On the applicability of sensitivity analysis formulas for traffic equilibrium models

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Abstract

The paper by Tobin and Friesz (1988) brought the classic nonlinear programming subject of sensitivity analysis to transportation science. It is still the most widely used device by which "gradients" of traffic equilibrium solutions (that is, flows and/or demands) are calculated, for use in bilevel transportation planning applications such as network design, origin–destination (OD) matrix estimation and problems where link tolls are imposed on the users in order to reach a traffic management objective. However, it is not widely understood that the regularity conditions proposed by them are stronger than necessary. Also, users of their method sometimes misunderstand its limitations and are not aware of the computational advantages offered by more recent methods. In fact, a more often applicable formula was proposed already in 1989 by Qiu and Magnanti, and Bell and Iida (1997) describe one of the cases in practice in which the formula by Tobin and Friesz would not be able to generate sensitivity information, because one of their regularity conditions fails to hold.

This paper provides a short overview of a sensitivity formula that provides directional derivatives of traffic equilibrium flows, route and link costs, and demands, exactly when they exist, and which are found in Patriksson and Rockafellar (2003) and Patriksson (2003). For the simplicity of the presentation, we provide the analysis for the simplest cases, where the link travel cost and demand functions are separable, so that we can work with optimization formulations; this specialization was first given in Josefsson and Patriksson (2003). The connection between directional derivatives and the gradient is that exactly when the directional derivative mapping of the traffic equilibrium solution is linear in the parameter, the solution is differentiable.

The paper then provides an overview of the formula of Tobin and Friesz (1988), and illustrates by means of examples that there are several cases where it is not applicable: First, the requirement that the equilibrium solution be strictly complementary is too strong—differentiable points may not be strictly complementary. Second, the special matrix invertibility condition implies a strong requirement on the topology of the traffic network being analyzed and which may not hold in practice, as noted by Bell and Iida (1997, page 97); moreover, the matrix condition may fail to hold at differentiable points.

The findings of this paper are hoped to motivate replacing the previous approach with the more often applicable one, not only because of this fact but equally importantly because it is intuitive and also can be much more efficiently utilized: the sensitivity problem that provides the directional derivative is a linearized traffic equilibrium problem, and the sensitivity information can be generated efficiently by only slightly modifying a state-of-the-art traffic equilibrium solver. This is essential for bringing the use of sensitivity analysis in transportation planning beyond the solution of only small problems.

Introduction

Performing a sensitivity analysis of traffic equilibria means evaluating the directions of change that occur in the flows and travel costs as parameters in the cost and demand functions change. A sensitivity analysis is particularly useful in control and pricing applications since, if we can anticipate the effects of a change in, say, the traffic infrastructure, on the behaviour of the travellers, then we can utilize this knowledge to optimize these changes according to some goal fulfillment, like a reduction in flows

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or delays, a higher revenue from congestion tolls, etc. Such problems constitute instances of *bilevel optimization* problems, or *mathematical programs with equilibrium constraints* (MPEC), which is the scientific field within operations research and mathematical programming that is associated with hierarchical optimization problems, and which also includes the origin–destination (OD) matrix estimation problem. (The monograph [LPR96] provides a good overview of MPEC models and methods.) Several algorithms for MPEC problems rely on efficiently and generally applicable sensitivity analysis tools; it is in this framework that are findings can best be utilized.

Recently, the authors have been involved in a project having the goal to provide a precise sensitivity analysis of elastic and fixed demand traffic equilibrium problems, focusing on general models involving possibly non-separable and non-invertible link cost and demand functions; cf. [PaR02, Pat03, PaR03, Pat04, JoP03]. Our focus here is on the special case of separable link cost and demand functions, the latter also being invertible, in which case we can work directly on optimization formulations. We illustrate how to perform a sensitivity analysis efficiently in practice by using a modification of state-of-the-art traffic equilibrium software.

In 1988 Tobin and Friesz did the transportation science community the great service of bringing to it the nonlinear programming topic of sensitivity analysis, with their publication [ToF88]. Their analysis is quite accessible to practitioners; for example, they utilize the rather intuitive implicit function theorem in their analysis. It also remains the most popular tool for producing sensitivity analysis information in traffic equilibrium problems.

We illustrate through examples how their formula is however less applicable in several ways. Moreover, it relies on calculations with very large matrices, and therefore cannot be applied to large-scale networks. Our sensitivity analysis problem is however quite structured and need not involve matrix calculations at all; it amounts to solving a perturbed, affine traffic equilibrium problem, which is no more difficult to solve than the original one.

1 The traffic model

Let $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ be a transportation network, where $\mathcal{N}$ and $\mathcal{L}$ are the sets of nodes and directed links, respectively. For certain ordered pairs of nodes, $(p, q) \in \mathcal{C}$, where node p is an origin, node q is a destination, and $\mathcal{C}$ is a subset of $\mathcal{N} \times \mathcal{N}$, there is a transport demand, which may be given by a function of the travel cost. We assume that the network is strongly connected, that is, that at least one route joins each origin–destination (OD) pair.

Wardrop's user equilibrium principle states that for every OD pair $(p, q) \in \mathcal{C}$, the travel costs of the routes utilized are equal and minimal for each individual user. We denote by $\mathcal{R}_{pq}$ the set of simple (loop-free) routes for OD pair (p, q), by h_r the flow on route $r \in \mathcal{R}_{pq}$, and by c_r the travel cost on the route as experienced by an individual user.

We introduce the parameter to be present in the sensitivity analysis: it is denoted ρ, and is assumed to be of dimension d. This parameter could be present in one or both of the travel cost and demand functions. We assume that the travel cost function has the form $c(\rho, \cdot) : \Re_+^{|\mathcal{R}|} \mapsto \Re^{|\mathcal{R}|}$ given a value of ρ, where $|\mathcal{R}|$ denotes the total number of routes in the network. Further, the demand function is given by $g(\rho, \cdot) : \Re^{|\mathcal{C}|} \mapsto \Re_+^{|\mathcal{C}|}$. (We introduce the notation $\Re_+ := \{ x \in \Re \mid x \geq 0 \}$ and $\Re_{++} := \{ x \in \Re \mid x > 0 \}$.)

In an application to OD estimation, d is in the order of $|\mathcal{C}|$, while $d \approx |\mathcal{L}|$ holds in equilibrium network design, pricing and control models.

We also introduce the matrix $\Gamma \in \Re^{|\mathcal{R}| \times |\mathcal{C}|}$, which is the route–OD pair incidence matrix (i.e., the element γ_{rk} is 1 if route r joins OD pair $k = (p, q) \in \mathcal{C}$, and 0 otherwise). Then, demand-feasibility is described by the conditions that $h \in \Re_+^{|\mathcal{R}|}$ and

$$\Gamma^{\mathrm{T}} h = g(\rho, \pi) \tag{1}$$

holds, while the Wardrop equilibrium conditions for the route flows are that

$$h_r > 0 \implies c_r(\rho, h) = \pi_{pq}, \qquad r \in \mathcal{R}_{pq}, \qquad (p, q) \in \mathcal{C}, \tag{2a}$$
$$h_r = 0 \implies c_r(\rho, h) \geq \pi_{pq}, \qquad r \in \mathcal{R}_{pq}, \qquad (p, q) \in \mathcal{C}, \tag{2b}$$

holds, where the value of $\pi_{pq} := \pi_{pq}(\rho, h)$ is the minimal (i.e., equilibrium) route cost in OD pair (p, q). By the non-negativity of the route flows, the system (1)–(2) can more compactly be written as the mixed complementarity problem (MCP)

$$0^{|\mathcal{R}|} \leq h \perp (c(\rho, h) - \Gamma\pi) \geq 0^{|\mathcal{R}|}, \tag{3a}$$

$$\Gamma^{\mathrm{T}} h = g(\rho, \pi), \tag{3b}$$

where $a \perp b$, for two arbitrary vectors $a, b \in \Re^n$, means that $a^{\mathrm{T}} b = 0$. (By nonnegativity, this implies that $a_j \cdot b_j = 0$ for all j.)

