (1.11) means that

$$\delta u_{1,0} = 1, \quad \delta u_{k+1,k} - \bar\partial u_{k,k-1} = 0, \quad 1 \le k \le n-2, \quad \bar\partial u_{n,n-1} = [z]. \tag{1.12}$$

Outside our fixed $z \in \mathbb{C}^n$ we can define the form

$$b = \frac{\partial |\zeta|^2}{2\pi i |\zeta|^2} = \frac{\sum_1^n \bar\zeta_j d\zeta_j}{2\pi i |\zeta|^2}.$$

It is clear that $\delta b = 1$, so if

$$B = \frac{b}{\nabla b} = \frac{b}{1 - \bar\partial b} = b + b \wedge \bar\partial b + \cdots + b \wedge (\bar\partial b)^{n-1}, \tag{1.13}$$

then $\nabla B = 1$ outside z. In fact,

$$\nabla \frac{b}{\nabla b} = \frac{\nabla b}{\nabla b} - \frac{b}{(\nabla b)^2} \nabla^2 b = 1$$

since $\nabla^2 = 0$. (Of course one can also use that $\delta \bar\partial b = -\bar\partial \delta b = -\bar\partial 1 = 0$ so that $\delta(b \wedge (\bar\partial b)^{k-1}) = (\bar\partial b)^{k-1}$. Applying ∇ to the right hand side of (1.13) one gets a telescoping sum.)

We have the following homogeneity property for the form $b \wedge (\bar\partial b)^{k-1}$. If ξ is a non-vanishing scalar function, then

$$\xi b \wedge (\bar\partial(\xi b))^{k-1} = \xi^\ell b \wedge (\bar\partial b)^{k-1} \tag{1.14}$$

since $b \wedge b = 0$. Taking $\xi = 2\pi i |\zeta - z|^2$ we get

$$B_{\ell,\ell-1} = b \wedge (\bar\partial b)^{\ell-1} = \frac{1}{(2\pi i)^\ell} \frac{\partial |\zeta - z|^2 \wedge (\bar\partial \partial |\zeta - z|^2)^{\ell-1}}{|\zeta - z|^{2\ell}} = \mathcal{O}\Big(\frac{1}{|\zeta - z|^{2\ell-1}}\Big).$$

Thus B is locally integrable and one can prove that B solves (1.11) in $\mathbb{C}^n$.

Notice that $B_{n,n-1}$ is the classical Bochner-Martinelli form with pole a z, so the equality $\bar\partial B_{n,n-1}=[z]$ is a compact formulation of the Bochner-Martinelli formula. If $D\subset\Omega\subset\mathbb{C}^n$ has reasonable boundary and $z\in D$, then as in case $n=1$ we have the representation

$$\phi(z)=\int_{\partial D} k\phi, \quad \phi\in\mathcal{O}(\Omega), \tag{1.15}$$

with $k=B_{n.n-1}=b\wedge(\bar\partial b)^{n-1}$.

Proposition 1.4 (The Cauchy-Fantappiè-Leray formula)**.** *Let $z\in D$ and assume that $\sigma=\sigma_1 d\zeta_1+\ldots+\sigma_n d\zeta_n$ is a smooth $(1,0)$-form defined on ∂D such that*

$$0\neq\delta\sigma=2\pi i\sum\sigma_j(\zeta_j-z_j)=2\pi i\langle\sigma,\zeta-z\rangle$$

on ∂D. Then we have the repesentation (1.15) *with*

$$k=\frac{1}{(2\pi i)^n}\frac{\sigma\wedge(d\sigma)^{n-1}}{\langle\sigma,\zeta-z\rangle^n}. \tag{1.16}$$

Proof. Let $\tilde\sigma$ be any smooth extension of σ to a neighborhood of ∂D. Then $\delta\tilde\sigma\neq 0$ in a neighborhood U of ∂D. If $s=\tilde\sigma/\delta\tilde\sigma$, then $\delta s=1$ in U. Thus $v=s/\nabla s$ is defined in U and, as for B above, we have that $\nabla v=1$ in U. Thus

$$\nabla(v\wedge B)=B-v,$$

so that $\bar\partial(v\wedge B)_{n,n-2}=v_{n,n-1}-B_{n,n-1}$. It follows that $\bar\partial(v\wedge B)_{n,n-2}\phi=v_{n,n-1}\phi-B_{n,n-1}\phi$ if $\phi\in\mathcal{O}(\Omega)$. For bidegree reasons we can replace $\bar\partial$ by d. Since ∂D is compact it follows that we can take $k=v_{n,n-1}$ instead of $k=B_{n,n-1}$ in (1.15). By a homogeneity argument as above we find that $v_{n,n-1}$ is equal to the right hand side of (1.16). □

One can also verify directly that $v=\tilde\sigma/\nabla\tilde\sigma$.

Example 1.5. If $D=\mathbb{B}$ is the unit ball, then for any $z\in\mathbb{B}$ we can take $\sigma=\partial|\zeta|^2$ on $\partial\mathbb{B}$. In fact,

$$\delta\sigma=2\pi i\langle\bar\zeta,\zeta-z\rangle=2\pi i\langle|\zeta|^2-\bar\zeta\cdot z\rangle=2\pi i(1-\bar\zeta\cdot z)$$

that is non-vanishing, and so we get (1.15) with

$$k=\frac{1}{(2\pi i)^n}\frac{\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{n-1}}{(1-\bar\zeta\cdot z)^n}.$$

Applying to $\phi\equiv 1$ and $z=0$ we see that

$$\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{n-1}/(2\pi i)^n$$

has total mass 1, and because of apparent rotation invariance it must be the normalized surface measure on $\partial\mathbb{B}$. □

2. Lecture 2

We say that a smooth form $g\in\mathcal{L}^0(\Omega)$ is a *weight* with respect to $z\in\Omega$ if $\nabla g=0$ and $g_{0,0}(z)=1$.

Example 2.1. If $w\in\mathcal{L}^{-1}(\Omega)$ is smooth, then

$$g=1+\nabla w \tag{2.1}$$

is a weight. □

Notice that if g^1 and g^2 are weights (with respect to z), then $g^1 \wedge g^2$ is a weight. More generally, if $g^1, \dots, g^m$ are weights and $h(\lambda_1, \dots, \lambda_m)$ is holomorphic on the image on $\zeta \mapsto (g^1_{0,0}(\zeta), \dots, g^m_{0,0}(\zeta))$ and $g(1, \dots, 1) = 1$, then $g = h(g^1, \dots, g^m)$ is a weight.

Example 2.2. If g is a weight in Ω and $\operatorname{Re} g_{0,0} > 0$, then g^α is a weight for any $\alpha \in \mathbb{C}$. $\square$

Example 2.3. If $w \in \mathcal{L}^{-1}(\Omega)$ and $G(\lambda)$ is holomorphic on the image of $\zeta \mapsto \delta w = \langle w, \zeta - z \rangle$, and $G(0) = 1$, then

$$g = G(\nabla w) = G(\langle w, \zeta - z \rangle - \bar\partial w) = \sum_{\ell=0}^{n} G^{(\ell)}(\langle w, \zeta - z \rangle) \frac{1}{\ell!} (-\bar\partial w)^\ell$$

is a weight in Ω with respect to z. $\square$

Proposition 2.4. *If g is a weight with respect to z, with compact support in Ω, then*

$$\phi(z) = \int g_{n,n} \phi, \quad \phi \in \mathcal{O}(\Omega).$$

Proof. Notice that

$$\nabla(B \wedge g) = (1 - [z]) \wedge g = g - [z] \wedge g = g - g_{0,0}[z] = g - [z]$$

since $g_{0,0}(z) = 1$. Thus $\bar\partial(B \wedge g)_{n,n-1} = [z] - g_{n,n}$. Since $B \wedge g$ has compact support in Ω it follows, cf., the proof of Proposition 1.4, that

$$0 = \int d((B \wedge g)_{n,n-1} \phi) = \int [z] \phi - \int g_{n,n} \phi = \phi(z) - \int g_{n,n} \phi.$$

$\square$

Example 2.5. Let $s = \partial |\zeta|^2 / 2\pi i (|z|^2 - \bar\zeta \cdot z)$, cf., Example 1.5, so that

$$s \wedge (\bar\partial s)^{k-1} = \sum_\ell \frac{1}{(2\pi i)^\ell} \frac{\partial |\zeta|^2 \wedge (\bar\partial \partial |\zeta|^2)^{\ell-1}}{(|z|^2 - \bar\zeta \cdot z)^\ell}.$$

Let χ be a smooth approximation of $\chi_{\mathbb{B}}$ such that $\chi \equiv 1$ on $\mathbb{B}$ and with support in, say, $(1+\epsilon)\mathbb{B}$. Then

$$v = \frac{s}{\nabla s}$$

is defined and $\delta v = 1$ on the support of $\bar\partial \chi$. Thus

$$g = \chi - \bar\partial \chi \wedge \frac{s}{\nabla s} = \chi - \bar\partial \chi \wedge v$$

is a weight with respect to $z \in \mathbb{B}$ with compact support in $\Omega = (1+\epsilon)\mathbb{B}$. $\square$

Notice that for any fixed $z \in D \subset\subset \Omega$ we can always choose $v = b/\nabla b$. The advantage with the choice in Example 2.5 is that it depends holomorphically on z.

