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Stochastic Structural Topology Optimization: Discretization and Penalty Function Approach

Anton Evgrafov and Michael Patriksson

Abstract We consider structural topology optimization problems including unilateral constraints arising from, for example, non-penetration conditions in contact mechanics or non-compression conditions for elastic ropes. To construct more realistic models and to hedge off possible failures or inefficient behaviour of optimal structures, we allow parameters (for example, loads) defining the problem to be stochastic. The resulting nonsmooth stochastic optimization problem is an instance of stochastic mathematical programs with equilibrium constraints (MPEC), or stochastic bilevel programs. We propose a solution scheme based first on the approximation of the given topology optimization problem with a sequence of simpler sizing optimization problems, and second on approximating the probability measure in the latter problems. For stress constrained weight minimization problems an alternative to ε-perturbation based on a new penalty function is proposed.

Key words Bilevel programming – stochastic programming – robust optimization – discretization – ε-perturbation – stress constraints

1 Introduction and notation

In this paper we propose a solution scheme for stochastic generalizations of two of the most natural and classic structural topology optimization problems: the finding of a maximally stiff truss under a volume constraint, and the finding of a truss of minimal weight under stress constraints. The reason for introducing stochasticity into the problem is that uncertainty due to several factors (such as loads unknown in advance, varying material properties, manufacturing errors, etc.) has to be taken into account to obtain robust optimal solutions. Capturing the uncertainty in the model through the use of probability theory allows us to construct arbitrarily general models.

To include a wide range of applications we allow mechanical structures to be *unilaterally* constrained, i.e., some parts of the structure might come into unilateral frictionless contact with rigid obstacles, while some other parts might sustain only tensile forces. Practical applications of unilateral contact include such machine elements as joints, hinges, press-fits, and examples of structures with tensile-only members include suspension bridges and cranes.

The resulting optimization problem is an instance of stochastic mathematical programming problems with equilibrium constraints (SMPEC), or stochastic bilevel programming problems (Patriksson and Wynter 1999; Evgrafov and Patriksson 2001). In context of structural optimization, this class of problems was studied for discrete probability measures by Christiansen et al. (2001); **?**, and for general probability measures by Evgrafov et al. (2002).

In the case of discrete probability measures there is a strong connection with the intuitive multiple load approach (Sved and Ginos (1968), Bendsøe (1995,p. 8), Stolpe and Svanberg (2001)), which is widely used to obtain designs that are more robust with respect to unknown loading conditions. In such an approach, an engineer picks up a few loading scenarios, empirically assigns them some "weights" and then solves, essentially, the stochastic optimization problem with a discrete probability measure, where each scenario has a probability proportional to its "weight". By considering our problems in the framework of stochastic programming, we can analyze the question of existence of the limiting optimal designs if the sequence of the empirical probability distributions converges, i.e., as the number of the load cases converges to infinity!

When the probability measure is not discrete, the problem is posed in an infinite-dimensional space, whence it is important to construct finite-dimensional approximations to the problem. The most popular approach to solving stochastic optimization problems is based on the idea of approximating the probability measure with a sequence of discrete probability measures with finite support, resulting in a desired sequence of finite-dimensional optimization problems.

Unfortunately, in our case the straightforward implementation of such strategy is impossible, owing to the lack of

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standard constraint qualifications by the feasible set of the problem. Therefore, we approximate the given topology optimization problem by a sequence of simpler sizing optimization problems (ε-perturbations), and then discretize the latter problems. The "naïve" ε-perturbation involving only a change in the lower bounds on design variables studied by Achtziger (1998); **?**); Evgrafov et al. (2002) is sufficient for solving stochastic compliance minimization problems. For stress constrained weight minimization we propose a new penalty function based alternative to the perturbation method by Cheng and Guo (1997) (see also Petersson 2001; **?**; Evgrafov et al. 2002).

The outline of this paper is as follows. After introducing the necessary mechanical principles and formulating the problems, in Section 2 we formulate the penalty function approach to the stress constrained weight minimization. We illustrate this new approximation scheme with small numerical examples. Section 3 is dedicated to the construction of the finite-dimensional discretizations of the sizing approximations of the topology compliance and weight minimization problems. We end the paper with a numerical example (Section 4), illustrating all the steps of the proposed solution approach.

1.1 Mechanical equilibrium

In this subsection we introduce the notation and mechanical principles necessary to state the problems we are going to analyze.

Given positions of the nodes the *design* (and topology in particular) of a truss can be described by the following sets of *design* variables:

- $x_i \geq 0$, $i = 1,\dots,m$, representing the volume of material, allocated to the bar i in the structure;
- $X_j \geq 0$, $j = 1,\dots,r_2$, representing the volume of material, allocated to the cable j.

We introduce two index sets of the present (or active) members in the structure: $\mathscr{I}(\mathbf{x}) = \{\, i = 1,\dots,m \mid x_i > 0\,\}$ and $\mathscr{J}(\mathbf{X}) = \{\, j = 1,\dots,r_2 \mid X_j > 0\,\}$.

Let $(\Omega,\mathscr{S},\mathrm{P})$ be a complete probability space. Given a particular design the status of the linear elastic mechanical system is governed by the principle of minimum complementary energy $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega)$ (in our case it is the $(\mathbf{x},\mathbf{X},\omega)$-parametric minimization problem):

$$\begin{cases} \min\limits_{(\mathbf{s},\mathbf{S},\lambda)} \mathscr{E}(\mathbf{x},\mathbf{X},\mathbf{s},\mathbf{S},\lambda,\omega) := \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} \frac{s_i^2}{E(\omega)x_i} + \mathbf{g}_1^T(\omega)\lambda \\ \qquad\qquad + \displaystyle\sum_{j=1}^{r_2}\Big(\frac{(L_j(\omega)S_j)^2}{2E_c(\omega)X_j} + (\mathbf{g}_2(\omega))_j S_j\Big), \\ \text{s.t.} \begin{cases} \mathsf{C}_1^T(\omega)\lambda + \displaystyle\sum_{i\in\mathscr{I}(\mathbf{x})} \mathsf{B}_i^T(\omega)s_i + \sum_{j\in\mathscr{J}(\mathbf{X})} S_j\gamma_j(\omega) = \mathbf{f}(\omega), \\ \lambda \geq 0, \\ \mathbf{S}_{\mathscr{J}(\mathbf{X})} \geq 0, \end{cases} \end{cases}$$

where the functions in the problem have the following meaning from a mechanical point of view:

- $E(\omega)$ and $E_c(\omega)$ are Young's moduli for the structure and cable materials respectively;
- $\mathsf{B}_i(\omega)$ is the kinematic transformation matrix for the bar i;
- $\gamma_j(\omega)$ is the unit direction vector of the cable j;
- $(\mathbf{g}_2(\omega))_j$ is the initial slack of the cable j;
- $L_j(\omega)$ is the length of the cable j;
- $\mathsf{C}_1(\omega)$ is the quasi-orthogonal kinematic transformation matrix for rigid obstacles;
- $\mathbf{g}_1(\omega) \geq 0$ is the vector of the initial gaps;
- $\mathbf{f}(\omega)$ is the vector of external forces.

For the problem to be tractable we assume that all functions listed above are $\mathscr{S}$-measurable. We further assume that the matrix C_1 is quasi-orthogonal, that is, that $\mathsf{C}_1\mathsf{C}_1^T = \mathsf{I}$, the unit matrix of the corresponding size. This condition is fulfilled if at each node either there is at most one rigid support or multiple supports "act" in orthogonal directions to each other.

The variables in the problem $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega)$ have the following interpretation:

- s_i is the tensile force in the bar i times its length;
- S_j is the tensile force in the cable j;
- λ is the vector of contact forces.

Note, that from the quasi-orthogonality of C_1 it follows that λ is uniquely determined by $(\mathbf{s},\mathbf{S})$ and depends continuously on them:

$$\lambda = \mathsf{C}_1(\omega)\Big(\mathbf{f}(\omega) - \sum_{i\in\mathscr{I}(\mathbf{x})} \mathsf{B}_i^T(\omega)s_i - \sum_{j\in\mathscr{J}(\mathbf{X})} S_j\gamma_j(\omega)\Big). \quad (1.1)$$

These facts will be used often without backward reference.

1.2 Stochastic minimum compliance problem

We are now ready to state the first problem considered in this paper — the general stochastic minimum compliance problem $(\mathscr{P}_1)$:

$$\begin{cases} \min\limits_{(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot))} c^f(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot),\lambda(\cdot)) := \\ \qquad\qquad \displaystyle\int_\Omega \mathscr{E}(\mathbf{x},\mathbf{X},\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega),\omega)\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}, \qquad \mathbf{1}_m^T\mathbf{x} \leq v, \\ \underline{\mathbf{X}} \leq \mathbf{X} \leq \overline{\mathbf{X}}, \quad \mathbf{1}_{r_2}^T\mathbf{X} \leq V, \\ (\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega)) \text{ solves } (\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega), \text{ P-a.s.,} \end{cases} \end{cases}$$

where v and V are the limits on the amount of cable and structure material correspondingly.

In topology optimization we set lower bounds $\underline{\mathbf{x}} = \mathbf{0}$ and $\underline{\mathbf{X}} = \mathbf{0}$.

1.3 Stochastic stress constrained weight minimization problem

We formulate problem $(\mathscr{P}_2)$ as follows:

$$\begin{cases} \min_{(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot))} w(\mathbf{x},\mathbf{X}) := \rho_1 \mathbf{1}_m^T\mathbf{x} + \rho_2 \mathbf{1}_{r_2}^T\mathbf{X} \\ \text{s.t.} \begin{cases} \underline{\mathbf{x}} \le \mathbf{x} \le \overline{\mathbf{x}}, \\ \underline{\mathbf{X}} \le \mathbf{X} \le \overline{\mathbf{X}}, \\ |s_i(\omega)| \le \overline{\sigma}_1 x_i, & i = 1,\dots,m, & \text{P-a.s.}, \\ L_j S_j(\omega) \le \overline{\sigma}_2 X_j, & j = 1,\dots,r_2, & \text{P-a.s.}, \\ (\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega)) \text{ solves } (\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega), & & \text{P-a.s.}, \end{cases} \end{cases}$$

where $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are the maximal allowable effective stresses in, and ρ_1 and ρ_2 the densities of, the structure and the cable materials, respectively.

In topology optimization we set lower bounds $\underline{\mathbf{x}} = \mathbf{0}$ and $\underline{\mathbf{X}} = \mathbf{0}$.

2 Penalization

A continuation algorithm based on so-called ε-perturbation is a popular numerical approach for solving stress constrained weight minimization problems. The method was first proposed by Cheng and Guo (1997) for trusses, and it corresponds to solving a sequence of sizing structural optimization problems obtained from the original one by substituting lower bound on the design variables with ε^2 and relaxing the stress constraints by ε. The perturbation parameter ε (whence the name — ε-perturbation) is eventually reduced to zero, and the optimal solution from the previous iteration is used as a starting point for the new iteration. Petersson (2001) established the convergence of the optimal solutions to perturbed problems towards an optimal solution of the original problem, **?** generalized the procedure for stochastic weight minimization of trusses including unilateral constraints and discrete probability measures, and Evgrafov et al. (2002) proved the convergence of the method for general probability measures in the special case of trusses without unilateral constraints under stochastic loading conditions.