As we are interested in the sensitivity of link flows, we will assume that the route cost is additive. For each link $l \in \mathcal{L}$, the travel cost has the form $t_l(\rho, v_l)$, where $v \in \Re^{|\mathcal{L}|}$ is the vector of link flows. The route and link travel costs and flows are related through a route–link incidence matrix, $\Lambda \in \{0,1\}^{|\mathcal{L}| \times |\mathcal{R}|}$, whose element λ_{lr} equals one if route $r \in \mathcal{R}$ utilizes link $l \in \mathcal{L}$, and zero otherwise. Route r has an additive route cost $c_r(\rho, h)$ if it is the sum of the costs of using all the links defining it. In other words, $c_r(\rho, h) = \sum_{l \in \mathcal{L}} \lambda_{lr} t_l(\rho, v_l)$. In short, then, $c(\rho, h) = \Lambda^{\mathrm{T}} t(\rho, v)$. Also, implicit in this relationship is the assumption that the pair (h, v) is consistent, in the sense that v equals the sum of the route flows: $v = \Lambda h$. We shall use the representation in terms of v, as it is an entity for which we can introduce conditions ensuring that uniqueness holds at equilibrium.

As could be noted above, the link travel cost was assumed to be separable. The same assumption is made with respect to the demand function, which is supposed to be of the form $g_k(\rho, \pi_k)$, $k \in \mathcal{C}$.

In order to be able to work with an optimization formulation, which furthermore admits a unique solution (v^*, d^*) and is such that we can apply sensitivity analysis theory, we introduce the following assumption, which is supposed to hold throughout:

Assumption 1 (properties of the network model).

(a) For each $l \in \mathcal{L}$, the link travel cost function $t_l(\cdot, \cdot)$ is continuously differentiable, and strictly increasing in its second argument.

(b) For each $k \in \mathcal{C}$, the demand funciton $g_k(\cdot, \cdot)$ is continuously differentiable, non-negative, upper bounded, and strictly decreasing in its second argument. The function $g_k(\rho, \cdot)$ is therefore invertible, and has a single-valued inverse, $\xi_k(\rho, \cdot)$, which also is continuously differentiable and strictly decreasing. □

The optimization formulation that we will work with is the following standard one for elastic demand traffic assignment (e.g., [Pat94]):

$$\underset{(v,d)}{\text{minimize}}\ \phi(v, d) := \sum_{l \in \mathcal{L}} \int_0^{v_l} t_l(\rho, s)\, ds - \sum_{k \in \mathcal{C}} \int_0^{d_k} \xi_k(\rho, s)\, ds, \tag{4a}$$

$$\text{subject to } \Gamma^{\mathrm{T}} h = d, \tag{4b}$$

$$v = \Lambda h, \tag{4c}$$

$$h \geq 0^{|\mathcal{R}|}. \tag{4d}$$

For future use, let C denote the set of feasible solutions to (4) in (h, v, d), that is,

$$C = \left\{ \begin{pmatrix} h \\ v \\ d \end{pmatrix} \in \Re_+^{|\mathcal{R}|} \times \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \,\middle|\, \Gamma^{\mathrm{T}} h = d; \quad v = \Lambda h \right\}.$$

The variational inequality problem, which characterizes the solution (h^*, v^*, d^*) to this problem, is stated as that of finding $(h^*, v^*, d^*) \in C$ such that

$$t(\rho, v^*)^{\mathrm{T}}(v - v^*) - \xi(\rho, d^*)^{\mathrm{T}}(d - d^*) \geq 0, \qquad (h, v, d) \in C. \tag{5}$$

To see that this expression characterizes the Wardrop conditions stated earlier in (3), we notice that (5) is equivalent to (h^*, v^*, d^*) solving the following linear program:

$$\underset{(v,d)}{\text{minimize}}\ t(\rho, v^*)^{\mathrm{T}} v - \xi(\rho, d^*)^{\mathrm{T}} d, \tag{6a}$$

$$\text{subject to } \Gamma^{\mathrm{T}} h - d = 0^{|\mathcal{C}|}, \tag{6b}$$

$$v - \Lambda h = 0^{|\mathcal{L}|}, \tag{6c}$$

$$h \geq 0^{|\mathcal{R}|}. \tag{6d}$$

Its LP dual is to

$$\begin{aligned}
&\underset{(\pi,\alpha)}{\text{maximize}}\ 0, && (7a)\\
&\text{subject to } \Gamma\pi - \Lambda^{\mathrm{T}}\alpha \leq 0^{|\mathcal{R}|}, && (7b)\\
&-\pi = -\xi(\rho, d^*), && (7c)\\
&\alpha = t(\rho, v^*), && (7d)
\end{aligned}$$

where π and α are, respectively, the LP dual variables for the constraints (6b) and (6c). The dual variable α is eliminated by using (7d). The complementarity conditions between the two LP problems can then be written as

$$0^{|\mathcal{R}|} \leq h^* \perp (\Lambda^{\mathrm{T}} t(\rho, v^*) - \Gamma\pi^*) \geq 0^{|\mathcal{R}|}, \tag{8}$$

which is identical to the Wardrop condition (3a). The condition (3b) is obtained as follows: from (6b) and (7c), $\Gamma^{\mathrm{T}} h^* = d^* = g(\rho, \pi^*)$. As $t(\rho, \cdot)$ and $-g(\rho, \cdot)$ both are strictly monotone, the objective function of (4) is strictly convex; therefore, the solution in (v^*, d^*) to (4), and equivalently to the variational inequality (5) and to the Wardrop conditions (3), is unique. We see that from (7c)–(7d), also the dual entities (π^*, α^*) are unique.