Corollary 2.6. *Let g be any weight in Ω with respect to $z \in D \subset\subset \Omega$, let χ be a smooth approximand of χ_D and take v such that $\nabla v = 1$ on* supp χ. *Then we have the representation*

$$\phi(z) = -\int \bar\partial \chi \wedge (v \wedge g)_{n,n-1} \phi + \int \chi g_{n,n} \phi, \quad \phi \in \mathcal{O}(\Omega). \tag{2.2}$$

In particular, in the ball we can choose v as in Example 2.5.

Letting $\chi \to \chi_D$ in (2.2) or by a direct argument we get

$$\phi(z) = \int_{\partial D} (v \wedge g)_{n,n-1}\phi + \int_D g_{n,n}\phi, \quad \phi \in \mathcal{O}(\Omega). \tag{2.3}$$

Notice that if we take $g \equiv 1$ and choose $v = s/\nabla s$ we get back Proposition 1.4.

Example 2.7. Let $z, \zeta \in \mathbb{B}$. Notice that

$$1 + \nabla_{\zeta - z} \frac{\partial |\zeta|^2}{2\pi i(1 - |\zeta|^2)} = \frac{1 - \bar{\zeta} \cdot z}{1 - |\zeta|^2} + \omega,$$

where ω is the positive $(1,1)$-form

$$\omega = \frac{i}{2\pi} \partial\bar{\partial} \log \frac{1}{1 - |\zeta|^2}.$$

Therefore, cf., Example 2.2,

$$g = \Big(\frac{1 - \bar{\zeta} \cdot z}{1 - |\zeta|^2} + \omega\Big)^{-\alpha}$$

is a weight for any $\alpha \in \mathbb{C}$. Notice that

$$g_{n,n} = c_{n,\alpha}\Big(\frac{1 - \bar{\zeta} \cdot z}{1 - |\zeta|^2}\Big)^{\alpha+n} \omega_n,$$

where $\omega_n = \omega^n/n!$ and

$$c_{\alpha,n} = (-1)^n n! \frac{1}{\pi^n} \frac{\Gamma(-\alpha + 1)}{\Gamma(n+1)\Gamma(-\alpha - n + 1)}.$$

Using that $\Gamma(n+1) = n!$ and $\Gamma(\tau + 1) = \tau\Gamma(\tau)$ we get

$$c_\alpha = \frac{1}{\pi^n} \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)}.$$

If $\operatorname{Re}\alpha \gg 0$, then g vanishes on $\partial\mathbb{B}$ so g has “compact supprt”, and and hence we have the representation

$$\phi(z) = \int_{|\zeta|<1} c_{n,\alpha}\Big(\frac{1 - \bar{\zeta} \cdot z}{1 - |\zeta|^2}\Big)^{\alpha+n} \omega_n \phi(\zeta), \quad z \in \mathbb{B}.$$

We claim that

$$\omega_n = \frac{dV(\zeta)}{(1 - |\zeta|^2)^{n+1}}. \tag{2.4}$$

To see this it is enough to check that both sides coincide after application by $\hat{\delta} = \delta_\zeta/2\pi i$, since $\hat{\delta}$ is injective on $(0,*)$-forms when $\zeta \neq 0$. Notice that

$$\hat{\delta}\partial\bar{\partial}|\zeta|^2 = \bar{\partial}|\zeta|^2, \quad \hat{\delta}\omega = \frac{i}{2} \frac{\bar{\partial}|\zeta|^2}{(1 - |\zeta|^2)^2}.$$

Thus

$$\hat{\delta}\omega_n = \hat{\delta}\omega \wedge \omega_{n-1} = \Big(\frac{i}{2}\Big)^n \frac{\bar{\partial}|\zeta|^2}{(1 - |\zeta|^2)^2} \wedge \partial \frac{\bar{\partial}|\zeta|^2}{(1 - |\zeta|^2)^2} = \Big(\frac{i}{2}\Big)^n \frac{\bar{\partial}|\zeta|^2 \wedge (\partial\bar{\partial}|\zeta|^2)^{n-1}}{(1 - |\zeta|^2)^{n+1}}$$
$$\Big(\frac{i}{2}\Big)^n \hat{\delta} \frac{(\partial\bar{\partial}|\zeta|^2)^n}{(1 - |\zeta|^2)^{n+1}} = \hat{\delta} \frac{dV}{(1 - |\zeta|^2)^{n+1}},$$

and thus (2.4) holds. □

There are a similar formulaa in a strictly pseudoconvex domains.

2.1. **Singular weights.** Fix $z \in \Omega$. We say that $g \in \mathcal{L}^0(\Omega)$ is a *singular weight* with respect to z if $\nabla g = 0$ and

$$\int g \wedge g' = 1 \tag{2.5}$$

for all smooth weights g' (with respect to z) with compact support in Ω.

When g is non-smooth the point evaluation we cannot use point evaluation of $g_{0,0}$ at z for normalization, so instead we use (2.5). For instance, $g = [z]$ is a weight with respect to z but $g_{0,0} \equiv 0$. Notice that if g is a smooth weight, then (2.5) holds, so the new normalization is consistent with the usual one in this case.

Example 2.8. If $w \in \mathcal{L}^{-1}(\Omega)$, then $g = 1 + \nabla w$ is a (singular) weight. In fact,

$$\int g \wedge g' = \int g' + \int \nabla w \wedge g' = 1 + \int \nabla(w \wedge g) = 1 + \int d(w \wedge g')_{n,n-1} = 1$$

since $w \wedge g'$ has compact support. □

One can prove that $g \in \mathcal{L}^0(\Omega)$ is a weight if and only if $g = 1 + \nabla w$ for some $w \in \mathcal{L}^{-1}(\Omega)$.

We have the analogue of Proposition 2.4 for singular weights.

Proposition 2.9. *If g is a (singular) weight with respect to $z \in \Omega$ with compact support, then*

$$\phi(z) = \int g_{n,n}\phi, \quad \phi \in \mathcal{O}(\Omega).$$

Proof. Take g' with compact support in Ω such that $g' \equiv 1$ on the support of g. If $\phi(z) = 1$, then $\phi g'$ is a smooth weight with compact support and so

$$\phi(z) = 1 = \int g \wedge g'\phi = \int g_{n,n}\phi.$$

The general case follows by scaling. □

To define the singular weights of interest for us we first have to recall the Poincaré residue.

2.2. **The Poincaré residue.** Let

$$i \colon X \to \Omega \subset \mathbb{C}^n$$

be a complex hypersurface in Ω defined by $X = \{f = 0\}$, where $df \neq 0$ on X. Then $1/f$ is locally integrable so $\bar\partial(1/f)$ is a well-defined $(0,1)$-current. We have the Poincaré-Lelong formula

$$\bar\partial\frac{1}{f} \wedge \frac{df}{2\pi i} = [X], \tag{2.6}$$

where $[X]$ is the current of integration over X. In fact, since $df \neq 0$ one can assume that $f = z_1$, and then (2.6) follows from the analogous one-variable statement, which is (1.1).

We claim that there is a unique holomorphic $(n-1,0)$-form ω on X such that

$$i_*\omega = \bar\partial\frac{1}{f} \wedge d\zeta, \tag{2.7}$$

where $d\zeta := d\zeta_1 \wedge \ldots \wedge d\zeta_n$. The uniqueness is clear since i_* is injective. Possibly up to a constant $2\pi i$ the form ω is the *Poincaré residue* of the meromorphic form $d\zeta/f$. For the existence, let α be a smooth form such that

$$\frac{1}{2\pi i} df \wedge \alpha = d\zeta. \tag{2.8}$$

For instance, one can take

$$\alpha = \frac{\sum_j \overline{\partial f/\partial \zeta_j}\widehat{d\zeta_j}}{2\pi i |df|^2}.$$

Locally one can also take

$$\alpha = \frac{\widehat{d\zeta_j}}{\partial f/\partial \zeta_j} \tag{2.9}$$

for a suitable j. We claim that $\omega = i^*\alpha$ will do. From the choice (2.9) of α it is clear that ω is holomorphic. In view of (2.8) and (2.6) we have

$$\int_\Omega i_*\omega \wedge \xi = \int_X \omega \wedge i^*\xi = \int_\Omega [X] \wedge \alpha\xi = \int_\Omega \bar\partial \frac{1}{f} \wedge \frac{df}{2\pi i} \wedge \alpha\xi = \int_\Omega \bar\partial \frac{1}{f} \wedge d\zeta \wedge \xi.$$

Thus (2.7) holds.

2.3. **Integral representation on X.** We keep the notation from Section 2.2. Let z be a point in Ω and let h be a $(1,0)$-form such that

$$\delta h = f - f(z),$$

where $f = f(\zeta)$. If Ω is pseudoconvex, then one can choose h that is holomorphic in both ζ and z. Since we are only interested in local questions we may assume that Ω is convex, say a ball.