Being an interesting theoretical result, it is difficult to transfer the ε-perturbation result to efficient algorithms. Stolpe and Svanberg (2001) discussed the possible discontinuity of the optimal solutions to ε-perturbed problems with respect to ε, which might cause the continuation approach to fail in practice. Furthermore, the presence of stress constraints in the problem does not allow us to use the many numerical algorithms that were designed for MPECs without state constraints (e.g., the BT algorithm (Outrata et al. 1998,Chapter 7) or the implicit programming algorithm (Luo et al. 1996,Section 6.1)). Specifically in the stochastic setting, the presence of stress constraints does not allow us to construct discretizations for the problem. Even though the stress constraints must hold with probability one they can be violated on some set of measure zero, which may happen to contain our discretization points.

The goal of this section is to introduce an alternative convergent scheme, which in addition to adding the small lower bounds on the design variables moves the stress constraints into the objective function using a convex penalty function. Let

$$G(\mathbf{x},\mathbf{X},\mathbf{s},\mathbf{S}) := \sum_{i=1}^{m} \frac{[|s_i| - \overline{\sigma}_1 x_i]_+^2}{x_i} + \sum_{j=1}^{r_2} \frac{[L_j S_j - \overline{\sigma}_2 X_j]_+^2}{X_j}.$$

Using the usual convention $0/0 = 0$ and $a/0 = \infty$ for any $a > 0$, the function G can be evaluated at any nonnegative design $(\mathbf{x},\mathbf{X})$. Furthermore, G is l.s.c. on $\mathbb{R}^m_+ \times \mathbb{R}^{r_2}_+ \times \mathbb{R}^m \times \mathbb{R}^{r_2}_+$.

Let $\mu : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be an arbitrary function such that $\lim_{\varepsilon\to+0} \mu(\varepsilon) = +\infty$ and $\lim_{\varepsilon\to+0} \varepsilon\mu(\varepsilon) = 0$. Consider the penalized problem $(\bar{\mathscr{P}}_2^\varepsilon)$:

$$\begin{cases} \min_{(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot))} w^\varepsilon(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot)) := \\ \qquad w(\mathbf{x},\mathbf{X}) + \mu(\varepsilon) \int_\Omega G(\mathbf{x},\mathbf{X},\mathbf{s}(\omega),\mathbf{S}(\omega))\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \varepsilon\mathbf{1}_m \le \mathbf{x} \le \overline{\mathbf{x}} + \varepsilon\mathbf{1}_m, \\ \varepsilon\mathbf{1}_{r_2} \le \mathbf{X} \le \overline{\mathbf{X}} + \varepsilon\mathbf{1}_{r_2}, \\ (\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega)) \text{ solves } (\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega), \quad \text{P-a.s.} \end{cases} \end{cases}$$

The problem $(\bar{\mathscr{P}}_2^\varepsilon)$ is an instance of a stochastic MPEC (or stochastic bilevel programming problem). Even the existence of optimal solutions for this class of problems is a nontrivial fact, which is caused by the implicit characterization of the feasible set. Examples of MPEC having non-closed feasible sets are easy to construct (Luo et al. 1996,Example 1.1.2).

The closedness properties of design-to-force mappings were investigated by Petersson (2001) for trusses, then generalized for trusses with unilateral constraints by **?**, and for stochastic settings by Evgrafov et al. (2002). We cite some auxiliary results from the latter paper, which will be used in the development that follows.

Proposition 2.1 (Evgrafov et al. 2002,Proposition 2.3) *Let a sequence of nonnegative designs $\{(\mathbf{x}_k,\mathbf{X}_k)\}$ converge to $(\bar{\mathbf{x}},\bar{\mathbf{X}})$. Suppose that $(\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))$ solves $(\mathscr{C})_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$, and that the sequence of energy expectations is bounded:*

$$\int_\Omega \mathscr{E}(\mathbf{x}_k,\mathbf{X}_k,\mathbf{s}_k(\omega),\mathbf{S}_k(\omega),\lambda_k(\omega),\omega)\,\mathrm{P}(d\omega) \le C < \infty.$$

Then there exists a solution $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$ to the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$, and $\{(\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))\}$ almost surely converges to $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$.

The following energy estimation will be useful to obtain closedness properties of the feasible set of the penalized problem $(\bar{\mathscr{P}}_2^\varepsilon)$ and to verify the existence of solutions.

Lemma 2.2 *Suppose that the functions $\mathsf{B}_i(\cdot)$, $\gamma_j(\cdot)$, $\mathsf{C}_1(\cdot)$, $\mathbf{f}(\cdot)$, $\mathbf{g}_1(\cdot)$, $\mathbf{g}_2(\cdot)$, $1/E(\cdot)$, $\mathbf{L}^2(\cdot)/E_c(\cdot)$ are essentially bounded* *by some constant $K > 0$. Let $(\mathbf{x},\mathbf{X})$ be a positive design such that $(\mathbf{x},\mathbf{X}) \le (\overline{\mathbf{x}},\overline{\mathbf{X}}) < \infty$ and the problem $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\cdot)$ has a solution $(\mathbf{s}(\cdot),\mathbf{S}(\cdot),\lambda(\cdot))$. Then there are constants $K_1 > 0$ and $K_2 > 0$, depending on K, $\overline{\mathbf{x}}$, $\overline{\mathbf{X}}$, $\overline{\sigma}_1$, $\overline{\sigma}_2$, m and r_2 only, such that*

$$\mathscr{E}(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot),\lambda(\cdot),\cdot) \le K_1 + K_2 G(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot)).$$

Proof The inequality follows from the assumed essential boundedness of the functions defining the problem $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\cdot)$ and inequalities of the following type:

$$\sum_{i=1}^{m} \frac{s_i^2(\cdot)}{2E(\cdot)x_i} \le \frac{1}{E(\cdot)} \sum_{i=1}^{m} \left[\overline{\sigma}_1^2 x_i + \frac{[|s_i(\cdot)| - \overline{\sigma}_1 x_i]_+^2}{x_i} \right],$$

following from the inequality $(a+b)^2 \le 2a^2 + 2b^2$, which holds for arbitrary $a,b \in \mathbb{R}$. In this way we can bound all the terms appearing in the definition of $\mathscr{E}$.

Proposition 2.3 *Suppose that the problem $(\bar{\mathscr{P}}_2^{\varepsilon})$ is feasible and proper. Then it possesses at least one optimal solution.*

Proof Lemma 2.2 gives us the energy estimation, necessary for the application of Proposition 2.1. Then we can follow the proof of Theorem 2.4 in Evgrafov et al. (2002) to reach the conclusion.

Now we come to the main result of this section, asserting that all limit points of the family of optimal solutions to $(\bar{\mathscr{P}}_2^{\varepsilon})$ as ε decreases to zero are in fact optimal solutions to the limiting problem $(\mathscr{P}_2)$. The proof of this result depends on the following assumption of locally directionally Lipschitz behaviour of the design-to-force mapping.

Assumption 1 *Let $(\mathbf{x},\mathbf{X}) \ge \mathbf{0}$ be a design for which the problem $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega)$ has a solution $(\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega))$ for almost any ω. Let $(\Phi,\Gamma) > \mathbf{0}$ be arbitrary in $\mathbb{R}^m \times \mathbb{R}^{r_2}$ and for $\varepsilon > 0$ set $(\mathbf{x}_\varepsilon,\mathbf{X}_\varepsilon) := (\mathbf{x},\mathbf{X}) + \varepsilon(\Phi,\Gamma)$. Denote by $(\mathbf{s}_\varepsilon(\omega),\mathbf{S}_\varepsilon(\omega),\lambda_\varepsilon(\omega))$ the corresponding optimal solution to $(\mathscr{C})_{(\mathbf{x}_\varepsilon,\mathbf{X}_\varepsilon)}(\omega)$. Then, there exist $\varepsilon_0 > 0$ and a nonnegative function $\tau(\omega) \in L_2(\Omega,\mathrm{P})$ such that the inequality*

$$\|(\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega)) - (\mathbf{s}_\varepsilon(\omega),\mathbf{S}_\varepsilon(\omega),\lambda_\varepsilon(\omega))\| \le \varepsilon\tau(\omega)$$

holds for almost all $\omega \in \Omega$ and all $0 < \varepsilon < \varepsilon_0$.

Assumption 1 is known to hold for trusses without unilateral constraints and general probability measures (Evgrafov et al. 2002,Lemma 3.3), and for trusses with unilateral contraints and discrete probability measures (**?**,Theorem 3.3).

We denote the optimal value of a problem $(\mathscr{P})$ by $\mathrm{val}(\mathscr{P})$.

Theorem 2.4 *Suppose that the functions $\mathsf{B}_i(\cdot)$, $\gamma_j(\cdot)$, $\mathsf{C}_1(\cdot)$, $\mathbf{f}(\cdot)$, $\mathbf{g}_1(\cdot)$, $\mathbf{g}_2(\cdot)$, $1/E(\cdot)$, $\mathbf{L}^2(\cdot)/E_c(\cdot)$ are essentially bounded, and let Assumption 1 hold at each feasible design in $(\mathscr{P}_2)$. Suppose further that for some $\varepsilon_0 > 0$ there is a solution $(\mathbf{x}_0,\mathbf{X}_0,\mathbf{s}_0(\cdot),\mathbf{S}_0(\cdot))$, which is feasible in $(\mathscr{P}_2)$ with $(\mathbf{x}_0,\mathbf{X}_0) \ge \varepsilon_0 \mathbf{1}_{m+r_2}$. Then for any $0 < \varepsilon \le \varepsilon_0$ the problem $(\bar{\mathscr{P}}_2^{\varepsilon})$ has an optimal solution $(\mathbf{x}_\varepsilon,\mathbf{X}_\varepsilon,\mathbf{s}_\varepsilon(\cdot),\mathbf{S}_\varepsilon(\cdot))$. Any limit point of the sequence $\{(\mathbf{x}_\varepsilon,\mathbf{X}_\varepsilon,\mathbf{s}_\varepsilon(\cdot),\mathbf{S}_\varepsilon(\cdot))\}$ as ε goes to zero is an optimal solution to $(\mathscr{P}_2)$.*