2 The basis for our sensitivity analysis

The basis of our sensitivity analysis is a result which is stated for a general variational inequality problem with a differentiable mapping, $f : \Re^d \times \Re^n \mapsto \Re^n$ in the parameters $\rho \in \Re^d$ and variables $x \in \Re^n$: find $x^* \in X$ such that

$$f(\rho, x^*)^{\mathrm{T}}(x - x^*) \geq 0, \qquad x \in X, \tag{9}$$

where $X \subseteq \Re^n$ is a polyhedral set. We let $S : \Re^d \rightrightarrows \Re^n$ denote the mapping that assigns to each vector $\rho \in \Re^d$ the set $S(\rho)$ of solutions to this problem:

$$S(\rho) := \{\, x^* \in X \mid f(\rho, x^*)^{\mathrm{T}}(x - x^*) \geq 0, \quad x \in X \,\}, \qquad \rho \in \Re^d. \tag{10}$$

Letting $\rho = \rho^*$ be the current value of the parameter vector, we are interested in the direction of change of the solution x^* as ρ^* is perturbed along a direction ρ'. This directional derivative of S is the solution to an auxiliary variational inequality, which has the following form: find $x' \in K$ such that

$$r(\rho', x')^{\mathrm{T}}(x - x') \geq 0, \qquad x \in K, \tag{11a}$$

where

$$\begin{aligned}
K &:= T_X(x^*) \cap f(\rho^*, x^*)^{\perp}, && (11b)\\
r(\rho', x') &:= \nabla_\rho f(\rho^*, x^*)\rho' + \nabla_x f(\rho^*, x^*)x'. && (11c)
\end{aligned}$$

We let $DS(\rho^*|x^*) : \Re^d \rightrightarrows \Re^n$ denote the mapping that assigns to each perturbation $\rho \in \Re^d$ the set $DS(\rho^*|x^*)(\rho')$ of solutions to this problem:

$$DS(\rho^*|x^*)(\rho') := \{\, x' \in K \mid r(\rho', x')^{\mathrm{T}}(x - x') \geq 0, \quad x \in K \,\}, \qquad \rho' \in \Re^d.$$

The set K denotes the set of variations around x^* that, roughly speaking, retains feasibility and optimality to the first order. T_X denotes the tangent cone to X, which means that if X is defined by linear constraints, we have that

$$X = \{x \in \Re^n |\, Ax \geq b;\ Bx = d\} \quad \Longrightarrow \quad T_X(x^*) = \left\{ z \in \Re^n |\, \bar{A}z \geq 0;\ Bz = 0 \right\},$$

where $\bar{A}$ consists of the rows A_i of A corresponding to the binding inequality constraints at x^*, that is, the indices i with $A_i x^* = b_i$. Further, for any vector $z \in \Re^n$, $z^{\perp} := \{\, y \in \Re^n \mid z^{\mathrm{T}} y = 0 \,\}$ is the orthogonal subspace associated with the vector z. The mapping r is a linearization of f around (ρ^*, x^*); it is an affine mapping in x'.

Suppose now that $f(\rho, \cdot)$ is monotone on X around $\rho = \rho^*$, and that the parameterization is such that $\operatorname{rank} \nabla_\rho f(\rho^*, x^*) = n$. (The latter result can always be fulfilled by including enough dummy parameters.) We say that the mapping S is *strongly regular* at ρ^* ([Rob80, Rob85]) if S is single-valued and Lipschitz continuous on some neighbourhood of ρ^*. Then, according to a result by Dontchev and Rockafellar [DoR01],

$$S \text{ is strongly regular at } \rho^* \quad \Longleftrightarrow \quad DS(\rho^*|x^*) \text{ is single-valued.} \tag{12}$$

Moreover, then the unique solution x' to (11) is the directional derivative of the solution x^* to (9) at ρ^*, in the direction of ρ'. A sufficient condition for the property of single-valuedness of DS in (12) to hold is, by [Kyp88, Lemma 2.1], that

$$\nabla_x f(\rho^*, x^*) \text{ is positive definite on } (K - K). \tag{13}$$

We refer to this as a sufficient *second-order* condition. A stronger result than directional differentiability can also be obtained under additional assumptions: according to a result of Kyparisis [Kyp90], under the strong regularity assumption above,

$$S \text{ is differentiable at } \rho^* \quad \Longleftrightarrow \quad DS(\rho^* \mid x^*)(\rho') \in -K, \qquad \rho' \in \Re^d.$$

Moreover, if further K is a subspace, that is, if $K = K \cap (-K)$, then the gradient can be represented as

$$\nabla_\rho x(\rho^*) = -Z \left[Z^{\mathrm{T}} \nabla_x f(\rho^*, x^*) Z \right]^{-1} Z^{\mathrm{T}} \nabla_\rho f(\rho^*, x^*), \tag{14}$$

for any $n \times \ell$ matrix Z such that $Z^{\mathrm{T}} Z$ is nonsingular and $z \in K \cap (-K)$ if and only if $z = Zy$ for some $y \in \Re^\ell$, where ℓ is the dimension of $K \cap (-K)$. This differentiability result is a kind of implicit function theorem; the relationship in (12) shows how the implicit function theorem naturally extends to more general cases.

We refer to this latter property not because we will establish sufficient conditions for its application in the present context (this has already been done in [Pat03, Pat04]), but to remark that the heuristic sensitivity analysis that is developed in the paper [ToF88] and its follow-up [CSF00] strives to utilize (14). Unfortunately, not only does the property $DS(\rho^* \mid x^*)(\rho') \in -K$ fail to hold in many cases (cf. [Pat03, Pat04], as well as below), but also there may not exist a nonsingular matrix of the kind that is referred to above (cf. [BeI97, page 97]).

3 Sensitivity analysis of separable traffic equilibria

We first identify the sensitivity problem in our notation. Let

$$x = \begin{pmatrix} h \\ v \\ d \end{pmatrix}; \qquad f(\rho, x) = \begin{pmatrix} 0^{|\mathcal{R}|} \\ t(\rho, v) \\ -\xi(\rho, d) \end{pmatrix}; \qquad X = C.$$

Then, we can identify the sensitivity problem through the following identifications:

$$K = \left\{ \begin{pmatrix} h' \\ v' \\ d' \end{pmatrix} \in \Re^{|\mathcal{R}|} \times \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \Gamma^{\mathrm{T}} h' = d'; \quad v' = \Lambda h'; \quad h' \in H' \right\},$$

where

$$H' = \left\{ h' \in \Re^{|\mathcal{R}|} \;\middle|\; \begin{array}{l} h'_r \text{ free if } h^*_r > 0 \\ h'_r \geq 0 \text{ if } h^*_r = 0 \text{ and } c_r(\rho^*, h^*) = \pi^*_k \\ h'_r = 0 \text{ if } h^*_r = 0 \text{ and } c_r(\rho^*, h^*) > \pi^*_k \\ [r \in \mathcal{R}_k,\ k \in \mathcal{C}] \end{array} \right\},$$

and

$$r(\rho', x') = \begin{pmatrix} 0^{|\mathcal{R}|} \\ \nabla_\rho t(\rho^*, v^*)\rho' + \nabla_v t(\rho^*, v^*) v' \\ -[\nabla_\rho \xi(\rho^*, d^*)\rho' + \nabla_d \xi(\rho^*, d^*) d'] \end{pmatrix}.$$

By the monotonicity and separability of t and $-\xi$, the resulting sensitivity variational inequality can be equivalently written as the following convex quadratic optimization problem to

$$\begin{aligned}
\underset{(v',d')}{\text{minimize}}\ \phi'(v',d') &:= [\nabla_\rho t(\rho^*,v^*)\rho']^{\mathrm{T}} v' + \frac{1}{2}\sum_{l\in\mathcal{L}} \frac{\partial t_l(\rho^*,v_l^*)}{\partial v_l}(v_l')^2 \\
&\quad - [\nabla_\rho \xi(\rho^*,d^*)\rho']^{\mathrm{T}} d' - \frac{1}{2}\sum_{k\in\mathcal{C}} \frac{\partial \xi_k(\rho^*,d_k^*)}{\partial d_k}(d_k')^2, &\text{(15a)}\\
\text{subject to } \Gamma^{\mathrm{T}} h' &= d', &\text{(15b)}\\
v' &= \Lambda h', &\text{(15c)}\\
h &\in H'. &\text{(15d)}
\end{aligned}$$