Remark 2.10. Such a form h is called a Hefer form for f. It can be obtained elementarily in a convex domain like the ball: In fact,

$$f(z) - f(\zeta) = \int_0^1 d_t f(z + t(\zeta - z)) = \sum_1^n (\zeta_j - z_j) \int_0^1 \frac{\partial f}{\partial w_j}(z + t(\zeta - z))dt$$

so one can take $h = h_1(\zeta, z)d\zeta_1 + \cdots + h_n(\zeta, z)d\zeta_n$ where

$$h_j(\zeta, z) = \int_0^1 \frac{\partial f}{\partial w_j}(z + t(\zeta - z))dt.$$

$\square$

Notice that $1 - \nabla(h/f) = 1 - \delta h/f + \bar\partial(1/f) \wedge h$ so that, cf., Example 2.8,

$$g' := \frac{f(z)}{f} + \bar\partial \frac{1}{f} \wedge h$$

is a singular weight. Let g be a (smooth) weight with respect to z with compact support in Ω. Then we have the representation

$$\Phi(z) = \int g' \wedge g\Phi, \quad \Phi \in \mathcal{O}(\Omega).$$

If now $z \in X$, then $f(z) = 0$ and if moreover $\phi = i^*\Phi$ we have

$$\phi(z) = \int \bar\partial \frac{1}{f} \wedge (h \wedge g)_{n,n-1}\Phi = \phi(z) = \int \bar\partial \frac{1}{f} \wedge h \wedge g_{n-1,n-1}\Phi.$$

For a form α of bidegree (n,q) we define the $(0,q)$-form $\{\alpha\}$ by the equality

$$\alpha = d\zeta \wedge \{\alpha\}.$$

Then

$$\phi(z) = \int \bar\partial \frac{1}{f} \wedge d\zeta \wedge \{h \wedge g_{n-1,n-1}\}\Phi = \int_X \omega \wedge \{h \wedge g_{n-1,n-1}\}\phi$$

in view of (2.7).

Proposition 2.11. *With the notation above we have the intrinsic representation*

$$\phi(z) = \int_X p^X(\zeta,z)\phi(\zeta), \quad \phi \in \mathcal{O}(X), \tag{2.10}$$

where

$$p^X(\zeta,z) = \omega \wedge i^*\{h \wedge g_{n-1,n-1}\}. \tag{2.11}$$

In fact, if Ω is pseudoconvex, then each $\phi \in \mathcal{O}(X)$ is $i^*\Phi$ for some $\Phi \in \mathcal{O}(\Omega)$ and so the proposition follows from the argument above. In the general case one has to compare with the formula obtained from a weight g'' with small support near z. Since we are only interested in the local case we leave the details to the reader.

Remark 2.12. If g is chosen such that it depends holomorphically of z in $\Omega' \subset \Omega$, then the formula (2.10) provides an explicit extension of ϕ to Ω'. $\square$

Example 2.13. If we are in the ball as above, and $n \geq 2$, then, cf., Example 2.5,

$$g_{n-1,n-1} = -\bar\partial\chi \wedge v_{n-1,n-1} = -\bar\partial\chi \wedge \frac{1}{(2\pi i)^{n-1}} \frac{\partial|\zeta|^2 \wedge (\bar\partial\partial|\zeta|^2)^{n-2}}{(|z|^2 - \bar\zeta \cdot z)^{n-1}}.$$

Thus the integration in (2.10) is performed over the "boundary" of $i^*\chi$ on X. $\square$

3. Lecture 3

In order to study the $\bar\partial$-equation we will now turn our attention to formulas that also involve $\bar\partial\phi$ for smooth forms, so-called Koppelman formulas. We first have to discuss operators defined by integral kernels.

3.1. **Integral operators.** Let $k(\zeta,z)$ be a current on $\Omega_\zeta \times \Omega'_z \subset \mathbb{C}^n \times \mathbb{C}^n$, where $\Omega' \subset \Omega \subset \mathbb{C}^n$. If ϕ is a smooth form in Ω_ζ such that $k \wedge \phi \otimes 1$ has compact support in Ω_ζ we define

$$\int_\zeta k(\zeta,z) \wedge \phi(\zeta) := \pi_*(k \wedge \phi \otimes 1), \tag{3.1}$$

where π is the natural projection $(\zeta,z) \mapsto z$. That is,

$$\int_z \Big(\int_\zeta k(\zeta,z) \wedge \phi(\zeta)\Big) \wedge \psi(z) = \int_{\zeta,z} k \wedge \phi \otimes 1 \wedge 1 \otimes \psi$$

for test forms ψ. Here we have used the natural orientation on $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z$. (Notice that the definition of push-forward is unaffected if we put ψ "on the left hand side" in this definition.) A moment of thought reveals that in practice the integral on the right is computed by first moving all differentials of ζ to the right (or to the left) and then performing the integration with respect to ζ. For instance, if $\psi(\zeta,z)$ is a function, then

$$\int_\zeta \psi(\zeta,z) d\zeta \wedge dz \wedge d\bar\zeta = -\Big[\int_\zeta \psi(\zeta,z) d\zeta \wedge d\bar\zeta\Big] dz.$$

3.2. **Weighted Koppelman formulas in $\mathbb{C}^n$.** Previously we have kept z fixed but now we want z to vary on a subset $\Omega' \subset \Omega$. We define $\mathcal{L}^k(\Omega, \Omega')$ as the direct sum of all $\mathcal{E}_{\ell,\ell+k}(\Omega_\zeta \times \Omega'_z)$ but where we only consider forms without any dz_j. That is, we only incorperate forms that has bidegree $(0, q)$ in z. It is convenient to use the notation $\eta_j = \zeta_j - z_j$. Notice that δ, i.e., interior multiplication with

$$\sum_j \eta_j \frac{\partial}{\partial \zeta_j},$$

as well as $\bar\partial$, are well-defined on $\mathcal{L}^k(\Omega, \Omega')$.

We say that a smooth form $g \in \mathcal{L}^0(\Omega, \Omega')$ is a weight with respect to $\Omega' \subset \Omega$ if $\nabla g = 0$ and $g_{0,0} = 1$ on the diagonal $\Delta \subset \Omega \times \Omega'$.

Notice that, as before, if w is any smooth form in $\mathcal{L}^{-1}(\Omega, \Omega')$, then $1 + \nabla w$ is a weight with respect to Ω'.

Example 3.1. If $\Omega = (1 + \epsilon)\mathbb{B}$, $\Omega = \mathbb{B}$ and s and $\chi = \chi(\zeta)$ are as in Example 2.5, then $g = \chi + \bar\partial\chi \wedge s/\nabla s$ is a weight with respect to Ω' that is holomorphic in $z \in \Omega'$. Moreover, g has compact support in Ω_ζ. □

We let b be as before. We define

$$B = b/\nabla b = b + b \wedge \bar\partial b + \cdots + b \wedge (\bar\partial b)^{n-1}$$

but let $\bar\partial$ now also act on the z-variable. One can prove that then

$$\nabla B = 1 - [\Delta]', \tag{3.2}$$

where $[\Delta]'$ is the component of $[\Delta]$ of full degree in $d\zeta$. This means that

$$\int_{\zeta,z} [\Delta]' \wedge \xi(\zeta) = \xi(z)$$

for test forms ξ of bidegree $(0, q)$. If g is a weight, then

$$\nabla(B \wedge g) = (1 - [\Delta]') \wedge g = g - [\Delta]'$$

and hence

$$\bar\partial(B \wedge g)_{n,n-1} = [\Delta]' - g_{n,n}. \tag{3.3}$$

We let

$$k := (B \wedge g)_{n,n-1}, \quad p := g_{n,n}.$$

If Φ is a smooth $(0, k)$-form with compact support (or such that $k \wedge \Phi \otimes 1$ and $p \wedge \Phi \otimes 1$ have compact support in ζ), then we get from (3.3) the Koppelman formula

$$\Phi(z) = \bar\partial_z \int k(\cdot, z) \wedge \Phi + \int k(\cdot, z) \wedge \bar\partial\Phi + \int p(\cdot, z) \wedge \Phi, \quad z \in \Omega'. \tag{3.4}$$

The signs are correct! There is a similar formula where the integration is over $D \subset \omega'$ and $z \in D$. Then also the boundary integral of $k \wedge \Phi$ appears in the formula (with plus sign!).

If Ω' is pseudoconvex, then one can choose g that it is holomorphic in $z \in \Omega'$; for the case of the ball, see Example 3.1 above. Then $p = 0$ if $k \geq 1$, or more precisely, the component of p that has bidegree $(0, k)$ in z vanishes, since no $d\bar z_j$ can appear in $p = g_{n,n}$. If in addition $\bar\partial\Phi = 0$ it follows from (3.4) that

$$\Phi(z) = \bar\partial \int k(\cdot, z) \wedge \Phi, \quad z \in \Omega',$$

so we get a solution to $\bar\partial v = \Phi$ in Ω'.

3.3. **A weighted Koppelman formula on** X**.** Let $X \subset \Omega$ be a smooth complex hypersurface as before, and let $X' = X \cap \Omega'$.

Assume that g is any weight with with respect to Ω' with compact support in Ω_ζ, and that g' is the singular weight from Section 2.3. If we formally replace g by $g' \wedge g$ in (3.4) and only consider $z \in X'$ we have that

$$k = (g' \wedge g \wedge B)_{n,n-1} = (\bar\partial \frac{1}{f} \wedge h \wedge g \wedge B)_{n,n-1} = \\ \bar\partial \frac{1}{f} \wedge h \wedge (g \wedge B)_{n-1,n-2} = \frac{1}{f} \wedge d\zeta \wedge \{h \wedge (g \wedge B)_{n-1,n-2}\}.$$

In particular, it is $\mathcal{O}(|\eta|^{-(2n-3)})$ and thus locally integrable on X.