Proof The point $(\mathbf{x}_0,\mathbf{X}_0,\mathbf{s}_0(\cdot),\mathbf{S}_0(\cdot))$ is feasible in $(\bar{\mathscr{P}}_2^{\varepsilon})$ for any $0 < \varepsilon \le \varepsilon_0$, with an objective value $w^{\varepsilon}(\mathbf{x}_0,\mathbf{X}_0,\mathbf{s}_0(\cdot),\mathbf{S}_0(\cdot)) = w(\mathbf{x}_0,\mathbf{X}_0)$. Proposition 2.3 implies the existence of a sequence $\{(\mathbf{x}_\varepsilon,\mathbf{X}_\varepsilon,\mathbf{s}_\varepsilon(\cdot),\mathbf{S}_\varepsilon(\cdot))\}$ of optimal solutions to $(\bar{\mathscr{P}}_2^{\varepsilon})$. The compactness of the design space implies the existence of a subsequence $\{(\mathbf{x}_{\varepsilon_k},\mathbf{X}_{\varepsilon_k})\}$ converging to a limit $(\bar{\mathbf{x}},\bar{\mathbf{X}}) \ge \mathbf{0}_{m+r_2}$. Since $w^{\varepsilon_k}(\mathbf{x}_{\varepsilon_k},\mathbf{X}_{\varepsilon_k},\mathbf{s}_{\varepsilon_k}(\cdot),\mathbf{S}_{\varepsilon_k}(\cdot)) \le \mathrm{val}(\mathscr{P}_2)$, we can apply Lemma 2.2 and Proposition 2.1 to conclude that the sequence $\{(\mathbf{s}_{\varepsilon_k}(\cdot),\mathbf{S}_{\varepsilon_k}(\cdot))\}$ converges to a limit $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot))$, solving the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$. The boundedness of $\{\mu(\varepsilon_k)\int_\Omega G(\mathbf{x}_{\varepsilon_k},\mathbf{X}_{\varepsilon_k},\mathbf{s}_{\varepsilon_k}(\cdot),\mathbf{S}_{\varepsilon_k}(\cdot))\mathrm{P}(d\omega)\}$ as ε_k goes to zero, the nonnegativity and l.s.c. property of G, and Fatou's Lemma, imply that the point $(\bar{\mathbf{x}},\bar{\mathbf{X}},\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot))$ satisfies the stress constraints, whence is feasible in $(\mathscr{P}_2)$. Then, the following inequalities hold:

$$\begin{aligned}\mathrm{val}(\mathscr{P}_2) \le w(\bar{x},\bar{X}) &\le \liminf_{k\to\infty} w^{\varepsilon_k}(\mathbf{x}_{\varepsilon_k},\mathbf{X}_{\varepsilon_k},\mathbf{s}_{\varepsilon_k}(\cdot),\mathbf{S}_{\varepsilon_k}(\cdot)) \\ &= \liminf_{k\to\infty} \mathrm{val}(\bar{\mathscr{P}}_2^{\varepsilon_k}).\end{aligned} \tag{2.1}$$

On the other hand, for any point $(\tilde{\mathbf{x}},\tilde{\mathbf{X}},\tilde{\mathbf{s}}(\cdot),\tilde{\mathbf{S}}(\cdot))$ that is feasible in $(\mathscr{P}_2)$ and for large enough k, Assumption 1 implies that the point $(\tilde{\mathbf{x}} + \varepsilon_k \mathbf{1}_m, \tilde{\mathbf{X}} + \varepsilon_k \mathbf{1}_{r_2}, \tilde{\mathbf{s}}_{\varepsilon_k}(\cdot), \tilde{\mathbf{S}}_{\varepsilon_k}(\cdot))$, where $(\tilde{\mathbf{s}}_{\varepsilon_k}(\cdot),\tilde{\mathbf{S}}_{\varepsilon_k}(\cdot),\tilde{\lambda}_{\varepsilon_k}(\cdot))$ solves $(\mathscr{C})_{(\tilde{\mathbf{x}}+\varepsilon_k\mathbf{1}_m,\tilde{\mathbf{X}}+\varepsilon_k\mathbf{1}_{r_2})}(\cdot)$, is feasible in the problem $(\bar{\mathscr{P}}_2^{\varepsilon_k})$ with the objective value $w^{\varepsilon_k}(\tilde{\mathbf{x}} + \varepsilon_k \mathbf{1}_m, \tilde{\mathbf{X}} + \varepsilon_k \mathbf{1}_{r_2}, \tilde{\mathbf{s}}_{\varepsilon_k}(\cdot), \tilde{\mathbf{S}}_{\varepsilon_k}(\cdot)) \le w(\tilde{\mathbf{x}},\tilde{\mathbf{X}}) + \varepsilon_k(m\rho_1 + r_2\rho_2) + K\varepsilon_k\mu(\varepsilon_k)$ for a suitable positive constant K, thus showing the reverse inequality:

$$\limsup_{k\to\infty} \mathrm{val}(\bar{\mathscr{P}}_2^{\varepsilon_k}) \le \mathrm{val}(\mathscr{P}_2);$$

together with (2.1) this establishes the optimality of $(\bar{\mathbf{x}},\bar{\mathbf{X}},\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot))$ in $(\mathscr{P}_2)$.

2.1 Numerical examples

We illustrate Theorem 2.4 with small numerical examples.

Example 2.5 (One-bar truss with a cable)
Figure 2.1 shows a simple one-dimensional structure introduced and analyzed in **?** that consists of a bar suspended with one cable. Suppose that $\Omega = [-1,2]$, P is the uniform distribution on $[-1/2,1]$, $f(\omega) = \omega$, $E = E_c = 1$, $\rho_1 = \rho_2 = 1$, $\overline{x} = 1$ and $\overline{X} = 2$. Every feasible design (x,X) must satisfy $x > 0$ (because $\mathrm{P}(f < 0) > 0$). Solving the equilibrium problem, we can see that the design-to-force mapping

$$s(\omega) = \begin{cases} \frac{x\omega}{x+X}, & \text{for } \omega \ge 0, \\ \omega, & \text{otherwise,} \end{cases} \qquad S(\omega) = \begin{cases} \frac{X\omega}{x+X}, & \text{for } \omega \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies the Assumption 1 for every feasible design, thus we can expect the convergence of solutions to penalized problems towards the solution of the original problem.

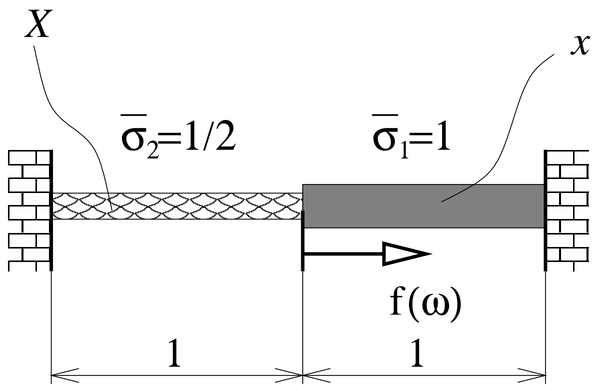


Fig. 2.1 The cable suspended one-bar truss.

After eliminating the state variables one obtains the following optimization problem:

$$\begin{cases} \min\limits_{(x,X)} x+X \\ \text{s.t.} \begin{cases} 0 \le x \le 1, & \\ 0 \le X \le 2, & \\ \dfrac{x\omega}{x+X} \le x, & \text{P-a.s.,} \\ -\omega \le x, & \text{P-a.s.,} \\ 0 \le \dfrac{X\omega}{x+X} \le X/2, & \text{P-a.s.} \end{cases} \end{cases}$$

Figure 2.2 shows the feasible domain for the design variables

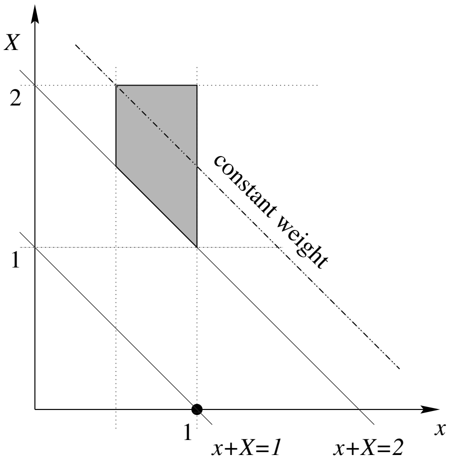


Fig. 2.2 The admissible design domain. The optimal solution is at the black circle.

(x,X). Note that this domain consists of the union of a two-dimensional, convex domain and the isolated optimal point $(x^*,X^*) = (1,0)$ with corresponding optimal weight $w^* = 1$.

Now we set $\mu(\varepsilon) = \varepsilon^{-0.8}$ and consider perturbation parameters $\varepsilon_k = 2^{-k}$. The corresponding penalized problem is:

$$\begin{cases} \min\limits_{(x,X)} x+X+\varepsilon_k^{-0.8}\Bigg(x \displaystyle\int_{\{x+X\le\omega\le 1\}} \left[\frac{\omega}{x+X}-1\right]^2 d\omega \\ \qquad + \displaystyle\int_{\{-1/2\le\omega\le -x\}} [-\omega-x]^2\, d\omega \\ \qquad + X \displaystyle\int_{\{(x+X)/2\le\omega\le 1\}} \left[\frac{\omega}{x+X}-1/2\right]^2 d\omega \Bigg) \\ \text{s.t.} \begin{cases} \varepsilon_k \le x \le 1+\varepsilon_k, \\ \varepsilon_k \le X \le 2+\varepsilon_k. \end{cases} \end{cases}$$

The behaviour of the optimal solutions to the penalized problems is shown in Figure 2.3. The sequence converges to the optimal solution of $(\mathscr{P}_2)$, as was predicted by Theorem 2.4. We note that this problem possesses uncountably many local minima, corresponding to the weight $x+X=2$, but our approximation scheme recovers the globally optimal solution.

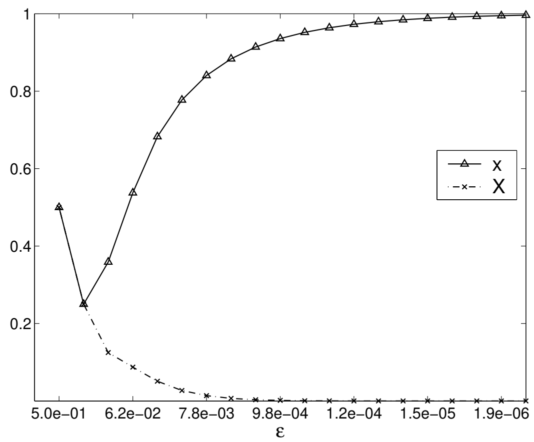


Fig. 2.3 The convergence of the optimal solutions to the penalized problems.

