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A POSTERIORI ERROR ESTIMATES FOR A COUPLED WAVE SYSTEM WITH A LOCAL DAMPING

M. Asadzadeh *
Chalmers University of Technology and University of Gothenburg
SE-41296 Gothenburg, Sweden
mohammad@chalmers.se

D. Rostamy
Imam Khomeini International University
Qazvin, Iran
rostamy@khayam.ut.ac.ir

F. Zabihi
Imam Khomeini International University
Qazvin, Iran
fatemh.zabihi@yahoo.com

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We study a finite element method applied to a system of coupled wave equations in a bounded smooth domain in $\mathbb{R}^d$, $d = 1, 2, 3$, *associated with a locally distributed damping function. We start with a spatially continuous finite element formulation allowing jump discontinuities in time. This approach yields,* $L_2(L_2)$ *and* $L_\infty(L_2)$, *a posteriori error estimates in terms of weighted residuals of the system. The proof of the a posteriori error estimates is based on the strong stability estimates for the corresponding adjoint equations. Optimal convergence rates are derived upon the maximal available regularity of the exact solution and justified through numerical examples. Bibliography:* 14 *titles. Illustrations*: 4 *figures.*

1 Introduction

The finite element study for the system of one dimensional damped wave equation was considered by several authors in various settings (cf., for example, [1, 2] and the references therein). The corresponding study for the multi-dimensional wave equation system is more involved, and a reasonable numerical analysis is possible only in very restrictive cases. Hence the approximate solution of the wave propagation in an arbitrary domain of higher dimensions, especially in

* To whom the correspondence should be addressed.

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the system form, and with a rigorous error analysis is of a vital interest. The advantages of a posteriori error bounds were based on the fact that they are expressed in terms of the residual of the computed (and therefore, known) approximate solutions, rather than norms of the unknown exact solution, which is the matter of the a priori error estimates. There are various approaches to a posteriori error estimates applied to a number of problems (cf., for example, [3]–[9]).

In this paper, we consider a system of coupled, multidimensional, wave equations associated with locally damping terms. Introducing vector quantities related to the solution, we reformulate this hyperbolic system as an elliptic system of equations. We also formulate a *streamline diffusion method* adequate for the finite element solution to the hyperbolic type partial differential equations. However, this will not be our main concern. We focus on a spatially continuous finite element scheme (with a *streamline-diffusion type* structure, but without the streamline-diffusion term) for a new elliptic system of equations, where jump discontinuities over certain time levels are allowed. For this system we derive a posteriori error estimates in the $L_2(L_2)$– and $L_\infty(L_2)$–norms. In our numerical examples, we do insert the streamline diffusion term into the scheme.

Studies of this type were considered by Gergoulus and co-workers [6], where, using the Galerkin finite element method for the linear wave equation without damping term, they obtained a posteriori error estimates in the $L_\infty(L_2)$–norm. Johnson [10] established the existence of a solution to the second order hyperbolic problem. He used the discontinues Galerkin method to obtain a priori and a posteriori error L_2–estimates.

We consider the following model problem which is of interest in computational fluid mechanics and plasma physics (cf. [11]): construct an algorithm for numerical solving a coupled wave system with energy decay such that the error between the exact and computed solutions, in a given norm, may be guaranteed to be below a given tolerance such that the computational work is nearly minimal. More specifically, we consider the following system of linear coupled wave equations:

$$\begin{cases} u_{tt} - \Delta u + \alpha(x)(u_t - v_t) = 0 \quad \text{in} \quad \Omega \times [0, \infty), \\ v_{tt} - \Delta v + \alpha(x)(v_t - u_t) = 0 \quad \text{in} \quad \Omega \times [0, \infty), \\ u = v = 0 \quad \text{in} \quad \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{in} \quad \Omega, \\ u_t(x, 0) = u_1(x), \quad v_t(x, 0) = v_1(x) \quad \text{in} \quad \Omega, \quad (u_t := \partial u/\partial t), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded domain with the smooth boundary $\partial\Omega$ (for $d = 2, 3$) and $\alpha \in W^{1,\infty}(\Omega)$ is a damping term such that $\alpha(x) \geqslant 0$ in Ω and

$$\alpha_0 := \int\limits_\Omega \alpha(x)\, dx > 0.$$

Hence $\alpha(x)$ may vanish at some points of the domain Ω, but its support is of positive measure. Here, Δ denotes the Laplace operator in the spatial variable x.

For the existence and uniqueness of a solution to the continuous problem (1.1) we refer to [12]. As was proved in [13], the solution to the problem (1.1) has an exponentially decaying energy associated with a locally distributed damping in a bounded smooth multidimensional domain. We use the vector form and reformulate the system (1.1) as the following *abstract elliptic system of partial differential equations*:

$$\begin{cases} \mathscr{L}u := \mathbf{u}_t + A\mathbf{u} = 0 \quad \text{in} \;\; \Omega \times [0, \infty), \\ \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \;\; \Omega, \end{cases} \tag{1.2}$$

where $\mathbf{u}(x,t) = (u(x,t), \phi(x,t), v(x,t), \psi(x,t))^T$, $\phi = u_t$, $\psi = v_t$, and the operator A is defined by the formula

$$A : [H_0^1 \times L_2]^2 \longrightarrow [H_0^1 \times L_2]^2$$

with the domain

$$D(A) = [(H_0^1 \cap H^2) \times H_0^1]^2,$$

and the matrix-operator form

$$A = \begin{bmatrix} 0 & -\mathscr{I} & 0 & 0 & \\ & -\Delta & \alpha(x)\mathscr{I} & 0 & -\alpha(x)\mathscr{I} \\ & 0 & 0 & 0 & -\mathscr{I} \\ & 0 & -\alpha(x)\mathscr{I} & -\Delta & \alpha(x)\mathscr{I} \end{bmatrix},$$

where $\mathscr{I}$ is the identity operator. We also introduce the initial data

$$\mathbf{u}_0(x) = (u_0(x), u_1(x), v_0(x), v_1(x))^T.$$

Let $L_2(\Omega \times [0,\infty)) := H^0(\Omega \times [0,\infty))$ be the usual Sobolev spaces of Lebesgue square integrable functions defined in $\Omega \times [0,\infty)$. By $H_0^1(\Omega \times [0,\infty))$ we mean a subspace of $H^1(\Omega \times [0,\infty))$ consisting of functions vanishing on $\partial\Omega \times [0,\infty)$, where $H_0^1(\Omega \times [0,\infty))$ consists of all functions in $H^0(\Omega \times [0,\infty))$ possessing all first order partial derivatives in $H^0(\Omega \times [0,\infty))$.

The paper is organized as follows. Section 2 contains preliminaries and a formulation of the finite element method for (1.2), considering space–time slabs: $S_n := \Omega \times I_n$, where $I_n = (t_{n-1}, t_n)$, $n = 1, 2, \ldots, n$, are subintervals of the time domain. In Subsection 3.1, we study a posteriori error estimates for (1.2) and derive the optimal $L_2(L_2)$– and $L_\infty(L_2)$–norm error bounds. The main ingredients of the proof are through a duality argument. In Subsection 3.2, we introduce projection operators and, again using duality, derive the interpolation estimates and complete the proof of the a posteriori error bounds. In Section 4, we prove the strong stability estimates for the dual problems. Some computational results are given in Section 5.

2 Notation and Preliminaries

In this section, we consider a time discontinuous Galerkin method for solving the problem (1.2) which is based on the use of finite elements over the space–time domain $\Omega \times [0,T]$. To describe this method, we consider a subdivision

$$0 = t_0 < t_1 < \ldots < t_N = T$$

of the time interval $[0,T]$ into the subintervals $I_n = (t_n, t_{n+1})$, with the time steps $k_n = t_{n+1} - t_n$, $n = 0, 1, \dots, N-1$, and introduce the corresponding space–time slabs

$$S_n = \{(x,t) : x \in \Omega,\ t_n < t < t_{n+1}\}, \quad n = 0, 1, \dots, N-1. \tag{2.1}$$

For notational convenience we denote by $k = k(t)$ the mesh function for the time discretization, where $k(t) = k_n$ for $t \in (t_n, t_{n+1})$. We also assume that Ω is a bounded open interval in the one-dimensional case or an open bounded subset of $\mathbb{R}^d$ with the piecewise smooth boundary $\partial\Omega$ in the case $d \geqslant 2$. We use the standard procedure partitioning Ω into subintervals for $d = 1$, quasiuniform triangular elements for $d = 2$, or tetrahedrons (with the corresponding minimal vertex angle conditions) for $d = 3$.