It follows, still by formal calulation, that

$$\int_\Omega k \wedge \Phi = \int_\Omega \frac{1}{f} \wedge d\zeta \wedge \{h \wedge (g \wedge B)_{n-1,n-2}\} \wedge \Phi = \int_X k^X(\cdot, z) \wedge \phi,$$

where

$$k^X = \omega \wedge i^*\{h \wedge (g \wedge B)_{n-1,n-2}\}. \tag{3.5}$$

If we define p^X as before, cf., (2.11), i.e.,

$$p^X = \omega \wedge i^*\{h \wedge g_{n-1,n-1}\}, \tag{3.6}$$

but this time $\bar\partial$ acts on z as well, then we formally get the Koppelman formula

$$\phi(z) = \bar\partial_z \int_X k^X(\cdot, z) \wedge \phi + \int_X k^X(\cdot, z) \wedge \bar\partial\phi + \int_X p^X(\cdot, z) \wedge \phi, \quad z \in X'. \tag{3.7}$$

Notice that all the integrals make sense since k is integrable on X. Moreover we have

Lemma 3.2. *If ξ is a smooth $(0,k)$-form on X then*

$$z \mapsto \int_X k^X(\cdot, z) \wedge \xi$$

is smooth on X'.

This follows from real analysis, see [2, Lemmas 6.1 and 6.2]. Intuitively it holds since the only singularity comes from B and it gives rise to an approximate convolution. The analogous statement for p^X is obvious.

Again, if we are in a situation where g is holomorphic in z, for instance, Ω is the ball, then p vanishes when $k \geq 1$. If in addition $\bar\partial\phi = 0$ we thus get

$$\phi(z) = \bar\partial_z \int_X k^X(\cdot, z) \wedge \phi(\zeta), \quad z \in X',$$

and thus a solution to $\bar\partial v = \phi$ on X'.

Proposition 3.3. *The Koppelman formula* (3.7) *holds.*

Koppelman formulas on a submanifold $X \subset \Omega$, with weights g of the form in Example 2.3, appeared in [4].

3.4. **Proof of Proposition 3.3.**

Lemma 3.4. *Let χ be a smooth approximation of the characteristic function for the interval $[1,\infty)$ and let $\chi_\epsilon = \chi(|f|^2/\epsilon)$. Then*

$$\bar\partial \frac{\chi_\epsilon}{f} \to \bar\partial \frac{1}{f}$$

weakly as measures when $\epsilon \to 0$.

Sketch of proof. Since $df \neq 0$ we may assume that $f = z_1$. Then the lemma follows from the corresponding one-variable statement. Thus it is enough to check that

$$\bar\partial \frac{\chi_\epsilon}{z} \wedge \frac{dz}{2\pi i} \to \bar\partial \frac{1}{z} \wedge \frac{dz}{2\pi i} = [0]$$

as measures. We leave the details to the reader. □

We now introduce the smooth regularization

$$g^\epsilon := 1 - \nabla\Big(\chi_\epsilon \frac{h}{f}\Big) = 1 - \chi_\epsilon + \chi_\epsilon \frac{f(z)}{f} + \bar\partial \frac{\chi_\epsilon}{f} \wedge h$$

of g'. In view of the lemma $g^\epsilon \to g'$ as measures.

We know that the Koppelman formula (3.4) holds with g replaced by $g^\epsilon \wedge g$, i.e., with k and p replaced by

$$k^\epsilon := (g^\epsilon \wedge g \wedge B)_{n,n-1}, \quad p^\epsilon := (g^\epsilon \wedge g)_{n,n}.$$

We thus have

$$\begin{aligned} (3.8) \quad & \int_{X'} \phi(z) \wedge \psi(z) = \\ & \pm \int_{\Omega\times X'} k^\epsilon \wedge \Phi \wedge \bar\partial\psi(z) + \int_{\Omega\times X'} k^\epsilon \wedge \bar\partial\Phi \wedge \psi(z) + \int_{\Omega\times X'} p^\epsilon \wedge \Phi \wedge \psi(z) \end{aligned}$$

for test forms ψ in X'.

Proposition 3.5. *The equation* (3.8) *tends to*

$$\int_{X'} \phi(z)\wedge\psi(z) = \pm \int_{X\times X'} k^X \wedge\phi\wedge\bar\partial\psi(z) + \int_{X\times X'} k^X \wedge\bar\partial\phi\wedge\psi(z) + \int_{X\times X'} p^X \wedge\phi\wedge\psi(z).$$

This is just a reformulation of Proposition 3.3.

Proof. The term with p^ϵ is comparatively simple to handle so we prove that

$$(3.9) \qquad \int_{\Omega\times X'} (g^\epsilon \wedge g \wedge B)_{n,n-1} \wedge \Phi \wedge \xi \to \int_{X\times X'} k^X \wedge \phi \wedge \xi$$

for each smooth $(0,*)$-form ξ.

Since $f(z) = 0$,

$$g^\epsilon = 1 - \chi_\epsilon + \bar\partial \frac{\chi_\epsilon}{f} \wedge h.$$

Since B is locally integrable on Ω,

$$(3.10) \qquad \int_{\Omega\times X'} ((1-\chi_\epsilon) \wedge g \wedge B)_{n,n-1} \wedge \Phi \wedge \xi \to 0$$

by dominated convergence. It remains to consider

$$I=\int_{\Omega\times X'}\Big(\bar\partial\frac{\chi_\epsilon}{f}\wedge h\wedge g\wedge B\Big)_{n,n-1}\wedge\Phi\wedge\xi=$$
$$\int_{\Omega\times X'}\bar\partial\frac{\chi_\epsilon}{f}\wedge h\wedge(g\wedge B)_{n-1,n-2}\wedge\Phi\wedge\xi.$$

Since $B_{k,k-1}$ is locally integrable on X if $k\le n-1$ is is not hard to see that

$$\zeta\mapsto\int_{X'}h\wedge(g\wedge B)_{n-1,n-2}\wedge\psi$$

is continuous in Ω. It follows from Lemma 3.4 that I tends to

$$\int_{\Omega\times X'}\bar\partial\frac{1}{f}\wedge h\wedge(g\wedge B)_{n-1,n-1}\wedge\Phi\wedge\xi=$$
$$\int_{X\times X'}\omega\wedge i^*\{h\wedge(g\wedge B)_{n-1,n-2}\}\wedge\phi\wedge\xi=\int_{X\times X'}k^X\wedge\phi\wedge\xi.$$

Together with (3.10) we get (3.9), and so the proposition is proved. $\square$

4. Lecture 4

If f is holomorphic in $\Omega\subset\mathbb{C}^n$ and we allow df to vanish, then in general $1/f$ is not locally integrable. Nevertheless there is a distribution U such that $fU=1$. Then $R=\bar\partial U$ is a $(0,1)$-current; clearly it vanishes outside the zero set $V(f)$ of f. It is easy to verify that a holomorphic function ϕ is in the ideal (f) generated by f if and only if $\phi R=0$. (If $0=\phi R=\phi\bar\partial U=\bar\partial(\phi U)$, then $a=\phi U$ is holomorphic and $af=\phi Uf=\phi$ so that $\phi\in(f)$. The converse is even simpler!) Thus, unless $V(f)=\emptyset$, R does not vanish identically, so there is something left, a residue, and therefore R is called a *residue current*.

However, U and R are by no means unique. There is however a canonical choice due to Herrera and Liebermann in the early 70's. See Section 4.1 below for a discussion of currents on a singular space.

Theorem 4.1. *If f is holomorphic and not vanishing identically on (the reduced pure-dimensional analytic space) X and v is a smooth strictly positive function, then*

$$\Big[\frac{1}{f}\Big]:=\lim_{\epsilon\to0}\chi\big(|f|^2v/\epsilon\big)\frac{1}{f}=\frac{|f|^{2\lambda}v^\lambda}{f}\Big|_{\lambda=0}$$

exists and $f[1/f]=1$. Moreover, if a is holomorphic and non-vanishing, then $[1/af]=(1/a)[1/f]$

Here χ is either the characterstic function for $[1,\infty)$ or a smooth apporoximand. The meaning of the rightmost member is the following: The function $\lambda\mapsto|f|^{2\lambda}v^\lambda/f$, that is elementarily defined for $\mathrm{Re}\,\lambda\gg0$, admits a current-valued analytic continuation to $\mathrm{Re}\,\lambda>-\delta$, and the expression is the value at $\lambda=0$. It is often more convenient to use the definition definition with λ. However, in these notes we stick to the ϵ-definition which is, perhaps, more conceptual.

It is clear that $f[1/f]=\lim\chi(|f|^2/\epsilon)=1$. The last statement follows by replacing v by $v|a|^2$. It implies that if f is a holomorphic section of a line bundle, then the principal value current $[1/f]$ has a well-defined meaning.

