THE DISCRETE ORDINATES METHOD FOR THE NEUTRON TRANSPORT EQUATION IN AN INFINITE CYLINDRICAL DOMAIN*

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Abstract. We prove a regularity result for a Fredholm integral equation with weakly singular kernel, arising in connection with the neutron transport equation in an infinite cylindrical domain. The theorem states that the solution has almost two derivatives in L_1, and is proved using Besov space techniques. This result is applied in the error analysis of the discrete ordinates method for the numerical solution of the neutron transport equation. We derive an error estimate in the L_1-norm for the scalar flux, and as a consequence, we obtain an error bound for the critical eigenvalue.

1. Introduction. We consider the numerical solution of the steady state isotropic mono-energetic neutron transport equation in a cylindrical domain in $\mathbf{R}^3$ with a polygonal cross section Ω. The restriction to the mono-energetic case means that we assume that the velocity space is the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$. The cylindrical symmetry reduces the problem to $\mathbf{R}^2$ by projection along the axis of the cylinder. Thus we study the neutron transport equation in a bounded polygonal domain $\Omega \subset \mathbf{R}^2$ with the velocity space equal to the unit disc $\mathbf{D} \subset \mathbf{R}^2$.

We analyze a semidiscrete numerical method, the discrete ordinates method, involving the discretization of a weighted integral over $\mathbf{D}$, described in polar coordinates, by means of an N-point Gaussian quadrature rule in the radial variable, and a uniform M-point quadrature rule in the angular variable.

For this method we give an L_1 error estimate for the scalar flux of order $N^{-4} + M^{-2+\epsilon}$, as well as an error bound of the same order for the critical eigenvalue. In order to prove these estimates we apply an error bound for weighted-L_1 polynomial interpolation due to De Vore and Scott [7], together with a new L_1 regularity result for the scalar flux. Loosely speaking, the latter result states that the scalar flux, which is the solution of a Fredholm integral equation with weakly singular kernel, has almost two derivatives in L_1. This is proved using Besov space techniques. The limited regularity of the solution strongly affects the error analysis. Although our main concern is L_1, we also derive some results in L_∞

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(regularity and error bounds), which are needed in a duality argument in the proof of the error estimate for the critical eigenvalue.

Problems of this type have been studied in various settings by several authors. The slab geometry, $\Omega \subset \mathbf{R}^1$ and velocity space $[-1,1]$, was considered by Pitkäranta and Scott [11], who proved L_p and eigenvalue estimates for both semidiscrete and fully discrete schemes. Two-dimensional geometry, $\Omega \subset \mathbf{R}^2$ and velocities in the unit circle $\mathbf{S}^1$, was considered by Johnson and Pitkäranta [8] and Asadzadeh [2]. In [8] semidiscrete and fully discrete schemes were analyzed in L_2, whereas [2] contains L_p and eigenvalue estimates for the discrete ordinates method.

In Asadzadeh [3] the discrete ordinates method was studied in L_2 in a fully three-dimensional setting, $\Omega \subset \mathbf{R}^3$ and velocity space $\mathbf{S}^2$. In Asadzadeh [1] the geometry is the same as in the present work, but the analysis takes place in L_2, this norm being more suitable for spatial discretization based on the finite element method. Due to the limited regularity of the exact solution, the error bound in [1] for the angular discretization was $O(N^{-1}+M^{-1})$. By using the L_1 norm together with our new regularity result (and a better analysis of the Gauss quadrature), we are able to improve this to $O(N^{-4}+M^{-2+\epsilon})$. Thus, due to regularity limitations, an error estimate in the L_1 norm for functions yields a better error bound for the critical eigenvalue. Moreover, the L_1 norm is the most relevant norm from a physical point of view, since the scalar flux represents a particle density.

The outline of the paper is as follows: In Section 2 we describe the continuous problem that we want to solve, and in Section 3 we formulate the semidiscrete approximation and state our main result. Section 4 is devoted to an analysis of the regularity of the scalar flux. Section 5 contains the error analysis for the scalar flux, and in Section 6 the results obtained in the previous section are used to prove an error bound for the critical eigenvalue.

2. The continuous problem. We consider the following model problem for one-velocity neutron transport in an infinite cylinder $\tilde{\Omega} \subset \mathbf{R}^3$ with boundary $\tilde{\Gamma}$:

$$\begin{aligned} &\mu \cdot \nabla u(x,\mu) + u(x,\mu) = \lambda \int_{\mathbf{S}^2} u(x,\eta)\, d\eta + f(x), \qquad (x,\mu) \in \tilde{\Omega} \times \mathbf{S}^2, \\ &u(x,\mu) = 0, \qquad (x,\mu) \in \tilde{\Gamma}_\mu^- \times \mathbf{S}^2. \end{aligned} \tag{2.1}$$

Here λ is a positive parameter and $u(x,\mu)$ is the density of neutrons at the point $x \in \tilde{\Omega}$ flowing in the direction $\mu \in \mathbf{S}^2 = \{\mu \in \mathbf{R}^3 : |\mu| = 1\}$. The boundary condition is specified on the inflow boundary

$$\tilde{\Gamma}_\mu^- = \{x \in \tilde{\Gamma} : \mu \cdot \hat{n}(x) < 0\}, \tag{2.2}$$

where $\hat{n}(x)$ is the unit outward normal.

We assume that the cross-section of the cylinder $\tilde{\Omega}$ is a bounded convex polygonal domain $\Omega \subset \mathbf{R}^2$ with boundary Γ. Assuming also that the source term is constant along the axial direction of the cylinder, we may project the integro-differential equation (2.1) onto the cross-section Ω:

$$\begin{aligned} &\mu \cdot \nabla u(x,\mu) + u(x,\mu) = \lambda \int_{\mathbf{D}} u(x,\eta)\,(1-|\eta|^2)^{-1/2}\, d\eta + f(x), \qquad (x,\mu) \in \Omega \times \mathbf{D}, \\ &u(x,\mu) = 0, \qquad (x,\mu) \in \Gamma_\mu^- \times \mathbf{D}, \end{aligned} \tag{2.3}$$

where now the velocities μ vary over the unit disc $\mathbf{D} = \{\mu \in \mathbf{R}^2 : |\mu| \le 1\}$ and Γ_μ^- is defined analogously to (2.2).

Let us introduce the so-called scalar flux U defined by

$$U(x) = \int_{\mathbf{D}} u(x, \mu)\,(1 - |\mu|^2)^{-1/2}\,d\mu. \tag{2.4}$$

In order to derive an equation for U, we consider the following hyperbolic partial differential equation: given $g = g(x)$ find $v(x, \mu)$ such that

$$\begin{aligned} \mu \cdot \nabla v + v &= g, \qquad (x, \mu) \in \Omega \times \mathbf{D}, \\ v &= 0, \qquad (x, \mu) \in \Gamma_\mu^- \times \mathbf{D}. \end{aligned} \tag{2.5}$$

The solution is

$$v(x, \mu) = (T_\mu g)(x) = \int_0^{d(x,\mu)/|\mu|} e^{-s} g(x - s\mu)\,ds, \tag{2.6}$$

where $d(x, \mu)$ is the distance from x to the inflow boundary in the direction $-\mu$:

$$d(x, \mu) = \inf\{s > 0 : x - s\,\frac{\mu}{|\mu|} \notin \Omega\}.$$

Using (2.3) and (2.6) we obtain the following equation for U:

$$(I - \lambda T)U = Tf, \tag{2.7}$$

where

$$Tg(x) = \int_{\mathbf{D}} T_\mu g(x)\,(1 - |\mu|^2)^{-1/2}\,d\mu. \tag{2.8}$$

We show in Section 4 below that T is a compact operator on $L_1(\Omega)$ by rewriting it as an integral operator with weakly singular kernel. The Fredholm integral equation (2.7) thus has a unique solution $U \in L_1(\Omega)$ for any $f \in L_1(\Omega)$ as long as λ^{-1} is not an eigenvalue of T. The solution u of (2.3) can then be computed from (2.6) with $g = \lambda U + f$.

