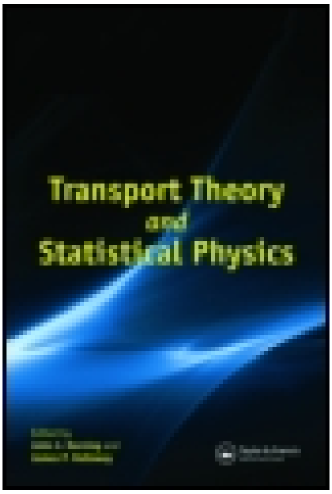

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Streamline diffusion methods for Fermi and Fokker-Planck equations

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STREAMLINE DIFFUSION METHODS FOR FERMI AND FOKKER-PLANCK EQUATIONS

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ABSTRACT. We derive error estimates in certain weighted L_2-norms for the streamline diffusion and discontinuous Galerkin finite element methods for steady state, energy dependent, Fermi and Fokker-Planck equations in two space dimensions, giving error bounds of order $\mathcal{O}(h^{k+1/2})$, for the weighted current function J, as in the convection dominated convection-diffusion problems, with $J \in H^{k+1}(\Omega)$ and h being the quasi-uniform mesh size in triangulation of our three dimensional phase-space domain $\Omega = I_x \times I_y \times I_z$, with z corresponding to the velocity variable. Our studies, in this paper, contain a priori error estimates for Fermi and Fokker-Planck equations with both piecewise continuous and piecewise discontinuous (in x and xy-directions) trial functions. The analyses are based on stability estimates which relay on an angular symmetry (not isotropy!) assumption. A continuation of this paper, the a posteriori error estimates for Fermi and Fokker-Planck equations, is the subject of a future work.

0. Introduction. This is the first part in a series of two papers on the streamline diffusion finite element methods for degenerate type convection dominated convection-diffusion problems arise, e.g., in asymptotic expansion of particle transport for narrowly focussed pencil beams. In a pencil beam the mean direction change of beam particles is assumed to be small. This assumption, which is realistic for photon transport and certain electron transport problems is referred to as *forward-peakedness* of the scattering. Under certain assumptions, including strong forward-peakedness of the scattering, the transport equation can be well approximated by the Fokker-Planck equation. Formally, the equation is obtained by approximating the particle transport equation so that the scattering kernel associated with the total cross section give rise, after an asymptotic expansion, to a diffusion term with respect to the angular variable. The Fermi equation

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is obtained either as an asymptotic limit of the Fokker-Planck equation as $\sigma_{tr} \to 0$ or as an asymptotic limit of the transport (linear Boltzmann) equation as $\sigma_{tr} \to 0$ and $\sigma_t \to \infty$ (the mean scattering angle is assumed to be small, $\bar{\mu}_0 \approx 1$, and the large-angle scattering is negligible). For details in derivations of Fermi and Fokker-Planck equations we refer to [5] and [10]. For our model problem a complete derivation of Fermi and Fokker-Planck equations in *"Flatland"* is given in [3].

There are some points of concern with these type of problems: The Fermi and Fokker-Planck equations considered in this paper are degenerate in both convection and diffusion terms in the sense that the intersection of differentiation variables in drift and diffusion terms is empty, especially no complete drift, in all directions, is expected and the diffusion term has a *very small* coefficient, i.e., σ_{tr}. Furthermore, the Fermi case, corresponds to a problem of describing a pencil beam of particles normally incident on a slab, $0 < x < L$, with the particles entering at a single point, say at $(x, y) = (0, 0)$, in the direction of the positive x-axis. This problem will have a boundary data in form of a δ function which is not suitable for the numerical considerations involving L_2-norms, we have therefore considered model problems with somewhat smoother data. We study the physically more relevant case: estimations in L_1 norm, in a future work. Moreover, in spite of the assumption of no back-scattering, i.e., the scattering angle $-\pi/2 \le \theta \le \pi/2$, we still need to restrict the range of θ, through focussing or filtering, and avoid small intervals in vicinity of the endpoints $\pm\pi/2$ in order to get, after scalings, bounded domain for our numerical studies.

Fermi equation has closed form solutions for σ_{tr} being constant or only a function of x. Analytic solutions of the Fokker-Planck equation can be found in i) one dimensional case, ii) linear drift and constant diffusion tensor case, iii) under detailed balance condition and in some very special cases. However, in general it is difficult to obtain analytic solutions for the Fokker-Planck equation especially in higher dimensional cases or if no separation of variables is possible.

The subject of this paper is error estimates for the stationary (steady state), energy dependent, two space dimensional Fermi and Fokker-Planck equations. In the present setting we have transformed and scaled the variables so that the x-direction, the direction of penetration of the beam, being perpendicular to the slab, may also be interpreted as the direction of the time variable so that the methods in here will be adequate even for the non-stationary case giving local in time estimates. To justify elimination of large-angle scatterings one needs to consider only small x-values corresponding to first few collisions. For large x-values the particles, undergoing several collisions, will have a large mean direction change. Therefore when the time variable replaces x, the results are only for small time values, this is relevant because the distribution gets almost immediately steady state. After scaling, the present technique treats all the variables as components of a multi-dimensional space variable. One could study other approaches, where no scaling is used.

In a forthcoming paper [2], we shall study a posteriori error estimates for these equations and also present numerical implementations for a variety of combined spatial and angular discretizations.

General theory of the Fokker-Planck equation, together with some solution techniques, can be found in [12]. Our methods in this paper will extend the results in [1] for the Vlasov-Poisson equation to a degenerate case. Here are some application areas for Fermi and Fokker-Planck equations: Cosmic rays penetrating the atmosphere, ion beams used to modify the properties of material, and electron or photon beams used for cancer therapy, (see the references in [5]).

Fokker-Planck equation is widely studied either in combination with Vlasov or other transport type equations or in the form of the forward Kolmogorov equation, see Risken [12]. In our knowledge, convergence rates and error analyses using the streamline diffusion finite element methods for these degenerate type problems are not considered elsewhere.

The method of our discretization; the streamline-diffusion method (SD-method) is a generalized form of the Galerkin method for the hyperbolic problems which gives good stability and high accuracy. The SD-method, used for our purpose in this paper, is obtained by modifying the test function through adding a multiple of the hyperbolic operator involved in the equation. This gives a weighted least square control of the residual of the finite element solution. See [1] and the references therein for further details in the SD-methods.

An outline of this paper is as follows: In Section 1, we introduce the continuous model problems and some notation. In Section 2, we give a priori error estimates for a discrete Fermi equation with $\sigma_{tr} = \sigma_{tr}(x, y)$. In Section 3, we extend results of section 2 to the corresponding two space dimensional Fokker-Planck equation. Section 4 is devoted to a SD-method for both equations, where we treat discontinuities in the x-direction. Finally in our concluding Section 5 we extend the results of Section 4 to a discontinuous Galerkin finite element method with trial functions being discontinuous in both x and y directions.

1. The Continuous Model Problem. The two dimensional *model problem* for the pencil beam transport ($\sigma_a \equiv 0$) can be formulated by the following integro-differential equation:

$$\mu \cdot \nabla\psi = \int_{S^1} \sigma_s(\mu \cdot \mu')[\psi(x, y, \mu') - \psi(x, y, \mu)]\, d\mu', \quad 0 < x < 1, \tag{1.1}$$

$$\psi(0, y, \mu) = \frac{1}{2\pi}\delta(y)\delta(1 - \mu_1), \qquad \mu_1 \geq 0, \tag{1.2}$$

$$\psi(1, y, \mu) = 0, \qquad \mu_1 \leq 0. \tag{1.3}$$

We use dimensionless spatial variables, scaled so that the slab width is 1. $\sigma_s > 0$ is the differential cross section defined as

$$\sigma_s(\omega) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\sigma_{sn}p_n(\omega),$$

with p_n being the nth Legendre polynomial and $\sigma_t = \int_{S^1} \sigma_s(\omega)\, d\omega$ is the total cross section. Here we have followed the conventional representation, otherwise the orthogonal cosine polynomials would be the most natural bases functions for this model case, see [3]. For a neutron travelling with a given speed through a given medium the probability of a collusion per unit path length is a constant and σ_t is this constant. The slab width in the units of *mean free paths* is σ_t^{-1}, see Davison [7] for details. Further, $\nabla = (\partial/\partial x\,, \partial/\partial y)$ and

$$S^1 = \{\mu \in \mathbb{R}^2 : |\mu| = 1\}, \qquad \mu \equiv (\mu_1\,, \mu_2) = (\cos\theta, \sin\theta), \quad 0 \leq \theta < 2\pi. \tag{1.4}$$

The Fokker-Planck approximation to this transport problem, which is also given in [3], [4], [5], [10] and [12], can be formulated in the following way:

For $0 < x < 1$ and $-\infty < y < \infty$, find $\psi^{FP} = \psi^{FP}(x, y, \theta)$ such that

$$\mu \cdot \nabla\psi^{FP} = \sigma_{tr}\psi^{FP}_{\theta\theta}, \qquad \theta \in (-\pi/2, \pi/2), \tag{1.5}$$

$$\psi^{FP}(0,y,\theta)=\frac{1}{2\pi}\delta(y)\delta(1-\cos\theta),\qquad \theta\in S^1_+, \tag{1.6}$$

$$\psi^{FP}(1,y,\theta)=0,\qquad \theta\in S^1_-, \tag{1.7}$$

where $\sigma_{tr}=\sigma_{s0}-\sigma_{s1}$ is the transport cross section, $S^1_+=\{\mu\in S^1:\mu_1>0\}$ and $S^1_-=S^1\setminus S^1_+$.