4.1. **Currents on a non-smooth reduced X of pure dimension n.** Locally there is an embedding

$$i\colon X \to \Omega \subset \mathbb{C}^N. \tag{4.1}$$

Let $\operatorname{Ker} i^*$ be the subsheaf of smooth forms ξ on Ω such that the pullback $i^*\xi$ to X_{reg} vanishes. Then $\mathcal{E}^X := \mathcal{E}^\Omega/\operatorname{Ker} i^*$ is the sheaf of smooth forms on X. Any embedding at a given point has the form $j\colon X \subset \Omega' \to \Omega' \times \mathbb{C}^{N-M}$, where j is a *minimal* embedding, and therefore unique. It follows that the definition of $\mathcal{E}^X$ is independent of the choice of embedding.

As in the smooth case the sheaf of currents $\mathcal{C}^X_{p,q}$ of bidegree (p,q) is the dual of $\mathcal{D}^X_{n-p,n-q}$. Given an embedding this means that each such current τ is represented by a unique $(N-n+p, N-n+q)$-current $T = i_*\tau$ in Ω such that $T.\xi$ vanishes for all ξ such that $i^*\xi = 0$.

If $p\colon X' \to X$ is proper and τ is a current on X', then $p_*\tau$, defined by $p_*\tau.\xi = \tau.\pi^*\xi$, is a current on X. For more details, see, e.g., [1].

4.2. **The case with monomials.** Let us first consider the one-variable case.

Proposition 4.2. *For each integer $m \geq 1$ and test function $\xi \in \mathcal{D}(\mathbb{C})$ the limit*

$$\frac{1}{z^m}.\xi dz \wedge d\bar{z} = \lim_{\epsilon \to 0} \int \chi(|z|^2/\epsilon) \frac{\xi dz \wedge d\bar{z}}{z^m} \tag{4.2}$$

exists and defines a current. We have the following equalities:

$$z\frac{1}{z^{m+1}} = \frac{1}{z^m}, \tag{4.3}$$

$$\frac{\partial}{\partial z}\frac{1}{z^m} = -m\frac{1}{z^{m+1}}, \quad m \geq 1, \tag{4.4}$$

$$\bar{\partial}\frac{1}{z^m}.\xi dz = \frac{2\pi i}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}\xi(0), \qquad m \geq 1, \tag{4.5}$$

$$\bar{z}\bar{\partial}\frac{1}{z^m} = 0, \qquad d\bar{z} \wedge \bar{\partial}\frac{1}{z^m} = 0, \quad m \geq 1. \tag{4.6}$$

$$\partial\frac{1}{z^m} = -m\frac{1}{z^{m+1}}dz, \qquad \partial\bar{\partial}\frac{1}{z^m} = m\bar{\partial}\frac{1}{z^{m+1}} \wedge dz, \quad m \geq 1 \tag{4.7}$$

$$\bar{\partial}\frac{1}{z^m} \wedge \frac{dz^m}{2\pi i} = m[0]. \tag{4.8}$$

This result is well-known and quite elementary, and we omit the proof. In $\mathbb{C}^n$, let $z^m := z_1^{m_1} \cdots z_r^{m_r}$, $r \leq n$, $m_j \geq 1$. We can form tensor product

$$\Big[\frac{1}{z^m}\Big] := \Big[\frac{1}{z_1^{m_1}}\Big] \otimes \cdots \otimes \Big[\frac{1}{z_r^{m_r}}\Big].$$

Proposition 4.3. *With χ as above we have*

$$\Big[\frac{1}{z^m}\Big] = \lim_{\epsilon \to 0} \chi\big(|z^{\widehat{m}}|^2 v/\epsilon\big)\frac{1}{z^m} \tag{4.9}$$

if each factor of the monomial z^m is a factor in $z^{\widehat{m}}$.

This proposition is not at all trivial. For a proof see, e.g., [1] or [5]. On the other hand, the analogous statement with the λ-definition is immediate in case $v \equiv 1$, and the general case follows then by a simple additional argument, see, e.g., [1].

4.3. **Proof of Theorem 4.1.** Since currents are locally defined, it follows from Proposition 4.3 that the principal value current

$$\Big[\frac{1}{f}\Big]=\lim_{\epsilon\to 0}\chi\big(|f|^2v/\epsilon\big)\frac{1}{f}$$

exists, and is independent of v and χ, as soon as locally one can choose coordinates with respect to which f is a monomial.

The case with a general f follows from the possibility to resolve singularities. Let $\pi\colon X'\to X$ be a modification, i.e., π is proper and a biholomorphism outside a set of positive codimension, so that X' is smooth and locally in X', π^*f is a monomial in suitable local coordinates. Then

$$\int_X\chi\big(|f|^2v/\epsilon\big)\frac{1}{f}\xi=\int_{X'}\chi\big(|\pi^*f|^2\pi^*v/\epsilon\big)\frac{1}{\pi^*f}\pi^*\xi\to\int_{X'}\Big[\frac{1}{\pi^*f}\Big]\pi^*\xi=\int_X\pi_*\Big[\frac{1}{\pi^*f}\Big]\xi.$$

Thus the limit exists, and moreover $[1/f]$ is equal to the push forward $\pi_*[1/\pi^*f]$ of the principal value current $[1/\pi^*f]$ in X'.

4.4. **The sheaf of pseudomeromorphic currents.** Assume that $m=(m',m'')=(m_1,\ldots,m_\nu,m_{\nu+1},\ldots,m_r)$ and let

$$\frac{1}{z^{m'}}\bar\partial\frac{1}{z^{m''}}:=\frac{1}{z^{m'}}\otimes\bar\partial\frac{1}{z_{\nu+1}^{m_{\nu+1}}}\wedge\ldots\wedge\bar\partial\frac{1}{z_r^{m_r}}.$$

If α is a smooth form with compact support, then we say that

$$\tau=\frac{1}{z^{m'}}\bar\partial\frac{1}{z^{m''}}\wedge\alpha$$

is an *elementary current*. Notice that

$$\bar z_j\tau=d\bar z_j\wedge\tau=0 \tag{4.10}$$

if z_j is a factor in $z^{m''}$.

Fix a point $x\in X$. We say that a germ μ of a current at x is *pseudomeromorphic* at x, $\mu\in\mathcal{PM}_x$, if it is a finite sum of currents of the form $\pi_*\tau=\pi^1_*\cdots\pi^m_*\tau$, where $\mathcal{U}$ is a neighborhood of x,

$$\mathcal{U}^m\xrightarrow{\pi^m}\cdots\xrightarrow{\pi^2}\mathcal{U}^1\xrightarrow{\pi^1}\mathcal{U}^0=\mathcal{U}, \tag{4.11}$$

each $\pi^j\colon\mathcal{U}^j\to\mathcal{U}^{j-1}$ is either a modification, a simple projection $\mathcal{U}^{j-1}\times Z\to\mathcal{U}^{j-1}$, or an open inclusion (i.e., $\mathcal{U}^j$ is an open subset of $\mathcal{U}^{j-1}$), and τ is elementary on $\mathcal{U}^m$.

By definition the union $\mathcal{PM}=\cup_x\mathcal{PM}_x$ is an open subset of the sheaf $\mathcal{C}=\mathcal{C}^X$ and hence it is a subsheaf, the sheaf of *pseudomeromorphic* currents, of $\mathcal{C}$. A section μ of $\mathcal{PM}$ over an open set $\mathcal{V}\subset X$, $\mu\in\mathcal{PM}(\mathcal{V})$, is then a locally finite sum

$$\mu=\sum(\pi_\ell)_*\tau_\ell, \tag{4.12}$$

where each π_ℓ is a composition of mappings as in (4.11) (with $\mathcal{U}\subset\mathcal{V}$) and τ_ℓ is elementary. The definition here is from [2] and it is in turn a slight elaboration of the definition introduced in [3].

If ξ is a smooth form, then $\xi\wedge\pi_*\tau=\pi_*\big(\pi^*\xi\wedge\tau\big)$. Thus $\mathcal{PM}$ is closed under exterior multiplication by smooth forms. Notice that if τ is an elementary current, then $\bar\partial\tau$ is a finite sum of elementary currents. Since moreover $\bar\partial$ commutes with push-forwards, $\mathcal{PM}$ is closed under $\bar\partial$.