2.1 The time discontinuous Galerkin scheme

For every n we denote by $\mathbf{U}^n$ a finite element subspace of $[H_0^1(S_n) \times L_2(S_n)]^2$. On each slab S_n we formulate the following spatially continuous problem: *for every $n = 0, \dots, N-1$ find $\mathbf{u}^n \in \mathbf{U}^n$ such that*

$$(\mathbf{u}_t^n + A\mathbf{u}^n, \mathbf{g})_n + \langle \mathbf{u}_+^n, \mathbf{g}_+ \rangle_n = \langle \mathbf{u}_-^n, \mathbf{g}_+ \rangle_n \quad \forall \quad \mathbf{g} \in \mathbf{U}^n, \tag{2.2}$$

where

$$(\mathbf{w}, \mathbf{g})_n = \int\limits_{S_n} \mathbf{w}^T \cdot \mathbf{g}\, dx dt,$$

$$\langle \mathbf{w}, \mathbf{g} \rangle_n = \int\limits_{\Omega} \mathbf{w}^T(x, t_n) \cdot \mathbf{g}(x, t_n) dx,$$

$$\mathbf{w}_{\pm}(x,t) = \lim_{s \to 0^{\pm}} \mathbf{w}(x, t+s).$$

The $\langle \cdot\,, \cdot \rangle$-term yields a jump which imposes a weakly enforced continuity condition across the slab interfaces at each time level $t = t_n$: a mechanism which governs the flow of information from one slab to adjacent one in the positive time direction. Note that we defined the inner product of $\mathbf{u}_j = (u_j, \phi_j, v_j, \psi_j)^T$, $j = 1, 2$, in the space $[H_0^1(S_n) \times L_2(S_n)]^2$, $n = 0, 1, \dots, N-1$, by the formula

$$(\mathbf{u}_1, \mathbf{u}_2)_n = \int\limits_{S_n} (\nabla u_1 \cdot \nabla u_2 + \nabla v_1 \cdot \nabla v_2 + \phi_1 \phi_2 + \psi_1 \psi_2)\, dx\, dt. \tag{2.3}$$

Summing over n, we get the function space

$$\mathbf{U} := \prod_{n=0}^{N-1} \mathbf{U}^n.$$

Thus we can write (2.2) in a more concise form as follows: *find $\widetilde{\mathbf{u}} \in \mathbf{U}$ such that*

$$B(\widetilde{\mathbf{u}}, \mathbf{g}) = L(\mathbf{g}) \quad \forall g \in \mathbf{U}, \tag{2.4}$$

where the bilinear form $B(\cdot, \cdot)$ and the linear form $L(\cdot)$ are defined by the formulas

$$B(\widetilde{\mathbf{u}}, \mathbf{g}) = \sum_{n=0}^{N-1} (\mathbf{u}_t^n + A\mathbf{u}^n, g)_n + \sum_{n=1}^{N-1} \langle [\mathbf{u}^n], \mathbf{g}_+ \rangle_n + \langle \mathbf{u}_+^n, \mathbf{g}_+ \rangle_0 \tag{2.5}$$

and

$$L(g) = \langle \mathbf{u}_0, \mathbf{g}_+ \rangle_0 \tag{2.6}$$

respectively. The corresponding weak variational formulation for the continuous problem (1.2) is as follows:

$$B(\mathbf{u}, \mathbf{g}) = L(\mathbf{g}) \quad \forall g \in [H_0^1(\Omega) \times L_2(\Omega)]^2, \tag{2.7}$$

where we replace $\widetilde{\mathbf{u}}$ in (2.5) by $\mathbf{u}$ and put the jumps $[\mathbf{u}] \equiv 0$. We set $\mathbf{u}^n = (u^n, \phi^n, v^n, \psi^n)^T$ and introduce the jump $[\mathbf{u}^n] = ([u^n], [\phi^n], [v^n], [\psi^n])^T$, where $[q] = q_+ - q_-$ for $q = u^n, \phi^n, v^n, \psi^n$. Finally, let $\mathscr{T}_h$ be a partition of Ω into quasiuniform triangular ($d = 2$) or tetrahedral ($d = 3$) domains of the maximal diameter h (the mesh size). We introduce

$$\mathbf{U}_h^n = \big\{\mathbf{u}^n \in [H_0^1(S_n) \times L_2(S_n)]^2 : \mathbf{u}^n|_K \in [P_\ell(K) \times P_\ell(K)]^2 \text{ for } K \in \mathscr{T}_h\big\},$$

where $P_\ell(K)$ denotes the set of polynomials in K of degree less than or equal to ℓ and define the discrete function space $\mathbf{U}_h$ by the formula

$$\mathbf{U}_h = \prod_{n=0}^{N-1} \mathbf{U}_h^n.$$

Thus, the problem (2.4) can be reformulated as follows: *find* $\mathbf{u}_h \in \mathbf{U}_h$ *such that*

$$B(\mathbf{u}_h, \mathbf{g}) = L(\mathbf{g}) \quad \forall \mathbf{g} \in \mathbf{U}_h. \tag{2.8}$$

Finally, subtracting (2.8) from (2.7), for $\mathbf{g} \in \mathbf{U}_h$ we end up with the Galerkin orthogonality relation

$$B(\mathbf{e}, \mathbf{g}) = 0 \quad \forall \mathbf{g} \in \mathbf{U}_h, \tag{2.9}$$

where $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ stands for the error.

3 A Posteriori Error Analysis

In this section, we estimate the error of a particular approximation of solution, in some weighted norms, by using the information from computation. The procedure is split in the following two steps.

3.1 Dual problem, stability, and error representation formula in $L_2(L_2)$

In this subsection, we state the dual problem for the weak (variational) formulation of the continuous problem (1.2) with jump discontinuities across time levels $t = t_n$: *find* $\mathbf{u}_h \in \mathbf{U}_h$ *such that for* $n = 0, 1, \dots, N-1$

$$\sum_{n=0}^{N-1} (\mathbf{u}_{h,t}^n + A\mathbf{u}_h^n, \mathbf{g})_n + \sum_{n=1}^{N-1} \langle [\mathbf{u}_h^n], \mathbf{g}_+ \rangle_n + \langle \mathbf{u}_{h,+}^n, \mathbf{g}_+ \rangle_0 = \langle \mathbf{u}_0, \mathbf{g}_+ \rangle_0, \tag{3.1}$$

where $g \in \mathbf{U}_h$ and $\mathbf{u}_{h,-}^0 = \mathbf{u}_0$. To obtain a representation of the error, we consider the dual problem: *find* $\Phi \in [H_0^1(\Omega \times [0,\infty)) \times L_2(\Omega \times [0,\infty))]^2$ *such that*

$$\begin{cases} \mathscr{L}^*\Phi \equiv -\Phi_t + A^T\Phi = \Psi^{-1}\mathbf{e} \quad \text{in} \quad \Omega \times [0,\infty), \\ \Phi(x,t)\Big|_{t=T} = 0, \quad x \in \Omega, \end{cases} \tag{3.2}$$

where $\mathscr{L}^*$ is the adjoint of the differential operator $\mathscr{L}$ describing the left-hand side of the first equation in (1.2) and Ψ is a positive weight function. Note that this problem is computed "backward", but with the corresponding change in sign. We introduce the weighted L_2-norm:

$$\| \mathbf{u} \|_{L_2^{\psi}(\Omega)} = (\mathbf{u}, \Psi\mathbf{u})_\Omega^{1/2}. \tag{3.3}$$

Multiplying (3.2) by $\mathbf{e}$ and integrating by parts over Ω, we obtain the following error representation formula:

$$\| \mathbf{e} \|_{L_2^{\Psi^{-1}}(\Omega)}^2 = (\mathbf{e}, \Psi^{-1}\mathbf{e})_\Omega = (\mathbf{e}, \mathscr{L}^*\Phi)_\Omega = \sum_{n=0}^{N-1} (\mathbf{e}, -\Phi_t)_n + \sum_{n=0}^{N-1} (\mathbf{e}, A^T\Phi)_n. \tag{3.4}$$

Further partial integration in t yields

$$(\mathbf{e}, -\Phi_t)_n = -\int_\Omega \left(\mathbf{e}^T(x,t) \cdot \Phi(x,t)\Big|_{t=t_n}^{t=t_{n+1}} \right) dx + (\mathbf{e}_t, \Phi)_n. \tag{3.5}$$

We recall that $\mathbf{e} = \mathbf{e}(x,t) = (e_1, e_2, e_3, e_4)^T$ and $\Phi = \Phi(x,t) = (\phi_1, \phi_2, \phi_3, \phi_4)^T$; moreover for $n = 0, 1, \dots, N-1$

$$(\mathbf{e}, A^T\Phi)_n = \int_{S_n} \mathbf{e}^T \begin{bmatrix} 0 & -\Delta & 0 & 0 \\ -\mathscr{I} & \alpha(x)\mathscr{I} & 0 & -\alpha(x)\mathscr{I} \\ 0 & 0 & 0 & -\Delta \\ 0 & -\alpha(x)\mathscr{I} & -\mathscr{I} & \alpha(x)\mathscr{I} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} dxdt$$

$$= \int_{S_n} [e_1, e_2, e_3, e_4] \begin{bmatrix} -\Delta\phi_2 \\ -\phi_1 + \alpha(x)\phi_2 - \alpha(x)\phi_4 \\ -\Delta\phi_4 \\ -\alpha(x)\phi_2 - \phi_3 + \alpha(x)\phi_4 \end{bmatrix} dxdt. \tag{3.6}$$