The derivation follows the same pattern as that in [PaR02, Pat03, PaR03, Pat04]. The sensitivity problem is closely related to the original model. Two main differences are notable: the link cost and demand functions are replaced by their linearizations, and the sign restrictions on h are replaced by individual restrictions on the route flow perturbations h_r' that depend on whether the route in question was used at equilibrium or not, cf. the set H'. Although the appearance of H' depends on the choice of route flow solution h^*, it is an interesting fact that the possible choices of v' in K does *not*; this is a general consequence of aggregation, which was also utilized in [PaR02, Pat03, PaR03, JoP03, Pat04]. In summary, it appears that the sensitivity problem can be solved using software similar to those for the original traffic equilibrium model, provided of course that route flow information can be extracted.

We now apply the result (12) in the previous section. The result states that the sensitivity problem provides directional derivatives, provided that the solution is unique. So, under what circumstances will the optimal solution to (15) be unique? Similarly, which entities in the solution to the problem (4) [flow, travel cost, demand] have directional derivatives?

Clearly, we cannot apply the theory of the previous section to the problem stated in (h, v, d), since h^* is not unique. As (v^*, d^*) is unique, we could project the problem onto this space. This is simply accomplished by redefining

$$C^{\text{proj}} = \left\{ \begin{pmatrix} v \\ d \end{pmatrix} \in \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \exists h \text{ with } \begin{pmatrix} h \\ v \\ d \end{pmatrix} \in C \right\};$$

further, we would let

$$x = \begin{pmatrix} v \\ d \end{pmatrix}; \qquad f(\rho, x) = \begin{pmatrix} t(\rho, v) \\ -\xi(\rho, d) \end{pmatrix}.$$

As we are also interested in the sensitivity of the travel costs, we will, for the first time, introduce yet another modification: we introduce a dummy variable, $s \in \Re^{|\mathcal{L}|}$, which will take on the (negative) value

$$s^* = -t(\rho^*, v^*)$$

of the link travel cost at equilibrium, and likewise a variable, $\pi_- \in \Re^{|\mathcal{C}|}$, to take on the (negative) value

$$\pi_-^* = -\xi(\rho^*, d^*)$$

of the equilibrium OD travel costs. [Note that $\pi_-^* = -\pi^*$, where π^* is given in (8).] In the sensitivity problem, then, s' and π_-' will equal the (negative of the) link and OD travel cost perturbation, respectively.

The problem which will be analyzed is the following: in (9), let

$$x = \begin{pmatrix} v \\ d \\ s \\ \pi_- \end{pmatrix}; \qquad f(\rho, x) = \begin{pmatrix} t(\rho, v) \\ -\xi(\rho, d) \\ s + t(\rho, v) \\ -\pi_- - \xi(\rho, d) \end{pmatrix}; \qquad X = C^{\text{proj}} \times \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|}.$$

The variational inequality corresponding to (9) states that, at ρ^*,

$$\begin{aligned}
t(\rho^*, v^*)^{\mathrm{T}}(v - v^*) - \xi(\rho^*, d^*)^{\mathrm{T}}(d - d^*) \geq 0, &\qquad (v, d) \in C^{\text{proj}}, \\
s^* = -t(\rho^*, v^*),& \\
\pi_-^* = -\xi(\rho^*, d^*),&
\end{aligned}$$

so it is entirely equivalent to the VIP in (5). The reason for introducing the last two rows of the problem, that is, the extra variables (s, π_-), is that by doing so, we have direct access to the sensitivity of the travel costs, through the corresponding elements (s', π'_-) of x'.

The sensitivity problem has the form of (11), with

$$r(\rho', x') = \begin{pmatrix} \nabla_\rho t(\rho^*, v^*)\rho' + \nabla_v t(\rho^*, v^*)v' \\ -[\nabla_\rho \xi(\rho^*, d^*)\rho' + \nabla_d \xi(\rho^*, d^*)d'] \\ s' + \nabla_\rho t(\rho^*, v^*)\rho' + \nabla_v t(\rho^*, v^*)v' \\ -\pi'_- - [\nabla_\rho \xi(\rho^*, d^*)\rho' + \nabla_d \xi(\rho^*, d^*)d'] \end{pmatrix}, \tag{16}$$

and

$$K = \left\{ \begin{pmatrix} v' \\ d' \\ s' \\ \pi'_- \end{pmatrix} \in \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \times \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \exists h' \in H' \text{ with } \Gamma^{\mathrm{T}} h' = d'; \quad v' = \Lambda h' \right\}.$$

The sensitivity optimization problem in (v', d') is (15), and the value of (s', π'_-) is then given by

$$\begin{aligned} s' &= -[\nabla_\rho t(\rho^*, v^*)\rho' + \nabla_v t(\rho^*, v^*)v'], \\ \pi'_- &= -[\nabla_\rho \xi(\rho^*, d^*)\rho' + \nabla_d \xi(\rho^*, d^*)d'], \end{aligned}$$

that is, the cost perturbations are given by a kind of chain rule. The result to follow establishes that this chain rule provides uniquely given values of (s', π'_-) even when v' is not unique.

Before we apply the sensitivity analysis results in the previous section to the present problem, we mention an important fact which allowed us in [JoP03] to provide stronger results for optimization formulations than for the general variational inequality models in [PaR02, Pat03, PaR03, Pat04]: for a differentiable, convex problem, the gradient of the objective function is invariant over the solution set (cf. [BuF91]). The following result stems from [JoP03].

THEOREM 2 (sensitivity of separable traffic equilibrium problems). *Let Assumption 1 hold, and consider an arbitrary vector $\rho^* \in \Re^d$. Then, the solution (v^*, d^*) to* (4) *is unique, and so are the (negative) travel cost entities $(s^*, \pi^*_-) = -(t(\rho^*, v^*), \xi(\rho^*, d^*))$. Let $\rho' \in \Re^d$ be an arbitrary perturbation.*

(a) *In the solution to* (15), *the travel cost perturbations (s', π'_-) are unique; therefore, the values*

$$\begin{aligned} -s' &= \nabla_\rho t(\rho^*, v^*)\rho' + \nabla_v t(\rho^*, v^*)v', \\ -\pi'_- &= \nabla_\rho \xi(\rho^*, d^*)\rho' + \nabla_d \xi(\rho^*, d^*)d', \end{aligned}$$

are the directional derivatives of, respectively, the equilibrium link and OD travel costs, at ρ^*, in the direction of ρ'.