Example 4.4. Let f be a holomorphic in X. By a partition of unity in X' (using the notation in the proof of Theorem 4.1) we see that $1/\pi^* f$ is a locally finite sum of elementary currents. It follows that $[1/f] = \pi_*[1/\pi^* f]$ is pseudomeromorphic in X. Thus also the residue current $\bar\partial[1/f]$ is pseudomeromorphic. □

Proposition 4.5. *If h is holomorphic and vanishes on $\operatorname{supp}\mu$, where μ is pseudomeromorphic, then $\bar h\mu = d\bar h \wedge \mu = 0$.*

Sketch of proof. Assume that μ has the representation (4.12). By further desingularizations we may assume as well that $\pi^* h$ locally is a monomial (with with respect to the same local coordinates that are involved in τ_ℓ). Now,

$$0 = \chi(|h|^2/\epsilon)\mu = \sum_\ell \pi_*(\chi(|\pi^* h|^2/\epsilon)\tau_\ell).$$

If $\operatorname{supp}\tau_\ell \subset \pi^{-1}V(h)$ then

$$T_\epsilon = \chi(|\pi^* h|^2/\epsilon)\tau_\ell$$

vanishes. Otherwise, we may assume that $\pi^* h$ is independent of the coordinates involved in the the residue factors of τ_ℓ. It follows from Proposition 4.3 that then $T_\epsilon \to \tau_\ell$. We conclude that

$$\mu = \sum_{\operatorname{supp}\tau_\ell \subset \pi^{-1}V(h)} \pi_*\tau_\ell.$$

It now follows from (4.10) that

$$\bar h\mu = \sum_{\operatorname{supp}\tau_\ell \subset \pi^{-1}V(h)} \pi_*(\overline{\pi^* h}\tau_\ell) = 0.$$

In a similar way $d\bar h \wedge \mu = 0$. □

By a standard argument, Proposition 4.3 leads to

Theorem 4.6 (Dimension principle)**.** *If T is pseudomeromorphic with bidegree $(*, p)$ and has support on a variety V with $\operatorname{codim} V > p$, then $T = 0$.*

Assume that μ is pseudomeromorphic and V is a subvariety. Let h be a tuple of holomorphic functions such that the common zero set is precisely V. Then the limit

$$\mathbf{1}_{X\setminus V}\mu := \lim_{\epsilon\to 0} \chi(|h|^2 v/\epsilon)\mu$$

exists, and it is a pseudomeromorphic current that is independent of χ and v. Clearly,

$$\mathbf{1}_V\mu := \mu - \mathbf{1}_{X\setminus V}\mu$$

has support on V, and $\mathbf{1}_V\mu = \mu$ if μ has support on V. The existence of the limit and the independence of h follow from the corresponding statements for elementary currents. If μ has the form (4.12), then

$$\mathbf{1}_V\mu = \sum_{\operatorname{supp}\tau_\ell \subset \pi^{-1}V} \pi_*\tau_\ell.$$

It follows that

$$\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V\cap W}\mu = \mathbf{1}_W\mathbf{1}_V\mu.$$

Notice also that

$$\mathbf{1}_V(\alpha\wedge\mu) = \alpha\wedge\mathbf{1}_V\mu \tag{4.13}$$

if α is smooth.

In general pseudomeromorphic currents cannot be multiplied. However, if f is holomorphic we can define a mapping

$$\mu \mapsto \Big[\frac{1}{f}\Big]\mu := \lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f}\mu$$

on $\mathcal{PM}$. Although written as a multiplication, one should be careful and notice that

$$f\Big[\frac{1}{f}\Big]\mu = \mathbf{1}_{f\neq 0}\mu.$$

We can also define

$$\bar\partial\Big[\frac{1}{f}\Big]\wedge\mu = \lim_{\epsilon\to 0}\bar\partial\chi(|f|^2 v/\epsilon)\frac{1}{f}\wedge\mu$$

so that the Leibniz rule

$$\bar\partial\Big(\Big[\frac{1}{f}\Big]\mu\Big) = \bar\partial\Big[\frac{1}{f}\Big]\wedge\mu + \Big[\frac{1}{f}\Big]\bar\partial\mu \tag{4.14}$$

holds. Notice that $\bar\partial[1/f]\wedge\mu$ has support on $\{f=0\}$.

Example 4.7. The current $\mu = \log|z|^2$ is not pseudomeromorphic. In fact, $\bar\partial\mu = d\bar z/\bar z$ in the current sense, but the limit

$$\lim \chi(|z|^2/\epsilon)\frac{1}{z}\frac{d\bar z}{\bar z}$$

does not exist. □

Example 4.8. Assume that f is a meromorphic form. Then locally $f = g/h$ where g is a holomorphic form and h is a holomorphic function, not vanishing identically (on any irreducible component of X). Let us define the pseudomeromorphic current

$$[f] := g\Big[\frac{1}{h}\Big].$$

If $f = g'/h'$ is another representation of f, then $g/h = g'/h'$ where $h\neq 0$ and $h'\neq 0$. Thus the pseudomeromorphic current

$$T = g\Big[\frac{1}{h}\Big] - g'\Big[\frac{1}{h'}\Big]$$

vanishes outside a set of positive codimension. By the dimension principle we conclude that T vanishes identically. Thus the pseudomeromorphic current $[f]$ is well-defined. □

Example 4.9. Let f, g be holomorphic functions such that $\operatorname{codim}\{f=g=0\}\geq 2$. We claim that then

$$\bar\partial\Big[\frac{1}{f}\Big]\wedge\bar\partial\Big[\frac{1}{g}\Big] = -\Big[\frac{1}{g}\Big]\wedge\bar\partial\Big[\frac{1}{f}\Big].$$

First notice that the pseudomeromorphic current

$$T = \Big[\frac{1}{f}\Big]\bar\partial\Big[\frac{1}{g}\Big] - \bar\partial\Big[\frac{1}{g}\Big]\cdot\Big[\frac{1}{f}\Big]$$

vanishes outside $\{f=0\}$, since $[1/f]$ is equal to the smooth function $1/f$ there. On the other hand, both terms vanish outside the set $g=0$. Thus T has support on $\{f=g=0\}$, and hence it vanishes identically in view of the dimension principle. □

4.5. **The Coleff-Herrera product.** By similar arguments, for any tuple $f_1, \dots, f_m$ such that $\operatorname{codim}\{f_1 = \dots = f_m = 0\} \geq m$, we have the pseudomeromorphic current

$$\Big[\bar\partial \frac{1}{f_m} \wedge \dots \wedge \bar\partial \frac{1}{f_1}\Big] := \bar\partial \Big[\frac{1}{f_m}\Big] \wedge \dots \wedge \bar\partial \Big[\frac{1}{f_1}\Big],$$

called the *Coleff-Herrera product*, that is alternating in the factors. It was introduced by Coleff and Herrera in [7]. It is easy to check that it is annihilated by each function f_j, that is, by the ideal $J = \langle f_1, \dots, f_m\rangle$. It was proved independently by Passare, [11], and by Dickenstein and Sessa, [8], that also the converse holds if X is smooth: If ϕ annihilates this current, then ϕ is indeed in the ideal J.

Remark 4.10. If X is not smooth, then this fails already when $m = 1$, as $0 = \phi\bar\partial[1/f] = \bar\partial(\phi[1/f])$ only implies that ϕ/f is holomorphic in the sense of Barlet-Henkin-Passare, see, e.g., [9] or [1]. □

Notice that by our definition

$$\Big[\bar\partial \frac{1}{f} \wedge \bar\partial \frac{1}{g}\Big] = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \frac{\bar\partial\chi(|f|^2/\epsilon) \wedge \bar\partial\chi(|g|^2/\delta)}{fg}. \tag{4.15}$$

As suggested above one can take either χ as the characteristic function of $[1, \infty)$ or a smooth approximand.

Coleff and Herrera originally took the limit when $0 < \epsilon \ll \delta$ or $0 < \delta \ll \epsilon$ and proved that one gets the same result in either way. It was an open question for more than a decade whether the limit exists unconditionally. In 1995 Passare and Tsikh found an example, with $\chi = \chi_{[1,\infty)}$, where the limit does not exist unconditiononally. Shortly after that Björk proved that this phenomenon occurs for generic choices of f and g. Thus the question was supposed to be settled, until Samuelsson 2004 surprisingly proved that if one instead uses a smooth χ, then indeed the function

$$(\epsilon, \delta) \mapsto \frac{\bar\partial\chi(|f|^2/\epsilon) \wedge \bar\partial\chi(|g|^2/\delta)}{fg}.$$

is even Lipschitz continuous on $[0, 1] \times [0, 1]$. The same holds true for $m > 2$, see [6].

Remark 4.11. If one eases the condition $\operatorname{codim}\{f = g = 0\} \geq 2$, then things break down. For instance,

$$\bar\partial \frac{1}{zw} \wedge \bar\partial \frac{1}{z} = 0,$$

whereas

$$\bar\partial \frac{1}{z} \wedge \bar\partial \frac{1}{zw} = \bar\partial \frac{1}{z} \wedge \bar\partial \frac{1}{w} \neq 0.$$

□

5. Lecture 5

Now let $i\colon X \to \Omega \subset \mathbb{C}^n$ be a hypersurface, $X = \{f = 0\}$, such that $df \neq 0$ on X_{reg}, and let $X' = X \cap \Omega'$, where $\Omega' \subset\subset \Omega$ as before, say Ω' is a ball.

We can choose h and g as in Section 3.3 and define approximations of the kernels k and p. Outside the singularities of $X \times X'$ we then get the Koppelman formula

(3.7) which thus can be formulated as

$$\begin{aligned}(5.1)\quad \phi(z) = \bar\partial_z \int_X k^X(\cdot, z)\wedge\phi + \int_X k^X(\cdot, z)\wedge\bar\partial\phi + \\ \int_X p^X(\cdot, z)\wedge\phi, \quad \phi\in\mathcal{D}_{0,k}(X_{reg}), \quad z\in X'_{reg},\end{aligned}$$

or alternatively as

$$\bar\partial k^X = [\Delta]' - p^X \tag{5.2}$$

on $X_{reg}\times X'_{reg}$, where $[\Delta]'$ is the current on $X_{reg}\times X'_{reg}$ such that, for test forms ψ in $X_{reg}\times X'_{reg}$,

$$\int_{X\times X'} [\Delta]'\wedge\psi(\zeta, z)$$

is equal to

$$\int_{X'} \psi(z, z)$$

if ψ has bidegree $(0,*)$ in ζ and 0 otherwise. Clearly it has an extension to a current on $X\times X'$, that we also denote by $[\Delta]'$, such that the same holds for test forms in $X\times X'$. In fact, we can define the extension by

$$[\Delta]' = \lim_\delta \chi_\delta[\Delta]',$$

where $\chi_\delta = \chi(|h|^2/\delta)$ and $\{h = 0\} = (X\times X')_{sing}$. For the moment it is not clear that this extended current $[\Delta]'$ is pseudomeromorphic in $X\times X'$, but this will follow a posteriori from Theorem 5.2 below.