Hence

$$\begin{aligned} (\mathbf{e}, A^T\Phi)_n &= \int_{S_n} (-e_1\Delta\phi_2) + e_2(-\phi_1 + \alpha(x)(\phi_2 - \phi_4)) - e_3\Delta\phi_4 + e_4(-\phi_3 + \alpha(x)(\phi_4 - \phi_2))dxdt \\ &= \int_{S_n} (\nabla e_1 \cdot \nabla\phi_2 - e_2\phi_1 + \alpha(x)(e_2\phi_2 - e_2\phi_4 - e_4\phi_2 + e_4\phi_4) + \nabla e_3 \cdot \nabla\phi_4 - e_4\phi_3)dxdt \\ &= \int_{S_n} (-\Delta e_1\phi_2 - e_2\phi_1 + \alpha(x)(e_2\phi_2 - e_2\phi_4 - e_4\phi_2 + e_4\phi_4) - \Delta e_3\phi_4 - e_4\phi_3)dxdt \\ &= \int_{S_n} (A\mathbf{e})^T \cdot \Phi dxdt = (A\mathbf{e}, \Phi)_n. \end{aligned}$$

Now, we compute the sum of the jumps on the right-hand side of (3.5):

$$J = \sum_{n=0}^{N-1} \int_{\Omega} \big(\mathbf{e}^T(x,t_{n+1}) \cdot \Phi(x,t_{n+1}) - \mathbf{e}^T(x,t_n) \cdot \Phi(x,t_n)\big)\, dx$$
$$= (\langle \mathbf{e}_-, \Phi_- \rangle_1 - \langle \mathbf{e}_+, \Phi_+ \rangle_0) + (\langle \mathbf{e}_-, \Phi_- \rangle_2 - \langle \mathbf{e}_+, \Phi_+ \rangle_1) + \dots$$
$$+ (\langle \mathbf{e}_-, \Phi_- \rangle_{N-1} - \langle \mathbf{e}_+, \Phi_+ \rangle_{N-2}) + (\langle \mathbf{e}_-, \Phi_- \rangle_N - \langle \mathbf{e}_+, \Phi_+ \rangle_{N-1}).$$

We rearrange the above sum by writing $\Phi_-^n = \Phi_-^n - \Phi_+^n + \Phi_+^n$, $n = 1, \dots, N-1$. Then we can write

$$-J = \langle \mathbf{e}_-, \Phi_- \rangle_N + \langle \mathbf{e}_+, \Phi_+ \rangle_0 + \sum_{n=0}^{N-1} \langle [\mathbf{e}], \Phi_+ \rangle_n + \sum_{n=0}^{N-1} \langle \mathbf{e}_-, [\Phi] \rangle_n.$$

According to (3.2), $\Phi(\cdot, t_N = T) = 0$. Since $\mathbf{e}_-^0 = [\mathbf{u}_0] = 0$, we get

$$J = \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n. \tag{3.7}$$

Substituting (3.5)–(3.7) into (3.4), we find

$$\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (\mathbf{e}_t, \Phi)_n + \sum_{n=0}^{N-1} (A\mathbf{e}, \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n$$
$$= \sum_{n=0}^{N-1} ((\mathbf{u} - \mathbf{u}_h)_t + A(\mathbf{u} - \mathbf{u}_h), \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n$$
$$= \sum_{n=0}^{N-1} (-\mathbf{u}_{h,t} - A\mathbf{u}_h, \Phi)_n - \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], \Phi_+ \rangle_n.$$

Recalling (3.1) and using the Galerkin orthogonality (2.9), we obtain the final form of the *error representation formula*

$$\| \mathbf{e} \|^2_{L_2^{\Psi^{-1}}(\Omega)} = \sum_{n=0}^{N-1} (\mathbf{u}_{h,t} + A\mathbf{u}_h, \widehat{\Phi} - \Phi)_n + \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], (\widehat{\Phi} - \Phi)_+ \rangle_n$$
$$\equiv I + II, \tag{3.8}$$

where $\widehat{\Phi} \in \mathbf{U}_h$ is an interpolant of Φ. The idea is now to estimate $\widehat{\Phi} - \Phi$ in terms of $\Psi^{-1}\mathbf{e}$ by using strong stability estimates for the solution Φ to the dual problem (3.2).

3.2 A posteriori error estimates for the dual solution in $L_2(L_2)$

In this subsection, for the interpolant $\widehat{\Phi} \in \mathbf{U}_h$ in (3.8), we consider a certain space–time L_2-projection of Φ. For this purpose, we define the projections

$$P_n : [H_0^1 \times L_2]^2 \Longrightarrow \mathbf{U}_h^n$$

and the local time averages

$$\pi_n : [L_2(S_n)]^4 \longrightarrow \Pi_{0,n} = \{\mathbf{u} \in [L_2(S_n)]^4 : \mathbf{u}(x,.) \ \text{ is constant on } \ I_n, \ x \in \Omega\}$$

such that

$$\int\limits_{\Omega} (P_n\Phi)^T \cdot \mathbf{u}\, dx = \int\limits_{\Omega} \Phi^T \cdot \mathbf{u}\, dx \quad \forall \mathbf{u} \in \mathbf{U}_h^n,$$

$$\pi_n \mathbf{u} \mid_{S_n} = \frac{1}{k_n} \int\limits_{I_n} \mathbf{u}(\cdot, t)\, dt \quad \forall \mathbf{u} \in \Pi_{0,n}.$$

Then we define $\widehat{\Phi} \mid_{S_n} \in \mathbf{U}_h^n$ as

$$\widehat{\Phi} \mid_{S_n} = P_n \pi_n \Phi = \pi_n P_n \Phi \in \mathbf{U}_h^n,$$

where $\Phi = \Phi \mid_{S_n}$. Further, introducing P and π by the formulas

$$(P\Phi) \mid_{S_n} = P_n(\Phi \mid_{S_n})$$

and

$$(\pi\Phi) \mid_{S_n} = \pi_n(\Phi \mid_{S_n})$$

respectively, we can choose $\widehat{\Phi} \in \mathbf{U}_h$ such that $\widehat{\Phi} = P\pi\Phi = \pi P\Phi$.

Now, we define the residuals for the computed solution $\mathbf{u}_h$ by the formula

$$R_0 = \mathbf{u}_{h,t} + A\mathbf{u}_h,$$

$$R_1 = (\mathbf{u}_{h,+}^n - \mathbf{u}_{h,-}^n)/k_n \quad \text{on} \quad S_n,$$

$$R_2 = (P_n - \mathscr{I})\mathbf{u}_{h,-}^n/k_n \quad \text{on} \quad S_n,$$

where $\mathscr{I}$ is the identity operator. Below, in our a posteriori approach, we will see how these residuals appear in a natural way.

To estimate I and II, we use stability estimates based on the following interpolation estimate for the projection operator P.

Lemma 3.1. *There is a constant C such that for a given residual $R \in L_2(\Omega)$*

$$| \, (R, \Phi - P\Phi)_\Omega \, | \leqslant C \parallel h^2(\mathscr{I} - P)R \parallel_{L_2^{\Psi^{-1}}(\Omega)} \parallel \Phi \parallel_{\dot{H}^{2,\Psi}(\Omega)}, \tag{3.9}$$

where $\parallel \cdot \parallel_{\dot{H}^{2,\Psi}}$ is the Ψ-weighted seminorm.

We omit the proof since the arguments generalize the proof in the case of one spatial dimension presented in [9] (cf. also [11]).

Now, we prove the a posteriori error estimates by bounding the terms I and II in the error representation formula (3.8). For this purpose, we introduce the stability factors (cf. [1] and [14]) associated with discretization in time and spatial variables and defined by the formulas

$$\gamma_{\mathbf{e}}^t = \frac{\parallel \Phi_t \parallel_{L_2^{\Psi}(\Omega)}}{\parallel \mathbf{e} \parallel_{L_2^{\Psi^{-1}}(\Omega)}} \tag{3.10}$$

and

$$\gamma_{\mathbf{e}}^{x} = \frac{\| \Phi \|_{\dot{H}^{2,\Psi}(\Omega)}}{\| \mathbf{e} \|_{L_2^{\Psi-1}(\Omega)}} \tag{3.11}$$

respectively. Combining (3.8), the interpolation estimate (3.9), the strong stability factors (3.10), and (3.11), we derive the $L_2(L_2)$ a posteriori error estimates for the finite element scheme (3.1).