Example 2.6 (4-bar truss) Consider the problem of minimizing the weight of the 4-bar structure shown in Figure 2.4 (this problem also appear in Evgrafov et al. (2002)). The stress limit for each bar is $\overline{\sigma} = 1$, and the Young's modulus is $E = 1$. Assume that the upper design bounds are inactive, and that the stochastic force is distributed as follows: the direction equals ω (where $\omega \in [0,\pi]$); the magnitude is defined for $0 \le \omega \le \pi/2$ (and is defined by symmetry for $\pi/2 \le \omega \le \pi$) as a parabola passing through the points $(0,1.0)$, $(\pi/6,0.85)$, $(\pi/2,1.75)$. The probability measure is the uniform one on $[0,\pi]$. Since the initial structural topology as well as the loading conditions are symmetric, we can expect symmetric optimal solutions (i.e., $x_1^* = x_4^*, x_2^* = x_3^*$). Figure 2.5 shows the projection of the subset of symmetric feasible designs (i.e., such

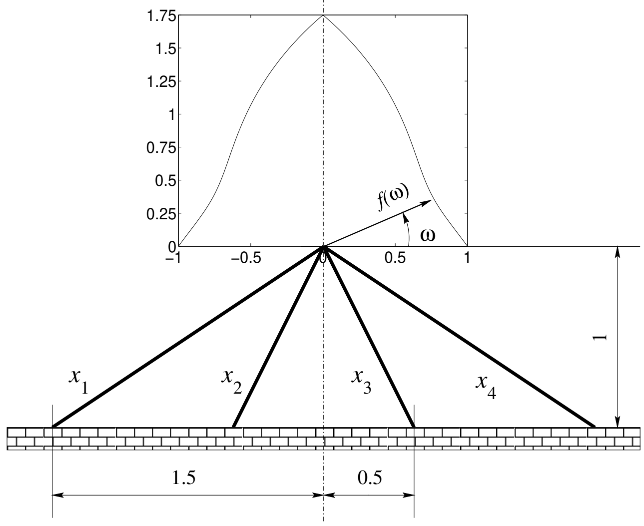


Fig. 2.4 The 4-bar truss problem.

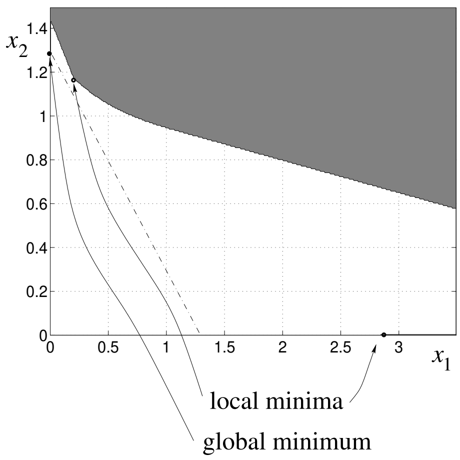


Fig. 2.5 The feasible design domain of the 4-bar truss problem.

that $x_1 = x_4$ and $x_2 = x_3$) onto the first two coordinates. Note that the feasible set is not *a finite union of polyhedra*, because we work with an infinite number of load cases (compare with the similar Problem 1 in Stolpe and Svanberg (2001)). Despite the large number of load cases, at the globally optimal solution, $x^* = (0.0, 1.294, 1.294, 0.0)$, the structural topology was modified (i.e., bars 1 and 4 were removed).

There are three local minima, two of which (including the globally optimal solution) are singular. The nonsingular, non-global, local minimum $x_l^* = (0.202, 1.173, 1.173, 0.202)$ to the original problem is in fact the global minimum for the "naïvely" perturbed problem for all small but positive values of ε. Therefore, we cannot approach the globally optimal solution by the use of a "naïve" ε-perturbation.

Our penalization scheme recovers the global optimal solution. Figure 2.6 shows the convergence of the optimal solutions to the ε-perturbed problems towards the solution to the original problem, as ε decreases to zero (variables x_3 and x_4 are not shown, owing to the symmetry of the optimal solutions calculated). We have used the nested formulation (with state variables eliminated), adaptive numerical quadratures to calculate the penalty function and a finite difference approximation of the derivatives to solve the problem using a sequential quadratic programming (SQP) algorithm. The penalty parameter was taken to be $\mu(\varepsilon) = \varepsilon^{-0.8}$.

Of course, applying a nonlinear programming algorithm to a penalized problem we can only expect a local solution, owing to the nonconvexity of the problem. For example, it is possible to obtain all three local minima for the 4-bar truss problem by providing different starting points to an SQP algorithm.

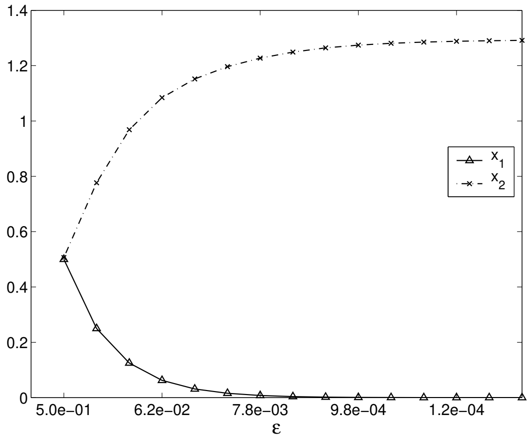


Fig. 2.6 Convergence of the ε-perturbations for the 4-bar truss problem.

3 Discretization

The most popular method to solve a stochastic programming problem involving a non-discrete probability measure is to approximate it with a sequence of finite-dimensional problems with discrete measures. To implement such a procedure one needs to construct two objects:

- a discrete measure $\widetilde{\mathrm{P}}$, in some sense close to the original one. The popular choice is to start from a finite partition $\mathscr{A} = \{\Omega_i \mid \cup_i \Omega_i = \Omega\}$ and define $\widetilde{\mathrm{P}}_i = \mathrm{P}(\Omega_i)$. This involves the calculation of probabilities $\mathrm{P}(\Omega_i)$, which, ideally, we would like to avoid and replace by some estimations of $\mathrm{P}(\Omega_i)$;
- an approximation $\widetilde{f}$ to a random element f of the original problem. One possible choice is to define $\widetilde{f} = E(f \mid \mathscr{A})$, which involves the computation of conditional means. In the case when Ω itself is a subset of a metric space we can define a vector $\widetilde{\omega} = E(\omega \mid \mathscr{A})$, and consider $\widetilde{f}_i = f(\widetilde{\omega}_i)$. This approximation still involves the computation of condi-

tional means, although simpler than in the previous case. In the computational scheme, we would instead like to choose sampling points $\widetilde{\omega}_i \in \Omega_i$ and set, as before, $\widetilde{f}_i = f(\widetilde{\omega}_i)$.

Examples of discretizations of stochastic programming problems with recourse (the most studied class of stochastic programming problems) based on the ideas of conditional means can be found in Olsen (1976b,a) and in Birge and Wets (1986). One generic scheme which allows us to avoid the computations of conditional means is the "method of mechanical quadratures" (Vainikko 1971), which we will use to approximate integrals in our problems. It was successfully used to discretize stochastic programming problems with complete recourse in Lepp (1990).

To characterize the convergence of the solutions to the discretized problems towards a solution to the original one, we use the general notion of discrete convergence, introduced by Stummel (1973) and Vainikko (1978). See also Vasin (1982); Lepp (1990, 1994) for the applications to stochastic programming problems, Pankov (1979); Liskovets (1987, 1990) for the application to variational inequalities in Banach spaces, and Lepp (1993, 1996) for applications to control problems. With some identifications we can embed the method of mechanical quadratures into the framework of discrete limit spaces. In spirit, the following analysis is close to that of Lepp (1990) for the stochastic programming problems with recourse.

It seems impossible to apply discretization schemes directly to the structural topology optimization problems, owing to the discontinuity of the lower-level objective function $\mathscr{E}$ at the points, where the topology changes. Therefore, we discretize sizing approximations of the topology optimization problems instead.

In this section we assume that all state variables are elements of $L_\infty(\Omega, \mathrm{P})$. Measurability of the state variables is necessary for the model to be tractable in the probabilistic setting; it is a very mild condition and holds under measurability assumptions on the problem's data (cf. Evgrafov et al. 2002,Corollary 2.2). A more questionable assumption is the essential boundedness of the state variables. But since we formulate our problems in the framework of linear elasticity, all loads and stresses considered must be bounded.

In the remainder of the section we list the main assumptions and results, while the interested reader can find the complete development with proofs in the appendix.

Suppose that Ω is a compact metric space with a metric denoted by $\rho(\cdot,\cdot)$. Let $\mathscr{S} \supset \mathscr{B}(\Omega)$, and $\mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) < r\}) = \mathrm{P}(\{\,\omega \mid \rho(\omega,\omega_0) \le r\}) > 0$ for any $\omega_0 \in \Omega$, $r > 0$.

Consider a sequence of *partitions* of Ω, $\mathscr{A}^k = \{A_1^k, \dots, A_k^k\}$, satisfying the following properties:

(M1) $\mathrm{P}(A_l^k) > 0$,
(M2) $\cup_{l=1}^k A_l^k = \Omega$,
(M3) $A_i^k \cap A_j^k = \emptyset$, $i \neq j$,
(M4) $\lim_{k\to\infty} \operatorname{diam}(A_l^k) = 0$,
(M5) $\mathrm{P}(\partial A_l^k) = 0$.

Note, that the collection of the sets $\{\mathscr{A}^k\}$, satisfying the properties (M1)–(M5) generates an algebra $\mathscr{S}_0 \subset \mathscr{S}$.

Define a sequence of discrete measures P_k with support $\operatorname{supp}\mathrm{P}_k = \{\,\omega_1^k, \dots, \omega_k^k\,\}$, satisfying the following properties:

(M6) $\omega_l^k \in A_l^k$,
(M7) $\lim_{k\to\infty} \max_{1\le l\le k} \mathrm{P}_k(\omega_l^k)/\mathrm{P}(A_l^k) = 1$.

We further assume the following:

(D1) the functions $E(\cdot)$, $E_c(\cdot)$, $\mathbf{g}_1(\cdot)$, $\mathbf{g}_2(\cdot)$, $\mathbf{L}_j(\cdot)$ and $\mathbf{f}(\cdot)$ are $\mathscr{S}_0$-measurable;
(D2) the functions $\mathbf{f}(\cdot)$, $\mathbf{g}_1(\cdot)$, $\mathbf{g}_2(\cdot)$, $1/E(\cdot)$, $\mathbf{L}_j^2(\cdot)/E_c(\cdot)$ are bounded;
(D3) the functions $\mathsf{C}_1(\cdot)$, $\mathsf{B}_i(\cdot)$, $\gamma_j(\cdot)$ are constants;
(D4) the functions $E(\cdot)$ and $E_c(\cdot)/\mathbf{L}_j^2(\cdot)$ are bounded.