3. The semidiscrete problem. Our aim in this section is to formulate a quadrature rule for the integral in (2.4) or (2.8) using a certain set of quadrature points $\Delta = \{\mu_1, \dots, \mu_n\} \subset \mathbf{D}$ and weights ω_μ, $\mu \in \Delta$, thereby obtaining a semidiscrete approximation U_n of the scalar flux U. Thus, if

$$\int_{\mathbf{D}} v(\mu)\,(1 - |\mu|^2)^{-1/2}\,d\mu \approx \sum_{\mu \in \Delta} v(\mu)\omega_\mu \equiv Q[v(\mu)] \tag{3.1}$$

is this quadrature rule, then we define U_n by

$$(I - \lambda T_n)U_n = T_n f, \tag{3.2}$$

where

$$T_n g(x) = \sum_{\mu \in \Delta} \omega_\mu T_\mu g(x).$$

Using polar coordinates $\mu = r\hat{\mu}(\varphi)$, $\hat{\mu}(\varphi) = (\cos\varphi, \sin\varphi)$, we want to define two quadrature rules

$$\int_0^1 v(r) \frac{r\,dr}{\sqrt{1-r^2}} \approx \sum_{k=1}^N A_k v(r_k) \equiv Q_r[v(r)], \tag{3.3}$$

$$\int_0^{2\pi} v(\varphi)\,d\varphi \approx \sum_{j=1}^M W_j v(\varphi_j) \equiv Q_\varphi[v(\varphi)], \tag{3.4}$$

and the quadrature rule in (3.1) will then be

$$Q[v(\mu)] = Q_\varphi[Q_r[v(r\hat{\mu}(\varphi))]]$$

with

$$\Delta = \{r_k \hat{\mu}(\varphi_j)\}_{k=1}^{N}{}_{j=1}^{M}, \quad \omega_{kj} = A_k W_j.$$

For convenience we introduce the notation

$$n = NM, \quad \bar{n} = \min(N, M).$$

Thus n is the total number of quadrature points, and if $\bar{n}$ is large then both N and M are large.

For the angular integral in (3.4) we choose the trapezoidal rule:

$$\varphi_j = \frac{2\pi j}{M}, \quad W_j = \frac{2\pi}{M}, \qquad j = 1, \dots, M. \tag{3.5}$$

For the radial integral in (3.3) we choose the Gauss rule associated with the measure $d\alpha(r) = (1-r^2)^{-1/2} r\,dr$:

$$r_k = \sin\theta_k, \quad A_k = \alpha(s_k) - \alpha(s_{k-1}), \qquad k = 1, \dots, N, \tag{3.6}$$

where $\alpha(r) = -\sqrt{1-r^2}$, and θ_k, s_k are certain points satisfying

$$\begin{aligned} &\theta_k \in \left[\frac{(2k-1)\pi}{4N+2}, \frac{2k\pi}{4N+2}\right], \\ &s_k \in (r_k, r_{k+1}), \quad s_0 = 0, \ s_N = 1, \end{aligned} \tag{3.7}$$

see Szegő [13], pp. 47–50.

In Lemma 5.7 below we demonstrate that equation (3.2) has a unique solution $U_n \in L_1(\Omega)$ for any $f \in L_1(\Omega)$, provided that λ^{-1} is not an eigenvalue of T and that $\bar{n}$ is large. The main result of this paper is the following error bound for U_n. Its proof will be given in Section 5.

Theorem 3.1. *Assume that λ^{-1} is not an eigenvalue of T. For each $\epsilon > 0$ there is C such that, for large $\bar{n}$,*

$$\|U - U_n\|_{L_1(\Omega)} \leq C\left(\frac{1}{N^4} + \frac{1}{M^{2-\epsilon}}\right)\left(\|U\|_{L_1(\Omega)} + \|f\|_{W_1^1(\Omega)}\right).$$

4. Regularity. In this section we study the mapping properties of the operator T defined in (2.8), which can be rewritten as

$$Tg(x) = \int_\Omega \int_0^{\frac{\pi}{2}} \frac{\exp(-|x-y|/\cos t)}{|x-y|}\, dt\, g(y)\, dy, \tag{4.1}$$

where Ω is our bounded polygonal domain in $\mathbf{R}^2$ and $g \in L_1(\Omega)$. Hence T can be viewed as an integral operator with a weakly singular kernel. We recall that T is associated with the scalar flux $U(x)$ via the Fredholm integral equation

$$U = \lambda T U + Tf. \tag{4.2}$$

Our main result in this section is the following estimate:

Theorem 4.1. *Let $f \in W_1^1(\Omega)$ and let $U \in L_1(\Omega)$ be a solution of (4.2). Then for all $\theta \in [0,1)$ and all $q \in [1,\infty]$ there is a constant $C = C(\lambda,\theta,q)$ such that*

$$\|U\|_{(L_1(\Omega),W_1^2(\Omega))_{\theta,q}} \leq C\left(\|U\|_{L_1(\Omega)} + \|f\|_{W_1^1(\Omega)}\right).$$

Here $(L_1(\Omega), W_1^2(\Omega))_{\theta,q}$ denotes the real interpolation space between $L_1(\Omega)$ and $W_1^2(\Omega)$, see Bergh and Löfström [5]. The value of the exponent q is irrelevant for our discussion and we let $q \in [1,\infty]$ be fixed throughout this section. Loosely speaking this result means that U has almost two derivatives in L_1. Since there is no restriction on λ, the result is also applicable to eigenfunctions of T. Furthermore we prove:

Theorem 4.2. *The operator T is a bounded linear operator from $L_1(\Omega)$ into $B_1^{s,q}(\Omega)$ for all $s \in [0,1)$, and from $L_\infty(\Omega)$ into $(L_\infty(\Omega), W_\infty^1(\Omega))_{\theta,q}$ for all $\theta \in [0,1)$.*

The Banach spaces $B_1^{s,q}(\Omega)$ appearing in the theorem are Besov spaces. We recall their definitions below. Employing the fact that $B_1^{s,q}(\Omega)$ is compactly imbedded in $L_1(\Omega)$ for $s > 0$, we conclude:

Corollary. *T is a compact operator on $L_1(\Omega)$.*

We remark that the regularity of solutions to (4.1) was also investigated by Pitkäranta [10] in terms of weighted Hölder spaces instead of Sobolev spaces as in the present work.

The main steps in the proof of Theorems 4.1 and 4.2 are the following:

(1) identifying T as a convolution operator on $\mathbf{R}^2$;
(2) imbeddings between interpolation spaces $(W_1^k, W_1^l)_{\theta,q}$ and Besov spaces;
(3) multilinear interpolation;
(4) a bootstrap argument.

Step 1. Set $h_\tau(x) = \exp(-|x|/\tau)/|x|$ for $x \in \mathbf{R}^2$, $\tau > 0$ and note that

$$Tg(x) = \int_\Omega \int_0^{\frac{\pi}{2}} \frac{\exp(-|x-y|/\cos t)}{|x-y|}\, dt\, g(y)\, dy = \int_0^{\frac{\pi}{2}} [h_{\cos t} * (\chi_\Omega\, g)](x)\, dt, \tag{4.3}$$

where χ_Ω is the characteristic function of the set $\Omega \subset \mathbf{R}^2$ and $*$ denotes convolution in $\mathbf{R}^2$. The functions $Tg(x)$ and $h_\tau(x)$ are defined for all $x \in \mathbf{R}^2$ and in the sequel we shall think of all functions in $L_1(\Omega)$ as being defined on $\mathbf{R}^2$ (extended by 0 on $\mathbf{R}^2 \setminus \Omega$ unless otherwise stated). We write $L_1 = L_1(\mathbf{R}^2)$, $B_1^{s,q} = B_1^{s,q}(\mathbf{R}^2)$, etc. In particular, by $(W_1^k, W_1^l)_{\theta,q}$ we mean $(W_1^k(\mathbf{R}^2), W_1^l(\mathbf{R}^2))_{\theta,q}$ although the final result refers to $(W_1^k(\Omega), W_1^l(\Omega))_{\theta,q}$. The justification for this is the existence of continuous linear extension operators

$$W_1^k(\Omega) \to W_1^k(\mathbf{R}^2), \qquad k \in \mathbf{N}, \tag{4.4}$$

(see Stein [12]), which are defined for bounded Lipschitz domains, and in particular, for the domain Ω above.

Step 2. All explicit norm calculations will be done in Besov spaces. The reason for this is that they have nice intrinsic norm definitions

$$\|f\|_{B_1^{s,q}} = \sum_{|\alpha|\le[s]} \|D^\alpha f\|_{L_1} + \sum_{|\alpha|=[s]} \left(\int_0^1 \left(t^{[s]-s}\omega(D^\alpha f)(t)\right)^q \frac{dt}{t}\right)^{1/q},$$

for $q \in [1,\infty]$ and noninteger $s > 0$, where $[s]$ denotes the integer part of s and

$$\omega(h)(t) = \sup_{|\eta|\le t} \|h(\cdot+\eta) - h\|_{L_1},$$

and with the usual modification for $q = \infty$, see Bergh and Löfström [5]. Moreover, they are interpolation spaces, i.e.,

$$(B_1^{s_1,q_1}, B_1^{s_2,q_2})_{\theta,q} = B_1^{\theta s_2 + (1-\theta)s_1, q},$$

for $0 \le \theta \le 1$, $s_1, s_2 > 0$, and they approximate the Sobolev spaces W_1^k in the sense that

$$B_1^{k+\epsilon,q} \subset W_1^k \subset B_1^{k-\epsilon,q},$$

for small $\epsilon > 0$, $k \in \mathbf{Z}_+$, see Nikolskii [9]. This trivially implies the imbeddings

$$B_1^{\theta l+(1-\theta)k+\epsilon,q_1} \subset (W_1^k, W_1^l)_{\theta,q} \subset B_1^{\theta l+(1-\theta)k-\epsilon,q_2}, \tag{4.5}$$

for small $\epsilon > 0$, $0 \le \theta \le 1$, $k, l \in \mathbf{N}$. We also note the Sobolev imbedding

$$B_1^{s,q} \subset L_p, \qquad \text{for } 1 - \frac{s}{2} < \frac{1}{p}. \tag{4.6}$$

In the argument below we need the following lemmas.