Theorem 3.2. *Let* $\mathbf{u}$ *be the solution to the continuous problem* (1.2), *and let* $\mathbf{u}_h$ *be its finite element approximation given by formula* (3.1)*. Then the error* $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$ *satisfies the estimate*

$$\begin{aligned} \| \mathbf{e} \|_{L_2^{\Psi-1}(\Omega)} &\leqslant C\gamma_{\mathbf{e}}^{x} \| h^2(\mathscr{I} - P)R_0 \|_{L_2^{\Psi-1}(\Omega)} + C\gamma_{\mathbf{e}}^{t} \| k_n R_1 \|_{L_2^{\Psi-1}(\Omega)} \\ &+ \gamma_{\mathbf{e}}^{x} \| h^2 R_2 \|_{L_2^{\Psi-1}(\Omega)} + \gamma_{\mathbf{e}}^{t} \| k_n R_2 \|_{L_2^{\Psi-1}(\Omega)} . \end{aligned} \tag{3.12}$$

Proof. Using the above notation, from (3.8) we have

$$\| \mathbf{e} \|_{L_2^{\Psi-1}(\Omega)}^2 = \sum_{n=0}^{N-1} (R_0, \widehat{\Phi} - \Phi)_n + \sum_{n=0}^{N-1} \langle [\mathbf{u}_h], (\widehat{\Phi} - \Phi)_+ \rangle_n := I + II.$$

We estimate I and II separately. Writing

$$\widehat{\Phi} - \Phi = \widehat{\Phi} - P\Phi + P\Phi - \Phi$$

and using the equality $\widehat{\Phi}_n = \pi_n P\Phi$, we find

$$\begin{aligned} I &= \sum_{n=0}^{N-1} (R_0, \widehat{\Phi}_n - P\Phi + P\Phi - \Phi)_n \\ &= \sum_{n=0}^{N-1} (R_0, (\pi_n - \mathscr{I})P\Phi)_n + \sum_{n=0}^{N-1} (R_0, P\Phi - \Phi)_n \\ &\leqslant C \| h^2(\mathscr{I} - P)R_0 \|_{L_2^{\Psi-1}(\Omega)} \| \Phi \|_{\dot{H}^{2,\Psi}(\Omega)}, \end{aligned}$$

where we used the fact that R_0 is constant in time and, by the definition of the projections, the contribution of the first term in the first sum is zero. To estimate II, we use (3.9) and

$$\Phi_+^n(x) = \Phi(x,t) - \int_{t_n}^{t} \frac{\partial}{\partial \tau} \Phi(x,\tau)\, d\tau.$$

Integrating over I_n, we find

$$k_n \Phi_+^n(x) = \int_{I_n} \Phi(x,t) dt - \int_{I_n} \int_{t_n}^{t} \Phi_\tau(x,\tau) d\tau dt, \tag{3.13}$$

where $\Phi_\tau = \dfrac{\partial \Phi}{\partial \tau}$ and $\widehat{\Phi}_n = \widehat{\Phi}(\cdot, t_n)$. Now, we write II as follows:

$$II = \sum_{n=0}^{N-1} \Big\langle k_n \frac{[\mathbf{u}_h]}{k_n}, (\widehat{\Phi} - \Phi)_+ \Big\rangle_n = \sum_{n=0}^{N-1} \Big\langle k_n \frac{[\mathbf{u}_h]}{k_n}, (\widehat{\Phi}_n - P\Phi + P\Phi - \Phi)_+ \Big\rangle_n$$

$$= \sum_{n=0}^{N-1} \Big\langle k_n \frac{[\mathbf{u}_h]}{k_n}\,,\, (\widehat{\Phi}_n - P\Phi)_+ \Big\rangle_n + \sum_{n=0}^{N-1} \Big\langle k_n \frac{[\mathbf{u}_h]}{k_n}\,,\, (P\Phi - \Phi)_+ \Big\rangle_n$$
$$:= II_1 + II_2.$$

To estimate II_1, we use (3.13) and get

$$II_1 = \sum_{n=0}^{N-1} \langle k_n R_1\,,\, (\widehat{\Phi}_n)_+ - P\Phi_+ \rangle_n = \sum_{n=0}^{N-1} \langle R_1\,,\, k_n \widehat{\Phi}_n - P k_n \Phi_+ \rangle_n$$
$$= \sum_{n=0}^{N-1} \Big\langle R_1\,,\, k_n \widehat{\Phi}_n - \int_{I_n} P\Phi(\cdot,t)dt + \int_{I_n}\int_{t_n}^{t} P\Phi_\tau(\cdot,\tau)d\tau dt \Big\rangle_n$$
$$= \sum_{n=0}^{N-1} \int_{I_n}\int_{t_n}^{t} \langle R_1\,,\, P\Phi_\tau(\cdot,\tau)\rangle_n d\tau dt$$
$$\leqslant \| k_n R_1 \|_{L_2^{\Psi^{-1}}(\Omega_T)} \| P\Phi_t \|_{L_2^{\Psi}(\Omega_T)}$$
$$\leqslant \| k_n R_1 \|_{L_2^{\Psi^{-1}}(\Omega_T)} \| \Phi_t \|_{L_2^{\Psi}(\Omega_T)},$$

were $\Omega_T := \Omega \times [0,T]$. As for the II_2-terms, we can write

$$II_2 = \sum_{n=0}^{N-1} \Big\langle k_n \frac{[\mathbf{u}_h]}{k_n}\,,\, (P\Phi - \Phi)_+ \Big\rangle_n = \sum_{n=0}^{N-1} \Big\langle \frac{\mathbf{u}_{h,+}^n - \mathbf{u}_{h,-}^n}{k_n}\,,\, (P_n - I)k_n \Phi_+ \Big\rangle_n$$
$$= \sum_{n=0}^{N-1} \Big\langle \frac{P_n \mathbf{u}_{h,-}^n - \mathbf{u}_{h,-}^n}{k_n}\,,\, (P_n - I)\Big(\int_{I_n} \Phi(\cdot,t)dt - \int_{I_n}\int_{t_n}^{t} \Phi_\tau(\cdot,\tau)d\tau dt \Big) \Big\rangle_n$$
$$= \sum_{n=0}^{N-1} \int_{I_n} \Big\langle \frac{(P_n - I)\mathbf{u}_{h,-}^n}{k_n}\,,\, (P_n - I)\Phi(\cdot,t) \Big\rangle_n dt$$
$$- \sum_{n=0}^{N-1} \int_{I_n}\int_{t_n}^{t} \Big\langle \frac{(P_n - I)\mathbf{u}_{h,-}^n}{k_n}\,,\, (P_n - I)\Phi_\tau(\cdot,t) \Big\rangle_n d\tau dt$$
$$\leqslant \| k_n R_2 \|_{L_2^{\Psi^{-1}}(\Omega_T)} \| \Phi \|_{\dot{H}^{2,\Psi}(\Omega_T)} + \| k_n R_2 \|_{L_2^{\Psi^{-1}}(\Omega_T)} \| \Phi_t \|_{L_2^{\Psi}(\Omega_T)}\ .$$

The final estimate is obtained by collecting the terms and using the definition of the stability factors (3.10) and (3.11). □

3.3 A posteriori error estimates in $L_\infty(L_2)$

We derive a posteriori error bounds in the $L_\infty(L_2)$-norm for the scheme (3.1). For this purpose, we introduce the dual problem

$$
\begin{aligned}
&L^*\Phi \equiv -\Phi_t + A^T\Phi = 0 \quad \text{in} \quad \Omega, \quad 0 < t < T, \\
&\Phi(x,T) = E, \quad x \in \Omega,
\end{aligned} \tag{3.14}
$$

where E satisfies the Poisson equation

$$
-\Delta E = \mathbf{e}, \quad e(x) = \mathbf{u}(x) - \mathbf{u}_h(x), \quad x \in \Omega., \tag{3.15}
$$

We introduce the *energy norm*

$$
||\mathbf{e}||_{L_2(\Omega)} = (\nabla E(T), \nabla E(T))_\Omega^{1/2}.
$$

Using (3.15) and integrating by parts, we get

$$
\begin{aligned}
||\mathbf{e}||^2_{L_2(\Omega)} &= ||\nabla_x E||^2_{L_2(\Omega)} \\
&= \langle \mathbf{e}_-\,,\,\Phi\rangle_N + \sum_{n=0}^{N-1}(\mathbf{e}, L^*\Phi)_n = \langle \mathbf{e}_-\,,\,\Phi\rangle_N + \sum_{n=0}^{N-1}(\mathbf{e}\,,\,-\Phi_t + A^T\Phi)_n \\
&= \langle \mathbf{e}_-\,,\,\Phi\rangle_N - \sum_{n=0}^{N-1}\mathbf{e}^T.\Phi|_{t_n}^{t_{n+1}} + \sum_{n=0}^{N-1}(\mathbf{e}_t\,,\,\Phi)_n + \sum_{n=0}^{N-1}(A\mathbf{e}, \Phi)_n \\
&= \sum_{n=0}^{N-1}(\mathbf{e}_t\,,\,\Phi)_n + \sum_{n=0}^{N-1}(A\mathbf{e}\,,\,\Phi)_n - \sum_{n=0}^{N-1}\langle[\mathbf{u}_h], \Phi_+\rangle_n \\
&= \sum_{n=0}^{N-1}(-\mathbf{u}_{h,t} - A\mathbf{u}_h\,,\,\Phi)_n - \sum_{n=0}^{N-1}\langle[\mathbf{u}_h]\,,\,\Phi_+\rangle_n.
\end{aligned}
$$