We denote by $(\mathscr{C})^k_{(\mathbf{x},\mathbf{X})}(\omega_l^k)$ the following equilibrium principle:

$$\begin{cases} \min\limits_{(\mathbf{s},\mathbf{S},\lambda)} \mathscr{E}(\mathbf{x},\mathbf{X},\mathbf{s},\mathbf{S},\lambda,\omega_l^k) := \dfrac{1}{2}\sum\limits_{i=1}^m \dfrac{s_i^2}{E(\omega_l^k)x_i} + \mathbf{g}_1^T(\omega_l^k)\lambda \\ \qquad\qquad + \sum\limits_{j=1}^{r_2} \Big(\dfrac{(L_j(\omega_l^k)S_j)^2}{2E_c(\omega_l^k)X_j} + (\mathbf{g}_2(\omega_l^k))_j S_j \Big), \\ \text{s.t.} \begin{cases} \mathsf{C}_1^T\lambda + \sum\limits_{i\in\mathscr{I}(\mathbf{x})} \mathsf{B}_i^T s_i + \sum\limits_{j\in\mathscr{J}(\mathbf{X})} S_j\gamma_j = \mathbf{f}(\omega_l^k), \\ \lambda \ge 0, \\ \mathbf{S}_{\mathscr{J}(\mathbf{X})} \ge 0. \end{cases} \end{cases}$$

Let $(\mathscr{P}_1^\varepsilon)$ be an ε-perturbation of the problem $(\mathscr{P}_1)$:

$$\begin{cases} \min\limits_{(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot))} c^f(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot),\lambda(\cdot)) \\ \text{s.t.} \begin{cases} \varepsilon\mathbf{1}_m \le \mathbf{x} \le \bar{\mathbf{x}}, \quad \mathbf{1}_m^T\mathbf{x} \le v, \\ \varepsilon\mathbf{1}_{r_2} \le \mathbf{X} \le \overline{\mathbf{X}}, \quad \mathbf{1}_{r_2}^T\mathbf{X} \le V, \\ (\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega)) \text{ solves } (\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega), \text{ P-a.s.} \end{cases} \end{cases}$$

Any limit point of the family of optimal solutions to $(\mathscr{P}_1^\varepsilon)$ as ε converges to zero is an optimal solution to the limiting problem $(\mathscr{P}_1)$, owing to (Evgrafov et al. 2002,Theorem 3.1).

In the following two theorems we construct discretizations for the problems $(\mathscr{P}_1^\varepsilon)$ and $(\bar{\mathscr{P}}_2^\varepsilon)$. The formal definition of weak* discrete convergence is given in the appendix (Definition A.2). We note that from the weak* discrete convergence of the sequence $\{(\mathbf{x}_k^*, \mathbf{X}_k^*, \mathbf{s}_k^*(\cdot), \mathbf{S}_k^*(\cdot))\}$ follows the (usual) convergence of the optimal designs.

Theorem 3.1 *Consider the following sequence* $\{(\mathscr{P}_1^\varepsilon)^k\}$ *of discretizations of the problem* $(\mathscr{P}_1^\varepsilon)$*:*

$$\begin{cases} \min\limits_{(\mathbf{x},\mathbf{X},\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot))} c^f(\mathbf{x},\mathbf{X},\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot)) := \\ \qquad \sum\limits_{l=1}^k \mathscr{E}(\mathbf{x},\mathbf{X},\mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k),\lambda(\omega_l^k),\omega_l^k)\,\mathrm{P}_k(\omega_l^k) \\ \text{s.t.} \begin{cases} \varepsilon\mathbf{1}_m \le \mathbf{x} \le \bar{\mathbf{x}}, \quad \mathbf{1}_m^T\mathbf{x} \le v, \\ \varepsilon\mathbf{1}_{r_2} \le \mathbf{X} \le \overline{\mathbf{X}}, \quad \mathbf{1}_{r_2}^T\mathbf{X} \le V, \\ (\mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k),\lambda_k(\omega_l^k)) \text{ solves } (\mathscr{C})^k_{(\mathbf{x},\mathbf{X})}(\omega_l^k), \\ \qquad l = 1,\dots,k. \end{cases} \end{cases}$$

Suppose that the assumptions (M1)–(M7) *and* (D1)–(D4) *hold. Suppose further that there exists an optimal solution* $(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot))$ *to the problem* $(\mathscr{P}_1^\varepsilon)$ *such that the energy functional* $\mathscr{E}(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot),\cdot)$ *is essentially bounded. Finally, assume that the problems* $(\mathscr{C})^k_{(\mathbf{x}^*,\mathbf{X}^*)}(\omega_l^k)$ *are feasible for any k,* $1 \le l \le k$.

Let $\{(\mathbf{x}_k^*,\mathbf{X}_k^*,\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot))\}$ *be a sequence of optimal solutions to* $\{(\mathscr{P}_1^\varepsilon)^k\}$. *Then any weak* discrete limit point of this sequence solves the limiting problem* $(\mathscr{P}_1^\varepsilon)$.

Theorem 3.2 *Consider the following sequence* $\{(\bar{\mathscr{P}}_2^\varepsilon)^k\}$ *of discretizations of the problem* $(\bar{\mathscr{P}}_2^\varepsilon)$:

$$\begin{cases} \min\limits_{(\mathbf{x},\mathbf{X},\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot))} w^\varepsilon(\mathbf{x},\mathbf{X},\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot)) := w(\mathbf{x},\mathbf{X}) + \\ \qquad\qquad \mu(\varepsilon)\sum\limits_{l=1}^{k} G(\mathbf{x},\mathbf{X},\mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k))\,\mathrm{P}_k(\omega_l^k) \\ s.t. \begin{cases} \varepsilon\mathbf{1}_m \le \mathbf{x} \le \overline{\mathbf{x}} + \varepsilon\mathbf{1}_m, \\ \varepsilon\mathbf{1}_{r_2} \le \mathbf{X} \le \overline{\mathbf{X}} + \varepsilon\mathbf{1}_{r_2}, \\ (\mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k),\lambda_k(\omega_l^k)) \text{ solves } (\mathscr{C})^k_{(\mathbf{x},\mathbf{X})}(\omega_l^k), \\ \qquad\qquad\qquad\qquad l = 1,\dots,k. \end{cases} \end{cases}$$

Suppose that the assumptions (M1)–(M7) *and* (D1)–(D4) *hold. Suppose further that there exists an optimal solution* $(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot))$ *to the problem* $(\bar{\mathscr{P}}_2^\varepsilon)$ *such that the energy functional* $\mathscr{E}(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot),\cdot)$ *is essentially bounded. Finally, assume that the problems* $(\mathscr{C})^k_{(\mathbf{x}^*,\mathbf{X}^*)}(\omega_l^k)$ *are feasible for any k,* $1 \le l \le k$.

Let $\{(\mathbf{x}_k^*,\mathbf{X}_k^*,\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot))\}$ *be a sequence of optimal solutions to* $\{(\bar{\mathscr{P}}_2^\varepsilon)^k\}$. *Then any weak* discrete limit point of this sequence solves the limiting problem* $(\bar{\mathscr{P}}_2^\varepsilon)$.

In Theorems 3.1 and 3.2 the requirement of feasibility of $(\mathscr{C})^k_{(\mathbf{x},\mathbf{X})}(\cdot)$ for positive designs may be related to a "richness" of the ground structure (a truss with the topology where all members are present). For example, we can start from a ground structure which is able to sustain any load and thus satisfy the assumption.

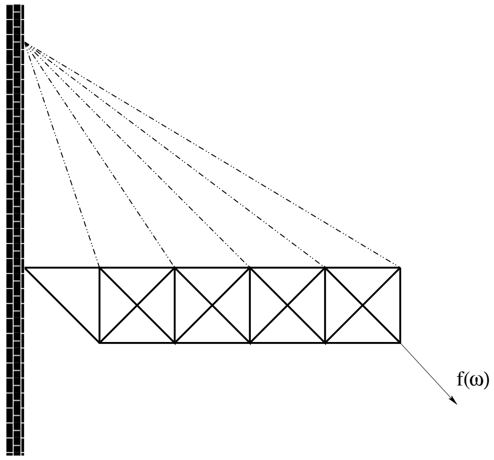


Fig. 4.1 The ground structure for the weight minimization problem.

k	w^*	$\lVert\mathbf{x}_k^* - \mathbf{x}_{625}^*\rVert$	$\lVert\mathbf{X}_k^* - \mathbf{X}_{625}^*\rVert$
1	10.3753	1.44	1.30
5	13.2174	$1.16 \cdot 10^{-1}$	$1.23 \cdot 10^{-1}$
25	13.3063	$5.29 \cdot 10^{-3}$	$2.26 \cdot 10^{-3}$
125	13.3118	$4.63 \cdot 10^{-4}$	$2.47 \cdot 10^{-4}$
625	13.312	-	-

Table 4.1 Convergence of the optimal desings and the objective values.

4 Numerical example

We consider the problem of finding a minimum weight of the cable suspended crane shown in Figure 4.1. In this example, the force is a unit vector with the direction uniformly distributed on $[-3\pi/4, -\pi/4]$. The number of bars is $m = 23$, and the number of cables is $r_2 = 5$. We set $\rho_1 = \rho_2 = 1.0$, the maximal cross-sectional area for both cables and bars equals 1.0, the maximal stresses are $\overline{\sigma}_1 = 1.4$ and $\overline{\sigma}_2 = 0.8$, Young's moduli are $E = E_c = 1.0$, and the initial slacks $\mathbf{g}_1 = \mathbf{0}$. For the penalty function, we use $\mu(\varepsilon) = \varepsilon^{-0.8}$, and then start with $\varepsilon = 0.05$ and successively multiply it by a factor of 0.6, until it gets as small as $5.0 \cdot 10^{-4}$.

We have solved the nested formulation of the problem $(\bar{\mathscr{P}}_2^\varepsilon)$, i.e., we have eliminated the state variables and treated them as functions of design. To solve for optimal solutions, we have used a sequential quadratic programming algorithm, and to obtain first-order information we have used finite-difference approximations. The starting point was the ground structure. This numerical approach can only be applied to trusses of small size, and the construction of efficient algorithms to solve problems of this type is one of the topics of the ongoing research.

In Table 4.1 we report the optimal weights for discretizations with k varying from 1 to 625. The norm of the differences between optimal designs for various values of k and the optimal design for $k = 625$ is also included. Further increases of k do not lead to any significant changes in the optimal design, therefore we conclude that the optimal design corresponding to $k = 625$ is a good approximation of the limiting optimal design. Table 4.2 shows the stress constraint violations, where $\sigma_{\alpha\%}$ denotes the event "at least one structure member (bar or cable) violates the stress constraint by at least $\alpha\%$". From the tables one can observe convergence of the optimal designs, even though we are not sure whether the Assumption 1 is satisfied in this problem. High values of $\mathrm{P}_{625}(\sigma_{1\%})$ even for $k = 625$ are due to our treatment of stress constraints via a penalty function, which allows small violations. In accordance with Theorem 2.4, this value reduces with a further reduction of the penalty parameter ε, as shown in Figure 4.2.

k	$P_{625}(\sigma_{1\%})$	$P_{625}(\sigma_{5\%})$	$P_{625}(\sigma_{10\%})$
1	0.9984	0.976	0.952
5	0.8512	0.168	0.12
25	0.432	0.0	0.0
125	0.430	0.0	0.0
625	0.4304	0.0	0.0

Table 4.2 Stress violations.

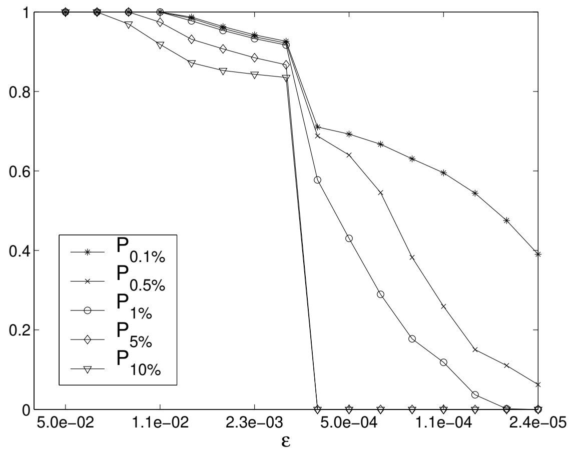


Fig. 4.2 The decrease of $P_{625}(\sigma_{\alpha\%})$ as ε decreases for various values of α ($k = 625$).