(b) *Assume that the link travel cost function $t(\rho^*, \cdot)$ is such that*

$$\frac{\partial t_l(\rho^*, v_l^*)}{\partial v_l} > 0, \qquad l \in \mathcal{L}. \tag{17}$$

Assume further that the demand function $g(\rho^*, \cdot)$ is such that

$$\frac{\partial g_k(\rho^*, \pi_k^*)}{\partial \pi_k} < 0, \qquad k \in \mathcal{C}. \tag{18}$$

Then, in the solution to (15), *the values of the link flow and demand perturbation v' and d' are unique; therefore, the value v' (respectively, d') is the directional derivative of the equilibrium link flow (respectively, demand), at ρ^*, in the direction of ρ'.* ☐

Note that the second-order condition (18) is equivalent to the condition that

$$\frac{\partial \xi_k(\rho^*, d_k^*)}{\partial d_k} < 0, \qquad k \in \mathcal{C}.$$

Obviously, this condition on the demand function derivative (or its inverse) is not needed in the case when we consider fixed demands.

An interesting aspect of this result is that the cost perturbations (s', π'_-), which are related to the perturbations (v', d') through a kind of chain rule, are *not* dependent on the perturbations (v', d') to be unique. This is in contrast to the type of analysis offered by Tobin and Friesz [ToF88], see also [BeI97, Section 5.4], where the sensitivity of the costs is considered an implication of that of the flows and demands. (Not to mention that it will sometimes fail, cf. [Pat03, JoP03, Pat04].)

4 An illustrative example

The following small numerical example, taken from [JoP03], illustrates the workings of our analysis.

The network of Braess [Bra68] is classic in the analysis of system optimal solutions. Consider the network in Figure 1.

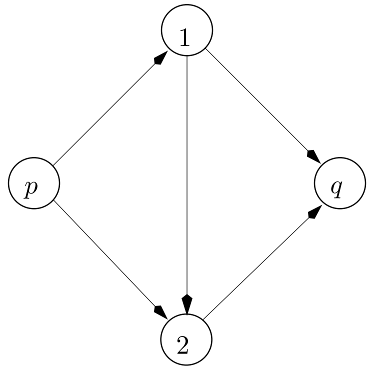

Figure 1: Braess' traffic network.

For this problem [where link $(1, 2)$ has a travel cost that is deliberately chosen so that the third route is not used but still has the same cost], we have the data given in Table 1.

Link	$t_{ij}(\rho, v_{ij})$	*OD pair*	d_{pq}
1: $(p, 1)$	$10v_{p1}$	1: (p, q)	6
2: $(p, 2)$	$50 + v_{p2}$		
3: $(1, q)$	$50 + v_{1q}$		
4: $(2, q)$	$10v_{2q}$		
5: $(1, 2)$	$23 + \rho + v_{12}$		

Table 1: Network data.

The data corresponds to an instance of the fixed (and unperturbed) demand traffic equilibrium problem, which can be written as that to

$$\underset{v}{\text{minimize}}\ \phi(v) := \sum_{l \in \mathcal{L}} \int_0^{v_l} t_l(\rho, s)\, ds, \tag{19a}$$

$$\text{subject to } \Gamma^{\mathrm{T}} h = d, \tag{19b}$$

$$v = \Lambda h, \tag{19c}$$

$$h \geq 0^{|\mathcal{R}|}, \tag{19d}$$

where $d \in \Re^{|\mathcal{C}|}_{++}$ is the vector of demands.

Solving the fixed demand traffic equilibrium problem with $\rho = \rho^* = 0$, we obtain the link flow solution $v^* = (3, 3, 3, 3, 0)^{\mathrm{T}}$. The cost of the three routes $\{1, 3\}$, $\{2, 4\}$, and $\{1, 5, 4\}$, are in fact the same, namely 83, but the route flows are 3 on each of the first two, and zero on the third. (This unique route flow solution is non-strictly complementary.)

Since the parameter ρ is present on link $(1, 2)$, which has link flow zero at equilibrium, it looks clear that a positive value of ρ' should lead to no changes [since the flow on link $(1, 2)$ cannot be negative] whereas a negative value of ρ' should imply that v'_{12} is positive. Indeed, that is the case.

In this special case where demand is fixed and unperturbed, the sensitivity problem is that to

$$\underset{v'}{\text{minimize}}\ \phi'(v') := [\nabla_\rho t(\rho^*, v^*)\rho']^{\mathrm{T}} v' + \frac{1}{2}\sum_{l\in\mathcal{L}} \frac{\partial t_l(\rho^*, v_l^*)}{\partial v_l}(v_l')^2, \tag{20a}$$

$$\text{subject to } \Gamma^{\mathrm{T}} h' = 0^{|\mathcal{C}|}, \tag{20b}$$

$$v' = \Lambda h', \tag{20c}$$

$$h \in H'. \tag{20d}$$

For any $\rho' \in \Re$, therefore, we have that

$$\phi'(v') = \rho' v_{12}' + 5(v_{p1}')^2 + \frac{1}{2}(v_{p2}')^2 + \frac{1}{2}(v_{1q}')^2 + 5(v_{2q}')^2 + \frac{1}{2}(v_{12}')^2,$$

and the constraints specify that

$$\begin{aligned} h_1' + h_2' + h_3' &= 0, \\ -v_{p1}' + h_1' + h_3' &= 0, \\ -v_{p2}' + h_2' &= 0, \\ -v_{1q}' + h_1' &= 0, \\ -v_{2q}' + h_2' + h_3' &= 0, \\ -v_{12}' + h_3' &= 0, \\ h_3' &\geq 0. \end{aligned}$$

Letting $\rho' = 1$, the optimal solution is $h' = 0^{\mathrm{T}}$ and $v' = 0^{\mathrm{T}}$.

Letting $\rho' = -1$, the optimal solution is $h' = \frac{1}{26}(-1, -1, 2)^{\mathrm{T}}$ and $v' = \frac{1}{13}(1, -1, -1, 1, 2)^{\mathrm{T}}$.

This is therefore also a case where the traffic equilibrium solution is non-differentiable, since clearly the directional derivative mapping is not linear.

5 A sensitivity analysis tool

In [JoP03], the disaggregate simplicial decomposition (DSD) method of Larsson and Patriksson [LaP92] for the fixed demand problem (19) was taken as the building block of the sensitivity analysis tool. (In the case of elastic demands, one can still solve a fixed demand problem, by first utilizing the fixed demand transformation of Gartner [Gar80].) It has the advantage of utilizing route flow information, and therefore the close resemblence between the original problem (19) and the sensitivity problem (15) can be utilized fully.

The DSD algorithm was recoded in Matlab for experimentational purposes, fully aware of the fact that the CPU time will be perhaps two orders of magnitude larger than a final C or Fortran implementation. We refer to the paper [LaP92] for the basics of the DSD algorithm, but remind the reader that the most central points of the algorithm are the following: at some iteration point (h^τ, v^τ) of consistent route and link flows, the current travel costs are used to solve a shortest route problem for each origin node. The routes that then are generated are compared to sets $\hat{\mathcal{R}}_{pq} \subset \mathcal{R}_{pq}$ that have been generated and stored previously, and those sets are augmented with any routes there were not known already. (This process is the column, or route, generation one.) With those subsets of the routes at hand, the restricted master problem (RMP)—which is the original problem (19) except that $\hat{\mathcal{R}}_{pq}$ replaces $\mathcal{R}_{pq}$ for each $(p, q) \in \mathcal{C}$—is solved using one of several methodologies implemented. The highly structured RMP is either solved by using a gradient projection method or a diagonalized Newton method. (In practice, it appears that the former is the best for smaller networks, but that the Newton method wins for large enough cases.)