Using the Galerkin orthogonality (2.3), we can subtract $\widehat{\Phi} \in \mathbf{U}_h$ from Φ on the right-hand side without changing the norm:

$$
||\mathbf{e}||^2_{L_2(\Omega)} = \sum_{n=0}^{N-1}(\mathbf{u}_{h,t} + A\mathbf{u}_h\,,\,\widehat{\Phi} - \Phi)_n + \sum_{n=0}^{N-1}\langle[\mathbf{u}_h]\,,\,(\widehat{\Phi} - \Phi)_+\rangle_n. \tag{3.16}
$$

Here, we again need to introduce the stability factors (cf. (3.10)–(3.11)), but this time in modified norms, adequate in the study of the fully discrete (space–time discretization) problem in the L_∞-norm:

$$
\gamma_E^t = \frac{\|\,\Phi_t\,\|_{L_1(L_2(\Omega))}}{\|\,\mathbf{e}\,\|_{L_2(\Omega)}} \tag{3.17}
$$

and

$$
\gamma_E^x = \frac{\|\,\Phi\,\|_{L_1(\dot{H}^2(\Omega))}}{\|\,\mathbf{e}\,\|_{L_2(\Omega)}}. \tag{3.18}
$$

Using the interpolation estimates (3.9) and arguing in a similar way as in the proof of Theorem 3.2, we get the following $L_\infty(L_2)$–estimate.

Theorem 3.3. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the same as in Theorem* 3.2. *Then the error* $\mathbf{e} : \mathbf{u} - \mathbf{u}_h$ *satisfies the estimate*

$$
\begin{aligned}
\|\,\mathbf{e}\,\|_{L_2(\Omega)} \leqslant{}& C\gamma_E^x \,\|\, h^2(\mathscr{I} - P)R_0 \,\|_{L_\infty(L_2(\Omega))} + C\gamma_E^t \,\|\, k_n R_1 \,\|_{L_\infty(L_2(\Omega))} \\
&+ \gamma_E^x \,\|\, h^2 R_2 \,\|_{L_\infty(L_2(\Omega))} + \gamma_E^t \,\|\, k_n R_2 \,\|_{L_\infty(L_2(\Omega))},
\end{aligned} \tag{3.19}
$$

where

$$||\Phi||_{L_\infty(L_2(\Omega))} = \sup_{0<t<T} ||\Phi(\cdot,t)||_{L_2(\Omega)},$$

$$||\Phi||_{L_1(L_2(\Omega))} = \int\limits_0^T ||\Phi(\cdot,t)||_{L_2(\Omega)} dt.$$

The proof of this theorem is a modification of that of Theorem 3.2 and, therefore, is omitted. The only difference consists in the use of the Hölder inequality $\|fg\|_1 \leqslant \|f\|_p \|g\|_q$, $1 \leqslant p, q \leqslant \infty$, $1/p + 1/q = 1$ ($p = q = 2$ in Theorem 3.2, whereas $p = 1$, $q = \infty$ in Theorem 3.3).

4 Analytical Strong Stability Estimates in $L_2(L_2)$

We need to estimate the strong stability factors used in the previous sections. Let us consider an a posteriori error estimate of type (3.12) in Theorem 3.2 which is based on the dual problem

$$\begin{aligned} &L^*\Phi \equiv -\Phi_t + A^T\Phi = \Psi^{-1}\mathbf{e} \quad \text{in} \quad \Omega, \\ &\Phi(x,T) = 0, \quad x \in \Omega. \end{aligned} \tag{4.1}$$

We prove a strong stability estimate for the dual problem (4.1).

Theorem 4.1. *For a given positive weight function $\Psi(x,t)$ the solution Φ to the dual problem* (4.1) *satisfies the estimate*

$$||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega = ||\mathbf{e}||_{L_2^{\Psi^{-1}}(\Omega)}.$$

Proof. We multiply the equation in (4.1) by $-\Psi(\Phi_t - A^T\Phi)$ and integrate over Ω to get

$$\int\limits_\Omega \Psi(\Phi_t - A^T\Phi)^2 dx = -\int\limits_\Omega \mathbf{e}(\Phi_t - A^T\Phi) dx \leqslant \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||_\Omega^2 + \frac{1}{2}||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2.$$

This yields

$$||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2 \leqslant ||\Psi^{-1/2}\mathbf{e}||_\Omega^2. \tag{4.2}$$

Similarly, multiplying the equation in (4.1) by $\mathbf{e}$ and integrating over Ω, we get

$$\int\limits_\Omega \mathbf{e}^2\Psi^{-1} dx = ||\Psi^{-1/2}\mathbf{e}||_\Omega^2 = -\int\limits_\Omega \mathbf{e}(\Phi_t - A^T\Phi)^2 dx \leqslant \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||_\Omega^2 + \frac{1}{2}||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2,$$

which yields

$$|||\Psi^{-1/2}\mathbf{e}||_\Omega^2 \leqslant ||\Psi^{1/2}(\Phi_t - A^T\Phi)||_\Omega^2. \tag{4.3}$$

Combining (4.2) and (4.3), we complete the proof. □

Theorem 4.2. *If $\Psi(x,t)$ is a positive weight function such that*

$$\Psi_t + A^T\Psi \geqslant -\Psi \quad in \;\; \Omega, \tag{4.4}$$

then the solution Φ to the problem (4.1) satisfies the estimate

$$||\Psi^{1/2}\Phi||_\Omega \leqslant C_T||\mathbf{e}||_{L_2^{\Psi^{-1}}(\Omega_T)}, \quad C_T = e^T.$$

Proof. Multiplying the equation in (4.1) by $\Psi\Phi$ and integrating over Ω, we get the equality

$$-(\Phi_t, \Psi\Phi(t)) + (A^T\Phi, \Psi\Phi(t)) = (\mathbf{e}, \Phi(t))$$

which can be written as

$$-\frac{1}{2}\frac{d}{dt}||\Psi^{1/2}\Phi(t)||^2 + \frac{1}{2}(\Psi_t, \Phi^2(t)) + (A^T\Phi, \Psi\Phi(t)) = (\mathbf{e}, \Phi(t)).$$

Integrating by parts in the spatial variables and then using (4.1) together with the Cauchy–Schwarz inequality, we get

$$-\frac{1}{2}\frac{d}{dt}||\Psi^{1/2}\Phi(t)||^2 + \frac{1}{2}(\Psi_t + A^T\Psi, \Phi^2(t)) \leqslant ||\Psi^{-1/2}\mathbf{e}||||\Psi^{1/2}\Phi|| \leqslant \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||^2 + \frac{1}{2}||\Psi^{1/2}\Phi||^2.$$

Using (4.4), we find

$$-\frac{1}{2}\frac{d}{dt}||\Psi^{1/2}\Phi(t)||^2 - \frac{1}{2}(\Psi\,,\,\Phi^2(t)) \leqslant \frac{1}{2}||\Psi^{-1/2}\mathbf{e}||^2 + \frac{1}{2}||\Psi^{1/2}\Phi||^2.$$

Integrating in the time variable over (t,T) and using the equality $\Psi(\cdot,T) = 0$, we get

$$||\Psi^{1/2}\Phi(t)||^2 \leqslant ||\Psi^{-1/2}\mathbf{e}||^2_{\Omega_T} + 2\int_t^T ||\Psi^{1/2}\Phi(t)||^2 ds.$$

By the Gronwall inequality, we arrive at the desired result

$$||\Psi^{1/2}\Phi(t)||^2 \leqslant \mathbf{e}^{2T}||\Psi^{-1/2}\mathbf{e}||^2_{\Omega_T}.$$

The theorem is proved. □

The proof for the analytical strong stability estimates in $L_\infty(L_2)$ is similar to the $L_2(L_2)$ case.

5 Numerical Experiments

In this section, we consider a general numerical scheme – the *streamline diffusion method* introduced and developed by Hughes and Brooks (cf., for example, [7]). The streamline diffusion method is a modification of the Galerkin method designed, basically, for numerical investigations of hyperbolic type problems (the system (1.1) rather than the system (1.2)). Roughly speaking, compared to the standard Galerkin method, the streamline diffusion method is a

Petrov–Galerkin type method, with modified test functions, that combines high accuracy and good stability properties. Generally, the convergence rate for the streamline diffusion method in the case of hyperbolic problems is of one order lower than that in the elliptic and parabolic cases. In the streamline diffusion scheme, an appropriate choice of the test functions gives rise to a weakly imposed diffusion term which improves the convergence rate of the streamline diffusion method for hyperbolic problems by $\sim \mathscr{O}(h^{1/2})$.