Lemma 4.3. *We have*

$$\sup_{\tau\in(0,1]} \|h_\tau\|_{B_1^{s,q}} < \infty, \qquad \text{for } s \in [0,1).$$

Proof. Recall that $h_\tau(x) = \exp(-|x|/\tau)/|x|$. Hence we have

$$\|h_\tau\|_{L_1} = \tau\|h_1\|_{L_1} \le C, \qquad \text{for } \tau \in (0,1],$$

and similarly

$$\omega(h_\tau)(t) = \sup_{|\eta|\le t} \|h_\tau(\cdot+\eta) - h_\tau\|_{L_1} = \tau \sup_{|\eta|\le t/\tau} \|h_1(\cdot+\eta) - h_1\|_{L_1} = \tau\omega(h_1)(t/\tau),$$

so that

$$\left(\int_0^1 \big(t^{-s}\omega(h_\tau)(t)\big)^q \,\frac{dt}{t}\right)^{1/q} = \tau^{1-s}\left(\int_0^{1/\tau} \big(t^{-s}\omega(h_1)(t)\big)^q \,\frac{dt}{t}\right)^{1/q}. \tag{4.7}$$

We shall show that

$$\omega(h_1)(t) \le C\min\big(t(1+|\log t|),1\big). \tag{4.8}$$

Then the expression in (4.7) is bounded by

$$\tau^{1-s}\left(\int_0^{1/\tau} \big(t^{-s}\omega(h_1)(t)\big)^q \,\frac{dt}{t}\right)^{1/q} \le C_{s,q},$$

for $0 < s < 1$, and the desired result follows. For the proof of (4.8) we write

$$\begin{aligned} h_1(x+\eta) - h_1(x) &= e^{-|x+\eta|}\Big(\frac{1}{|x+\eta|} - \frac{1}{|x|}\Big) + \big(e^{-|x+\eta|} - e^{-|x|}\big)\frac{1}{|x|} \\ &\equiv g_1(x,\eta) + g_2(x,\eta). \end{aligned}$$

Setting, for fixed η,

$$R = R(\eta) = \max\big(1, 3|\eta|, \frac{1}{|\eta|}\big),$$

we have

$$\|g_1(\cdot,\eta)\|_{L_1} = \int_{A_1} |g_1(x,\eta)|\,dx + \int_{A_2} |g_1(x,\eta)|\,dx,$$

with

$$A_1 = \{x \in \mathbf{R}^2 : 2|\eta| \le |x| \le R\}, \quad A_2 = \mathbf{R}^2 \setminus A_1.$$

Straightforward calculations give

$$\int_{A_1} |g_1(x,\eta)|\,dx \le C|\eta|\big(1+\big|\log|\eta|\big|\big)$$

and

$$\int_{A_2} |g_1(x,\eta)|\,dx \le C|\eta|.$$

Moreover, we obtain

$$\begin{aligned}\|g_2(\cdot,\eta)\|_{L_1} &\le \int_{|x|\le R} \frac{1}{|x|}|e^{-|x+\eta|}-e^{-|x|}|\,dx + C|\eta| \\ &\le C\int_{|x|\le R} \frac{e^{-|x|}}{|x|}\min(1,e^{|\eta|}-1)\,dx + C|\eta| \le C|\eta|.\end{aligned}$$

Together these estimates prove (4.8). □

Lemma 4.4. *Let $s\in(0,1)$ and let Ω be the polygonal domain above. Then the following estimates are true:*

$$\|\chi_\Omega\, g\|_{B_1^{s,q}} \le C\|g\|_{B_1^{2s+\epsilon,q}}, \qquad \epsilon>0, \tag{4.9}$$

$$\|\chi_\Omega\, g\|_{B_1^{s,q}} \le C\|g\|_{W_1^1}, \tag{4.10}$$

where C depends only on s,q,ϵ and s,q, respectively.

Proof. To prove the first estimate we note that

$$\omega(\chi_\Omega\, g)(t) \le \omega(g)(t) + \sup_{|\eta|\le t}\int_{\mathbf{R}^2} |g(x)|\,|\chi_\Omega(x+\eta)-\chi_\Omega(x)|\,dx.$$

Hölder's inequality applied to the second term on the right-hand side together with the Sobolev imbedding (4.6) yields ($\frac{1}{p'}\equiv s+\delta = 1-\frac{1}{p}$)

$$\begin{aligned}\omega(\chi_\Omega\, g)(t) &\le \omega(g)(t) + \|g\|_{L_p}\sup_{|\eta|\le t}\Big(\int_{\mathbf{R}^2}|\chi_\Omega(x+\eta)-\chi_\Omega(x)|^{p'}\,dx\Big)^{1/p'} \\ &\le \omega(g)(t) + Ct^{1/p'}\|g\|_{L_p} \le \omega(g)(t) + Ct^{s+\delta}\|g\|_{B_1^{2s+\epsilon,q}},\end{aligned}$$

for any $\epsilon>0,\ 0<\delta<\epsilon/2$. This proves (4.9).

The estimate (4.10) follows from

$$\|f\|_{B_1^{s,q}} \le C\big(\|f\|_{L_1} + \sum_{i=1}^{2}\|\partial_{x_i} f\|_{B_1^{s-1,q}}\big) \tag{4.11}$$

applied to $f = \chi_\Omega g$ and the estimate

$$\|\partial_{x_i}(\chi_\Omega g)\|_{B_1^{s-1,q}} \le C\|g\|_{W_1^1}, \qquad i = 1,2. \tag{4.12}$$

Proof of (4.12): For $g \in W_1^1$ we have by the divergence theorem that

$$\partial_{x_i}(\chi_\Omega g) = \chi_\Omega \partial_{x_i} g - g n_i \, dS, \qquad \text{in } \mathcal{D}'(\mathbf{R}^2), \tag{4.13}$$

where $n = (n_1, n_2)$ is the unit outer normal on Γ and dS denotes the arclength on Γ. Here we note that $gn_i \, dS$ is well-defined in $\mathcal{D}'(\mathbf{R}^2)$, since g has an L_1-trace on Γ. The first term on the right-hand side is an L_1-function. For the second term we use an equivalent form of the Besov norm,

$$\|f\|_{B_1^{s,q}} = \left(\sum_{j=0}^{\infty} (2^{sj}\|\mathcal{F}^{-1}(\phi_j \mathcal{F} f)\|_{L_1})^q\right)^{1/q},$$

where $s \in \mathbf{R}$, $\mathcal{F}$ denotes the Fourier transform and

$$\phi_j(\xi) = \phi(2^{-j}\xi), \qquad j = 1, 2, \dots,$$
$$\phi_0(\xi) = 1 - \sum_{j=1}^{\infty} \phi_j(\xi),$$

where $\phi \in C_0^\infty(\mathbf{R}^2)$ satisfies

$$\text{i) } \operatorname{supp}\phi = \{\xi : \tfrac{1}{2} \le |\xi| \le 2\},$$
$$\text{ii) } \phi > 0, \qquad \text{in } \{\xi : \tfrac{1}{2} < |\xi| < 2\},$$
$$\text{iii) } \sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1, \qquad \text{for } \xi \ne 0.$$

To prove

$$\|gn_i \, dS\|_{B_1^{s-1,q}} \le C\|g\|_{W_1^1}$$

it suffices to show

$$\|\mathcal{F}^{-1}(\phi_j \mathcal{F}(gn_i \, dS))\|_{L_1} \le C\|g\|_{W_1^1}, \qquad j = 0, 1, 2, \dots,$$

with C independent of j. The latter bound follows from

$$\begin{aligned}
\|\mathcal{F}^{-1}(\phi_j \mathcal{F}(gn_i \, dS))\|_{L_1} &= \| \int_\Gamma 2^{2j}(\mathcal{F}^{-1}\phi)(2^j(\cdot - y))\,(gn_i \, dS)(y)\|_{L_1} \\
&\le \int_{\mathbf{R}^2}\int_\Gamma 2^{2j}|\mathcal{F}^{-1}\phi(2^j(x-y))|\,|g(y)|\,dS(y)\,dx \\
&\le \int_{\mathbf{R}^2} |\mathcal{F}^{-1}\phi(x)|\,dx \int_\Gamma |g(y)|\,dS(y) \\
&\le C\|g\|_{L_1(\Gamma)} \le C\|g\|_{W_1^1}, \qquad j = 1, 2, \dots,
\end{aligned}$$

where in the last step we used the trace estimate. The case $j = 0$ follows in a similar way and completes the proof of (4.12).