Based on physical properties of the pencil beams, in the time dependent case the distribution gets "almost immediately" steady state, this motivates our study of the stationary problems in this paper, extension of the techniques presented in here to the non-stationary case, as stated in the introduction, is straightforward.

We introduce the *current*

$$j=(\cos\theta)\,\psi^{FP}. \tag{1.8}$$

Using the scaling substitution $z=\tan\theta$, $\theta\in(-\pi/2,\pi/2)$, and the obvious relation

$$\frac{\partial^2\Phi}{\partial\theta^2}=(1+z^2)\frac{\partial}{\partial z}\left((1+z^2)\frac{\partial\Phi}{\partial z}\right), \tag{1.9}$$

we get the Fokker-Planck equation for the current j:

$$j_x+zj_y=\sigma_{tr}(1+z^2)\frac{\partial}{\partial z}\left((1+z^2)\frac{\partial}{\partial z}(\sqrt{1+z^2}j)\right), \tag{1.10}$$

where we have used $\psi^{FP}=\sqrt{1+z^2}\,j$. Now we consider the identification

$$\begin{aligned}\int\int j(x,y,\theta)\,dy\,d\theta&=\int\int j(x,y,\tan^{-1}z)\,\frac{\partial(y,\theta)}{\partial(y,z)}\,dy\,dz\\&=\int\int\frac{j(x,y,\tan^{-1}z)}{1+z^2}\,dy\,dz\equiv\int\int J(x,y,z)\,dy\,dz,\end{aligned}$$

and define the function J as

$$J(x,y,z)\equiv\frac{j(x,y,\tan^{-1}z)}{1+z^2}. \tag{1.11}$$

Then (1.10) is equivalent to write

$$J_x+zJ_y=\sigma_{tr}\frac{\partial}{\partial z}\left((1+z^2)\frac{\partial}{\partial z}[(1+z^2)^{3/2}J]\right). \tag{1.12}$$

In this paper we consider the general case of $\sigma_{tr}=\sigma_{tr}(x,y)$, (actually $\sigma_{tr}=\sigma_{tr}[E(x,y)]$ is indicating the energy dependent character of the problem), and study the following model problem for two dimensional Fermi and Fokker-Planck equations on a bounded polygonal phase-space domain $\Omega\subset\mathbf{R}^3$ with the homogeneous "inflow" boundary conditions and the data $f\in L_2(\Omega_0)$, with $\Omega_0=\Omega\cap\{x=0\}$;

$$\frac{\partial J}{\partial x}+z\frac{\partial J}{\partial y}=\epsilon A\,J,\qquad \epsilon=\sigma_{tr}, \tag{1.13}$$

$$\frac{\partial J}{\partial z}(0,y,+z_0)=\frac{\partial J}{\partial z}(0,y,-z_0)=0,\qquad y\in[-y_0,y_0], \tag{1.14}$$

$$J(0, y_0, z) = 0, \qquad z < 0, \tag{1.15}$$

$$J(0, -y_0, z) = 0, \qquad z > 0, \tag{1.16}$$

$$J(0, y, z) = f(y, z), \qquad (y, z) \in [-y_0, y_0] \times [-z_0, z_0], \tag{1.17}$$

where we assumed "cut-offs" in y and θ directions as, $-y_0 \le y \le y_0$ and $-\pi/2 < -\theta_1 \le \theta \le \theta_1 < \pi/2$ corresponding to bounded positive y_0 and z_0 values, respectively. Observe that in our model problem we have replaced the incident boundary condition, i.e., the product of two δ functions; $\delta(y)\delta((\sqrt{1+z^2}-1)/(\sqrt{1+z^2}))$ by a smoother function f. For the Fermi equation, due to defining stretch variables asymptotic expansion will lead to; (see [4]),

$$A\,J = \frac{\partial^2 J}{\partial z^2}, \tag{1.18}$$

while for the Fokker-Planck equation

$$A\,J = \frac{\partial}{\partial z}\left[a(z)\frac{\partial}{\partial z}(b(z)J)\right], \tag{1.19}$$

with $a(z) = (1+z^2)$ and $b(z) = (1+z^2)^{3/2}$.

Here are some frequently used notation: Throughout the paper C will denote a general constant not necessarily the same at each occurrence, mostly depending on the size of the domain Ω and independent of the parameters h, ε, β and κ, unless otherwise specifically specified. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the usual L_2-inner product and L_2-norm, respectively, H^s, for s positive integer, is the usual Sobolev space with the norm $\|\cdot\|_s$ and the seminorm $|\cdot|_s$, with the maximal available derivatives:

$$|v|_s = \left(\int_\Omega [(\partial_x^s v)^2 + (\partial_y^s v)^2 + (\partial_z^s v)^2]\, d\mathbf{x}\right)^{1/2}.$$

Moreover

$$< v, w > = \int_\Gamma vw(\mathbf{n}\cdot\beta)\, ds,$$

$$< v, w >_+ = \int_{\Gamma^+} vw(\mathbf{n}\cdot\beta)\, ds,$$

$$< v, w >_- = \int_{\Gamma^-} vw(\mathbf{n}\cdot\beta)\, ds,$$

$$|v|_\Gamma = \left(\int_\Gamma v^2|\mathbf{n}\cdot\beta|\, ds\right)^{1/2},$$

$$|v|_{\Gamma_{\pm z_0}} = \left(\int_{\Gamma_{\pm z_0}} v^2\, ds\right)^{1/2},$$

$$\Gamma_{\pm z_0} = \{\mathbf{x} = (x, y, z) \in \Gamma = \partial\Omega : z = -z_0 \vee z = z_0\},$$

$$V_h = \{v \in H^1(\Omega) : v|_K \in \mathcal{P}_r(K), \quad \forall K \in \mathcal{C}_h\},$$
$$\Gamma^- = \{\mathbf{x} \in \Gamma = \partial\Omega : \mathbf{n}(\mathbf{x}) \cdot \beta < 0\},$$

where $\Gamma = \partial\Omega$ is the boundary of the domain Ω, $\mathbf{n}(\mathbf{x})$ is the outward unit normal to Γ at the phase-space point $\mathbf{x} = (x, y, z) \in \Gamma$, $\beta = (1, z, 0)$ in our cases, and $\Gamma^+ = \Gamma \setminus \Gamma^-$. Further $\mathcal{C}_h = \{K\}$ is a family of quasiuniform triangulation of $\Omega = I_x \times I_y \times I_z$ satisfying the minimal angle condition with $h = \text{diam}(K)$ as the mesh parameter and $\mathcal{P}_r(K)$ is the set of all polynomials in x, y and z of degree at most r on K. Finally $H_0^s(\Omega)$ and V_h^0 will denote the subsets of H^s and V_h vanishing on Γ. Let $v_\beta = \beta \cdot \nabla_{\mathbf{x}} v$ with $\nabla_{\mathbf{x}} := (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, we will frequently use the Green's formula

$$(1.20) \qquad (v_\beta, v) = \frac{1}{2} < v, v >,$$

which is a consequence of the divergence theorem $\int_\Omega \text{div}(\Psi)\, d\mathbf{x} = \int_\Gamma (\Psi \cdot \mathbf{n})\, ds$, applied to $\Psi = (vw, 0, 0)$ and $\Psi = (0, zvw, 0)$ i.e.

$$\int_\Omega a_i \left(\frac{\partial v}{\partial x_i} w + v \frac{\partial w}{\partial x_i} \right) d\mathbf{x} = \int_\Gamma vw\, n_i\, ds, \qquad i = 1, 2,$$

where $(x_1, x_2) = (x, y)$, $(a_1, a_2) = (1, z)$ and n_i are components of the outward unit normal $\mathbf{n}$. Adding the above formula for $i = 1$ and 2 and letting $w = v$ we obtain (1.20). Finally we shall need the following *Angular Balance Condition* :

$$(1.21) \qquad \left(\frac{d}{dz} |v|^2_{\Gamma_{xy}} \right) \Big|_{z=z_0} = \left(\frac{d}{dz} |v|^2_{\Gamma_{xy}} \right) \Big|_{z=-z_0}$$

This in a symmetry condition which is natural, e.g., in the case of a radially symmetric and compactly supported, positive, source term having, for each $x \in I_x$, a fixed value C_r on each circle $C_r(x, 0, 0) \subset I_y \times I_z \times \{x\}$ with centre at the point $(x, 0, 0)$ and radius r. Observe that the source term is unisotropic, however, the distribution of the intesity of the beam is symmetric about the x-axis and, of course, decreases as the scattering angle bifurcation of the beam from the x-axis and the value of x increase. By $\mathcal{H}^s_{z_0}(\Omega)$, s positive integer, we mean the set of all functions $v \in H^s(\Omega)$ satisfying the boundary conditions (1.14)-(1.15) of J, for all x values (not only for $x = 0$) and the angular balance condition (1.21).