We implement a general scheme (both with and without a streamline diffusion modification) applied to solve a one-dimensional time dependent coupling of two hyperbolic equations. For the sake of simplicity, we denote

$$\Upsilon^T := (1,1,1,1), \quad U := (u_1,u_2,u_3,u_4)^T, \quad V := (v_1,v_2,v_3,v_4)^T.$$

For an arbitrary operator Γ we introduce the vector and componentwise products by the formulas

$$\Gamma(U \otimes \Upsilon) = \Gamma U = (\Gamma u_1, \Gamma u_2, \Gamma u_3, \Gamma u_4)^T$$

and

$$U \odot V = (u_1v_1, u_2v_2, u_3v_3, u_4v_4)^T$$

respectively. Below, we will discuss computational aspects of the approximate solution to the system (1.1) by using the streamline diffusion method for (1.2): *for* $n = 0, \dots, N-1$ *find* $\mathbf{u}^n \in \mathbf{U}^n$ *such that*

$$(\mathbf{u}^n_{h,t} + A\mathbf{u}^n_h\,,\ \mathbf{g}_h + \delta(\mathbf{g}_{h,t} + A\mathbf{g}_h))_n + \langle \mathbf{u}^n_{h,+}\,,\ \mathbf{g}_{h,+}\rangle_n = \langle \mathbf{u}^n_{h,-}\,,\ \mathbf{g}_{h,+}\rangle_n, \tag{5.1}$$

where δ is the streamline diffusion parameter (usually ,$\delta \sim h$). Since (1.2) is a parabolic problem, the improving potential of the δ-term ($\delta \neq 0$ in (5.1)) is rather minimal. Nevertheless, the scheme removes the oscillatory behavior near the boundary layers. We use finite element approximation on a space–time slab with the trial functions that are piecewise polynomial in spatial variables and piecewise linear in time, i.e., for $(x,t) \in S_n$. We look for the approximate solution

$$\mathbf{u}^n_h(x,t) = \sum_{i=1}^{M} \left\{\varphi_i(x)(\theta_1(t)\widetilde{\mathbf{u}}^n_i + \theta_2(t)\mathbf{u}^{n+1}_i)\right\} \otimes \Upsilon$$

$$= \begin{pmatrix} u^n_h(x,t) = \sum_{i=1}^{M} \varphi_i(x)(\theta_1(t)\widetilde{u}^n_i + \theta_2(t)u^{n+1}_i) \\ \phi^n_h(x,t) = \sum_{i=1}^{M} \varphi_i(x)(\theta_1(t)\widetilde{\phi}^n_i + \theta_2(t)\phi^{n+1}_i) \\ v^n_h(x,t) = \sum_{i=1}^{M} \varphi_i(x)(\theta_1(t)\widetilde{v}^n_i + \theta_2(t)v^{n+1}_i) \\ \psi^n_h(x,t) = \sum_{i=1}^{M} \varphi_i(x)(\theta_1(t)\widetilde{\psi}^n_i + \theta_2(t)\psi^{n+1}_i) \end{pmatrix} \tag{5.2}$$

where $\varphi_i(x_j) = \delta_{ij}$ $(j = 1, \dots, M)$ is the spatial shape function at node i $(i = 1, \dots, M)$ and $\theta_1(t)$, $\theta_2(t)$ are the piecewise linear basis functions for the subinterval $(t_n, t_{n+1}]$ in the time discretization:

$$\theta_1(t) = \frac{t_{n+1} - t}{t_{n+1} - t_n} = \frac{t_{n+1} - t}{k}$$

and

$$\theta_2(t) = \frac{t - t_n}{t_{n+1} - t_n} = \frac{t - t_n}{k}.$$

Also, the nodal value of $\mathbf{u}$ for node i at $(t_n)_+$ and $(t_{n+1})_-$ are denoted by $\widetilde{\mathbf{u}}_i^n$ and $\mathbf{u}_i^{n+1}$ respectively. Then, on each slab S_n, the test function $\mathbf{g}_h^n$ is defined as a linear combination of $\varphi_j(x)\theta_1(t)$ and $\varphi_j(x)\theta_2(t)$ for $j = 1, \ldots, M$. Then (5.1) is equivalent to the following system of equations with the unknowns $\widetilde{\mathbf{u}}_i^n$ and $\mathbf{u}_i^{n+1}$: *for* $n = 0, 1, \ldots, N-1$ *and all* $j = 0, 1, \ldots M$ *find* $\widetilde{\mathbf{u}}_i^n$ *and* $\mathbf{u}_i^{n+1}$ *such that*

$$\sum_{i=1}^{M} \int_{S_n} \Bigg\{ \Bigg[(\varphi_i(x)(\frac{\mathbf{u}_i^{n+1} - \widetilde{\mathbf{u}}_i^n}{k}) \odot \Upsilon) + A\left(\varphi_i(x)(\theta_1(t)\widetilde{\mathbf{u}}_i^n + \theta_2(t)\mathbf{u}_i^{n+1}) \odot \Upsilon\right) \Bigg]$$
$$\odot \Bigg[(\varphi_j(x)\theta_1(t))\Upsilon + (\delta((\frac{-1}{k})\varphi_j(x))\Upsilon + A(\varphi_j(x)\theta_1(t)\Upsilon) \Bigg] \odot (dxdt\Upsilon) \Bigg\} \otimes \Upsilon = 0 \qquad (5.3)$$

and

$$\sum_{i=1}^{M} \int_{S_n} \Big\{ \Big[\varphi_i(x)(\frac{\mathbf{u}_i^{n+1} - \widetilde{\mathbf{u}}_i^n}{k}) \odot \Upsilon + A(\varphi_i(x)(\theta_1(t)\widetilde{\mathbf{u}}_i^n + \theta_2(t)\mathbf{u}_i^{n+1}) \odot \Upsilon) \Big]$$
$$\odot \Big[\varphi_j(x)\theta_2(t)\Upsilon + \delta((\frac{1}{k})\varphi_j(x))\Upsilon + A(\varphi_j(x)\theta_2(t)\Upsilon) \Big] \odot (dxdt\Upsilon) \Big\} \otimes \Upsilon$$
$$+ \sum_{i=1}^{M} \int_{\Omega} \{\varphi_j(x)\varphi_i(x)(\widetilde{\mathbf{u}}_i^n - \mathbf{u}_i^n) \odot (\Upsilon dx)\} \otimes \Upsilon = 0. \qquad (5.4)$$

For φ_i we take the *hat-function* (such functions form a basis for piecewise linear functions)

$$\varphi_i(x) = \frac{1}{h} \begin{cases} x - x_{i-1}, & x \in [x_{i-1}, x_i], \\ x_{i+1} - x, & x \in [x_i, x_{i+1}], \\ 0 & \text{elsewhere} \end{cases}$$

defined on a uniform partition $\mathscr{T}_h$ of $\Omega = [a, b]$, with the mesh size $h := x_{i+1} - x_i$. Thus, we can compute the entries of the coefficient matrices as follows:

$$M_{ij} = \left(\int_{\Omega} \varphi_i(x)\varphi_j(x) dx \right) \Upsilon = \frac{h}{6}\Upsilon \begin{cases} 4, & j = i, \\ 1, & j = i+1, j = i-1, \\ 0 & \text{elsewhere}, \end{cases}$$

$$B_{ij} = \int_{\Omega} (A(\varphi_i\varphi_j\Upsilon) \odot (dx\Upsilon)) \otimes \Upsilon = \int_{\Omega} \{(\varphi_i\Upsilon) \odot A(\varphi_j\Upsilon) \odot (\Upsilon dx)\} \otimes \Upsilon,$$

$$F_{ij} = \int_{\Omega} \{A(\varphi_i\Upsilon) \odot A(\varphi_j\Upsilon) \odot (dx\Upsilon)\} \otimes \Upsilon.$$

Further, using the trivial identities

$$\int_{t_n}^{t_{n+1}} \theta_1(t)\theta_2(t)dt = \frac{k}{6},$$

$$\int_{t_n}^{t_{n+1}} \theta_i^2(t)dt = \frac{k}{3}, \quad i = 1, 2,$$

$$\int_{t_n}^{t_{n+1}} \theta_i(t)dt = \frac{k}{2}, \quad i = 1, 2,$$

we can write (5.3) and (5.4) in the following equivalent form:

$$\sum_{i=1}^{M} \Big[\Big(\frac{1}{2} - \frac{\delta}{k^2}\Big)M_{ij} + \frac{k}{6}B_{ij} + \frac{\delta k}{6}F_{ij}\Big]^T \mathbf{u}_i^{n+1} + \sum_{i=1}^{M} \Big[\Big(\frac{\delta}{k^2} - \frac{1}{2}\Big)M_{ij} + \Big(\frac{k}{3} + \delta\Big)B_{ij} + \frac{k\delta}{3}F_{ij}\Big]^T \widetilde{\mathbf{u}}_i^n = 0 \tag{5.5}$$

and

$$\sum_{i=1}^{M} \Big[\Big(\frac{1}{2} + \frac{\delta}{k^2}\Big)M_{ij} + \Big(\frac{k}{3} + \delta\Big)B_{ij} + \frac{\delta k}{3}F_{ij}\Big]^T \mathbf{u}_i^{n+1} + \sum_{i=1}^{M} \Big[\Big(\frac{1}{2} - \frac{\delta}{k^2}\Big)M_{ij} + \frac{k}{6}B_{ij} + \frac{k\delta}{6}F_{ij}\Big]^T \widetilde{\mathbf{u}}_i^n + \sum_{i=1}^{M} M_{ij}^T \mathbf{u}_i^n = 0. \tag{5.6}$$