Proof of (4.11): For $i = 1, 2$ set

$$\begin{aligned} \psi_i(\xi) &= \xi_i \phi(\xi), \\ \psi_{i,j}(\xi) &= \psi_i(2^{-j}\xi), \qquad j = 1, 2, \dots, \\ \psi_{i,0}(\xi) &= \xi_i - \sum_{j=1}^{\infty} \psi_{i,j}(\xi). \end{aligned}$$

With this notation we have

$$\|\partial_{x_i} f\|_{B_1^{s-1,q}} = \Big(\sum_{j=0}^{\infty} (2^{sj} \|\mathcal{F}^{-1}(\psi_{i,j}\mathcal{F}f)\|_{L_1})^q \Big)^{1/q}, \qquad i = 1, 2,$$

and so it suffices to show

$$\|\mathcal{F}^{-1}(\phi_j \mathcal{F}f)\|_{L_1} \le C \sum_{i=1}^{2} \|\mathcal{F}^{-1}(\psi_{i,j}\mathcal{F}f)\|_{L_1}, \qquad j = 1, 2, \dots, \tag{4.14}$$

and

$$\|\mathcal{F}^{-1}(\phi_0 \mathcal{F}f)\|_{L_1} \le C \|f\|_{L_1}. \tag{4.15}$$

Inequality (4.14) is equivalent to (not the same f)

$$\|\mathcal{F}^{-1}(\phi \mathcal{F}f)\|_{L_1} \le C \sum_{i=1}^{2} \|\mathcal{F}^{-1}(\psi_i \mathcal{F}f)\|_{L_1}.$$

Let $\chi_i \in C_0^\infty(\mathbf{R}^2)$, $\chi_i \ge 0$, be supported away from the ξ_i-axis for $i = 1, 2$, and such that $\chi_1 + \chi_2 = 1$ on $\operatorname{supp}\phi$. We obtain

$$\begin{aligned} \|\mathcal{F}^{-1}(\phi \mathcal{F}f)\|_{L_1} &\le \sum_{i=1}^{2} \|\mathcal{F}^{-1}(\frac{1}{\xi_i}\chi_i \psi_i \mathcal{F}f)\|_{L_1} \\ &= \sum_{i=1}^{2} \|\mathcal{F}^{-1}(\frac{1}{\xi_i}\chi_i) * \mathcal{F}^{-1}(\psi_i \mathcal{F}f)\|_{L_1} \\ &\le C \sum_{i=1}^{2} \|\mathcal{F}^{-1}(\psi_i \mathcal{F}f)\|_{L_1}. \end{aligned}$$

Finally for (4.15) we find

$$\|\mathcal{F}^{-1}(\phi_0 \mathcal{F}f)\|_{L_1} = \|(\mathcal{F}^{-1}\phi_0) * f\|_{L_1} \le C \|f\|_{L_1},$$

and (4.11) is proved. □

Remark. Actually the right-hand side of of (4.11) is an equivalent norm for $B_1^{s,q}$.

Step 3. Consider the bilinear mapping $(f_1, f_2) \mapsto f_1 * f_2$. We observe that $D^\alpha(f_1 * f_2) = (D^{\alpha_1} f_1) * (D^{\alpha_2} f_2)$ for any multi-index $\alpha = \alpha_1 + \alpha_2$. This yields

Lemma 4.5. *The following estimates hold true:*

$$\|f_1 * f_2\|_{B_1^{s,q}} \leq C\|f_1\|_{B_1^{s_1,q}} \|f_2\|_{B_1^{s_2,q}}, \quad 0 < s < s_1 + s_2, \tag{4.16}$$

$$\|f_1 * f_2\|_{(L_\infty, W_\infty^k)_{\theta,q}} \leq C\|f_1\|_{(L_\infty, W_\infty^k)_{\theta,q}} \|f_2\|_{L_\infty}, \quad k \geq 0,\ 0 \leq \theta \leq 1. \tag{4.17}$$

Proof. For fixed s, s_1, s_2 satisfying the hypotheses in (4.16) we choose $k > s_1, l > s_2$ and $\bar{\theta} \in (0,1)$ such that $k\bar{\theta} < s_1$, $l\bar{\theta} < s_2$, $(k+l)\bar{\theta} > s$. Multilinear interpolation applied to

$$\|f_1 * f_2\|_{W_1^{k+l}} \leq \|f_1\|_{W_1^k} \|f_2\|_{W_1^l},$$
$$\|f_1 * f_2\|_{L_1} \leq \|f_1\|_{L_1} \|f_2\|_{L_1},$$

yields

$$\|f_1 * f_2\|_{(L_1, W_1^{k+l})_{\theta,q}} \leq C\|f_1\|_{(L_1, W_1^k)_{\theta,q}} \|f_2\|_{(L_1, W_1^l)_{\theta,q}},$$

for $\theta \in [0,1]$. An application of (4.5) now proves (4.16). The remaining estimate (4.17) is proved analogously. □

We can now prove Theorem 4.2. Let $s \in [0,1)$ and choose $s_1 \in (s,1)$. Using the representation (4.3) and Lemma 4.5 together with Lemma 4.3 we obtain

$$\begin{aligned} \|Tg\|_{B_1^{s,q}(\Omega)} &\leq \|Tg\|_{B_1^{s,q}} \leq \int_0^{\frac{\pi}{2}} \|h_{\cos t} * (\chi_\Omega\, g)\|_{B_1^{s,q}}\, dt \\ &\leq C \int_0^{\frac{\pi}{2}} \|h_{\cos t}\|_{B_1^{s_1,q}}\, dt\, \|\chi_\Omega\, g\|_{L_1} \leq C\|g\|_{L_1(\Omega)}, \end{aligned} \tag{4.18}$$

which is the first claim of Theorem 4.2. The second assertion is proved in the same way using (4.17) instead of (4.16).

Step 4. We now prove Theorem 4.1 by means of Theorem 4.2 and a bootstrap argument. Consider the integral equation (4.2), i.e.,

$$U = \lambda TU + Tf.$$

Assume that $f \in W_1^1(\Omega)$ and that $U \in L_1(\Omega)$ is a solution of (4.2). We shall show that

$$\|U\|_{B_1^{s,q}(\Omega)} \leq C\big(\|U\|_{L_1(\Omega)} + \|f\|_{W_1^1(\Omega)}\big), \qquad \text{for } 0 \leq s < 2, \tag{4.19}$$

from which the desired bound follows in view of the imbedding (4.5). If $0 \leq s < 1$, then (4.19) follows directly from (4.18). Thus let $1 \leq s < 2$. Let $\tilde{U}$ denote the extension of U

to $\mathbf{R}^2$ by $\lambda TU + Tf$. Extend $f \in W_1^1(\Omega)$ to $\tilde{f} \in W_1^1(\mathbf{R}^2)$ as in (4.4). Using again the representation (4.3) together with the bounds in Lemmas 4.4 and 4.3 we obtain

$$
\begin{aligned}
\|\tilde{U}\|_{B_1^{s,q}} &\le C \int_0^{\frac{\pi}{2}} \|h_{\cos t}\|_{B_1^{1-\epsilon,q}}\, dt\, \|\chi_\Omega(\lambda\tilde{U} + \tilde{f})\|_{B_1^{s-1+\epsilon',q}} \\
&\le C\big(\|\chi_\Omega \tilde{U}\|_{B_1^{s-1+\epsilon',q}} + \|\chi_\Omega \tilde{f}\|_{B_1^{s-1+\epsilon',q}}\big) \\
&\le C\big(\|\tilde{U}\|_{B_1^{2(s-1+\epsilon')+\epsilon'',q}} + \|\tilde{f}\|_{W_1^1}\big),
\end{aligned}
\tag{4.20}
$$

for small $\epsilon, \epsilon', \epsilon'' > 0$. Choosing ϵ', ϵ'' such that

$$s - \big(2(s-1+\epsilon') + \epsilon''\big) > \frac{2-s}{2},$$

and repeating the argument in (4.20) a finite number of times we arrive at

$$\|\tilde{U}\|_{B_1^{s,q}} \le C\big(\|\tilde{U}\|_{B_1^{s_1,q}} + \|\tilde{f}\|_{W_1^1}\big),$$

with $0 < s_1 < 1$. Since $\|U\|_{B_1^{s,q}(\Omega)} \le \|\tilde{U}\|_{B_1^{s,q}}$, and since (4.18) shows

$$\|\tilde{U}\|_{B_1^{s_1,q}} \le C\big(\|U\|_{L_1(\Omega)} + \|f\|_{L_1(\Omega)}\big),$$

and by (4.4)

$$\|\tilde{f}\|_{W_1^1} \le C\|f\|_{W_1^1(\Omega)},$$

this completes the proof of (4.19) and Theorem 4.1 is proved.