2. Discrete Fermi equation. For σ_{tr} constant or $\sigma_{tr} = \sigma_{tr}(x)$ one can obtain closed form analytic solutions for the Fermi equation. The idea is to use scaling in order to have both y and the angular variable vary over the entire real line and then take Fourier transforms with respect to y and z, (see [4] for the exact solution and the corresponding variable scalings in three dimensions). Below we use a variational formulation, with the test functions consisting of the sum of a trial function $v \in V_h$ and an extra streaming term: κv_β, where κ is a small coefficient, i.e., we use test functions different from the trial functions, therefore we are dealing with a kind of "Petrov-Galerkin" method. We prove a stability Lemma for the continuous problem in general two dimensional case, i.e., with $\sigma_{tr} = \sigma_{tr}(x, y)$, using also the corresponding discrete variational formulation we derive our first a priori error estimate. Now we write the equation (1.13) as

$$(2.1) \qquad J_\beta - \varepsilon J_{zz} = 0.$$

Equation (2.1) combined with the boundary condition (1.17) gives rise to the variational formulation

$$(2.2) \qquad (J_\beta, v + \kappa v_\beta) - (\varepsilon J_{zz}, v + \kappa v_\beta) - < J, v >_- = - < f, v >_-, \qquad \forall v \in H^1(\Omega)$$

where we shall choose $\kappa \cong \bar{C} \min_K h_K/|\beta|$ with h_K being the local mesh size in the triangulation C_h and $\bar{C}$ is sufficiently small (see proof of Lemma 2.1 below.) Introducing the bilinear form

$$(2.3) \qquad B(w, v) = (w_\beta, v + \kappa v_\beta) - (\varepsilon w_{zz}, v + \kappa v_\beta) - < w, v >_-,$$

and the linear form

$$(2.4) \qquad L(v) = - < f, v >_-,$$

we may write (2.2) as

$$(2.5) \qquad B(J, v) = L(v), \qquad \forall v \in H^1(\Omega).$$

Stability Lemma 2.1. *There is a constant $C = C(\Omega)$ such that*

$$\|\varepsilon^{1/2} v_z\|^2 + \frac{1}{2}|v|^2 + \|\kappa^{1/2} v_\beta\|^2 \le C(\Omega)|f|^2_{\Omega_0}, \qquad \forall v \in \mathcal{H}^1_{z_0}(\Omega).$$

Proof. We have using Green's formula (1.20) and the condition (1.21), that

$$(2.6) \qquad \begin{aligned} B(v, v) &= \|\kappa^{1/2} v_\beta\|^2 + \frac{1}{2} < v, v > -(\varepsilon v_{zz}, v + \kappa v_\beta) - < v, v >_- \\ &= \|\kappa^{1/2} v_\beta\|^2 + \frac{1}{2}|v|^2 + \|\varepsilon^{1/2} v_z\|^2 - (\kappa \varepsilon v_{zz}, v_\beta), \end{aligned}$$

where we use the notation $|v| = |v|_\Gamma$, see Section 1. Observe that, in the assertion of Lemma, we require a low regularity on v, i.e., $v \in H^1(\Omega)$, while the equation in (2.6) contains v_{zz}. Therefore, (2.6) should be interpreted as a weak formulation where an integration by parts is performed for the terms involving v_{zz}. This works for the worst term: (v_{zz}, v_β), because β has *zero* z–component. We assume $v \in H^1(\Omega)$, mainly because of the numerical considerations we want to have as general continuous variational test function v as possible. Now since $\varepsilon = \sigma_{tr}$ is independent of the scaled angular variable $z = \tan\theta$, we may use the inverse estimate to obtain

$$(2.7) \qquad |(\kappa \varepsilon v_{zz}, v_\beta)| \le \frac{1}{2}\|\varepsilon^{1/2} v_z\|^2 + \frac{1}{2}\bar{\varepsilon}\bar{\kappa} C^2 h^{-2} \|\kappa^{1/2} v_\beta\|^2,$$

where $\bar{\varepsilon} = \max \varepsilon \le \min_K h_K|\beta|$ and $\bar{\kappa} = \max_{K \in C_h} \kappa \le \bar{C}\frac{h_K}{|\beta|}$. Thus our bilinear form will satisfy

$$(2.8) \qquad \begin{aligned} B(v, v) &\ge \|\kappa^{1/2} v_\beta\|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}\|\varepsilon^{1/2} v_z\|^2 - \frac{1}{2}\bar{\varepsilon}\bar{\kappa} C^2 h^{-2} \|\kappa^{1/2} v_\beta\|^2 \\ &= \frac{1}{2}\|\varepsilon^{1/2} v_z\|^2 + \frac{1}{2}|v|^2 + (1 - \frac{1}{2}\bar{\varepsilon}\bar{\kappa} C^2 h^{-2})\|\kappa^{1/2} v_\beta\|^2 \end{aligned}$$

$$\geq \frac{1}{2}\left(\|\varepsilon^{1/2}v_x\|^2 + |v|^2 + \|\kappa^{1/2}v_\beta\|^2\right),$$

the last inequality is a consequence of the fact that $\bar{C}$ is assumed to be sufficiently small, so that

$$\bar{\varepsilon}\bar{\kappa}C^2h^{-2} \leq \bar{\varepsilon}\frac{\bar{C}h_K}{|\beta|}C^2 \times \frac{1}{h_K^2} \leq \bar{C}C^2 < 1.$$

This together with

$$L(v) = - < f, v >_- = -\int_{\Gamma^-} fv(\mathbf{n}\cdot\beta)\,ds \leq |f|^2 + \frac{1}{4}|v|^2,$$

completes the proof. □

Remark 2.1. Lemma 2.1 is a somewhat "weak stability" result. Here are some features of problem (2.1): (i) The lack of pure current term for the beam problem, i.e., no absorption on the left hand side of the equation, will lead to stability inequalities with no explicit L_2−norm control. Besides, the semi-norms, (L_2−norms of partial derivatives), appear with small coefficients of order $\mathcal{O}(\varepsilon^{1/2})$ and $\mathcal{O}(\kappa^{1/2})$, i.e., of order $\mathcal{O}(\sqrt{h})$. Usually, the Poincare-Friedricks inequality:

$$\|v\| \leq C(\Omega)|v|_1, \qquad \forall v \in H_0^1(\Omega) \tag{2.9}$$

is used to include L_2-norms on the left hand side of such stability results as Lemma 2.1. However, our test functions are not vanishing on the whole boundary of Ω (they are not in the space $H_0^1(\Omega)$). Below, in Lemma 2.2, we shall assume that the test functions are in $\mathcal{H}_{x_0}^1(\Omega)$, and prove a similar result as (2.9) with these new boundary conditions (see definition of $\mathcal{H}_{x_0}^1(\Omega)$ at the end of section 1). Combining Lemmas 2.1 and 2.2, the coefficients ε and κ will appear in the L_2-norm estimates as well. Therefore, in implementations, one should expect an actual rate of convergence of order $\mathcal{O}(h^k)$, i.e., a reduced rate of order $h^{1/2}$, compared to the theory. This is in the nature of the problem and cannot be avoided.

The stability estimates for the pencil beam problems gain an obvious advantage of the fact that the source term is an incoming flow from only a part of the inflow boundary and although the stability constant, in Lemma 2.1, depends on the size of the domain Ω (this Ω dependence appears in definition of κ and since Ω is bounded, in some of our estimates throughout the paper, the stability constant is replaced by 1), the stability norm on the right hand side will, inevitably, be only over a part of the boundary: Here, over $\Omega_0 := \Omega \cap \{x = 0\}$. This gives a more desirable result, when Ω is not very large and also provided that f has the required regularity. We could use the trace inequality:

$$|v|_\Gamma \leq C(\Omega)\|v\|_1,$$

and obtain an estimate involving H^1 norm of v on the right hand side. However, again because of the lack of absorption in the equation, the H^1 norm of v can not be hidden in the seminorms on the left hand side and therefore the result remains less sharp in this case. In Lemma 4.2, below, we demonstrate another way of including improved L_2-norm control in stability estimates using somewhat more involved norms. See also Remark 4.1. □

Lemma 2.2. *There is a constant $C = C(\omega, \Omega)$ such that*

$$\|v\| \leq C(\omega, \Omega)|v|_1, \qquad \forall v \in H^1(\Omega), \quad v|_\omega = 0,$$

where $\omega = \Gamma_{y_0}^{z^-} \cup \Gamma_{-y_0}^{z^+}$, *with*

$$\Gamma_{y_0}^{z^-} = \{\mathbf{x} = (x, y, z) \in \Gamma : y = y_0 \quad and \quad z < 0\},$$
$$\Gamma_{-y_0}^{z^+} = \{\mathbf{x} = (x, y, z) \in \Gamma : y = -y_0 \quad and \quad z > 0\}.$$

We may replace ω by any subset of $\partial\Omega$ with a positive Lebesgue measure.