We write these equations in the matrix form: for $n = 0, 1, \ldots, N-1$

$$\Big[\Big(\frac{1}{2} - \frac{\delta}{k^2}\Big)M + \frac{k}{6}B + \frac{\delta k}{6}F\Big]\mathbf{U}^{n+1} + \Big[\Big(\frac{\delta}{k^2} - \frac{1}{2}\Big)M + \Big(\frac{k}{3} + \delta\Big)B + \frac{k\delta}{3}F\Big]\widetilde{\mathbf{U}}^n = 0 \tag{5.7}$$

and

$$\Big[\Big(\frac{1}{2} + \frac{\delta}{k^2}\Big)M + \Big(\frac{k}{3} + \delta\Big)B + \frac{\delta k}{3}F\Big]\mathbf{U}^{n+1} + \Big[\Big(\frac{1}{2} - \frac{\delta}{k^2}\Big)M + \frac{k}{6}B + \frac{k\delta}{6}F\Big]\widetilde{\mathbf{U}}^n + M\mathbf{U}^n = 0, \tag{5.8}$$

where

$$\mathbf{U}^n = [\mathbf{u}_1^n, \ldots, \mathbf{u}_M^n]^T, \quad \mathbf{U}^{n+1} = [\mathbf{u}_1^{n+1}, \ldots, \mathbf{u}_M^{n+1}]^T, \quad \widetilde{\mathbf{U}}^n = [\widetilde{\mathbf{u}}_1^n, \ldots, \widetilde{\mathbf{u}}_M^n].$$

5.1 Test problem

We carried out experimental computations to solve (5.7) and (5.8) by using an AMD Opteron computer with 15 Gigabytes RAM memory with 2.2 GHz CPU. For each slab S_n we choose a partition of the spatial interval into the subinterval $J_i^n = (x_{i-1}^n, x_i^n)$ with $h_i^n = x_i^n - x_{i-1}^n$. For small $h > 0$ we denote by T_h^n a triangulation of the slab S_n into quasiuniform space–time triangular elements K (cf. Figure 1) satisfying the minimum angle condition. The triangulation for S_n may be chosen independently of that of S_{n-1}. Then a projection (from one face to the other) is necessary. For the sake of simplicity we assume the same quasiuniformity in all slabs and small shape variations. We use finite element approximations on space and time with trial functions that are piecewise polynomial in the spatial variables and piecewise linear in time. First, we compute a numerical solution for a given δ with $\Delta x := h = 0.01$, $\Delta t := k = 0.0005$, and discretize (1.1) by assuming that $\Omega = [-1, 1]$ and $\alpha(x) = x^2$:

$$u_0 = v_0 = \begin{cases} 0, & |x| \geqslant 1, \\ \dfrac{x(x+1)}{2+x}, & -1 \leqslant x \leqslant 0, \\ \dfrac{x(x-1)}{2+x}, & 0 \leqslant x \leqslant 1. \end{cases}$$

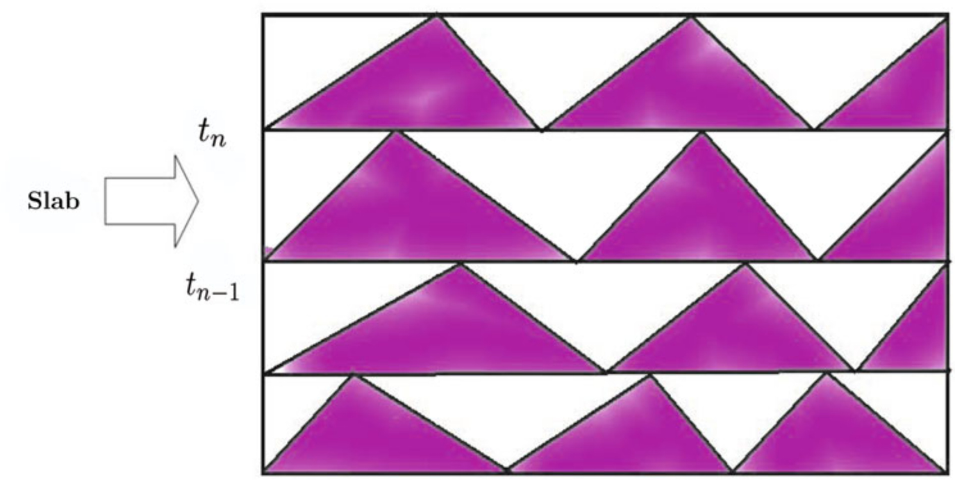

Figure 1. Slabs on a rectangle.

Figure 2 shows the results of our numeric computations of the rate of convergence of the error, in the L_2-norm, for u and v, i.e, for $\|u - u_h\|$ and $\|v - v_h\|$. The results are shown in even time steps at $t_j = j \times \Delta t$ and a uniform partition of the spatial domain $\Omega = [-1, 1]$, namely, the partition $x_0 = -1.00$, $x_i = x_0 + i \times \Delta x$. For example, under the above choice, the time is between $t_0 = 0$ and $t_N = 200 \times \Delta t = 10^{-1}$. We plot the absolute error for

$$u_{ij} := u(i\Delta x, j\Delta t), \quad v_{ij} := v(i\Delta x, j\Delta t)$$

in a grid with the discrete times

$$t = 0 \times \Delta t,\ 10^{-3} = 2 \times \Delta t, \ldots, 10^{-1} = 200 \times \Delta t$$

and the spatial nodes

$$x_0 = -1.00,\ x_1 = -0.99, \ldots, x_M = 0,\ x_{M+1} = 0.01, \ldots, x_{2M} = 1.00.$$

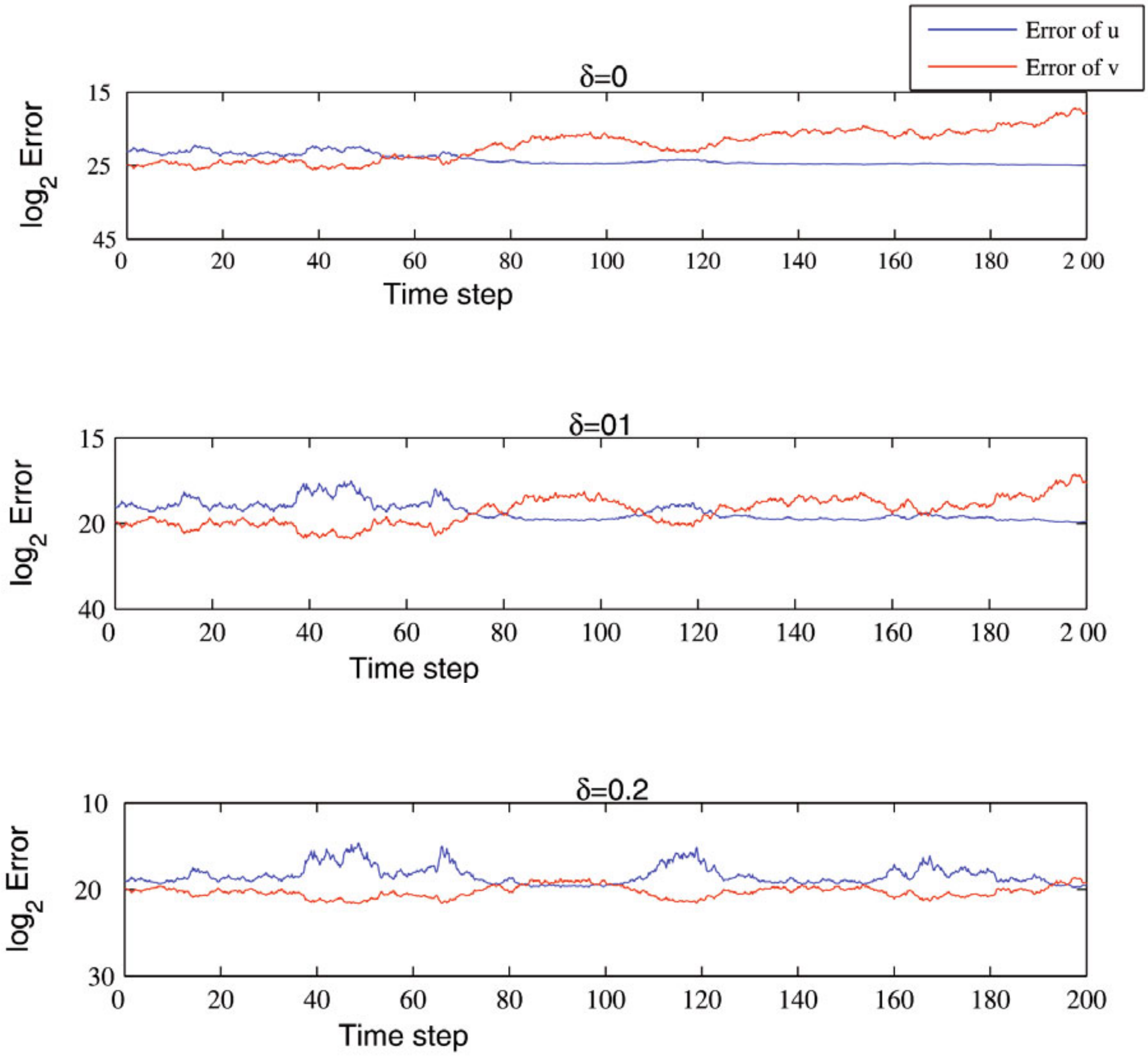

Figure 2. The behavior of error in time step for u and v.
Here, $x = -1.00,\ -0.99, \ldots, 0, 0.01, \ldots,\ 1.00$.