5. Error analysis. The aim of this section is to prove Theorem 3.1. The proof is based on the observation that, since

$$(I - \lambda T)U = Tf, \quad (I - \lambda T_n)U_n = T_n f,$$

we have

$$(I - \lambda T_n)(U - U_n) = (T - T_n)(\lambda U + f).$$

We show in Lemma 5.7 below that $(I - \lambda T_n)^{-1}$ is uniformly bounded on $L_1(\Omega)$. This implies that

$$\|U - U_n\|_{L_1(\Omega)} \le C\|(T - T_n)(\lambda U + f)\|_{L_1(\Omega)}. \tag{5.1}$$

We therefore need to estimate $(T - T_n)g$, which can be viewed as the quadrature error

$$(T - T_n)g(x) = \int_{\mathbf{D}} v(x,\mu) \frac{d\mu}{\sqrt{1 - |\mu|^2}} - Q[v(x,\mu)], \quad \text{with } v(x,\mu) = T_\mu g(x).$$

This will also be the key to the proof of the boundedness of $(I - \lambda T_n)^{-1}$.

Using polar coordinates $\mu = r\hat{\mu}(\varphi)$, $\hat{\mu}(\varphi) = (\cos\varphi, \sin\varphi)$, we split the quadrature error as follows

(5.2)

$$
\begin{aligned}
(T - T_n)g(x) &= \int_0^{2\pi} \Big(\int_0^1 v(x, r\hat{\mu}(\varphi)) \frac{r}{\sqrt{1-r^2}}\, dr - Q_r[v(x, r\hat{\mu}(\varphi))]\Big)\, d\varphi \\
&\quad + Q_r\Big[\int_0^{2\pi} v(x, r\hat{\mu}(\varphi))\, d\varphi - Q_\varphi[v(x, r\hat{\mu}(\varphi))]\Big].
\end{aligned}
$$

We estimate the latter two terms in the following sequence of lemmas.

Lemma 5.1. *Let Q_φ be defined by the trapezoidal rule (3.5). Then*

$$\left|\int_0^{2\pi} u(\varphi)\,d\varphi - Q_\varphi[u(\varphi)]\right| \le \frac{C}{M^s}\int_0^{2\pi} |u^{(s)}(\varphi)|\,d\varphi, \qquad s = 1, 2.$$

Proof. This is a standard result. □

Lemma 5.2. *Let Q_r be defined by the Gauss rule (3.6). Then*

$$\left|\int_0^1 u(r)\frac{r}{\sqrt{1-r^2}}\,dr - Q_r[u(r)]\right| \le \frac{C_s}{N^s}\int_0^1 |u^{(s)}(r)|\big(r(1-r)\big)^{\frac{s-1}{2}}\,r\,dr, \qquad s = 1, 2, \dots.$$

Proof. We modify the argument of De Vore and Scott [7] so as to handle an integral with weight $w(r) = r/\sqrt{1-r^2}$. By Taylor's formula and using the fact that N-point Gauss quadrature is exact for polynomials of degree $\le 2N-1$, we can represent the error as

$$e[u] = \int_0^1 u(r)\frac{r}{\sqrt{1-r^2}}\,dr - Q_r[u(r)] = \int_0^1 u'(r)E(r)\,dr,$$

where

$$E(r) = \int_r^1 \frac{t}{\sqrt{1-t^2}}\,dt - \sum_{r_k > r} A_k.$$

We show below that

$$|E(r)| \le \frac{C}{N} r. \tag{5.3}$$

Hence

$$\big|e[u]\big| = \inf_{P\in\mathcal{P}_{2N-1}} \big|e[u-P]\big| \le \frac{C}{N}\inf_{P\in\mathcal{P}_{2N-2}} \int_0^1 |u'(r) - P(r)|\,r\,dr,$$

and our task is now reduced to a problem of weighted-L_1 polynomial approximation. Applying a general result from [7] we obtain

$$\inf_{P\in\mathcal{P}_{2N-2}} \int_0^1 |u'(r) - P(r)|\,r\,dr \le \frac{C_s}{N^{s-1}}\int_0^1 |u^{(s)}(r)|\big(r(1-r)\big)^{\frac{s-1}{2}}\,r\,dr,$$

which proves the desired result.

It remains to prove (5.3). Consider first the situation that $r \in [0, r_N]$. We may assume, without loss of generality, that $r_{l-1} < r < r_l$. Then, by (3.6), we have

$$E(r) = \sqrt{1-r^2} - \sum_{r_k > r} A_k = \sqrt{1-r^2} - \alpha(s_N) + \alpha(s_{l-1}) = \sqrt{1-r^2} - \sqrt{1-s_{l-1}^2},$$

and, since $r_{l-1} < s_{l-1} < r_l$, by Taylor expansion we get

$$|E(r)| \le \sqrt{1-r_{l-1}^2} - \sqrt{1-r_l^2} = \cos\theta_{l-1} - \cos\theta_l \le \frac{C}{N} r.$$

Finally, for $r \in [r_N, 1]$ we have

$$E(r) = \sqrt{1-r^2} \le \sqrt{1-r_N^2} \le \cos\frac{(2N-1)\pi}{4N+2} \le \frac{C}{N}. \qquad \square$$

We now need to consider the regularity of $v(x,\mu)$ with respect to r and φ.

Lemma 5.3. *Let S_i, $i = 1, ..., P$, denote the sides of the polygonal domain Ω and let $\hat{\tau}_i = (\cos\psi_i, \sin\psi_i)$ be a tangent to S_i. Let $v(\cdot,\mu) = T_\mu g$ with $\mu = r\hat{\mu}(\varphi)$, $\hat{\mu}(\varphi) = (\cos\varphi, \sin\varphi)$. Then for $0 \le r \le 1$, $0 \le \varphi \le 2\pi$, we have*

$$\left\|\frac{\partial v(\cdot,\mu)}{\partial\varphi}\right\|_{L_1(\Omega)} \le C\|g\|_{W_1^1(\Omega)}, \tag{5.4}$$

$$\left\|\frac{\partial v(\cdot,\mu)}{\partial\varphi}\right\|_{L_\infty(\Omega)} \le \frac{C}{\min_{1\le i\le P}|\sin(\varphi-\psi_i)|}\|g\|_{W_\infty^1(\Omega)}, \tag{5.5}$$

$$\left\|\frac{\partial^2 v(\cdot,\mu)}{\partial\varphi^2}\right\|_{L_1(\Omega)} \le \frac{C}{\min_{1\le i\le P}|\sin(\varphi-\psi_i)|}\|g\|_{W_1^2(\Omega)}, \tag{5.6}$$

$$\left\|\frac{\partial^2 v(\cdot,\mu)}{\partial r^2}\right\|_{L_\infty(\Omega)} \le \frac{C}{r^2}\|g\|_{L_\infty(\Omega)}. \tag{5.7}$$

$$\left\|\frac{\partial^4 v(\cdot,\mu)}{\partial r^4}\right\|_{L_1(\Omega)} \le \frac{C}{r^3}\|g\|_{W_1^1(\Omega)}, \tag{5.8}$$

Proof. In order to prove (5.4) we recall from (2.6) that $v(x,\mu) = T_\mu g(x)$ is given by

$$v(x,\mu) = \int_0^{d/r} e^{-s} g(x - sr\hat{\mu})\,ds, \tag{5.9}$$

where $d = d(x,\mu)$, $\mu = r\hat{\mu}$, $\hat{\mu} = (\cos\varphi, \sin\varphi)$. Making the transformation of variables $\sigma = sr$ we find that

$$v(x,\mu) = \frac{1}{r}\int_0^d e^{-\sigma/r} g(x - \sigma\hat{\mu})\,d\sigma, \tag{5.10}$$

so that

$$\frac{\partial v}{\partial\varphi} = \frac{1}{r}\frac{\partial d}{\partial\varphi}e^{-d/r}g(x - d\hat{\mu}) - \int_0^d \frac{\sigma}{r}e^{-\sigma/r}\frac{\partial\hat{\mu}}{\partial\varphi}\cdot\nabla g(x - \sigma\hat{\mu})\,d\sigma.$$