Proof. Suppose that the assertion of the Lemma is false. Then, there is a sequence of functions $\{v_n\}$ converging to v such that

$$|v_n|_1 \to 0, \quad \text{as } n \to \infty, \qquad \text{while } \|v\| = 1.$$

Thus, $\{v_n\}$ is bounded in H^1 and hence there is a subsequence of $\{v_n\}$ called, for simplicity, again $\{v_n\}$ such that

$$v_n \rightharpoonup v \quad \text{in } H^1 \qquad \text{and} \qquad v_n \to v \quad \text{in } L_2,$$

where "$\rightharpoonup$" denotes weak convergence. Hence, we have using the lower semicontinuity

$$|v|_1 \leq \lim \inf_{n\to\infty} |v_n|_1 = 0.$$

Therefore, v is constant and since $v|_\omega = 0$, thus, $v \equiv 0$. This is in contradiction with $\|v\| = 1$ and the proof is complete. □

Now recall that our continuous variational problem is: Find $J \in H^s(\Omega)$, s integer, $s > 1$, such that

$$B(J, v) = - < f, v >_-, \qquad \forall v \in H^1(\Omega). \tag{2.10}$$

A more desirable situation is to use the same interpretation as in the proof of Lemma 2.1 and consider $J \in H^1(\Omega)$. This is mainly because in this case both J and the continuous variational test functions will be in the same function space. However, dealing with convergence rates, we need higher regularity assumptions on J and therefore we shall consider $J \in H^s(\Omega)$, $s > 1$.

The corresponding discrete variational formulation reads as: Find $J^h \in V_h$ such that

$$\begin{aligned} B(J^h, v) &= - < f, v >_-, \qquad \forall v \in V_h, \\ B(J^h, v) &= (J^h_\beta, v + \kappa v_\beta) - \sum_K (\varepsilon J^h_{zz}, v + \kappa v_\beta)_K - < J^h, v >_-, \end{aligned} \tag{2.11}$$

where $(\cdot, \cdot)_K$ is the L_2-product over K. We get from (2.10) and (2.11) that

$$B(e, v) = 0, \qquad \forall v \in V_h, \tag{2.12}$$

where $e = J - J^h$ is the error. We shall also need the following interpolation error estimates, see Ciarlet [6]: Let $J \in H^{r+1}(\Omega)$ then there exists an interpolant $\tilde{J}^h \in V_h$ such that

$$\|J - \tilde{J}^h\| \leq Ch^{r+1}\|J\|_{r+1}, \tag{2.13}$$

$$(2.14) \qquad \|J - \tilde{J}^h\|_1 \le Ch^r \|J\|_{r+1},$$

$$(2.15) \qquad |J - \tilde{J}^h| \le Ch^{r+1/2} \|J\|_{r+1}.$$

Now our main result in this section is:

Theorem 2.1. *There is a constant $C = C(\Omega)$ such that for J and J^h satisfying in (2.10) and (2.11), respectively, and $J_z = J_z^h = 0$ on $\Gamma_{\pm z_0}$, we have*

$$(2.16) \qquad \|J - J^h\|_\beta \le C(\Omega) h^{r+1/2} \|J\|_{r+1},$$

where

$$(2.17) \qquad \|v\|_\beta = \frac{1}{2}\left(\|\varepsilon^{1/2} v_z\|^2 + |v|^2 + \|\kappa^{1/2} v_\beta\|^2\right)^{1/2}$$

Proof. Let $\tilde{J}^h \in V_h$ be an interpolant of J satisfying (2.13)-(2.15). We write $\eta^h = J - \tilde{J}^h$ and $e^h = J^h - \tilde{J}^h$. Using the bilinear form $B(v,v)$ with $v \equiv e = J - J^h$, the relation $B(e, e^h) = 0$ (since $e^h \in V_h$), our assumption: $J_z = J_z^h = 0$, on $\Gamma_{\pm z_0}$ and (2.8), we have that

$$\begin{aligned}
\|e\|_\beta^2 \le & B(e,e) = B(e,\eta^h) - B(e,e^h) = B(e,\eta^h) \\
= & (e_\beta, \eta^h) + (\kappa e_\beta, \eta_\beta^h) + (\varepsilon e_z, \eta_z^h) - (\varepsilon e_z, \eta^h)\big|_{z=-z_0}^{z=z_0} - \sum_K (\kappa \varepsilon e_{zz}, \eta_\beta^h)_K - < e, \eta^h >_- \\
\le & \frac{1}{8}\|e_\beta\|^2 + 2\|\eta^h\|^2 + \frac{1}{4}\|\kappa^{1/2} e_\beta\|^2 + \kappa\|\eta_\beta^h\|^2 + \frac{1}{8}\|\varepsilon^{1/2} e_z\|^2 \\
& + 2\bar{\varepsilon}\|\eta_z^h\|^2 + \frac{1}{8}\|\varepsilon^{1/2} e_z\|^2 + 2\bar{\varepsilon}\bar{\kappa}^2 h^{-2}\|\eta_\beta^h\|^2 + \frac{1}{4}|e|^2 + |\eta^h|^2.
\end{aligned}$$

Where, in $(\cdot,\cdot)|_{z=-z_0}^{z=z_0}$, the integration is over x, y-variables and vanishes because of our assumption. We now use (2.13)-(2.15), $\bar{\kappa} < Ch$, $\bar{\varepsilon} < h$ and a standard kick-back argument and obtain the desired result. □

3. The Fokker-Planck equation. Recall the scaled Fokker-Planck equation in two space dimensions

$$(3.1) \qquad J_\beta - \varepsilon A\, J = 0,$$

where A is as in (1.19), $J_\beta = \beta \cdot \nabla_{\mathbf{x}} J$, with $\beta = (1, z, 0)$ and $\mathbf{x} = (x, y, z)$ is the phase-space variable. Further, Γ^- denotes the inflow boundary defined in Section 1 so that

$$\Gamma^- = \Gamma_0 \cup \Gamma_{-y_0}^{z^+} \cup \Gamma_{y_0}^{z^-},$$
$$\Gamma_0 = \{\mathbf{x} = (x, y, z) \in \Gamma : x \equiv 0\},$$

with $\Gamma_{-y_0}^{z^+}$ and $\Gamma_{y_0}^{z^-}$, defined as in Lemma 2.2. For this problem we define a modified variational formulation as: Find $J \in H^1(\Omega)$ such that

$$\begin{aligned}
(3.2) \qquad (J_\beta, bv + \kappa v_\beta) - (\varepsilon A\, J, bv + \kappa v_\beta) - < J, bv >_- + < J, \varepsilon\Lambda(z_0) v >_{\Gamma_{\pm z_0}} = \\
= - < f, bv >_- + < f, \varepsilon\Lambda(z_0) v >_{\Gamma_{\pm z_0}},
\end{aligned}$$

where $b = b(z) = (1 + z^2)^{3/2}$, $\Lambda(z_0) = 6z_0(1 + z_0^2)^3$ and the terms of the form (J_{zz}, v) and (J_{zz}, v_β) are considered after performing an integration by parts with respect to z variable as in the case for $v \in H^1(\Omega)$ in the proof of Lemma 2.1. The term involving the parameter $\Lambda(z_0)$ is for cancelling a negative $\Gamma_{\pm z_0}$ boundary term in the proof of the stability inequality, i.e., Lemma 3.1 below. We now introduce the corresponding bilinear form

(3.3) $$B(w, v) = (w_\beta, bv + \kappa v_\beta) - (\varepsilon A w, bv + \kappa v_\beta) - < w, bv >_- + < w, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}} .$$

Thus, by (3.2)

(3.4) $$B(J, v) = L(v), \qquad \forall v \in H^1(\Omega)$$

where

$$L(v) = - < f, bv >_- + < f, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}} .$$

Lemma 3.1. *There is a constant $C = C(\Omega)$ such that*

$$B(v, v) \geq C(\Omega)\|v\|^2_{\beta,b}, \qquad \forall v \in \mathcal{H}^1_{z_0}(\Omega)$$

where

$$\|v\|_{\beta,b} = \frac{1}{\sqrt{2}}\left(|v|_b^2 + \Lambda(z_0)|\varepsilon^{1/2}v|^2_{\Gamma_{\pm z_0}} + \|\kappa^{1/2}v_\beta\|^2 + \int_\Omega |\varepsilon^{1/2}\frac{\partial}{\partial z}(bv)|^2 a(z)\right)^{1/2},$$

$$|v|_b = \left(\int_\Gamma v^2 b|\mathbf{n}\cdot\beta|\, ds\right)^{1/2}$$

Proof. We use (3.3) with the same interpretation of the terms having higher derivatives on v as in the proof of Lemma 2.1 and write

$$\begin{aligned} B(v, v) =& (v_\beta, bv) + \|\kappa^{1/2}v_\beta\|^2 - (\varepsilon Av, bv) - (\varepsilon Av, \kappa v_\beta) - \\ & - < v, bv >_- + < v, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}} \equiv: \sum_{i=1}^{6} T_i. \end{aligned}$$

Below, we estimate each T_i separately. By a Green's formula approach, since $\beta = (1, z, 0)$, and $b = b(z) = (1 + z^2)^{3/2}$ is independent of x and y-variables, we have that

$$(v_\beta, bv) = \int_\Gamma bv^2(\mathbf{n}\cdot\beta)\, ds - (v_\beta, bv) =< v, bv > -(v_\beta, bv).$$

Thus,

(3.5) $$T_1 \equiv (v_\beta, bv) = \frac{1}{2} < v, bv > .$$

Further, since $\varepsilon = \sigma_{tr}(x, y)$ does not depend on z we use partial integration and write

$$(\varepsilon Av, bv) = -\left(\varepsilon a(z)\frac{\partial}{\partial z}(bv), \frac{\partial}{\partial z}(bv)\right) + \int_{I_x\times I_y}\left[\varepsilon a(z)\frac{\partial}{\partial z}(bv)(bv)\right]_{z=-z_0}^{z=z_0} := I_1 + I_2.$$