Remark 5.1. Assume that a is a current in $Y\backslash V$, and that χ_δ cuts out the analytic set $V\subset Y$ of positive codimension. If the limit $A = \lim_\delta \chi_\delta a$ exists, and is independent of the choice of χ_δ, one says that A is the *standard extension* of a. Notice that if A is pseudomeromorphic, then A is the standard extension of a across V if and only if $A = a$ in $X\setminus V$ and $\mathbf{1}_V A = 0$. □

Theorem 5.2. *The standard extensions of the currents k^X and p^X to $X\times X'$ exist and are pseudomeromorphic in $X\times X'$, and* (5.2) *holds on $X\times X'$.*

It follows that (5.1) holds in the current sense for all $\phi\in\mathcal{E}(X)$.

Lemma 5.3. *If ϕ is smooth on X, then*

$$v(z) = \int_X k^X(\cdot, z)\wedge\phi$$

is smooth on X'_{reg}.

If we have chosen g so that the component of bidegree $(0,k)$ in z of p^X vanishes when $k\ge 1$ in $X_{reg}\times X'_{reg}$, then also its standard extension vanishes. If $\phi\in\mathcal{E}_{0,k}(X)$ and $\bar\partial\phi = 0$ we thus get a current v on X' that is smooth on X'_{reg} and such that $\bar\partial v = \phi$ in the current sense on X'.

5.1. **Extension of ω.** We first discuss the extension of ω across X_{sing}. In view of (2.9) we see that $\omega := i^*\alpha$ is meromorphic on X and hence we have an extension of ω across X_{sing} as a principal value current, cf., Example 4.8. However, for simplicity we write ω rather than $[\omega]$ even for the extension. We claim that

$$i_*\omega = \bar\partial\Big[\frac{1}{f}\Big] \wedge d\zeta. \tag{5.3}$$

Proof of the claim. We already know that the equality holds outside X_{sing} that has codimension at least 2, so if we also a priori knew that the left hand side is pseudomeromorphic, then the claim would follow from the dimension principle. It is true, but a quite deep fact, that indeed i_* maps $\mathcal{PM}^X$ into $\mathcal{PM}^\Omega$, see [1]. However, there is a simpler way out: Assume that $X_{sing} = \{h = 0\}$ so that $\chi_\delta = \chi(|h|^2/\delta)$ cuts out X_{sing} in Ω. Then $i^*\chi_\delta$ cuts out X_{sing} in X and thus

$$\chi_\delta i_*\omega = i_*(i^*\chi_\delta \cdot \omega) \to i_*\mathbf{1}_{X\setminus X_{sing}}\omega = i_*\omega$$

since $\mathbf{1}_{X_{sing}}\omega = 0$ by the dimension principle. On the other hand we also have that

$$\chi_\delta i_*\omega = \chi_\delta\bar\partial\Big[\frac{1}{f}\Big] \wedge d\zeta \to \mathbf{1}_{\Omega\setminus X_{sing}}\bar\partial\Big[\frac{1}{f}\Big] \wedge d\zeta = \bar\partial\Big[\frac{1}{f}\Big] \wedge d\zeta,$$

since $\mathbf{1}_{X_{sing}}\bar\partial[1/f] \wedge d\zeta = 0$ by the dimension principle, as already noted above. $\square$

From (5.3) we have that

$$i_*\bar\partial\omega = \bar\partial i_*\omega = \bar\partial\Big(\bar\partial\Big[\frac{1}{f}\Big] \wedge d\zeta\Big) = 0,$$

where the last equality follows from the Leibniz rule (4.14). Since i_* is injective we conclude that

$$\bar\partial\omega = 0 \tag{5.4}$$

on X.

Remark 5.4. We claim that if ϕ is a smooth $(n-1,*)$-form on X, then there is a (unique) smooth form $\hat\phi$ on X, such that $\phi = \hat\phi \wedge \omega$. To see this, assume that $\phi = i^*\Phi$ where Φ is a smooth $(n-1,*)$-form in Ω. Then there is a smooth $(0,*)$-form $\hat\Phi$ in Ω such that $\Phi \wedge df = 2\pi i\hat\Phi \wedge dz$. If $2\pi i dz = df \wedge \alpha$ (close to X), then

$$\Phi \wedge df = 2\pi i\hat\Phi \wedge dz = \pm\hat\Phi \wedge \alpha \wedge df,$$

and hence $\phi = i^*\Phi = \pm i^*\hat\Phi \wedge i^*\alpha = \hat\phi \wedge \omega$. Since ω is non-vanishing on X_{reg}, the form $\hat\phi$ must be unique. $\square$

5.2. **Extension of B.** Let E be a trivial rank n-bundle over $X \times X'$ with global frame $e_1, \ldots, e_n$, and let us form the exterior algebra over $E \oplus T^*_{0,1}(X \times X')$. Then we can define the form

$$b' = \frac{\sum_j \bar\eta_j \wedge e_j}{|\zeta|^2},$$

and let

$$B' = b' + b' \wedge \bar\partial b' + \cdots + b' \wedge (\bar\partial b')^{n-1}.$$

Similarly we let h' and g' be the forms obtained from h and g, respectively, by replacing $d\zeta_j$ by e_j. Notice that we then have the equality

$$\{h \wedge (B \wedge g)_{n-1,n-2}\} = \{h' \wedge (B' \wedge g')_{n-1,n-2}\}$$

where the right hand side is defined by the equality

$$\{h' \wedge (B' \wedge g')_{n-1,n-2}\} \wedge e_1 \wedge \ldots \wedge e_n = h' \wedge (B' \wedge g')_{n-1,n-2}.$$

Thus it is enough to describe how the coefficients in B' are extended pseudomeromorphically across X_{sing}.

Let $\pi\colon Y \to X \times X'$ be a principalization of the ideal sheaf on $X \times X'$ generated by the η_j such that π is a biholomorphism $Y \setminus \pi^{-1}\Delta \simeq X \times X' \setminus \Delta$. Then there is a a holomorphic section σ on Y of a certain line bundle $L \to Y$ and a smooth section a on Y of $L \otimes \pi^* E$ such that $\pi^* b' = a/\sigma$ in $Y \setminus \pi^{-1}\Delta$. We thus have that

$$\pi^*(b' \wedge (\bar\partial b')^{\ell-1}) = \frac{a_\ell}{\sigma^\ell},$$

where a_ℓ is a smooth section of $L^\ell \otimes \Lambda^\ell \pi^* E$. In view of Theorem 4.1 there is a canonical pseudomeromorphic extension A of a_ℓ/σ^ℓ across $Y \setminus \pi^{-1}\Delta$. We get the desired pseudomeromorphic extension of $B'_\ell = b' \wedge (\bar\partial b')^{\ell-1}$ as $\pi_* A$. It coincides with B'_ℓ outside Δ since π is a biholomorphism there by assumption.

Notice that this extension of B' is unique, in view of the dimension principle, since since the highest bidegree of B' that occurs is $(0, n-2)$ and Δ has codimension $n-1$ in $X \times X'$. Let B' also denote this extension.

5.3. **The heart of residue theory.** Assume that a and b are currents in $Y \setminus V$ that have standard extensions A and B, respectively, to Y. Moreover, assume that

$$\bar\partial a = b$$

in $Y \setminus V$. It is natural to ask whether

$$\bar\partial A = B \tag{5.5}$$

holds.

Proposition 5.5. *Let χ_δ be a sequence as above that cuts out V. Then the limit*

$$\lim_\delta \bar\partial \chi_\delta \wedge a \tag{5.6}$$

exists, and it is zero if and only if (5.5) *holds.*

If A is pseudomeromorphic, then

$$\lim_\delta \bar\partial \chi_\delta \wedge a = \mathbf{1}_V \bar\partial A. \tag{5.7}$$

If A is pseudomeromorphic, therefore (5.5) holds if and only if no *residue* appears at V when applying $\bar\partial$ to A.

Proof. By assumption, $\chi_\delta a \to A$ and $\chi_\delta b \to B$, so

$$\bar\partial \chi_\delta \wedge a = \bar\partial(\chi_\delta a) - \chi_\delta \bar\partial a = \bar\partial(\chi_\delta a) - \chi_\delta b \to \bar\partial A - B.$$

Thus the first part follows.