We now divide Ω into strips

$$\Omega_{\mu,i} = \{x \in \Omega : x - d\hat{\mu} \in S_i\},$$

where the S_i are the sides of Ω. In each $\Omega_{\mu,i}$ we use a transformation of coordinates

$$x = P_i + \xi_1\hat{\mu} + \xi_2\hat{\tau}_i, \qquad 0 \le \xi_1 \le B_i(\xi_2),\ 0 \le \xi_2 \le L_i,$$

where P_i is an endpoint of S_i and $\hat{\tau}_i = (\cos\psi_i, \sin\psi_i)$ is a tangent of S_i. Thus $d = \xi_1$ and we compute that

$$\frac{\partial d}{\partial\varphi} = \frac{\partial\xi_1}{\partial\varphi} = -\frac{\cos(\varphi-\psi_i)}{\sin(\varphi-\psi_i)}\xi_1.$$

Since the area element is $dx = |\sin(\varphi - \psi_i)|\, d\xi$ we thus have

$$\begin{aligned}
&\left\| \frac{\partial v(\cdot,\mu)}{\partial\varphi} \right\|_{L_1(\Omega_{\mu,i})} \\
&\le \int_0^{L_i} \int_0^{B_i(\xi_2)} \left| \frac{\cos(\varphi - \psi_i)}{\sin(\varphi - \psi_i)} \right| \frac{\xi_1}{r} e^{-\xi_1/r} |g(P_i + \xi_2\hat{\tau}_i)|\, |\sin(\varphi - \psi_i)|\, d\xi_1 d\xi_2 \\
&\quad + \int_0^{L_i} \int_0^{B_i(\xi_2)} \int_0^{\xi_1} \frac{\sigma}{r} e^{-\sigma/r} \left| \frac{\partial\hat{\mu}}{\partial\varphi} \right| |\nabla g(P_i + (\xi_1 - \sigma)\hat{\mu} + \xi_2\hat{\tau}_i)|\, d\sigma\, |\sin(\varphi - \psi_i)|\, d\xi_1 d\xi_2 \\
&\le C \left(\|g\|_{L_1(S_i)} + \|g\|_{W_1^1(\Omega_{\mu,i})} \right).
\end{aligned}$$

Using also the trace theorem and summing over i we obtain (5.4). The maximum norm bound (5.5) is proved in a similar way.

In order to prove (5.6) we differentiate $v(x,\mu)$ once more:

$$\begin{aligned}
\frac{\partial^2 v}{\partial\varphi^2} &= \left[\frac{1}{r}\frac{\partial^2\xi_1}{\partial\varphi^2} - \frac{1}{r^2}\left(\frac{\partial\xi_1}{\partial\varphi}\right)^2 \right] e^{-\xi_1/r} g(P_i + \xi_2\hat{\tau}_i) \\
&\quad + \frac{1}{r}\frac{\partial\xi_1}{\partial\varphi} e^{-\xi_1/r} \frac{\partial\xi_2}{\partial\varphi} \hat{\tau}_i \cdot \nabla g(P_i + \xi_2\hat{\tau}_i) \\
&\quad - \frac{\xi_1}{r}\frac{\partial\xi_1}{\partial\varphi} e^{-\xi_1/r} \frac{\partial\hat{\mu}}{\partial\varphi} \cdot \nabla g(P_i + \xi_2\hat{\tau}_i) \\
&\quad - \int_0^{\xi_1} \frac{\sigma}{r} e^{-\sigma/r} \frac{\partial^2\hat{\mu}}{\partial\varphi^2} \cdot \nabla g(P_i + (\xi_1 - \sigma)\hat{\mu} + \xi_2\hat{\tau}_i)\, d\sigma \\
&\quad + \int_0^{\xi_1} \frac{\sigma^2}{r} e^{-\sigma/r} \left(\frac{\partial\hat{\mu}}{\partial\varphi} \cdot \nabla \right)^2 g(P_i + (\xi_1 - \sigma)\hat{\mu} + \xi_2\hat{\tau}_i)\, d\sigma.
\end{aligned}$$

One easily computes that

$$\frac{\partial\xi_2}{\partial\varphi} = \frac{\xi_1}{\sin(\varphi - \psi_i)}, \qquad \frac{\partial^2\xi_1}{\partial\varphi^2} = \frac{1 + \cos^2(\varphi - \psi_i)}{\sin^2(\varphi - \psi_i)}\xi_1.$$

Hence

$$\begin{aligned}
\left\| \frac{\partial^2 v(\cdot,\mu)}{\partial\varphi^2} \right\|_{L_1(\Omega_{\mu,i})} &\le C \int_0^{L_i} \int_0^{B_i(\xi_2)} \frac{\left(\frac{\xi_1}{r} + (\frac{\xi_1}{r})^2 \right) e^{-\xi_1/r}}{|\sin(\varphi - \psi_i)|} |g(P_i + \xi_2\hat{\tau}_i)|\, d\xi_1 d\xi_2 \\
&\quad + C \int_0^{L_i} \int_0^{B_i(\xi_2)} \left(\frac{\frac{\xi_1}{r} e^{-\xi_1/r}}{|\sin(\varphi - \psi_i)|} + \frac{\xi_1^2}{r} e^{-\xi_1/r} \right) |\nabla g(P_i + \xi_2\hat{\tau}_i)|\, d\xi_1 d\xi_2 \\
&\quad + C \int_0^{L_i} \int_0^{B_i(\xi_2)} \int_0^{\xi_1} \left(\frac{\sigma}{r} e^{-\sigma/r} |\nabla g(P_i + (\xi_1 - \sigma)\hat{\mu} + \xi_2\hat{\tau}_i)| \right. \\
&\quad \left. + \frac{\sigma^2}{r} e^{-\sigma/r} |D^2 g(P_i + (\xi_1 - \sigma)\hat{\mu} + \xi_2\hat{\tau}_i)| \right) d\sigma\, d\xi_1 d\xi_2
\end{aligned}$$

$$\leq \frac{C}{|\sin(\varphi-\psi_i)|}\left(\|g\|_{L_1(S_i)}+\|\nabla g\|_{L_1(S_i)}\right)+C\|g\|_{W_1^2(\Omega_{\mu,i})}$$
$$\leq \frac{C}{|\sin(\varphi-\psi_i)|}\|g\|_{W_1^2(\Omega)} \leq \frac{C}{\min_j|\sin(\varphi-\psi_j)|}\|g\|_{W_1^2(\Omega)},$$

where $|D^2 g|$ denotes the sum of the absolute values of all second order partial derivatives, and where we have used the trace theorem. Summation with respect to i now leads to (5.6).

We now turn to the proof of (5.8). We want to take four derivatives of v with respect to r, placing as few derivatives as possible on g. To achieve this we first differentiate (5.10) three times, using the fact that d is independent of r. With $P(s)$ denoting various polynomials, not necessarily the same at each occurrence, we obtain

$$\frac{\partial^3 v}{\partial r^3} = r^{-4}\int_0^d P(\sigma/r)e^{-\sigma/r}g(x-\sigma\hat\mu)\,d\sigma = r^{-3}\int_0^{d/r} P(s)e^{-s}\,g(x-sr\hat\mu)\,ds.$$

Differentiating once more we get

$$\frac{\partial^4 v}{\partial r^4} = r^{-4}P\left(\frac{d}{r}\right)e^{-d/r}\,g(x-d\hat\mu) + r^{-4}\int_0^{d/r} P(s)\,e^{-s}\,g(x-sr\hat\mu)\,ds$$
$$+\,r^{-3}\int_0^{d/r} P(s)\,e^{-s}\,\hat\mu\cdot\nabla g(x-sr\hat\mu)\,ds,$$

or, after integration by parts in the middle term,

$$\frac{\partial^4 v}{\partial r^4} = r^{-4}P\left(\frac{d}{r}\right)e^{-d/r}\,g(x-d\hat\mu) + r^{-3}\int_0^{d/r} P(s)\,e^{-s}\,\hat\mu\cdot\nabla g(x-sr\hat\mu)\,ds \equiv I_1+I_2.$$

Here

$$\|I_1\|_{L_1(\Omega_{\mu,i})} \leq r^{-4}\int_0^{L_i}\int_0^{B_i(\xi_2)}\left|P\left(\frac{\xi_1}{r}\right)\right| e^{-\xi_1/r}\,|g(P_i+\xi_2\hat\tau_i)|\,d\xi_1\,d\xi_2$$
$$\leq r^{-4}\int_0^{\infty}\left|P\left(\frac{\xi_1}{r}\right)\right| e^{-\xi_1/r}\,d\xi_1\,\|g\|_{L_1(S_i)} \leq C\,r^{-3}\|g\|_{L_1(\Gamma)},$$

and

$$\|I_2\|_{L_1(\Omega_{\mu,i})} \leq r^{-3}\int_0^{L_i}\int_0^{B_i(\xi_2)}\int_0^{\xi_1/r}|P(\xi_1)|\,e^{-\xi_1}|\nabla g(P_i+(\xi_1-s)\hat\mu+\xi_2\hat\tau_i)|\,ds\,d\xi_1\,d\xi_2$$
$$\leq C\,r^{-3}\|g\|_{W_1^1(\Omega)}.$$

Using also the trace theorem and summing over i we obtain (5.8). The maximum norm bound (5.7) is proved in a similar way. □

Lemma 5.4. *(a) Let Q_r be defined by the Gauss rule (3.6). Then*

$$\left\| \int_0^1 v(\cdot,\mu)\frac{r}{\sqrt{1-r^2}}\,dr - Q_r[v(\cdot,\mu)] \right\|_{L_1(\Omega)} \le \frac{C}{N^4}\|g\|_{W_1^1(\Omega)},$$

$$\left\| \int_0^1 v(\cdot,\mu)\frac{r}{\sqrt{1-r^2}}\,dr - Q_r[v(\cdot,\mu)] \right\|_{L_\infty(\Omega)} \le \frac{C}{N^2}\|g\|_{L_\infty(\Omega)},$$

independently of φ.