Two optimal designs corresponding to $k = 1$ and $k = 625$ are shown in the Figure 4.3, and their behaviour under various loading conditions is shown in the Figure 4.4.

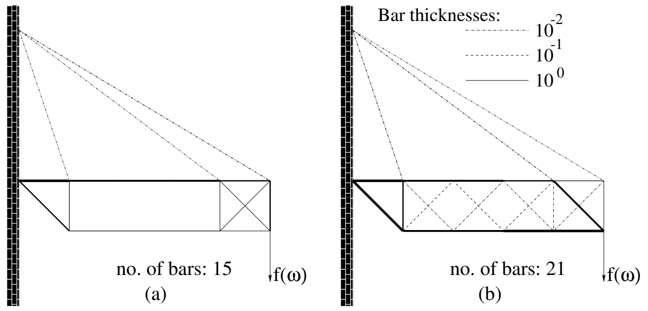
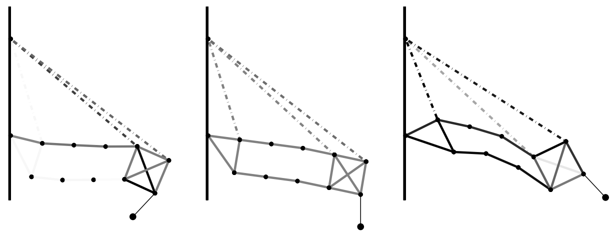


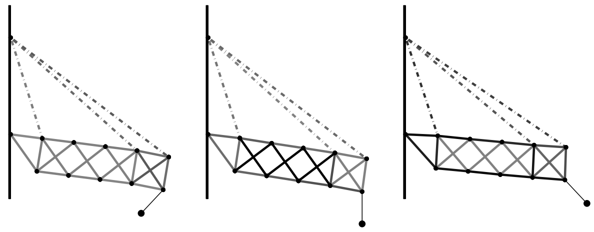
Fig. 4.3 The optimal designs for the weight minimization problem corresponding to (a) $k = 1$ and (b) $k = 625$. Line thicknesses are proportional to cross-sectional areas.

5 Concluding remarks and further research

In Section 2 we have introduced a principally new approach to solve stress constrained topology weight minimization problems. Even though we do not get an a priori estimate of the



(a)



(b)

Fig. 4.4 Stresses and displacements for various random forces for optimal designs, corresponding to (a) $k = 1$ and (b) $k = 625$. Note: for the sake of better vizualization of stresses, line thicknesses are *not* proportional to cross-sectional areas. Lighter color means bigger stress.

violation of the stress constraints in this way (as we do while using an ordinary ε-perturbation approach), this approach has several practical advantages. The penalized formulation does not have any state constraints, which makes it suitable for the discretization of the "continuous" stochastic programming problem. Furthermore, many numerical algorithms for bilevel programming problems, which work only for problems without state constraints, can be applied.

It is still an open question whether the *simultaneous* decrease of the perturbation parameter ε and increase of the number of discretization points leads to the convergence of the optimal solutions to discretized sizing optimization problems towards optimal solutions to stochastic topology compliance minimization problems. In the case of stress constrained weight minimization such convergence does not necessarily hold because of the known discontinuity of solutions with respect to changes in probability measure (Evgrafov et al. 2002). Thus, for stochastic weight minimization, an actual question is the finding of an alternative formulation, which has stable solutions with respect to modelling errors.

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A
The proof of convergence of discretizations

A.1
Discrete limit spaces

Consider a sequence of finite-dimensional spaces $\{\ell_\infty^k(\mathrm{P}_k)\}$ and a Banach space $L_\infty(\mathrm{P})$. Further consider a sequence of linear bounded *connection* operators $\mathrm{p}_k : L_\infty(\mathrm{P}) \to \ell_\infty^k(\mathrm{P}_k)$ satisfying the following property:

$$\forall\, s(\cdot) \in L_\infty(\mathrm{P}) : \lim_{k\to\infty} \|\mathrm{p}_k(s(\cdot))\| = \|s(\cdot)\|.$$

In the same way, we can consider a sequence of finite-dimensional spaces $\{\ell_1^k(\mathrm{P}_k)\}$ and a Banach space $L_1(\mathrm{P})$, together with a sequence of linear connection operators $\mathrm{q}_k : L_1(\mathrm{P}) \to \ell_1^k(\mathrm{P}_k)$, satisfying the corresponding property.

Definition A.1 *The sequence* $\{s_k(\cdot) \in \ell_\infty^k(\mathrm{P}_k)\}$ *(strongly) discretely converges to a limit* $s(\cdot) \in L_\infty(\mathrm{P})$, *if* $\lim_{k\to\infty} \|s_k(\cdot) - \mathrm{p}_k(s(\cdot))\| = 0$.

A completely analogous definition could be given in the case of L_1-spaces.

Definition A.2 *The sequence* $\{s_k(\cdot) \in \ell_\infty^k(\mathrm{P}_k)\}$ *weakly* discretely converges to a limit* $s(\cdot) \in L_\infty(\mathrm{P})$, *if for any sequence* $\{u_k(\cdot) \in \ell_1^k(\mathrm{P}_k)\}$ *(strongly) discretely converging to a limit* $u(\cdot) \in L_1(\mathrm{P})$, *it holds that* $\lim_{k\to\infty}\langle u_k(\cdot), s_k(\cdot)\rangle = \langle u(\cdot), s(\cdot)\rangle$, *where* $\langle\cdot,\cdot\rangle$ *denotes the standard pairing between the corresponding dual spaces.*

For further details about discrete limit spaces, the reader is referred to the original works of Stummel (1973) and Vainikko (1978).

A.2 Method of mechanical quadratures

The main reason for introducing the partitions $\mathscr{A}^k$ and the measures P_k with the properties (M1)–(M7) (cf. Section 3) is the following result:

Proposition A.3 (Vainikko 1971,Corollary 1) *The equality*

$$\lim_{k\to\infty} \sum_{l=1}^{k} f(\omega_l^k)\mathrm{P}_k(\omega_l^k) = \int_\Omega f(\omega)\,\mathrm{P}(d\omega)$$

holds for an arbitrary bounded $\mathrm{P}|_{\mathscr{S}_0}$*-integrable function function* $f : \Omega \to \mathbb{R}$ *iff the properties* (M1)–(M7) *hold.*

Thus we can approximate any integral involving $\mathrm{P}|_{\mathscr{S}_0}$-integrable functions.

To put the method into a framework of discrete limit spaces, we need to define corresponding connection operators. We define $\mathrm{p}_k : L_\infty(\Omega,\mathrm{P}) \to \ell_\infty^k(\mathrm{P}_k)$ in the following way:

$$[\mathrm{p}_k(s(\cdot))](\omega_l^k) := \mathrm{P}(A_l^k)^{-1}\int_{A_l^k} s(\omega)\,\mathrm{P}(d\omega).$$

Then we define operators $\mathrm{q}_k : L_1(\Omega,\mathrm{P}) \to \ell_1^k(\mathrm{P}_k)$ on the everywhere dense subset $S_0 \subset L_1(\Omega,\mathrm{P})$ of $\mathscr{S}_0$-measurable functions by:

$$[\mathrm{q}_k(u(\cdot))](\omega_l^k) := u(\omega_l^k),$$

and extend them onto the whole space $L_1(\Omega,\mathrm{P})$ by continuity. For the proof of the norm-consistency and the properties of the connection operators, the reader is referred to Lepp (1988, 1990).

A.3 Auxiliary results

In this section we adapt two auxiliary results due to Lepp (1990) to our notation. First we restate some assumptions for further reference.

Consider a function $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ and suppose that the following assumptions hold:

(A3) The function $f(\cdot,\cdot,\omega)$ is convex and differentiable for each $\omega \in \Omega$. Suppose further that $f(\mathbf{x},\mathbf{s},\cdot)$ is $\mathscr{S}_0$-measurable and bounded for each $(\mathbf{x},\mathbf{s}) \in \mathbb{R}^n \times \mathbb{R}^m$. Moreover, for each bounded set $B \subset \mathbb{R}^n \times \mathbb{R}^m$ there corresponds a constant $\beta \in \mathbb{R}$ such, that $|f(\mathbf{x},\mathbf{s},\omega)| \le \beta$ for all $(\mathbf{x},\mathbf{s}) \in B$.

(A4) The functions $f'_{\mathbf{x}}(\cdot,\cdot,\omega)$, $f'_{\mathbf{s}}(\cdot,\cdot,\omega)$ are continuous, and the functions $f'_{\mathbf{x}}(\mathbf{x},\mathbf{s},\cdot)$, $f'_{\mathbf{s}}(\mathbf{x},\mathbf{s},\cdot)$ are bounded and $\mathscr{S}_0$-measurable. Moreover, to each bounded set $B \subset \mathbb{R}^n \times \mathbb{R}^m$ there correspond bounded and $\mathscr{S}_0$-measurable functions $\gamma : \Omega \to \mathbb{R}$, $\delta : \Omega \to \mathbb{R}$, such that $|f'_{\mathbf{x}}(\mathbf{x},\mathbf{s},\cdot)| \le \gamma(\cdot)$, $|f'_{\mathbf{s}}(\mathbf{x},\mathbf{s},\cdot)| \le \delta(\cdot)$ for all $(\mathbf{x},\mathbf{s}) \in B$.

Proposition A.4 (Lepp 1990,Proposition 3.1) *Suppose that a function f satisfies assumptions* (A3) *and* (A4). *Suppose further that* $(\mathbf{x}_k,\mathbf{s}_k(\cdot)) \in \mathbb{R}^n \times \ell_\infty^k(\mathrm{P}_k)$ *is a bounded sequence such that* $f(\mathbf{x}_k,\mathbf{s}_k(\cdot),\cdot) \le 0$, P_k*-a.s. Then any weak* discrete limit point* $(\mathbf{x},\mathbf{s}(\cdot)) \in \mathbb{R}^n \times L_\infty(\mathrm{P})$ *satisfies* $f(\mathbf{x},\mathbf{s}(\cdot),\cdot) \le 0$, P-*a.s.*

Proposition A.5 (Lepp 1990,Proposition 3.2) *Suppose that a function f satisfies assumptions* (A3) *and* (A4).