The similarity of the sensitivity problem to the equilibrium problem meant that much of the code from the DSD algorithm implementation could be reused. Of course, in the sensitivity context, flow and cost derivatives take the place of flows and costs themselves. For simplicity, these derivatives can be considered "virtual" costs and flows. The restricted master problem solver code then only had to be altered slightly to allow the subset of the routes that were used at equilibrium to take on negative "flow" values.

In order to set up the sensitivity problem, except for the flows and costs, the main part concerns the routes to be included. Remember that only least-cost routes are valid, some of which have a non-negativity requirement (if it is unused). In order to construct this set of routes, we first included only those routes that were used in the equilibrium solution. In order to compensate for the possibility that the equilibrium solution might not perfectly identify these routes, a "fuzz"-factor was used. In other words, to determine whether a route was used, the route flow was compared not to zero but to a very small positive number, obtained by multiplying the OD-pair demand by a tiny factor. In other words, a sign restriction may be included for a route that has a very small amount of flow at the terminal flow of the DSD algorithm. Any remaining set of routes that are potentially interesting for the sensitivity problem could then be included based on a final shortest route calculation and a graph search, together with a "fuzz"-factor similar to the above, allowing for near-shortest routes to be included as well.

This set of routes then is the one that defines the sensitivity problem; no further route generation is necessary, and so the only problem left is a convex quadratic RMP with some variables being free and some being sign restricted. Virtually the same algorithm as in the RMP for the original problem can be used; the only special case stems from the sign restrictions. Further details on the implementation of this algorithm can be found in the first author's master's thesis [Jos03].

6 A dissection of the sensitivity analysis of Tobin and Friesz

6.1 The analysis

We show, by means of both analytical and numerical tools, some examples in which the sensitivity analysis presented in Tobin and Friesz [ToF88] requires too strong assumptions.

6.1.1 The strong monotonicity condition

The analysis is performed on a problem similar to (19) but where the fixed demand is also perturbed. In order to ensure local uniqueness, they introduce the following condition:

(Condition 1—strong monotonicity) $t(\rho, \cdot)$ is strongly monotone in a neighbourhood of ρ^*.

This condition is stronger than necessary, as we have already seen.

6.1.2 The strict complementarity condition

The analysis is based on first selecting a particular equilibrium route flow solution. Among the conditions stated, the route flow is supposed to be strictly complementary. The definition is however not the one commonly used, it being the following: a route flow solution h^* is *strictly complementary* if and only if that it is complementary (that is, that $0 \leq h_r^* \perp [c_r(\rho^*, h^*) - \pi_{pq}(\rho^*)] \geq 0$ holds for all $r \in \mathcal{R}_{pq}$, $(p,q) \in \mathcal{C}$), and

$$h_r^* + [c_r(\rho^*, h^*) - \pi_{pq}(\rho^*)] > 0, \qquad r \in \mathcal{R}_{pq}, \quad (p,q) \in \mathcal{C}. \tag{21}$$

In other words, our use of the term strict complementarity means that for an arbitrary route $r \in \mathcal{R}_{pq}$, it is either used ($h_r^* > 0$) or it is more expensive than the route used in the OD pair $[c_r(\rho^*, h^*) > \pi_{pq}(\rho^*)]$.

Tobin and Friesz state a definition of traffic equilibrium in terms of *total* link flows v only, and which unfortunately is not consistent with the standard definition, given in (2). Their definition of a user equilibrium in terms of the vector v is that there exists a vector $\lambda^* \in \Re^{|\mathcal{N}|}$ such that for every link $l = (i,j) \in \mathcal{L}$,

$$\begin{aligned} v_l^* = 0 \qquad &\Longrightarrow \qquad t_l(v^*) \geq \lambda_j^* - \lambda_i^*, \\ v_l^* > 0 \qquad &\Longrightarrow \qquad t_l(v^*) = \lambda_j^* - \lambda_i^*. \end{aligned}$$

An inherent problem with this definition is that the *aggregated* potential differences, $\lambda_j^* - \lambda_i^*$, are not consistent with our node price vectors π_k, $k \in \mathcal{C}$, associated with the shortest route problem for OD pair k at v^*; in other words, at a traffic equilibrium, $\pi_{jk}^* - \pi_{ik}^* \neq \pi_{j\kappa}^* - \pi_{i\kappa}^*$ may hold for two OD pairs k and κ. (For example, it can happen as soon as link (i,j) lies on a shortest route in the OD pair k but not in κ.)

Based on the equilibrium definition, however, the authors define a strict complementarity criterion for the perturbed problem:

(Condition 2—strict complementarity) For each link $l = (i, j) \in \mathcal{L}$, $v_l^* = 0 \implies t_l(\rho^*, v^*) > \lambda_j^* - \lambda_i^*$ holds.

So, whenever the total link flow vector v^* is positive, this condition is satisfied. Clearly, it is therefore not compatible with the strict complementarity condition (21).

In any case, the strict complementarity condition is not a necessary condition for the differentiability of the traffic equilibrium solution, although our strict complementarity condition is *sufficient*. An example below will illustrate this fact.

6.1.3 The linear independence condition

Next, we are asked to restrict the network $\mathcal{G}$ to $\mathcal{G}_+ = (\mathcal{N}, \mathcal{L}_+)$, where $l \in \mathcal{L}_+$ if and only if $v_l^* > 0$, that is, to the network corresponding to the links having a positive flow given ρ^*. Consequently, there are possibly some routes that will be removed as well. The + notation to follow reflects this restriction.

Under the assumptions stated sofar, the set $H_+^*(\rho^*)$ of equilibrium route flows is a bounded polyhedron. The next condition states that an equilibrium route flow vector h_+^* is selected such that it is a "non-degenerate extreme point" of $H_+^*(\rho^*)$:

(Condition 3—linear independence) An equilibrium route flow h_+^* is chosen such that it is an extreme point of $H_+^*(\rho^*)$ which has exactly as many routes with a positive flow as the rank of the matrix $[\Lambda_+^{\mathrm{T}} \mid \Gamma_+]$.

The rank of this matrix is never higher than the number of links with a positive flow at v^* plus $|\mathcal{C}|$. The authors state an LP problem that can be used to generate such a point, but also remark in their Theorem 6 that the sensitivity values do not depend on this choice, as long as it is an extreme point of $H_+^*(\rho^*)$.

A final restriction is then made, such that we remove all the indices in the vector h_+^* for which the flow is zero. (We do not change the notation to reflect this restriction.) The sensitivity problem is then finally set up as follows:

$$\begin{pmatrix} \nabla_\rho h_+ \\ \nabla_\rho \pi \end{pmatrix} = \begin{pmatrix} \nabla_h c_+(\rho^*, h_+^*) & -\Lambda_+^{\mathrm{T}} \\ \Lambda_+ & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\nabla_\rho c_+(\rho^*, h_+^*) \\ \nabla_\rho g(\rho^*) \end{pmatrix}. \tag{22}$$

6.2 Examples

6.2.1 A case of differentiability where strictly complementary does not hold

To show that strict complementarity is not necessary for differentiability, we consider the network depicted in Figure 2.