Here, $v \in \mathcal{H}^1_{z_0}$ (we could assume, instead, that we have $v_z(-z_0) = v_z(z_0) = 0$) and consequently since a and b are even and b' is an odd function of z we get

$$\begin{aligned} I_2 &= a(z_0)b'_z(z_0)b(z_0)\int_0^1 dx \int_{-y_0}^{y_0} \varepsilon(x,y)[v^2(x,y,z_0)+v^2(x,y,-z_0)]\,dy \\ &= 3z_0(1+z_0^2)^3|\varepsilon^{1/2}v|^2_{\Gamma_{\pm z_0}}. \end{aligned}$$

Hence

$$T_3 = -(\varepsilon Av, bv) = \int_\Omega \left|\varepsilon^{1/2}\frac{\partial}{\partial z}(bv)\right|^2 a(z)\,dx - \frac{1}{2}\Lambda(z_0)|\varepsilon^{1/2}v|^2_{\Gamma_{\pm z_0}} \tag{3.6}$$

Finally, using the inverse inequality and interpreting the $(\cdot,\cdot)$ as a sum of integrals over the elements

$$\begin{aligned} |(\varepsilon Av, \kappa v_\beta)| &\le \left(\int|\varepsilon Av|^2\right)^{1/2}\left(\int|\kappa v_\beta|^2\right)^{1/2} \le \left(h^{-2}\int\left|\varepsilon a(z)\frac{\partial}{\partial z}(bv)\right|^2\right)^{1/2}\|\kappa v_\beta\| \\ &\le \frac{1}{2}\int_\Omega\left|\varepsilon^{1/2}\frac{\partial}{\partial z}(bv)\right|^2 a(z) + \frac{\bar\varepsilon}{2}h^{-2}\bar\kappa a_{\max}\|\kappa^{1/2}v_\beta\|^2, \end{aligned}$$

where $a_{\max} = \max a(z) = 1 + z_0^2$. Thus,

$$T_4 = -(\varepsilon Av, \kappa v_\beta) \ge -\frac{1}{2}\int_\Omega\left|\varepsilon^{1/2}\frac{\partial}{\partial z}(bv)\right|^2 a(z)\,dx - \frac{\bar\varepsilon}{2}h^{-2}\bar\kappa a_{\max}\|\kappa^{1/2}v_\beta\|^2.$$

Moreover

$$T_1 + T_5 = \frac{1}{2} < v, bv > - < v, bv >_- = \frac{1}{2} < v, bv >_+ + \frac{1}{2} < v, bv >_- - < v, bv >_- = \frac{1}{2}|v|^2_b,$$

and $-I_2 + T_6 = \frac{1}{2}T_6$. Hence, summing up we get

$$B(v,v) \ge \frac{1}{2}|v|_b^2 + \left(1-\frac{\bar\varepsilon}{2}h^{-2}\bar\kappa a_{\max}\right)\|\kappa^{1/2}v_\beta\|^2 + \frac{1}{2}\Lambda(z_0)|\varepsilon^{1/2}v|^2_{\Gamma_{\pm z_0}} + \frac{1}{2}\int_\Omega\left|\varepsilon^{1/2}\frac{\partial}{\partial z}(bv)\right|^2 a(z).$$

Now assuming $\bar\varepsilon h^{-2}\bar\kappa a_{\max} < 1$, i.e., for $C_1 h < \bar\kappa < C\frac{h}{a_{\max}}$ and $\bar\varepsilon < h$, we have the desired stability estimate. □

We recall that, the presence of the constant $C(\Omega)$ in Lemma 3.1 is to emphasize the Ω dependence in the definition of the parameter κ. We define the trial function space $V_h^{z_0} \subset \mathcal{H}^1_{z_0}$, as

$$V_h^{z_0} = \{v \in V_h : v \text{ satisfies the angular balance condition (1.21)}\}.$$

The finite element formulation of our variational problem, this time, would be: Find $J^h \in V_h^{z_0}$ such that with B defined as in (3.3),

$$B(J^h, v) = - < f, bv >_- + < f, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}}, \qquad \forall v \in V_h^{z_0}. \tag{3.7}$$

Subtracting (3.7) from its continuous counterpart (3.4) we get the equation for the error $e = J - J^h$, (3.8) $$B(e,v)=0, \qquad \forall v\in V_h^{z_0}.$$

Our convergence theorem is as follows:

Theorem 3.1. *There is a constant $C=C(\Omega)$ such that for $J\in\mathcal{H}_{z_0}^{r+1}$ satisfying (1.13)-(1.15) and solving (3.1) and $J^h\in V_h^{z_0}$ solving (3.7), we have*

$$\|J-J^h\|_{b,\beta}\le C(\Omega)h^{r+1/2}\|J\|_{r+1}.$$

Proof. Let $\tilde{J}^h$ be an interpolant of J satisfying (1.21) and (2.13)-(2.15), (i.e., in particular, $\tilde{J}^h\in V_h^{z_0}$), where the Fermi solution is replaced by the Fokker-Planck solution. We write as in the previous section $\eta^h=J-\tilde{J}^h$ and $e^h=J^h-\tilde{J}^h$. Using (1.14) we have that $e_z|_{z=-z_0}^{z=z_0}\equiv 0$, thus, $\int_{I_x\times I_y}\varepsilon a(z)\frac{\partial}{\partial z}(be)(b\eta^h)|_{z=-z_0}^{z=z_0}=\frac{1}{2}<e,\varepsilon\Lambda(z_0)\eta^h>_{\Gamma_{\pm z_0}}$. Hence, using Lemma 3.1,

(3.9)
$$\begin{aligned}\|e\|_{b,\beta}^2&\le B(e,e)=B(e,\eta^h)-B(e,e^h)=B(e,\eta^h)\\&=(e_\beta,b\eta^h)+(\kappa e_\beta,\eta_\beta^h)+(\varepsilon a(z)\frac{\partial}{\partial z}(be),\frac{\partial}{\partial z}(b\eta^h))-\frac{1}{2}<e,\varepsilon\Lambda(z_0)\eta^h>_{\Gamma_{\pm z_0}}\\&-(\kappa\varepsilon Ae,\eta_\beta^h)-<e,b\eta^h>_-+<e,\varepsilon\Lambda(z_0)\eta^h>_{\Gamma_{\pm z_0}}.\end{aligned}$$

Thus, by a similar argument as in the proof of Theorem 2.1, using the inverse estimate and combining some of the terms we have

$$\begin{aligned}\|e\|_{b,\beta}^2&\le\frac{1}{8}\|\kappa^{1/2}e_\beta\|^2+2\underline{\kappa}^{-1}\|b\eta^h\|^2+\frac{1}{8}\|\kappa^{1/2}e_\beta\|^2+2\bar{\kappa}\|\eta_\beta^h\|^2\\&+\frac{1}{4}\int_\Omega|\varepsilon^{1/2}\frac{\partial}{\partial z}(be)|^2a(z)+2\bar{\varepsilon}\int_\Omega|\frac{\partial}{\partial z}(b\eta^h)|^2a(z)+\frac{1}{4}\Lambda(z_0)|\varepsilon^{1/2}e|_{\Gamma_{\pm z_0}}^2\\&+\frac{1}{4}|\eta^h|^2+2h^{-2}\bar{\kappa}^2\bar{\varepsilon}\int_\Omega|\eta_\beta^h|^2a(z)+\frac{1}{4}|e|_b^2+|\eta^h|_b^2,\end{aligned}$$

where $\underline{\kappa}=\min_{C_h}\kappa$. Hiding the e terms from the right hand side in $\|e\|_{b,\beta}^2$ we get

$$\|e\|_{b,\beta}^2\le C(\Omega)\times\left[\underline{\kappa}^{-1}\|\eta^h\|^2+\bar{\kappa}\|\eta_\beta^h\|^2+\bar{\varepsilon}\|\eta_z^h\|^2+\bar{\varepsilon}\|\eta^h\|^2+\bar{\varepsilon}h^{-2}\bar{\kappa}^2\|\eta_\beta^h\|^2+|\eta^h|_b^2\right],$$

where $C(\Omega)=C\max(b^2a,bb'a)$. Thus, for $Ch<\kappa<\frac{h}{a_{\max}}$ and $\bar{\varepsilon}<h$ and using the interpolation errors (2.13)-(2.15), we get the desired error estimate. □

Recall that the crucial step in this argument is that we have considered the problem in a bounded domain especially in the z-direction. In the remaining Sections: 4 and 5, some analyses similar to those in [1] are not carried out in full details. Interested reader is asked to see Sections 3 and 4 in [1].