(*i*) *Suppose that* $(\mathbf{x}_k,\mathbf{s}_k(\cdot)) \in \mathbb{R}^n \times \ell_\infty^k(\mathrm{P}_k)$ *weakly* discretely converges to a limit* $(\mathbf{x},\mathbf{s}(\cdot)) \in \mathbb{R}^n \times L_\infty(\mathrm{P})$. *Then the following inequality holds:*

$$\liminf_{k\to\infty} \sum_{l=1}^{k} f(\mathbf{x}_k,\mathbf{s}_k(\omega_l^k),\omega_l^k)\,\mathrm{P}_k(\omega_l^k) \ge \int_\Omega f(\mathbf{x},\mathbf{s}(\omega),\omega)\,\mathrm{P}(d\omega).$$

(*ii*) *Suppose that* $(\mathbf{x}_k,\mathbf{s}_k(\cdot)) \in \mathbb{R}^n \times \ell_\infty^k(\mathrm{P}_k)$ *(strongly) discretely converges to a limit* $(\mathbf{x},\mathbf{s}(\cdot)) \in \mathbb{R}^n \times L_\infty(\mathrm{P})$. *Then the following inequality holds:*

$$\limsup_{k\to\infty} \sum_{l=1}^{k} f(\mathbf{x}_k,\mathbf{s}_k(\omega_l^k),\omega_l^k)\,\mathrm{P}_k(\omega_l^k) \le \int_\Omega f(\mathbf{x},\mathbf{s}(\omega),\omega)\,\mathrm{P}(d\omega). \tag{A.1}$$

Remark A.6 Almost the same arguments as those in the proof of Proposition A.5 (*ii*) can be used to prove the following fact: if a function f satisfies assumptions (A3) and (A4) and $(\mathbf{x}_k,\mathbf{s}_k(\cdot)) \in \mathbb{R}^n \times \ell_1^k(\mathrm{P}_k)$ (strongly) discretely converges to a limit $(\mathbf{x},\mathbf{s}(\cdot)) \in \mathbb{R}^n \times L_1(\mathrm{P})$ such that $\mathbf{s}(\cdot)$ is essentially bounded, then inequality (A.1) holds.

A.4 Discretization of $(\mathscr{C})_{(x,X)}$

In this section we study the convergence of the solutions to discretizations of the subproblem $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\cdot)$ appearing as a constraint in our problems. We note that we cannot apply Theorem 3.1 in Lepp (1990) because the Slater condition is not necessarily fulfilled in our problem. We use Hoffman's error bound (Hoffman 1952) instead.

In the same way as the notion of discrete convergence generalizes the notion of convergence onto sequences of discrete spaces, we may generalize the notion of closedness of point-to-set mappings. The following two propositions assert such "discrete closedness" and "discrete continuity" of the mapping $(\mathbf{x},\mathbf{X}) \to \{\text{solution of } (\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\cdot)\}$ on a set $(\mathbf{x},\mathbf{X}) \geq \varepsilon\mathbf{1}_{m+r_2}$, for fixed $\varepsilon > 0$, with respect to weak* discrete convergence.

Proposition A.7 *Suppose that the assumptions* (M1)–(M7) *and* (D1)–(D3) *(cf. Section 3) hold. Consider an arbitrary sequence of positive designs* $(\mathbf{x}_k,\mathbf{X}_k) \geq \varepsilon\mathbf{1}_{m+r_2}$ *converging to* $(\bar{\mathbf{x}},\bar{\mathbf{X}})$*. Suppose that the sequence* $\{(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))\}$ *of solutions to* $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$ *weakly* discretely converges to* $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$*. Suppose further that the optimal solution to the problem* $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ *exists. Then* $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$ *is the solution of* $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$.

Proof The proof is in three steps. First, we need to establish the feasibility of the limit $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$ in the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$. Then we show that the objective value at the limit point is no greater than the lower limit of the sequence of optimal values of discrete problems. Finally, we prove that any feasible solution is a strong discrete limit of the feasible solutions to discrete problems, thus showing that the limit point is indeed optimal.

Assumptions (D1)–(D3) imply assumptions (A3) and (A4) for the constraints. Thus we can apply Proposition A.4 to verify the feasibility of $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$ in the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$.

Denote the optimal solution to the limit problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ by $(\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot))$. Since the inequality $0 \leq \mathscr{E}(\bar{\mathbf{x}},\bar{\mathbf{X}},\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot),\cdot) \leq \mathscr{E}(\bar{\mathbf{x}},\bar{\mathbf{X}},\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot),\cdot)$ holds and the latter function is essentially bounded, so is the optimal value of the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$. For each $\mathscr{S}_0$-measurable set D consider the problem $(\mathscr{C})_{(x,X;D)}$:

$$\begin{cases} \min\limits_{(\mathbf{s}(\cdot),\mathbf{S}(\cdot),\lambda(\cdot))} c_D^f(\mathbf{x},\mathbf{X},\mathbf{s}(\cdot),\mathbf{S}(\cdot),\lambda(\cdot)) := \\ \qquad\qquad \displaystyle\int_\Omega \chi_D(\omega)\mathscr{E}(\mathbf{x},\mathbf{X},\mathbf{s}(\omega),\mathbf{S}(\omega),\lambda(\omega),\omega)\,\mathrm{P}(d\omega) \\ \text{s.t.} \begin{cases} \mathsf{C}_1^T\lambda(\omega) + \displaystyle\sum_{i=1}^m \mathsf{B}_i^T\mathbf{s}_i(\omega) + \sum_{j=1}^{r_2}\mathbf{S}_j(\omega)\gamma_j = \mathbf{f}(\omega), \text{P-a.s.}, \\ \lambda(\omega) \geq 0, \quad \text{P-a.s.}, \\ \mathbf{S}(\omega) \geq 0, \quad \text{P-a.s.} \end{cases} \end{cases}$$

Owing to the integrability of the objective value $\mathscr{E}(\bar{\mathbf{x}},\bar{\mathbf{X}},\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot),\cdot)$, $(\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot))$ solves the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}};D)}$ for any $D \in \mathscr{S}_0$. This property also holds for the discretizations $(\mathscr{C})^k_{(\mathbf{x},\mathbf{X})}(\cdot)$ and $(\mathscr{C})^k_{(\mathbf{x},\mathbf{X};D)}$ (the latter is defined in a similar way). On the other hand, the optimal value of the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ is measurable with respect to the completion of the σ-algebra, generated by $\mathscr{S}_0$ (cf. Castaing and Valadier 1977,Lemma III.39). Thus if $(\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot))$ solves $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}};D)}$ for each $\mathscr{S}_0$-measurable D then it solves $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ as well.

Thus we assume that $(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))$ solves the problem $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k;D)}$, and we will prove that $(\bar{\mathbf{s}}(\cdot),\bar{\mathbf{S}}(\cdot),\bar{\lambda}(\cdot))$ solves $(\mathscr{C})^k_{(\bar{\mathbf{x}},\bar{\mathbf{X}};D)}$ for a fixed, but arbitrary $\mathscr{S}_0$-measurable D.

Assumptions (D1), (D2), $\mathscr{S}_0$-measurability of D and inequality $(\mathbf{x}_k,\mathbf{X}_k) \geq \varepsilon\mathbf{1}_{m+r_2}$ (and thus $(\bar{\mathbf{x}},\bar{\mathbf{X}}) \geq \varepsilon\mathbf{1}_{m+r_2}$) allow us to invoke Proposition A.5(i) to conclude:

$$\begin{aligned} &\operatorname{val}(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}};D)} \\ &\leq \int_\Omega \chi_D(\omega)\mathscr{E}(\bar{\mathbf{x}},\bar{\mathbf{X}},\bar{\mathbf{s}}(\omega),\bar{\mathbf{S}}(\omega),\bar{\lambda}(\omega),\omega)\,\mathrm{P}(d\omega) \\ &\leq \liminf_{k\to\infty} \sum_{l=1}^k \chi_D(\omega_l^k)\mathscr{E}(\mathbf{x}_k,\mathbf{X}_k, \\ &\qquad\qquad \mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k),\lambda_k(\omega_l^k),\omega_l^k)\,\mathrm{P}_k(\omega_l^k) \\ &= \liminf_{k\to\infty} \operatorname{val}(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k;D)}. \end{aligned} \tag{A.2}$$

Now, let $(\widetilde{\mathbf{s}}_k(\omega_l^k),\widetilde{\mathbf{S}}_k(\omega_l^k),\widetilde{\lambda}_k(\omega_l^k))$ be a projection of $\mathrm{p}_k\{(\widetilde{\mathbf{s}}(\cdot),\widetilde{\mathbf{S}}(\cdot),\widetilde{\lambda}(\cdot))\}(\omega_l^k)$ onto the set:

$$\{(\mathbf{s},\mathbf{S},\lambda) \in \mathbb{R}^{m+r_2+r_1} \mid \mathsf{C}_1^T\lambda + \sum_{i=1}^m \mathsf{B}_i^T\mathbf{s}_i + \sum_{j=1}^{r_2}\mathbf{S}_j\gamma_j = \mathbf{f}(\omega_l^k), \\ \mathbf{S} \geq 0, \lambda \geq 0\},$$

for an arbitrary point $(\widetilde{\mathbf{s}}(\cdot),\widetilde{\mathbf{S}}(\cdot),\widetilde{\lambda}(\cdot))$ that is feasible in $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}};D)}$. Then $(\widetilde{\mathbf{s}}_k(\omega_l^k),\widetilde{\mathbf{S}}_k(\omega_l^k),\widetilde{\lambda}_k(\omega_l^k))$ is feasible in $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k;D)}$. Since assumption (D3) holds we can use Hoffman's error bound for linear systems (Hoffman 1952) to obtain the following estimation:

$$\begin{aligned} \|(\widetilde{\mathbf{s}}_k(\omega_l^k),\widetilde{\mathbf{S}}_k(\omega_l^k),\widetilde{\lambda}_k(\omega_l^k)) - \mathrm{p}_k\{(\widetilde{\mathbf{s}}_k(\cdot),\widetilde{\mathbf{S}}_k(\cdot),\widetilde{\lambda}_k(\cdot))\}(\omega_l^k)\| \\ \leq K\|\mathbf{f}(\omega_l^k) - \mathrm{p}_k\{\mathbf{f}(\cdot)\}(\omega_l^k)\|, \end{aligned}$$

for some constant $K > 0$ independent from k, l and $\mathbf{f}$. Both sequences $\{\mathrm{p}_k(\mathbf{f})\}$ and $\{\mathrm{q}_k(\mathbf{f})\}$ strongly discretely converge to $\mathbf{f}$ in discrete L_1-sense (cf. Vainikko 1978,p. 649), and thus we can establish the strong discrete convergence of $\{(\widetilde{\mathbf{s}}_k(\cdot),\widetilde{\mathbf{S}}_k(\cdot),\widetilde{\lambda}_k(\cdot))\}$ to $(\widetilde{\mathbf{s}}_k(\cdot),\widetilde{\mathbf{S}}_k(\cdot),\widetilde{\lambda}_k(\cdot))$:

$$\begin{aligned} \lim_{k\to\infty} \|(\widetilde{\mathbf{s}}_k(\cdot),\widetilde{\mathbf{S}}_k(\cdot),\widetilde{\lambda}_k(\cdot)) - \mathrm{p}_k(\widetilde{\mathbf{s}}_k(\cdot),\widetilde{\mathbf{S}}_k(\cdot),\widetilde{\lambda}_k(\cdot))\|_{\ell_1^k} \\ \leq \lim_{k\to\infty} K\|\mathrm{q}_k(\mathbf{f}(\cdot)) - \mathrm{p}_k(\mathbf{f}(\cdot))\|_{\ell_1^k} = 0. \end{aligned}$$

Then, we use Remark A.6 to get:

$$\begin{aligned}
&\operatorname{val}(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}};D)}\\
&\geq \limsup_{k\to\infty}\sum_{l=1}^{k}\chi_D(\omega_l^k)\mathscr{E}(\mathbf{x}_k,\mathbf{X}_k,\\
&\qquad\qquad \widetilde{\mathbf{s}}_k(\omega_l^k),\widetilde{\mathbf{S}}_k(\omega_l^k),\widetilde{\lambda}_k(\omega_l^k),\omega_l^k)\,\mathrm{P}_k(\omega_l^k)\\
&\geq \limsup_{k\to\infty}\operatorname{val}(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k;D)},
\end{aligned}$$

which together with (A.2) completes the proof.