(b) Let Q_φ be defined by the trapezoidal rule (3.5). Then

$$\left\| \int_0^{2\pi} v(\cdot,\mu)\,d\varphi - Q_\varphi[v(\cdot,\mu)] \right\|_{L_1(\Omega)} \le \frac{C}{M}\|g\|_{W_1^1(\Omega)},$$

$$\left\| \int_0^{2\pi} v(\cdot,\mu)\,d\varphi - Q_\varphi[v(\cdot,\mu)] \right\|_{L_1(\Omega)} \le \frac{C}{M^2}\log M\,\|g\|_{W_1^2(\Omega)},$$

$$\left\| \int_0^{2\pi} v(\cdot,\mu)\,d\varphi - Q_\varphi[v(\cdot,\mu)] \right\|_{L_\infty(\Omega)} \le \frac{C}{M}\log M\,\|g\|_{W_\infty^1(\Omega)},$$

independently of r.

Proof. Part (a) and the first estimate of part (b) are immediate consequences of Lemmas 5.2, 5.3, and Lemmas 5.1, 5.3, respectively. For the second estimate of part (b) we have to pay special attention to angles φ that are close to an angle of direction ψ_i of a side S_i. For this purpose we define intervals

$$I_k = \left[\frac{2\pi(k-1)}{M}, \frac{2\pi k}{M}\right] = [\varphi_{k-1}, \varphi_k], \qquad k = 1, \ldots, M,$$

and for each ψ_i we let J_i be the union of the interval I_k that contains ψ_i and the adjacent interval $I_{k\pm1}$ closest to ψ_i. Further, we define

$$S_0 = \cup_{i=1}^P J_i,$$

where the union is taken over all the sides of Ω, and

$$S = [0, 2\pi], \qquad S_1 = \overline{S \setminus S_0}.$$

Corresponding to the division $\int_S = \int_{S_0} + \int_{S_1}$ of the integral we divide the trapezoidal rule over S into the trapezoidal rule over S_0 and S_1. For this we have the weights

$$\tilde{W}_j = \begin{cases} W_j = 2\pi/M, & \text{if} \quad \varphi_j \notin S_0 \cap S_1, \\ W_j/2 = \pi/M, & \text{if} \quad \varphi_j \in S_0 \cap S_1. \end{cases}$$

Using the standard error estimates for the trapezoidal rule from Lemma 5.1 together with Lemma 5.3 we now have

$$\begin{aligned}
\left\| \int_0^{2\pi} v(\cdot,\mu)\,d\varphi - Q_\varphi[v(\cdot,\mu)] \right\|_{L_1(\Omega)} &\le \left\| \int_{S_0} v(\cdot, r\hat{\mu}(\varphi))\,d\varphi - \sum_{\varphi_j\in S_0} \tilde{W}_j v(\cdot, r\hat{\mu}(\varphi_j)) \right\|_{L_1(\Omega)} \\
&+ \left\| \int_{S_1} v(\cdot, r\hat{\mu}(\varphi))\,d\varphi - \sum_{\varphi_j\in S_1} \tilde{W}_j v(\cdot, r\hat{\mu}(\varphi_j)) \right\|_{L_1(\Omega)} \\
&\le \frac{C}{M}\int_{S_0} \left\| \frac{\partial v(\cdot, r\hat{\mu}(\varphi))}{\partial\varphi} \right\|_{L_1(\Omega)} d\varphi + \frac{C}{M^2}\int_{S_1} \left\| \frac{\partial^2 v(\cdot, r\hat{\mu}(\varphi))}{\partial\varphi^2} \right\|_{L_1(\Omega)} d\varphi \\
&\le \frac{C}{M}\int_{S_0} d\varphi\, \|g\|_{W_1^1(\Omega)} + \frac{C}{M^2}\int_{S_1} \frac{1}{\min_i |\sin(\varphi-\psi_i)|}\,d\varphi\, \|g\|_{W_1^2(\Omega)} \\
&\le \frac{C}{M^2}\|g\|_{W_1^1(\Omega)} + \frac{C}{M^2}\log M\,\|g\|_{W_1^2(\Omega)} \le \frac{C}{M^2}\log M\,\|g\|_{W_1^2(\Omega)}.
\end{aligned}$$

The maximum norm bound is proved in a similar way. □

Lemma 5.5. *We have*

$$\|(T-T_n)g\|_{L_1(\Omega)} \le C\left(\frac{1}{N^4}+\frac{1}{M}\right)\|g\|_{W_1^1(\Omega)},$$

$$\|(T-T_n)g\|_{L_1(\Omega)} \le C\,\frac{1}{N^4}\,\|g\|_{W_1^1(\Omega)} + C\,\frac{1}{M^2}\log M\,\|g\|_{W_1^2(\Omega)}, \tag{5.11}$$

$$\|(T-T_n)g\|_{L_\infty(\Omega)} \le C\,\frac{1}{N^2}\,\|g\|_{L_\infty(\Omega)} + C\,\frac{1}{M}\log M\,\|g\|_{W_\infty^1(\Omega)}. \tag{5.12}$$

Proof. This is an immediate consequence of the previous lemma and (5.2). □

We can now prove the boundedness of $(I-\lambda T_n)^{-1}$. We need the following simple lemma, the proof of which can be found in Johnson and Pitkäranta [8].

Lemma 5.6. *Let $T : L_1(\Omega) \to L_1(\Omega)$ be a bounded linear operator such that for some C*

$$\|(I-\lambda T)g\|_{L_1(\Omega)} \ge C\|g\|_{L_1(\Omega)} \qquad \forall g \in L_1(\Omega),$$

and let $\{T_n\}_{n=1}^\infty$ be a bounded sequence of continuous linear operators on $L_1(\Omega)$ such that for some positive integer m

$$\|(T-T_n)T_n^m\| \to 0 \qquad \text{as } n\to\infty. \tag{5.13}$$

Then there is C_1 such that for n large

$$\|(I-\lambda T_n)g\|_{L_1(\Omega)} \ge C_1\|g\|_{L_1(\Omega)} \qquad \forall g \in L_1(\Omega). \tag{5.14}$$

Lemma 5.7. *Assume that λ^{-1} is not an eigenvalue of T. Then the operator $I - \lambda T_n$ is invertible on $L_1(\Omega)$ for $\bar{n}$ large and, moreover, its inverse is uniformly bounded.*

Proof. We only sketch the proof, which is a simple modification of the corresponding argument in [1]. The idea is to show that (5.13) holds with $m = 2$. In order to do this one splits T_n^2 into two terms:

$$T_n^2 = \sum_{(\mu,\nu)\in\Delta^2} \omega_\mu\omega_\nu T_\mu T_\nu = \sum_{(\mu,\nu)\in\Delta_\epsilon^2} \omega_\mu\omega_\nu T_\mu T_\nu + \sum_{(\mu,\nu)\in\Delta^2\setminus\Delta_\epsilon^2} \omega_\mu\omega_\nu T_\mu T_\nu, \tag{5.15}$$

where the first sum on the right is over all velocities $\mu, \nu \in \Delta$,

(1) whose directions are bounded away from each other;
(2) whose directions are bounded away from the directions of the sides of Ω; and
(3) whose magnitudes are bounded away from 0;

closeness being quantified by the positive parameter ϵ. One can then show that

$$\|T_\mu T_\nu g\|_{W_1^1(\Omega)} \leq C\epsilon^{-a}\|g\|_{L_1(\Omega)}, \qquad \forall(\mu,\nu)\in\Delta_\epsilon^2, \tag{5.16}$$

for some $a > 0$. The proof of this is based on direct estimates on the formula (5.9). For example one has

$$\|\mu\cdot\nabla T_\mu g\|_{L_1(\Omega)} + \|T_\mu g\|_{L_1(\Omega)} + \int_\Gamma |T_\mu g|\,|\mu\cdot\hat{n}|\,ds \leq C\|g\|_{L_1(\Omega)}, \tag{5.17}$$

where $\hat{n}$ is the outward unit normal to Γ. The negative power of ϵ in (5.16) comes from the fact that the operator T_μ is smoothing in the direction of μ but not in other directions.