4. Streamline diffusion with discontinuity in x. In order to study the distribution of the particle beams in a certain depth, e.g., $x=x_d$, a reasonable initial guess would be obtained using the information in some previous distinct depths $x=x_i$, $i=1,\dots,n$ with $x_i<x_{i+1}$. One may assume having various filters instalated in different depths in order to control or adjust the beam intensity. This corresponds considering discontinuities in x-direction. (Observe that, to deal with the diffusion term $-\varepsilon J_{zz}$, the trial functions should be continuous in the z-variable). In this section we consider a method where the trial functions are continuous in y and z-directions and discontinuous in the x-direction, based on a "triangulation" of each phase-space slab, with quasiuniform triangular bases

in yz and heights $h_n = x_{n+1} - x_n$, being equal to the corresponding slab's width. We let also the maximum of the sides of the triangular bases of the elements in yz in all slab levels to be $h \sim \max_n h_n$. To define such a method let $0 = x_0 < x_1 < ... < x_N = L$, be a quasiuniform subdivision of the interval $[0, L]$, (scaling in $x-$variable, we may choose $L = 1$), and introduce the strips

$$S_n = I_n^x \times I_y \times I_z, \quad n = 1, 2, ..., N,$$
$$I_n^x = \{x : x_{n-1} < x < x_n\}, \quad n = 1, 2, ..., N.$$

For each n, let W^n be a finite element subspace of $H^1(S_n)$ based on the triangulation $\mathcal{C}_h$ of the strip S_n with the elements of size $h > \varepsilon$. (Because of the relation between the parameters h and n, in this Section, we only include the index h in the discrete function spaces and hence in our discrete functions.) Let $\mathcal{T}_h$ be a triangulation of $I_y \times I_z$, with elements $\tau \in \mathcal{T}_h$, and define

$$\mathcal{W}_h = \{u \in \tilde{\mathcal{H}}_{z_0}^1 : u|_K \in \mathcal{P}_k(\tau) \times \mathcal{P}_k(I_n^x); \forall K = \tau \times I_n^x \in \mathcal{C}_h\}, \quad \text{here,} \quad \tilde{\mathcal{H}}_{z_0}^1 = \prod_{n=1}^{N} \mathcal{H}_{z_0}^1(S_n).$$

If we now apply the streamline diffusion method successively on each strip S_n for the Fermi problem given by (1.14)-(1.18) and impose the boundary conditions at the points $x = x_{n-1}$ weakly, we obtain the following method: Find $J^h \in \mathcal{W}_h$ such that for $n = 1, 2, ..., N$,

$$(J_\beta^h, v + \kappa v_\beta)_n + (\varepsilon J_z^h, v_z)_n - \int_{I_n^x \times I_y} \varepsilon J_z^h v\Big|_{z=-z_0}^{z=z_0} - (\varepsilon\kappa J_{zz}^h, v_\beta)_n$$
$$+ < J_+^h, v_+ >_{n-1} - < J_+^h, v_+ >_{\Gamma_n^-} = < J_-^h, v_+ >_{n-1}, \quad \forall v \in \mathcal{W}_h,$$

where the term with J_{zz}^h is again interpreted after that a partial integration on z is performed and where we have used the following notation:

$$J_-^0 = f,$$
$$(v, w)_n = \int_{S_n} vw,$$
$$< v, w >_n = \int_{-y_0}^{y_0} \int_{-z_0}^{z_0} v(x_n, y, z) w(x_n, y, z),$$
$$< v, w >_{\Gamma_n^-} = < v, w >_{\Gamma^- \cap \partial S_n},$$
$$v_\pm(x, y, z) = \lim_{s \to 0\pm} v(x + s, y, z),$$
$$[v] = v_+ - v_-.$$

The suitable continuous bilinear form for this method is:

(4.1)

$$B(J, v) = \sum_{n=1}^{N} \left[(J_\beta, v + \kappa v_\beta)_n + (\varepsilon J_z, v_z)_n - \int_{I_n^x \times I_y} \varepsilon J_z v\Big|_{z=-z_0}^{z=z_0} - (\varepsilon\kappa J_{zz}, v_\beta)_n \right]$$
$$- \sum_{n=1}^{N} < J_+, v_+ >_{\Gamma_n^-} + \sum_{n=1}^{N-1} < [J], v_+ >_n + < J_+, v_+ >_0,$$

with the discrete counterpart, where all J's are replaced by J^h. The corresponding linear form is:

$$L(v) = < f, v_+ >_0 = \int_{\Gamma_0} f v \, ds.$$

Thus, we have

$$B(J, v) = L(v), \qquad \forall v \in \mathcal{W}_h. \tag{4.2}$$

We shall use a stability estimate for (4.1) in a norm $|||\cdot|||$ defined by

$$|||v|||^2 = \frac{1}{2}[|v|_N^2 + |v|_0^2 + \sum_{n=1}^{N-1} |[v]|_n^2 + \|\kappa^{1/2} v_\beta\|^2 + \|\varepsilon^{1/2} v_z\|^2 + \| |z|^{1/2} v|_{\Gamma_{\pm y_0}}^2],$$

where

$$[v]_n = v(x_n+) - v(x_n-), \quad \text{and} \quad |w|_{\Gamma_{\pm y_0}} = \left(\int_0^L \int_{-z_0}^{z_0} [w^2(-y_0) + w^2(y_0)]\right)^{1/2},$$

with $w(\pm y_0) =: w(x, \pm y_0, z)$, we use similar convention for the other variables.

Lemma 4.1. *We have that*

$$B(v, v) \geq |||v|||^2, \qquad \forall v \in \tilde{\mathcal{H}}^1_{z_0}.$$

Proof. We use the definition of B in (4.1) and write

$$B(v, v) = \|\kappa^{1/2} v_\beta\|^2 + \|\varepsilon^{1/2} v_z\|^2 - (\varepsilon\kappa v_{zz}, v_\beta) - < v_+, v_+ >_{\Gamma^-} + \sum_{n=1}^{N} (v_\beta, v)_n + \sum_{n=1}^{N-1} < [v], v_+ >_n + < v_+, v_+ >_0, \tag{4.3}$$

where we have used the angular balance condition (1.21) which leads, after summation, to the elimination of the third term in the first sum in (4.1). Integrating by parts we have

$$(v_x, v)_n = \frac{1}{2} < v, v >|_{x_{n-1}^+}^{x_n^-} = \frac{1}{2} \int_{I_y \times I_z} \left[v^2(x_n^-)(\mathbf{n} \cdot \beta) - v^2(x_{n-1}^+)(\mathbf{n} \cdot \beta)\right],$$

and since, for $x = x_n^-$, $(\mathbf{n} \cdot \beta) \equiv 1$ while for $x = x_{n-1}^+$, $(\mathbf{n} \cdot \beta) \equiv -1$, we get

$$(v_x, v)_n = \frac{1}{2} \int_{I_y \times I_z} [v^2(x_n^-) + v^2(x_{n-1}^+)].$$

Thus, rearranging the terms we may write the last three terms in (4.3) as

$$\sum_{n=1}^{N} (v_\beta, v)_n + \sum_{n=1}^{N-1} < [v], v_+ >_n + < v_+, v_+ >_0 = \sum_{n=1}^{N} (z v_y, v)_n + \frac{1}{2} \left(\sum_{n=1}^{N-1} |[v]|_n^2 + |v|_N^2 + |v|_0^2\right). \tag{4.4}$$

Further, using the equalities:

$$(zv_y, v)_n = \frac{1}{2}\int_{x_{n-1}}^{x_n}\int_{-z_0}^{z_0} z[v^2(y_0) - v^2(-y_0)],$$

$$< v_+, v_+ >_{\Gamma_n^-} = \int_{x_{n-1}}^{x_n} dx \left(\int_{-z_0}^{0} zv_+^2(y_0)\,dz + \int_0^{z_0} -zv_+^2(-y_0)\,dz\right),$$

it is easy to check that

$$\begin{aligned}(4.5)\qquad (zv_y, v)_n - &< v_+, v_+ >_{\Gamma_n^-} \\ &= -\frac{1}{2}\int_{x_{n-1}}^{x_n}\int_{-z_0}^{0} zv_+^2(y_0) + \frac{1}{2}\int_{x_{n-1}}^{x_n}\int_0^{z_0} zv_+^2(-y_0) \\ &= \frac{1}{2}\int_{x_{n-1}}^{x_n}\int_{-z_0}^{z_0} |z|[v_+^2(y_0) + v_+^2(-y_0)].\end{aligned}$$

To estimate the term involving v_{zz} we use the same inverse estimate technique as in the previous sections to obtain

$$(4.6)\qquad \sum_{n=1}^{N}(\kappa\varepsilon v_{zz}, v_\beta)_n \le \frac{1}{2}\|\varepsilon^{1/2}v_z\|^2 + \frac{1}{2}\bar{\varepsilon}\bar{\kappa}C^2h^{-2}\|\kappa^{1/2}v_\beta\|^2.$$

Now (4.3)-(4.6) together with the fact that the parameters are chosen so that $\bar{\varepsilon}\bar{\kappa}C^2h^{-2} < 1$, will give us the desired result. □

Theorem 4.1. *There is a constant $C = C(\Omega)$ independent of the mesh size h and the parameter ε such that for $\bar{\varepsilon} < h$ and with κ and $\bar{\kappa}$ satisfying the stability condition of Lemma 2.1, we have the following error estimate valid for the solution of the Fermi equation:*

$$|||J - J^h||| \le Ch^{r+1/2}\|J\|_{r+1}.$$

Proof. The proof is similar to that of Theorem 2.1. Here, we need to control some additional jump and boundary terms. Let $\tilde{J}^h \in \mathcal{W}_h$ be an interpolant of the exact solution J so that $\tilde{J}_z^h\big|_{\pm z_0} = J_z|_{\pm z_0}$, and $\eta = J - \tilde{J}^h$. The error term can be written as

$$e := J - J^h = (J - \tilde{J}^h) - (J^h - \tilde{J}^h) \equiv \eta - \xi.$$

Now since $\xi \in \mathcal{W}_h,\ B(e,\xi) = 0$. Thus, we have using Lemma 4.1, that

$$\begin{aligned}(4.7)\qquad &|||\xi|||^2 \le B(\xi,\xi) = B(\eta - e, \xi) = B(\eta,\xi) \\ &= \sum_{n=1}^{N}\left[(\eta_\beta, \xi + \kappa\xi_\beta)_n + (\varepsilon\eta_z, \xi_z)_n - \int_{I_n^x\times I_y}\varepsilon\eta_z\xi\Big|_{z=-z_0}^{z=z_0} - (\varepsilon\kappa\eta_{zz}, \xi_\beta)_n\right] \\ &\quad - < \eta_+, \xi_+ >_{\Gamma^-} + \sum_{n=1}^{N-1} < [\eta], \xi_+ >_n + < \eta_+, \xi_+ >_0 .\end{aligned}$$