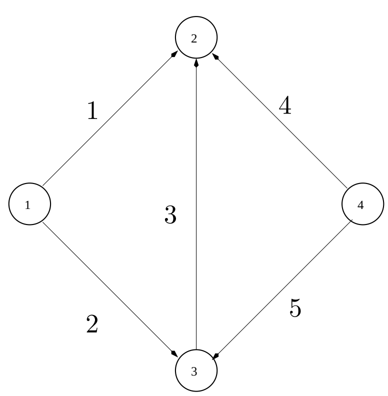

Figure 2: A traffic network.

There are two OD pairs, $(1, 2)$ and $(4, 2)$, with a fixed and unperturbed demand of 2 and 1 units of flow, respectively. The link cost functions are given by

$$t_1(v_1, \rho) := 2v_1 + \rho;\ t_2(v_2) := v_2;\ t_3(v_3) := 1;\ t_4(v_4) := v_4 + 2;\ t_5(v_5) = v_5.$$

We have four routes: $\{1\}$, $\{2,3\}$, $\{4\}$, and $\{5,3\}$, two for each OD pair.

With $\rho^* = 0$, the unperturbed traffic equilibrium solution is $v^* = (1,1,1,1,1)^{\mathrm{T}}$. The route flow is unique: $h^* = (1,1,0,1)^{\mathrm{T}}$. We see that the travel cost on route 3 is 2, as is the case for route 4, so this is a non-strictly complementary equilibrium solution. Since it is the unique route flow, we do not comply with the conditions (21) for strict complementarity.

In order to check if the solution v^* is nevertheless differentiable at $\rho^* = 0$, we solve the sensitivity problem for both $\rho' := 1$ and $\rho' := -1$. For $\rho' = 1$, we obtain the following unique solution to the sensitivity problem (20), thus being the directional derivative of v^* with respect to the direction $\rho' = 1$ at $\rho^* = 0$: $v' = \frac{1}{3}(-1,1,1,0,0)^{\mathrm{T}}$. The effect, as we can see, of perturbing link 1's cost such that it becomes more expensive, is that of sending flow in the cycle $\{-1,2,3\}$, where the minus reflects that flow is send backwards on link 1. When solving the sensitivity problem for $\rho' := -1$, we obtain the directional derivative $v' = \frac{1}{3}(1,-1,-1,0,0)^{\mathrm{T}}$, that is, the negative of the directional derivative of v^* in the direction of $\rho' := 1$. This proves that the directional derivative mapping is linear, and thus that the derivative of v^* with respect to ρ' at $\rho^* = 0$ equals

$$\frac{\mathrm{d}\,v^*}{\mathrm{d}\,\rho} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix}.$$

At the same time, we have here shown that the sufficient matrix condition (13) indeed is only sufficient; it is not satisfied here because the set of feasible route flow perturbations is the entire space and the partial Jacobian of t with respect to v at the pair (v^*, ρ^*) is the non-positive definite diagonal matrix with diagonal entries $(2,1,0,1,1)$; yet the equilibrium solution is even differentiable.

This is then an example where the analysis formula (22) is not applicable, although the solution is differentiable.

6.2.2 A case of differentiability where the formula (22) fails

Consider the network shown in Figure 3.

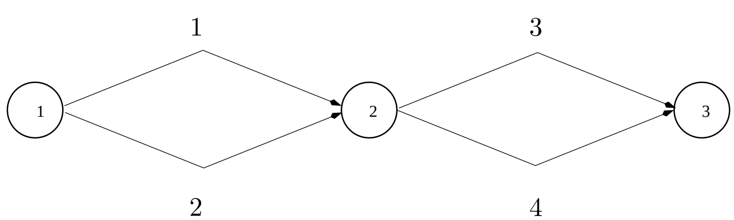

Figure 3: Network for the first counter-example.

There is a single OD pair, $(1,3)$, with a fixed demand of 2 units of flow. The link cost functions are given by

$$t_1(v_1,\rho) := v_1 + \rho; \quad t_2(v_2) := v_2; \quad t_3(v_3) := v_3; \quad t_4(v_4) := v_4.$$

We have four routes: $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, and $\{2,4\}$.

With $\rho^* = 0$, the unperturbed traffic equilibrium solution is $v^* = (1,1,1,1)^{\mathrm{T}}$. We can easily see that the solution is differentiable; it is strictly complementary even. The derivative with respect to ρ at ρ^* moreover is

$$\begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

This is intuitive: if the value of ρ increases, then the flow on link 1 should decrease, whence link 2 must increase its flow with the same amount. If, on the other hand, the value of ρ decrease, the reverse should happen.

Consider then the workings of the formula (22) outlined above. We obviously fulfill Condition 1 on the travel cost functions. We also satisfy Condition 2, because $v^* > 0^{|\mathcal{L}|}$. Also, then, $\mathcal{G}_+ = \mathcal{G}$. We last try to comply with the linear independence Condition 3, by choosing the right equilibrium route flow solution. Note then that

$$[\Lambda^{\mathrm{T}} \mid \Gamma] = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

has rank 3. So, we should find a route flow solution, h^*, in which exactly 3 routes have a positive flow. This is however impossible; the only alternatives are 2 or 4. To see why, suppose that the flow on the first route, $\{1, 3\}$, is $\alpha \in [0, 1]$. Then, the flows on the routes $\{1, 4\}$ and $\{2, 3\}$ must both be $1 - \alpha$, in order to comply with the total flow on the links. This implies that the flow on route $\{2, 4\}$ is α. This shows that for any value of $\alpha \in [0, 1]$, the number of routes having a non-zero flow is either 2 or 4. Since we cannot comply with Condition 3, the formula (22) fails, even though the gradient exists.

The problems regarding the applicability of the formula (22) associated with the rank Condition 3 was first observed and commented on by Bell and Iida [BeI97, p. 97]; our example however seems to be the first that has been worked out in detail.

6.2.3 A case of non-differentiability where the formula (22) may provide a result

Consider the network shown in Figure 4.

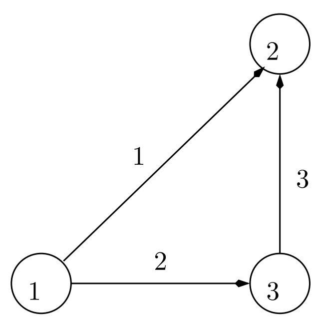

Figure 4: Network for the second counter- example.

In this example, there are three OD pairs, with the following fixed demands:

$$d_{12} := 1; \qquad d_{13} := 1; \qquad d_{32} := 1.$$

The link cost functions are given by

$$t_1(v_1, \rho) := 2v_1 + \rho; \qquad t_2(v_2) := v_2; \qquad t_3(v_3) := v_3.$$

(We thereby comply with Condition 1.) With $\rho^* = 0$, the unique equilibrium link volume is $v^* = (1, 1, 1)^{\mathrm{T}}$. In this case, the route flow is unique: the flow on route $\{(1, 2)\}$ is 1; the flow on route $\{(1, 3), (3, 2)\}$ is 0; the flow on route $\{(1, 3)\}$ is 1; and the flow on route $\{(3, 2)\}$ is 1 as well.