Finally, in Tables 1 and 2, we show the error for the approximate solution to the problem (5.1). The order of error is computed by using the logarithmic division:

$$\text{Order of error for } w \approx \ln \frac{E_{h_i}(w)}{E_{h_{i+1}}(w)}, \quad w = u, v,$$

where

$$E_{h_i}(w) = \|w(x,0) - w_{h_i}(x,0)\|_\infty, \quad w = u, v, \quad i = 1, 2, 3, 4, 5.$$

The small values for the errors are indicating the efficiency of the method. We may also observe the behavior in initial error (the error made by the approximation in the initial data $u(x,0)$ and $v(x,0)$) in two independent variables x and δ (cf. Figures 3 and 4).

Table 1. $E_{h_i}(u)$ and order of error for u obtained by the streamline diffusion method at $\delta = 0.1$ and $k = 0.01$.

x	$h_1 = 0.15$	$h_2 = 0.10$	$h_3 = 0.05$	$h_4 = 0.01$	$h_5 = 0.005$
-1.0	0.481e-5	0.362e-6	0.431e-8	0.701e-8	0.401e-9
0.0	0.436e-6	0.911e-8	0.454e-7	0.983e-9	0.932e-12
1.0	0.734e-6	0.743e-7	0.713e-10	0.801e-9	0.210e-9
order	-	2.587	2.076	1.868	3.508

Table 2. $E_{h_i}(u)$ and order of error for v obtained by the streamline diffusion method at $\delta = 0.1$ and $k = 0.01$.

x	$h_1 = 0.15$	$h_2 = 0.10$	$h_3 = 0.05$	$h_4 = 0.01$	$h_5 = 0.005$
-1.0	0.4231e-5	0.362e-5	0.913e-7	0.634e-9	0.421e-9
0.0	0.206e-4	0.201e-6	0.934e-8	0.785e-9	0.762e-9
1.0	0.134e-6	0.176e-7	0.903e-8	0.701e-8	0.401e-10
order	-	1.739	3.680	2.569	2.812

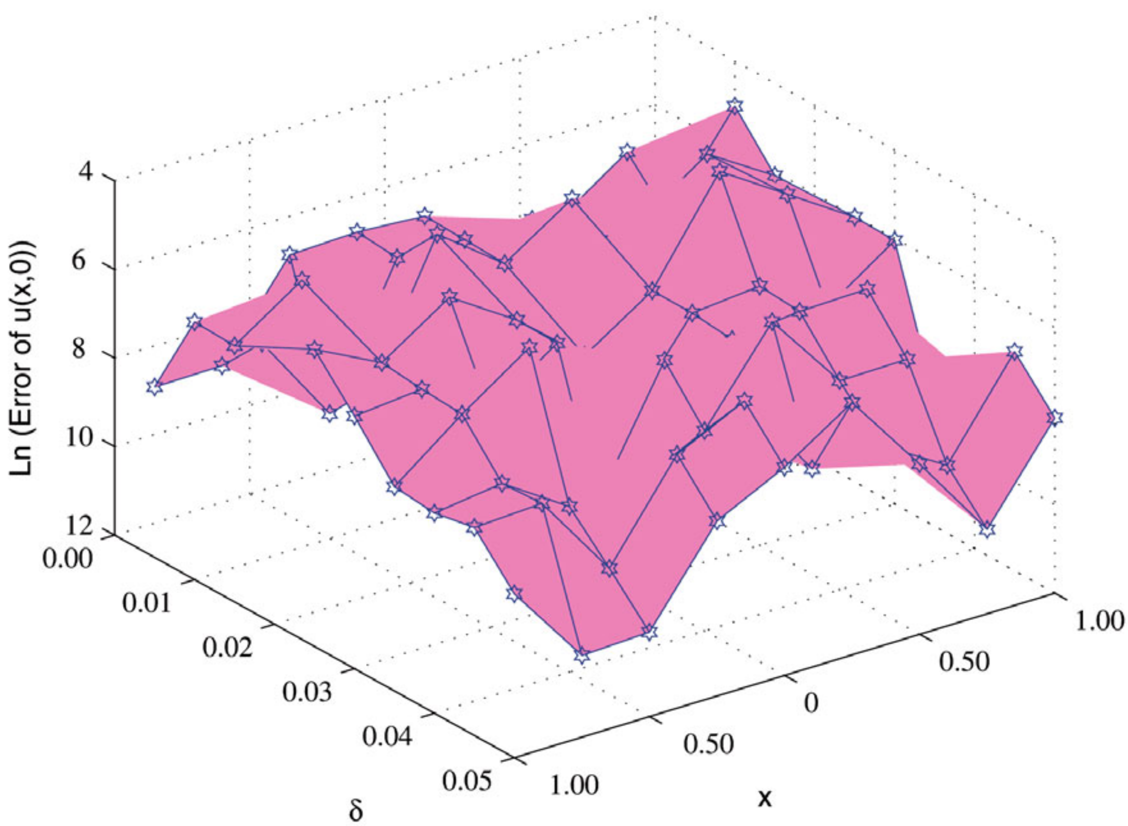

Figure 3: The error of the streamline diffusion method for $u(x, 0)$.

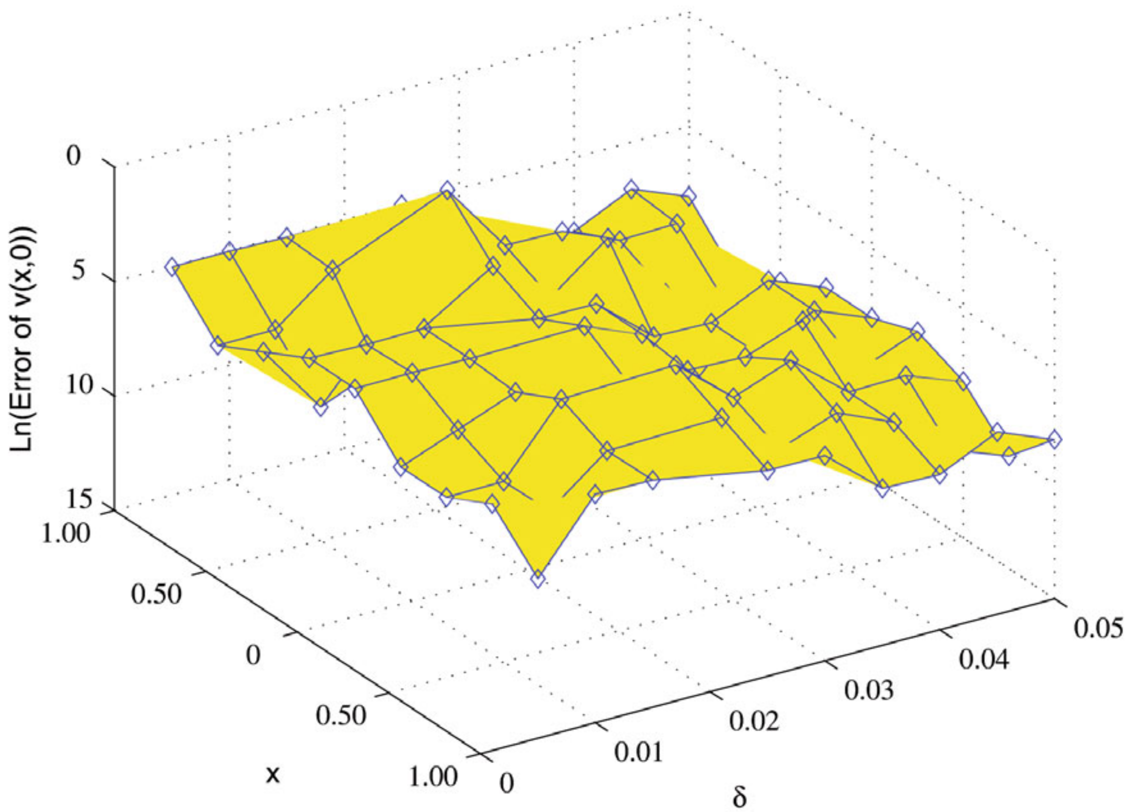

Figure 4: The error of the streamline diffusion method for $v(x, 0)$.

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