If A is pseudomeromorphic, then also $\bar\partial A$ is, and we have

$$\mathbf{1}_V \bar\partial A = \bar\partial A - \mathbf{1}_{Y\setminus V} \bar\partial A = \bar\partial A - \lim_\delta \chi_\delta \bar\partial A = \bar\partial A - \lim_\delta \bar\partial(\chi_\delta A) + \lim_\delta \bar\partial \chi_\delta \wedge A =$$
$$\bar\partial A - \bar\partial A + \lim_\delta \bar\partial \chi_\delta \wedge A = \lim_\delta \bar\partial \chi_\delta \wedge a.$$

□

5.4. **Proof of Theorem 5.2 and Lemma 5.3.** We first define extensions of k^X and p^X. In view of the preceding sections

$$\{h \wedge (B \wedge g)_{n-1,n-2}\}$$

is pseudomeromorphic on $X \times X'$ and since ω is meromorphic,

$$k^X := \omega \wedge \{h \wedge (B \wedge g)_{n-1,n-2}\} \tag{5.8}$$

is a well-defined pseudomeromorphic current on $X \times X'$ that coincides with (3.5) on $X_{reg} \times X'_{reg}$. In the same way we define the extension

$$p^X := \omega \wedge \{h \wedge g_{n-1,n-1}\}$$

of p^X in $X_{reg} \times X'_{reg}$.

We are to prove that they are the standard extensions across $(X \times X')_{sing}$. In view of Remark 5.1 we thus have to check that

$$\mathbf{1}_{(X\times X')_{sing}} k^X = 0, \quad \mathbf{1}_{(X\times X')_{sing}} p^X = 0. \tag{5.9}$$

We will also prove that

$$\mathbf{1}_{(X\times X')_{sing}} \bar\partial k^X = 0. \tag{5.10}$$

We focus on the latter equality; (5.9) is obtained in the same way.

We use the equality

$$\mathbf{1}_{(X\times X')_{sing}} \bar\partial k^X = \mathbf{1}_{(X\times X')_{sing}} \mathbf{1}_{X\times X'\setminus\Delta} \bar\partial k^X + \mathbf{1}_{(X\times X')_{sing}} \mathbf{1}_{\Delta} \bar\partial k^X =: I + II.$$

Let us write $k^X = \omega \wedge A$, cf., (5.8). Outside Δ the second factor A is smooth, and so

$$\bar\partial k^X = \omega \wedge \bar\partial A,$$

since $\bar\partial\omega = 0$. Since $\bar\partial A$ is smooth thus

$$I = \mathbf{1}_{(X\times X')_{sing}} \mathbf{1}_{X\times X'\setminus\Delta} (\omega \wedge \bar\partial A) = \mathbf{1}_{(X\times X')_{sing}} \mathbf{1}_{X\times X'\setminus\Delta} \omega \wedge \bar\partial A = 0.$$

The last holds equality because $\mathbf{1}_{(X\times X')_{sing}}(\omega \otimes 1)$ vanishes by the dimension principle. We now consider II. Notice that $\Delta \simeq X'$ and that the current in II has support on Δ_{sing} which has codimension $(n-1)+1 = n$ in $X \times X'$. Since the current has bidegree $(*, n-1)$ it therefore vanishes by the dimension principle.

In view of Section 5.3 Theorem 5.2 follows from (5.9) and (5.10).

Proof of Lemma 5.3. Fix a small neighborhood $U \subset X_{reg}$ of a point in X_{reg}. If ϕ has support close to X_{sing}, then v is smooth in U since B is smooth outside the diagonal. If ϕ has support in X_{reg}, then v is smooth by Lemma 3.2. $\square$

5.5. **The strong $\bar\partial$-operator on X.** Assume that v and ϕ are pseudomeromorphic in X, smooth in X_{reg}, and that

$$\mathbf{1}_{X_{sing}} v = 0 = \mathbf{1}_{X_{sing}} \phi. \tag{5.11}$$

We then say that $\bar\partial_X v = \phi$ if

$$\bar\partial(\omega \wedge v) = \omega \wedge \phi. \tag{5.12}$$

Lemma 5.6. *If $\bar\partial_X v = \phi$, then $\bar\partial v = \phi$.*

Proof. Notice that (5.12) means that

$$\int v \wedge \omega \wedge \bar\partial\xi = \pm \int \phi \wedge \omega \wedge \xi \tag{5.13}$$

for test forms ξ of bidegree $(0,*)$. By Remark 5.4, for any test form η of bidegree $(n-1,0)$ there is a smooth form ξ such that $\eta = \xi\omega$ in X_{reg}. Thus $\bar\partial\eta = \bar\partial\xi \wedge \omega$ in X_{reg}. Because of the assumption (5.11), (5.13) implies that

$$\int v \wedge \bar\partial\eta = \pm \int \phi \wedge \eta.$$

We conclude that $\bar\partial v = \phi$. □

We say that v as above is in the *domain of* $\bar\partial_X$ if $\bar\partial_X v = \phi$ for some ϕ as above. One can check that this holds if and only if $\mathbf{1}_{X_{sing}}\bar\partial(v\wedge\omega) = 0$. This is an intrinsic boundary condition at X_{sing}.

With precisely the same arguments as in the proof of Theorem 5.2 we actually get the equality

$$\bar\partial(\omega_z \wedge k^X) = \omega_z \wedge [\Delta]' - \omega_z \wedge p^X.$$

This leads to a solution to $\bar\partial_X v = \phi$ in X' if ϕ is smooth on X and $\bar\partial\phi = 0$.

If ϕ is as above and $\bar\partial_X\phi = 0$, then

$$v = K\phi := \int_X k^X \wedge \phi$$

is a solution to $\bar\partial v = \phi$ in X_{reg}. One can prove that $\mathbf{1}_{X_{sing}} v = 0$ but we do not know whether $\bar\partial v = 0$ or $\bar\partial_X v = \phi$ on X in general.

5.6. **A fine resolution of $\mathcal{O}^X$.** We already mentioned that $v = K\phi_1$ is in the domain of $\bar\partial_X$ if ϕ_1 is smooth. With similar arguments one can prove that also $K(\phi_2 \wedge v)$ is in the domain of $\bar\partial_X$ if ϕ_2 is smooth. Proceeding in this way we obtain fine sheaves $\mathcal{A}_k$ of currents that are smooth on X_{reg} and in the domain of $\bar\partial_X$, and one can prove that

$$0 \to \mathcal{O}^X \to \mathcal{A}_0 \xrightarrow{\bar\partial} \mathcal{A}_1 \xrightarrow{\bar\partial}$$

is an exact sequence of sheaves, thus a fine resolution of the structure sheaf $\mathcal{O}^X$.

5.7. **The case with a general reduced analytic space X.** Let us conclude with a few words about the case when X is an arbitrary reduced analytic space of pure dimension n. Locally we have an embedding $i\colon X \to \Omega \subset \mathbb{C}^N$, and the analogues of $[1/f]$ and the associated residue current $\bar\partial[1/f]$ are currents U and R obtained from a free resolution of the ideal sheaf $\mathcal{J}_X$ in Ω associated with X.

In case X is locally Cohen-Macaulay, R is a vector-valued $\bar\partial$-closed $(0, N-n)$-current, and there is an associated form ω as before, called a *structure form* in [2], such that $i_*\omega = R \wedge dz$ in Ω. Thus ω is a tuple of $(n,0)$-forms ω_j such that $\bar\partial\omega_j = 0$. These forms actually generate the $\mathcal{O}^X$-module of all $\bar\partial$-closed $(n,0)$-forms. The form g is as before, and h corresponds to a certain holomorphic matrix that acts on the tuple ω. Besides various technicalities this case works pretty much in the same way as for the hypersurface discussed in these lectures.

When we go beyond the Cohen-Macaualay case, several substantially new difficulties arise. We still get solutions to $\bar\partial$ and we get a fine resolution of $\mathcal{O}^X$ but things are more involved, and we refer to the paper [2] for details.

References

[1] M. Andersson: *Residue theory and integral formulas*, In preparation.http://www.math.chalmers.se/ matsa/3introresidy.pdf

[2] M. Andersson, H. Samuelsson: A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas, *Inventiones math.*, **190** (2012), 261–297.

[3] M. Andersson, E. Wulcan: Decomposition of residue currents. *J. Reine Angew. Math.*, **638** (2010), 103–118.

[4] B. Berndtsson: *A formula for division and interpolation*, Math Ann. **263** (1983).

[5] J-E Björk: *Residues and $\mathcal{D}$-modules*, The legacy of Niels Henrik Abel, 605–651, Springer, Berlin, 2004.

[6] J-E. Björk, H. Samuelsson: Regularizations of residue currents. *J. Reine Angew. Math.*, **640** (2010), 101–115.

[7] N.r. Coleff & M.e. Herrera: *Les courants résiduels associés à une forme méromorphe*, Lect. Notes in Math. **633**, Berlin-Heidelberg-New York (1978).

[8] A. Dickenstein & C. Sessa: *Canonical representatives in moderate cohomology*, Invent. Math. **80** (1985), 417–434..

[9] G. Henkin, M. Passare: Abelian differentials on singular varieties and variations on a theorem of Lie-Griffiths. *Invent. Math.*, **135(2)** (1999), 297–328.

[10] G. Henkin, P. Polyakov: The Grothendieck-Dolbeault lemma for complete intersections. *C. R. Acad. Sci. Paris Sér I Math*, **308(13)** (1989), 405–409.

[11] M. Passare: *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), 75–152.

Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, S-412 96 GÖTEBORG, SWEDEN

E-mail address: matsa@math.chalmers.se