Integrating by parts we have

$$(4.8)\qquad (\eta_\beta, \xi)_n = -(\eta, \xi_\beta)_n + < \eta_-, \xi_- >_n - < \eta_+, \xi_+ >_{n-1} + < \eta, \xi >_{\Gamma_n^-\cup\Gamma_n^+} .$$

Inserting (4.8) in (4.7) we get

$$B(\eta,\xi) = -(\eta,\xi_\beta) + (\kappa\eta_\beta,\xi_\beta) + (\varepsilon\eta_z,\xi_z) - (\varepsilon\kappa\eta_{zz},\xi_\beta) + <\eta_-,\xi_->_N - \sum_{n=1}^{N-1} <\eta_-,[\xi]>_n + <\eta,\xi>_{\Gamma^+_{L^c}}, \tag{4.9}$$

where, $\Gamma^+_{L^c} = \Gamma^+ \setminus \{(x,y,z)\in\Gamma : x\equiv L\}$, is the outflow boundary except the top surface for $x = L$. Thus using the same technique as in the proof of Lemma 4.1,

$$|B(\eta,\xi)| \le \frac{1}{4}|||\xi|||^2 + \left[\underline{\kappa}^{-1}\|\eta\|^2 + \bar{\kappa}\|\eta_\beta\|^2 + \bar{\varepsilon}\|\eta_z\|^2 + \bar{\varepsilon}\bar{\kappa}^2h^{-2}\|\eta_z\|^2 + \sum_{n=1}^{N}|\eta_-|^2_n + |\eta|^2_{\Gamma^+_{L^c}}\right].$$

Now by standard interpolation theory we have (see Ciarlet [5], p. 123),

$$\left[h|\eta|^2_{\Gamma^+_{L^c}} + \|\eta\|^2 + h^2\|\eta_\beta\|^2 + h^2\|\eta_z\|^2 + h\sum_{n=1}^{N}|\eta_-|^2_n\right]^{1/2} \le Ch^{k+1}\|J\|_{k+1}.$$

Thus

$$|||\xi|||^2 \le Ch^{2k+1}, \tag{4.10}$$

and since $|||\eta|||$, the interpolation error, is of the same order as $|||\xi|||$, we have the desired result. □

Remark 4.1. Once again, we emphasize that even in this Section, as in Section 2, the stability Lemma 4.1 will not, explicitly, give a control of the L_2-norm. We could again use a version of the Poincare-Friedricks inequality as in Lemma 2.2 and obtain an estimate for the L_2-norm with the same coefficients as for the semi-norms involved in the weighted stability norm, i.e., we could add a L_2-norm with a coefficient of order $\mathcal{O}(\sqrt{h})$ to the $|||\cdot|||$ norm in Lemma 4.1. However, a better approach would be through Lemma 4.2 below, (see also Lemma 3.2 in [1]), in a situation where jump discontinuities are introduced and included in the stability norm $|||\cdot|||$. This approach improves the L_2-norm estimate regaining the factor $h^{1/2}$. Therefore, in all of the norms $|||\cdot|||$, in here and in the following Section, we can insert the L_2-norm of the function without any small coefficients of order $\mathcal{O}(\sqrt{h})$. □

Lemma 4.2. *For any constant $C_1 > 0$, we have for $v \in \tilde{\mathcal{H}}^1_{x_0}$,*

$$\|v\|^2 \le \left[\frac{1}{C_1}\|v_\beta\|^2 + \sum_{n=1}^{N}|v_-|^2_n + |z^{1/2}v|^2_{\Gamma_{\pm v_0}}\right]he^{C_1h}.$$

Proof. For $x_n < x < x_{n+1}$, we have using Green's formula and a relation of the type (4.5), that

$$\|v(x)\|^2_{L_2(I_y\times I_z)} = |v_-|^2_{n+1} - \int_x^{x_{n+1}} \frac{d}{dx}\|v(s)\|^2_{L_2(I_y\times I_z)}ds$$
$$= |v_-|^2_{n+1} - 2\int_x^{x_{n+1}}\left[(v_x + zv_y, v)_{\Omega_s} - <v_+,v_+>_{\Gamma_s^-} - \frac{1}{2}\int_{\Gamma_s} zv^2|\mathbf{n}\cdot\beta|\right]$$

$$\leq |v_-|_{n+1}^2 + \frac{1}{C_1}\|v_\beta\|_n^2 + C_1 \int_x^{x_{n+1}} \|v(s)\|_{\Omega_s}^2 ds + |z^{1/2}v|_{\Gamma_{\pm\nu_0}^n}^2,$$

where, for simplicity, we have used the notation $\Omega_x := I_y \times I_z \times \{x\}$, $\Gamma_x := \partial\Omega_x$, similarl, $\Gamma_x^- := \partial\Omega_x^-$ and $|\cdot|_{\Gamma_{\pm\nu_0}^n}$ is defined as $|\cdot|_{\Gamma_{\pm\nu_0}}$, with the x-integration over $I_n = [x_n, x_{n+1}]$. Thus, Grönwall's inequality for $x_n < x < x_{n+1}$ yields

$$\|v(x)\|_{\Omega_x}^2 \leq \left[\frac{1}{C_1}\|v_\beta\|_n^2 + |v_-|_{n+1}^2 + |z^{1/2}v|_{\Gamma_{\pm\nu_0}^n}^2\right] e^{C_1 h}. \tag{4.11}$$

Integrating over $x_n < x < x_{n+1}$ and summing over $n = 0, 1, ..., N-1$, we obtain the desired result. □

The Streamline Diffusion method for the Fokker-Planck equation is defined in an analogue way: Find $J^h \in \mathcal{W}_h$ such that

(4.12)

$$\begin{aligned} B(J^h, v) = &\sum_{n=1}^{N} \left[(J_\beta^h, bv + \kappa v_\beta)_n - (\varepsilon A J^h, bv + \kappa v_\beta)_n\right] - < J_+^h, bv_+ >_{\Gamma^-} \\ &+ \sum_{n=1}^{N-1} < [J^h], bv_+ >_n + < J^h, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}^n} + < J_+^h, bv_+ >_0 \\ = &< f, bv_+ >_0 + < f, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}^n}, \qquad \forall v \in \mathcal{W}_h, \end{aligned}$$

where, $\Gamma_{\pm z_0}^n := \Gamma_{\pm z_0}|_{I_n}$. We introduce the norm

$$\begin{aligned} |||v|||_b^2 = &\frac{1}{2}[\|\kappa^{1/2}v_\beta\|^2 + \|(\varepsilon a)^{1/2}\frac{\partial}{\partial z}(bv)\|^2 + |b^{1/2}v|_N^2 + |b^{1/2}v|_0^2 \\ &+ \sum_{n=1}^{N-1} |b^{1/2}[v]|_n^2 + \Lambda(z_0)|\varepsilon^{1/2}v|_{\Gamma_{\pm z_0}}^2 + ||z|^{1/2}v|_{\Gamma_{\pm\nu_0}}^2]. \end{aligned}$$

Then, our stability Lemma is:

Lemma 4.3. *There is a constant $C = C(\Omega)$ such that for $v \in \mathcal{W}_h$ we have*

$$B(v, v) \geq C|||v|||_b^2.$$

The convergence theorem is now:

Theorem 4.2. *Let J and J^h be the solutions of the continuous and discrete Fokker-Planck equations satisfying (3.4) and (4.12), respectively, further assume that J and J^h satisfy the boundary conditions (1.13)-(1.15) and the angular balance condition (1.21), then there is a constant $C = C(\Omega)$ such that for sufficiently small h and for $J \in H^{k+1}(\Omega)$ we have*

$$|||J - J^h|||_b \leq C(\Omega)h^{k+1/2}\|J\|_{k+1}.$$

The proofs of Lemma 4.3 and Theorem 4.2 are lenghty, however, straightforward combinations of the proofs in here and the previous Section. We omit these tedious details.