This solution is non-strictly complementary by our definition, since the route $\{(1, 3), (3, 2)\}$ is of least cost but it cannot be used. It is however strictly complementary according to Condition 2, which we thereby satisfy.

We also see that a small negative perturbation in ρ would not affect the equilibrium solution, since the link $\{(1, 2)\}$ (that is, the first route in the first OD pair) is already utilized to send all the demand in the first OD pair. But if the perturbation is positive, we see that the flow in route $\{(1, 2)\}$ would decrease, and the flow on the route $\{(1, 3), (3, 2)\}$ would increase. This is then a case where the directional derivative (which of course exists) is not linear, so $\rho^* = 0$ is a point of non-differentiability.

What happens if we wish to apply the formula (22)? We here have that

$$[\Lambda^{\mathrm{T}} \mid \Gamma] = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

which has rank 4. Since the flow on the route $\{(1,3),(3,2)\}$ is restricted to zero, in all fairness, the formula then breaks down. But it does not really do so for the right reason; there is every possibility of believing that the formula might still work if we either still include the route, or if we were to delete it. In both cases, the formula (22) does produce a result, which in none of the two cases can be interpreted as the value of the gradient at ρ^*.

6.3 The gradient formula of Cho, Smith, and Friesz

The sensitivity analysis of Cho, Smith, and Friesz [CSF00] is somewhat related to that in [ToF88]. It replaces all the three conditions 1–3 mentioned in the previous section with weaker ones, and further provides an analysis entirely in link flows. We briefly discuss this analysis below.

(Condition 1—strict monotonicity) $t(\rho, \cdot)$ is strictly monotone in a neighbourhood of ρ^*. Further, the Jacobian matrix $\nabla_v t(\rho^*, v^*)$ is positive definite.

(Condition 2—strict complementarity) There exists a strictly complementary route flow $h^* \in H^*(\rho^*)$.

Notice that these two conditions together imply differentiability, but that they are stronger then necessary; the latter utilizes the classic definition of strict complementarity, as we do in this paper.

We are first asked to consider, as in [ToF88], the graph $\mathcal{G}_+$, which only includes links $l \in \mathcal{L}$ with $v_l^* > 0$ at ρ^*. In order to state the differences between the analysis in [CSF00] and [ToF88] more clearly, we do not introduce the $+$ notation here, and assume, from now on, that $\mathcal{G} = \mathcal{G}_+$.

Next, suppose that we, for each OD pair $(p,q) \in \mathcal{C}$, remove the routes $r \in \mathcal{R}_{pq}$ whose cost is higher than π^*_{pq}. Thus, we reach a network which we may denote by $\mathcal{G}_0$, in which the set $\mathcal{R}$ is replaced by the subset $\mathcal{R}_0$ of least-cost routes at v^*. By Condition 2, they must also be the routes with positive flow at h^*.

The sensitivity analysis proceeds with a further reduction:

(Condition 3—linear independence) Select a subset of the rows of Λ_0, such that the resulting matrix $[(\Lambda_0')^{\mathrm{T}} \mid \Gamma_0]$ has full (column) rank.

Note that there is no requirement on the rank itself, and therefore this condition is milder than the Condition 3 in [ToF88].

The sensitivity formula is similar to that in (22), but provides the sensitivity in the link flow space directly, and therefore does not require the selection of a particular equilibrium route flow solution. It is however much more complicated in the sense that the translation between the spaces in h and v implies that several submatrices of, for example, Λ_0' must be constructed, collected, inverted and multiplied:

$$\begin{pmatrix} \nabla_\rho v \\ \nabla_\rho \pi \end{pmatrix} = \begin{pmatrix} \nabla_v t(\rho^*, v^*) & -[\Lambda_0'' N_1, -I]^{\mathrm{T}} \\ [\Lambda_0'' N_1, -I] & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\nabla_\rho t(\rho^*, v^*) \\ -\Lambda_0'' N_2 \nabla_\rho g(\rho^*) \end{pmatrix}, \tag{23a}$$

where Λ_0'' contains the rows of Λ_0 which are not present in Λ_0', and

$$\begin{aligned} N_1 :=\, & (\Lambda_0')^{\mathrm{T}}[\Lambda_0'(\Lambda_0')^{\mathrm{T}} - \Lambda_0'\Gamma_0(\Gamma_0^{\mathrm{T}}\Gamma_0)^{-1}\Gamma_0^{\mathrm{T}}(\Lambda_0')^{\mathrm{T}}]^{-1} \\ & - \Gamma_0[\Gamma_0^{\mathrm{T}}\Gamma_0 - \Gamma_0^{\mathrm{T}}(\Lambda_0')^{\mathrm{T}}[\Lambda_0'(\Lambda_0')^{\mathrm{T}}]^{-1}\Lambda_0'\Gamma_0]^{-1}\Gamma_0^{\mathrm{T}}(\Lambda_0')^{\mathrm{T}}[\Lambda_0'(\Lambda_0')^{\mathrm{T}}]^{-1}, \end{aligned} \tag{23b}$$

$$\begin{aligned} N_2 :=\, & -(\Lambda_0')^{\mathrm{T}}[\Lambda_0'(\Lambda_0')^{\mathrm{T}}]^{-1}\Lambda_0'\Gamma_0[\Gamma_0^{\mathrm{T}}\Gamma_0 - \Gamma_0^{\mathrm{T}}(\Lambda_0')^{\mathrm{T}}[\Lambda_0'(\Lambda_0')^{\mathrm{T}}]^{-1}\Lambda_0'\Gamma_0]^{-1} \\ & + \Gamma_0[\Gamma_0^{\mathrm{T}}\Gamma_0 - \Gamma_0^{\mathrm{T}}(\Lambda_0')^{\mathrm{T}}[\Lambda_0'(\Lambda_0')^{\mathrm{T}}]^{-1}\Lambda_0'\Gamma_0]^{-1}. \end{aligned} \tag{23c}$$

6.4 Conclusion

While being applicable to a wider selection of situations than the analysis in [ToF88], the analysis in [CSF00] is still not valid for problems with a non-strictly complementary equilibrium solution, since it relies on the implicit function theorem. Its main drawback is however its complexity; the formula (23) reached in [CSF00] is rather unintuitive and computationally burdensome to use. It is also clear from the papers that have been written and referred to during the past two–three years that the analysis in [ToF88] is the one favoured, despite the fact that it is less generally applicable.

Interestingly, shortly after Tobin and Friesz published their paper, Qiu and Magnanti [QiM89] published the first paper which develops a sensitivity analysis of traffic equilibria based on Robinson's strong regularity condition (albeit under sligthly stronger asssumptions than necessary; cf. [Pat03]); it is based on a linearized traffic equilibrium model which is similar to (15) in the case of a separable problem. Their paper did however not get much attention from the transportation science community. (See [Pat03] for an account of the history of the sensitivity analysis of traffic equilibria.) One of the few who has utilized their results is Denault [Den94], who applied it in the context of OD matrix estimation. He compared numerically the Qiu/Magnanti sensitivity analysis formulas to that of Tobin/Friesz, and found that the former was significantly more robust and efficient to use. Our above findings clearly is supportive of that claim.

The paper [JoP03] provides a first application of our sensitivity formulas to network design problems, with encouraging results.

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