Applying Lemma 5.5, using (5.16) and (5.17), we thus obtain

$$\begin{aligned}
\|(T - T_n)T_n^2 g\|_{L_1(\Omega)} &\leq \|(T - T_n)\sum_{(\mu,\nu)\in\Delta_\epsilon^2} \omega_\mu\omega_\nu T_\mu T_\nu g\|_{L_1(\Omega)} \\
&\quad + \|(T - T_n)\sum_{(\mu,\nu)\in\Delta^2\setminus\Delta_\epsilon^2} \omega_\mu\omega_\nu T_\mu T_\nu g\|_{L_1(\Omega)} \\
&\leq C\left(\frac{1}{N^4} + \frac{1}{M}\right)\sum_{(\mu,\nu)\in\Delta_\epsilon^2} \omega_\mu\omega_\nu \|T_\mu T_\nu g\|_{W_1^1(\Omega)} \\
&\quad + C\sum_{(\mu,\nu)\in\Delta^2\setminus\Delta_\epsilon^2} \omega_\mu\omega_\nu \|g\|_{L_1(\Omega)} \\
&\leq C\left(\left(\frac{1}{N^4} + \frac{1}{M}\right)\epsilon^{-a} + \sum_{(\mu,\nu)\in\Delta^2\setminus\Delta_\epsilon^2} \omega_\mu\omega_\nu\right)\|g\|_{L_1(\Omega)}.
\end{aligned}$$

Choosing ϵ appropriately and using the fact that $\sum_{(\mu,\nu)\in\Delta^2\setminus\Delta_\epsilon^2} \omega_\mu\omega_\nu \to 0$ as $\epsilon \to 0$, we find that

$$\|(T - T_n)T_n^2\| \to 0 \qquad \text{as } \bar{n} \to \infty. \tag{5.18}$$

This proves (5.14), and hence that $I - \lambda T_n$ is one-to-one. Since T_n is not compact we may not conclude directly that $I - \lambda T_n$ is onto, but this can be shown by using (5.14) together with the fact that (5.15) splits T_n^2 as $T_n^2 = A_n + B_n$, where $A_n : L_1(\Omega) \to W_1^1(\Omega)$, i.e., A_n is compact, and $\|B_n\| \to 0$ as $\bar{n} \to \infty$. We refer to [8] for the details. □

We can now prove Theorem 3.1. Interpolating between (5.11) and the bound

$$\|(T - T_n)g\|_{L_1(\Omega)} \leq C\|g\|_{L_1(\Omega)},$$

obtained by stability, cf. (5.17), we get

$$\|(T - T_n)g\|_{L_1(\Omega)} \leq C\,\frac{1}{N^4}\,\|g\|_{W_1^1(\Omega)} + C\,\frac{1}{M^{2-\epsilon}}\,\|g\|_{(L_1(\Omega),W_1^2(\Omega))_{\theta,q}}, \tag{5.19}$$

for $0 \leq 1 - \epsilon/2 < \theta \leq 1$. Applying this with $g = \lambda U + f$, and recalling Theorem 4.1, we have in view of (5.1),

$$\|U - U_n\|_{L_1(\Omega)} \leq C\left(\frac{1}{N^4} + \frac{1}{M^{2-\epsilon}}\right)\left(\|U\|_{L_1(\Omega)} + \|f\|_{W_1^1(\Omega)}\right).$$

This completes the proof of Theorem 3.1.

6. An error estimate for the critical eigenvalue. The kernel of the compact integral operator T is symmetric and positive, see (4.1). Hence T is self-adjoint (on $L_2(\Omega)$), and therefore has only real eigenvalues. Moreover, by the Krein-Rutman theory, its largest eigenvalue is positive and simple. Without loss of generality we may assume that M is even so that the set of quadrature points Δ is symmetric, i.e., $\mu \in \Delta$ implies $-\mu \in \Delta$. Then it follows that T_n is self-adjoint and its eigenvalues are real.

Theorem 6.1. *Assume that M is even and let κ and κ_n be the largest eigenvalues of T and T_n, respectively. For any $\epsilon > 0$ there is $C = C(\epsilon, \kappa)$ such that, for $\bar{n}$ sufficiently large, we have*

$$|\kappa - \kappa_n| \leq C\left(\frac{1}{N^4} + \frac{1}{M^{2-\epsilon}}\right).$$

Proof. We shall show that

$$\|T^3 - T_n^3\| \to 0, \qquad \text{as } \bar{n} \to \infty, \tag{6.1}$$

i.e., T_n^3 converges to T^3 in the norm of bounded linear operators on $L_1(\Omega)$. It then follows that the family $\{T_n^3\}$ is collectively compact, see [6, p. 251], and a result of Atkinson [4] yields

$$|\kappa^3 - \kappa_n^3| \leq C\|(T^3 - T_n^3)\chi\|_{L_1(\Omega)},$$

where χ is the eigenfunction of T associated with κ, normalized by $\|\chi\|_{L_1(\Omega)} = 1$. Using the boundedness of T and T_n, the error bound (5.19), and the regularity estimate from Theorem 4.1, we thus have

$$\begin{aligned} |\kappa^3 - \kappa_n^3| &\le C\left(\frac{1}{N^4} + \frac{1}{M^{2-\epsilon}}\right) \|\chi\|_{(L_1(\Omega), W_1^2(\Omega))_{\theta,q}} \\ &\le C\left(\frac{1}{N^4} + \frac{1}{M^{2-\epsilon}}\right) \|\chi\|_{L_1(\Omega)} = C\left(\frac{1}{N^4} + \frac{1}{M^{2-\epsilon}}\right), \end{aligned}$$

which implies the desired result, since $\kappa > 0$.

It remains to prove (6.1). We first note that

$$T^3 - T_n^3 = T^2(T - T_n) + T(T - T_n)T_n + (T - T_n)T_n^2.$$

Here, by (5.18), the last term tends to zero in the operator norm. For the remaining terms we note that, with $(\cdot,\cdot)$ denoting the duality pairing between $L_1(\Omega)$ and $L_\infty(\Omega)$, and using the symmetry of T and T_n,

$$|(T(T - T_n)f, g)| = |(f, (T - T_n)Tg)| \le \|f\|_{L_1(\Omega)} \|(T - T_n)Tg\|_{L_\infty(\Omega)}.$$

By interpolation, (5.12), stability in L_∞ and Theorem 4.2 we get

$$\begin{aligned} \|(T - T_n)Tg\|_{L_\infty(\Omega)} &\le C\left(\frac{1}{N^2} + \frac{1}{M}\log M\right)^\theta \|Tg\|_{(L_\infty(\Omega), W_\infty^1(\Omega))_{\theta,q}} \\ &\le C\left(\frac{1}{N^2} + \frac{1}{M}\log M\right)^\theta \|g\|_{L_\infty(\Omega)}, \end{aligned}$$

with $0 < \theta < 1$. Hence

$$\|T(T - T_n)\| \le C\left(\frac{1}{N^2} + \frac{1}{M}\log M\right)^\theta,$$

and (6.1) follows. □

References

1. M. Asadzadeh, *Analysis of a fully discrete scheme for neutron transport in two-dimensional geometry*, SIAM J. Numer. Anal. **23** (1986), 543–561.
2. M. Asadzadeh, *L_p and eigenvalue error estimates for two-dimensional neutron transport*, SIAM J. Numer. Anal. **26** (1989), 66–87.
3. M. Asadzadeh, *L_2 error estimates for discrete ordinates method for three-dimensional neutron transport*, Transport Theory and Statistical Physics **17** (1988), 1–24.
4. K. E. Atkinson, *Convergence rates for approximate eigenvalues of compact integral operators*, SIAM J. Numer. Anal. **12** (1975), 213–222.
5. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, 1976.
6. F. Chatelin, *Spectral Approximation of Linear Operators*, Academic Press, 1983.

7. R. A. De Vore and L. R. Scott, *Error bounds for Gaussian quadrature and weighted-L_1 polynomial approximation*, SIAM J. Numer. Anal. **21** (1984), 400–412.
8. C. Johnson and J. Pitkäranta, *Convergence of a fully discrete scheme for two-dimensional neutron transport*, SIAM J. Numer. Anal. **20** (1983), 951–966.
9. S. M. Nikolskii, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, 1975.
10. J. Pitkäranta, *Estimates for the derivatives of solutions to weakly singular Fredholm integral equations*, SIAM J. Math. Anal. **11** (1980), 952–968.
11. J. Pitkäranta and L. R. Scott, *Error estimates for the combined spatial and angular approximations of the transport equation in slab geometry*, SIAM J. Numer. Anal. **20** (1983), 922–950.
12. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
13. G. Szegő, *Orthogonal Polynomials*, AMS Colloquium Publications 23, American Mathematical Society, 1957.

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