5. Discontinuous Galerkin. Now we assume trial functions with discontinuities in both x and y-directions (we cannot impose this condition in z-direction, because of the presence of ∂_{zz}^2 in the operator A). We let $\mathcal{C}_h$ be a family of quasiuniform triangulation of

Ω and $\forall z \in I_z$ let $\mathcal{T}_h^z := \mathcal{C}_h \cap \{z\}$. Then $\mathcal{T}_h^z$ is a family of quasiuniform triangulation of $I_x \times I_y \times \{z\}$ and we may define for $\tau \in \mathcal{T}_h^z$,

$$\begin{aligned}\partial\tau_\pm(\beta) &= \{(x,y) \in \partial\tau : \mathbf{n}(x,y)\cdot\beta' \gtrless 0\},\\ \tilde{\mathcal{W}}_h &= \{v \in L_2(\Omega) : v|_K \in P_k(K),\ \forall K \in \mathcal{C}_h\},\\ \mathcal{W}_h{}^z &= \{v \in L_2(\Omega_z) : v|_\tau \in P_k(\tau),\ \forall \tau \in \mathcal{T}_h^z\},\end{aligned}$$

where $\beta' \equiv (1,z)$. Now, consider the quasiuniform gridpoints $z_i, i = 1,\dots,m$ in the z-direction, with m chosen so that the three dimensional triangulation $\mathcal{C}_h$ remains quasiuniform with the size h. Then supressing the superscript z of $\mathcal{T}_h^z$ and interpreting $\mathcal{T}_h$ as triangulations in the z-direction, the discontinuous (in x, y) Galerkin finite element method for Fermi equation can now be formulated as follows: Find $J^h \in \tilde{\mathcal{W}}_h$ such that $J^h|_{z=z_i} \in \mathcal{W}_h{}^{z_i}$, $i = 1,\dots m$, and

$$(5.1)\qquad \begin{aligned}(J^h_\beta, v+\kappa v_\beta) + (\varepsilon J^h_z, v_z) - \sum_{\tau\in\mathcal{T}_h}\varepsilon\int_\tau (J^h_z v)\Big|_{z=-z_0}^{z=z_0} - (\kappa\varepsilon J^h_{zz}, v_\beta)\\ + \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)\times I_z}[J^h]v_+|\mathbf{n}\cdot\beta| = 0, \quad \forall v \in \tilde{\mathcal{W}}_h,\end{aligned}$$

here, $(\cdot,\cdot) = \sum_{K\in\mathcal{C}_h}(\cdot,\cdot)_K$, $[J^h] = J^h_+ - J^h_-$ and

$$J^h_\pm = \lim_{s\to 0\pm} J^h((x,y,z)+s\beta) = \lim_{s\to 0\pm} J^h(x+s, y+sz, z).$$

Recall that since $\beta = (1,z,0)$ is divergent free, $\mathbf{n}\cdot\beta$ is continuous across the inter-element boundaries of $\mathcal{T}_h$ and thus $\partial\tau_\pm(\beta)$ is well defined. Problem (5.1) can be formulated as

(5.2)

$$\begin{aligned}B(J^h, v) = (J^h_\beta, v+\kappa v_\beta) + (\varepsilon J^h_z, v_z) - \sum_{\tau\in\mathcal{T}_h}\varepsilon\int_\tau (J^h_z v)\Big|_{z=-z_0}^{z=z_0} - (\kappa\varepsilon J^h_{zz}, v_\beta)\\ + \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}[J^h]v_+|\mathbf{n}\cdot\beta| + < J^h_+, v_+ >_0 = < f, v_+ >_0, \qquad \forall v \in \tilde{\mathcal{W}}_h,\end{aligned}$$

where $\partial\tau_-(\beta)' = \partial\tau_-(\beta)\setminus\Omega_0$. Our stability Lemma for this problem is:

Lemma 5.1. *We have for B as in (5.2) and κ and ε as in the proof of Lemma 2.1,*

$$B(v,v) \geq |||v|||^2, \qquad \forall v \in (\tilde{\mathcal{W}}_h) \cap \tilde{\mathcal{H}}^1_{z_0}(\Omega),$$

where

$$|||v|||^2 = \frac{1}{2}[||\kappa^{1/2}v_\beta||^2 + ||\varepsilon^{1/2}v_z||^2 + \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}[v]^2|\mathbf{n}\cdot\beta| + \int_{\Gamma^+_{\pm z_0}\times I_z} v^2|\mathbf{n}\cdot\beta|].$$

Proof. The proof is similar to that of Lemma 4.1 and is a consequence of the inequality

$$\sum_{\tau\in\mathcal{T}_h}\left[(v_\beta,v)_{\tau\times I_z}+\int_{\partial\tau_-(\beta)'\times I_z}[v]v_+|\mathbf{n}\cdot\beta|\right]+|v|_0^2$$
$$\geq\frac{1}{2}\left[\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}[v]^2|\mathbf{n}\cdot\beta|+\int_{\Gamma^+_{\pm z_0}\times I_z}.v^2|\mathbf{n}\cdot\beta|\right],$$

We omit the details. □

Our next convergence result is:

Theorem 5.1. *Let J and J^h be as in Theorem 4.1, then for $|||\cdot|||$ defined as in Lemma 5.1, we have the following error estimate for the problem (5.1),*

$$|||J-J^h|||\leq C(\Omega)h^{k+1/2}||J||_{k+1}.$$

Proof. We use the same notation as in the proof of Theorem 4.1 and with B given by (5.2), to write

$$|||\xi|||^2\leq B(\xi,\xi)=B(\eta,\xi),$$

where using the same technique as in the proof of Theorem 4.1 we finally need to control a term of the form

$$T=\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}[\xi]\eta_+|\mathbf{n}\cdot\beta|.$$

Using Cauchy's inequality we have for $\lambda>0$, that

$$|T|\leq\frac{C}{\lambda}\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}[\xi]^2|\mathbf{n}\cdot\beta|+C\lambda\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}|\eta_+|^2|\mathbf{n}\cdot\beta|.$$

Here, the first sum can be hidden in $|||\xi|||^2$ and we estimate the last one as

$$\int_{\partial\tau_-(\beta)'\times I_z}|\eta_+|^2|\mathbf{n}\cdot\beta|\leq||\eta||^2_{\tau,\infty}\left[\int_{\partial\tau_-(\beta)'\times I_z}|\mathbf{n}\cdot\beta|^2+\int_{\partial\tau_-(\beta)'\times I_z}ds\right]$$
$$\leq C||\eta||^2_{\tau,\infty}\left[Ch^{-1}||\beta||^2_{\tau\times I_z}+2z_0h^2\right],$$

summing over τ and using the interpolation error

$$||\eta||_\infty=||J-\tilde{J}^h||_\infty\leq Ch^{k+1}||J||_{k+1,\infty},$$

thus

$$|T|\leq\frac{1}{8}|||\xi|||^2+Ch^{2k+2}||J||_{k+1,\infty}\times[C(\Omega)h^{-1}+h^2],$$

and hence

$$|T|\leq Ch^{2k+1}+\frac{1}{8}|||\xi|||^2,$$

and the proof is complete □

The analogue of this method is valid for the Fokker-Planck equation where the discrete problem is: Find $J^h\in\tilde{\mathcal{W}}_h$ such that $\forall v\in\tilde{\mathcal{W}}_h$,

$$B(J^h, v) = (J^h_\beta, bv + \kappa v_\beta) - (\varepsilon AJ, bv + \kappa v_\beta) + \sum_{\tau \in \mathcal{T}_h} \int_{\partial\tau_-(\beta)' \times I_z} [J^h](bv_+)|\mathbf{n}\cdot\beta|$$
$$+ < J_+, bv_+ >_0 + < J, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}} = < f, bv_+ >_0 + < f, \varepsilon\Lambda(z_0)v >_{\Gamma_{\pm z_0}}$$

Introducing the norm

$$|||v|||_b^2 = \frac{1}{2}\left[\|\kappa^{1/2}v_\beta\|^2 + \|(\varepsilon a)^{1/2}\frac{\partial}{\partial z}(bv)\|^2 + \Lambda(z_0)|\varepsilon^{1/2}v|^2_{\Gamma_{\pm z_0}}\right]$$
$$+\frac{1}{2}\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau_-(\beta)'\times I_z}|b^{1/2}[v]|^2|\mathbf{n}\cdot\beta| + \frac{1}{2}\int_{\Gamma^+_{\pm z_0}\times I_z} bv^2|\mathbf{n}\cdot\beta|,$$

we have the following stability and convergence results for the Fokker-Planck equation:

Lemma 5.2. *For B and $|||\cdot|||_b$ as above we have*

$$B(v,v) \geq |||v|||_b^2.$$

Theorem 5.2. *For the discontinuous Galerkin solution J^h for the Fokker-Planck equation we have the error estimate*

$$|||J - J^h|||_b \leq C(\Omega)h^{k+1/2}\|J\|_{k+1}.$$

The proofs are again straightforward, however, lenghty and similar to those of this and the previous Sections and therefore are omitted.

Conclusion. Our analyses extend the results of [1] to a degenerate type convection-dominated convection-diffusion problem with a small and variable diffusion coefficient. The results in this paper are affected by the degenerate character of the equation and also by the absence of a pure current term in the original problem characterizing the beam, therefore the convergence rates are given in different weighted stability norms with no explicit L_2-norm control. As we mentioned earlier, either using a version of the Poincare's inequality, (Lemma 2.2), or a simple estimate as in Lemma 4.2, (relevant for the discontinuous cases), we can include L_2-norms in our stability estimates. However, using Lemma 2.2, would not lead to any better covergence rate for the L_2-norms than what we have for our weighted semi-norms. Lemma 4.2 improves the L_2-norm controls in much more involved weighted norms. These results are of order $\mathcal{O}(h^{k+1/2})$ and sharp in the sense that omitting any power of the diffusion coefficient on the left hand side of our stability norms will cause the same amount of reduced convergence rate. □

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