Proposition A.8 *Suppose that the assumptions* (M1)–(M7) *and* (D1)–(D4) *hold. Consider an arbitrary sequence of positive designs* $(\mathbf{x}_k,\mathbf{X}_k)\geq \varepsilon\mathbf{1}_{m+r_2}$ *converging to* $(\bar{\mathbf{x}},\bar{\mathbf{X}})$*, and suppose that the problems* $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$ *and* $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ *are feasible. Suppose further that the optimal value of the problem* $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ *is essentially bounded.*

Then the sequence of solutions $(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))$ *to* $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}$ *strongly discretely converges to the solution* $(\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot))$ *of the problem* $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$.

Proof The additional assumption (D4) implies the strong convexity of the objective function with respect to $(\mathbf{s},\mathbf{S})$ locally uniformly with respect to $(\mathbf{x},\mathbf{X})$. Since λ depends linearly on $(\mathbf{s},\mathbf{S})$ (cf. (1.1)) we may assume the strong convexity of $\mathscr{E}$ with respect to state variables on a feasible set locally uniformly with respect to $(\mathbf{x},\mathbf{X})$. According to Lepp (1994,Remark 2) under the strong convexity of the objective function weak* discrete convergence of the solutions to the problems $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$ implies strong discrete convergence. Since the solution to the problem $(\mathscr{C})_{(\bar{\mathbf{x}},\bar{\mathbf{X}})}(\cdot)$ is unique, the weak* discrete compactness of the sequence $\{(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))\}$ and Proposition A.7 imply that the sequence $\{(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))\}$ weakly* discretely converges to $(\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot))$. Since weak* discrete compactness follows from boundedness (cf. Vainikko 1978,Proposition 10), it is sufficient to show the boundedness of $\{(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))\}$.

From the proof of Proposition A.7 we know that there exists a sequence $\{(\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))\}$ feasible in $\{(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)\}$, and strongly discretely converging to $(\mathbf{s}^*(\cdot),\mathbf{S}^*(\cdot),\lambda^*(\cdot))$ in L_1-sense. Owing to the optimality of $(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))$ in $(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$ the inequality $\mathscr{E}(\mathbf{x}_k,\mathbf{X}_k,\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot),\cdot)\leq \mathscr{E}(\mathbf{x}_k,\mathbf{X}_k,\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot),\cdot)$ holds. For any $\mathscr{S}_0$-measurable set D according to Remark A.6 we then have:

$$\begin{aligned}
&\limsup_{k\to\infty}\sum_{i=1}^{k}\chi_D(\omega_l^k)\,\mathscr{E}(\mathbf{x}_k,\mathbf{X}_k,\\
&\qquad\qquad \mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k),\lambda_k(\omega_l^k),\omega_l^k)\mathrm{P}_k(\omega_l^k)\\
&\quad\leq \int_\Omega \chi_D(\omega)\mathscr{E}(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}^*(\omega),\mathbf{S}^*(\omega),\lambda^*(\omega),\omega)\,\mathrm{P}(d\omega),
\end{aligned}$$

from which we deduce that the sequence $\{\mathscr{E}(\mathbf{x}_k,\mathbf{X}_k,\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot),\cdot)\}$ is bounded. Since $\mathscr{E}$ is weakly coercive in $(\mathbf{s},\mathbf{S},\lambda)$ locally uniformly w.r.t. $(\mathbf{x},\mathbf{X})$ on a set $\{(x,X)\geq \varepsilon\mathbf{1}_{m+r_2}\}$, and always nonnegative, the boundedness of $(\mathbf{s}_k^*(\cdot),\mathbf{S}_k^*(\cdot),\lambda_k^*(\cdot))$ follows.

A.5 Discretization of $(\mathscr{P}_1)$

Proof (of Theorem 3.1) Applying Proposition A.8 to the sequence $\{(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))\}$, where $(\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))$ solves $(\mathscr{C})^k_{(\mathbf{x}^*,\mathbf{X}^*)}(\cdot)$, and then using Proposition A.5(*ii*), we obtain the following inequality:

$$\begin{aligned}
&\limsup_{k\to\infty}\operatorname{val}(\mathscr{P}_1^\varepsilon)^k\\
&\leq \limsup_{k\to\infty}\sum_{l=1}^{k}\mathscr{E}(\mathbf{x}^*,\mathbf{X}^*,\mathbf{s}_k(\omega_l^k),\mathbf{S}_k(\omega_l^k),\lambda_k(\omega_l^k),\omega_l^k)\,\mathrm{P}(\omega_l^k)\\
&\leq \operatorname{val}(\mathscr{P}_1^\varepsilon).
\end{aligned}\tag{A.3}$$

We may assume that the original sequence weakly* discretely converges to a limit $(\widetilde{\mathbf{x}},\widetilde{\mathbf{X}},\widetilde{\mathbf{s}}(\cdot),\widetilde{\mathbf{S}}(\cdot))$. Its feasibility is implied by Proposition A.7 and the continuity of the design constraints. From Proposition A.5(*i*) we get the reverse inequality:

$$\begin{aligned}
&\operatorname{val}(\mathscr{P}_1^\varepsilon)\leq c^f(\widetilde{\mathbf{x}},\widetilde{\mathbf{X}},\widetilde{\mathbf{s}}(\cdot),\widetilde{\mathbf{S}}(\cdot),\widetilde{\lambda}(\cdot))\\
&\leq \liminf_{k\to\infty}\sum_{l=1}^{k}\mathscr{E}(\mathbf{x}_k^*,\mathbf{X}_k^*,\mathbf{s}_k^*(\omega_l^k),\mathbf{S}_k^*(\omega_l^k),\lambda_k^*(\omega_l^k),\omega_l^k)\,\mathrm{P}(\omega_l^k)\\
&= \liminf_{k\to\infty}\operatorname{val}(\mathscr{P}_1^\varepsilon)^k,
\end{aligned}$$

which, together with (A.3), completes the proof.

A.6 Discretization of $(\mathscr{P}_2)$

Proof (of Theorem 3.2) The proof is similar to the one of Theorem 3.1 and is therefore omitted.

A.7 Existence of weak* limit points

To establish the convergence of the discretization scheme it remains to establish the existence of weak* discrete limit points for the sequence of solutions to discretized problems. A sufficient condition for such an existence is the boundedness of the sequence.

Proposition A.9 *Suppose that the assumptions* (M1)–(M7) *and* (D1)–(D4) *hold. Suppose further that the functions* $E(\cdot)$*,* $E_c(\cdot)$*, and* $\mathbf{L}(\cdot)$ *are independent from* ω*. Let* $\{(\mathbf{x}_k,\mathbf{X}_k)\}$ *be a bounded sequence of designs such that* $(\mathbf{x}_k,\mathbf{X}_k)\geq\varepsilon\mathbf{1}_{m+r_2}$ *for some* $\varepsilon>0$ *and all k. Suppose that the problem* $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\omega)$

is feasible for any $\omega \in \Omega$ *for* $(\mathbf{x},\mathbf{X}) > 0$*. Then the sequence of solutions* $\{(\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))\}$ *to the problems* $\{(\mathscr{C})^k_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)\}$ *is bounded.*

Proof Owing to (Bank et al. 1983,Theorem 5.5.2) the solution $(\widetilde{\mathbf{s}}_1(\cdot),\widetilde{\mathbf{S}}_1(\cdot),\widetilde{\lambda}_1(\cdot))$ of $(\mathscr{C})_{(\mathbf{x}_1,\mathbf{X}_1)}(\cdot)$ is a piecewise affine function of $(\mathbf{f}(\cdot),\mathbf{g}_1(\cdot),\mathbf{g}_2(\cdot))$, whence it is bounded.

Then we can use the locally Lipschitz continuity result (**?**,Theorem 3.4) to bound the sequence $\{(\widetilde{\mathbf{s}}_k(\cdot),\widetilde{\mathbf{S}}_k(\cdot),\widetilde{\lambda}_k(\cdot))\}$ of solutions to $(\mathscr{C})_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$:

$$\|(\widetilde{\mathbf{s}}_k(\omega),\widetilde{\mathbf{S}}_k(\omega),\widetilde{\lambda}_k(\omega)) - (\widetilde{\mathbf{s}}_1(\omega),\widetilde{\mathbf{S}}_1(\omega),\widetilde{\lambda}_1(\omega))\| \\ \leq \tau(\omega)\|(\mathbf{x}_k,\mathbf{X}_k) - (\mathbf{x}_1,\mathbf{X}_1)\| \cdot \|(\mathbf{u}_k(\omega),\mathbf{e}_k(\omega)\|,$$

and thus the boundedness of $\{(\mathbf{s}_k(\cdot),\mathbf{S}_k(\cdot),\lambda_k(\cdot))\}$ will follow if we can show the boundedness of $\tau(\omega)$ and of the sequence of Lagrange multipliers $\{(\mathbf{u}_k(\cdot),\mathbf{e}_k(\cdot))\}$ for the constraints of $(\mathscr{C})_{(\mathbf{x}_k,\mathbf{X}_k)}(\cdot)$.

For positive designs we can assume that both the optimal solutions and Lagrange multipliers for the problem $(\mathscr{C})_{(\mathbf{x},\mathbf{X})}(\cdot)$ are unique. Therefore, owing to (Bank et al. 1983,Theorem 5.5.1) they continuously depend on $(\mathbf{f}(\cdot),\mathbf{g}_1(\cdot),\mathbf{g}_2(\cdot))$. Proposition 3 in (Luo and Tseng 1997) then implies that τ can be choosen independent from ω.

The boundedness of $\{(\mathbf{u}_k(\cdot),\mathbf{e}_k(\cdot))\}$ follows from (Hager 1979,Theorem 3.1), which asserts the Lipschitz dependence of solutions to quadratic problems on a bounded set of parameters, provided a uniform strong convexity of the objective function and a uniform linear independence constraint qualification conditions hold. We note that the assumptions of the theorem follow from the boudedness of $\{(\mathbf{x}_k,\mathbf{X}_k)\}$ and $(\mathbf{f}(\cdot),\mathbf{g}_1(\cdot),\mathbf{g}_2(\cdot))$, the quasi-orthogonality of C_1, and the boundedness away from zero of $\{(\mathbf{x}_k,\mathbf{X}_k)\}$.