TMA 682 Lecture Notes

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Contents

Chapter 1

Laplace Transformation

Laplace transformation is a powerful technique for solving differential equations with constant coefficients. Areas of application are widespread but traditional fields include mechanics, electronics, and automatic control engineering.

Before the advent of computers it was a tedious task to multiply numbers such as 1.4142 and 3.1416. Therefore logarithms were used to transform the complicted operation of multiplication into the simpler operation of addition via the formula

$$\log(1.4142 \cdot 3.1416) = \log(1.4141) + \log(3.1416).$$

By consulting tables of precomputed logarithms and exponentials one obtained the result 4.4429. Roughly speaking, Laplace transformation works analoguosly and reduces problems of calculus into simple algebraic problems via tables and general properties of the transform.

1.1 The Laplace Transform

1.1.1 Definition

Let $f(t)$ be a function defined for all $t \geq 0$. If the improper integral

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt, \tag{1.1.1}$$

converges for any s, then $F(s)$ is said to be the *Laplace transform*[1] of $f(t)$.

[1] Pierre Simon de Laplace (1749-1827) French mathematician.

Example 1. *Find the Laplace transform of the Heaviside step function,*

$$\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \tag{1.1.2}$$

Solution. By the definition (1.1.1) *we get, for* $s > 0$

$$F(s) = \int_0^\infty \theta(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s}\,e^{-st}\right]_0^\infty = \frac{1}{s}. \tag{1.1.3}$$

A common shorthand notation for the operation of taking the Laplace transform of a function $f(t)$ is $\mathcal{L}[f(t)]$. For example, $\mathcal{L}[\theta(t)] = \frac{1}{s}$.

Since the integral (1.1.1) has the limits 0 and ∞, it follows that $F(s)$ is not influenced by $f(t)$ when $t < 0$. As a result, if f_1 and f_2 are two functions such that $f_1 = f_2$ for $t \geq 0$, then these functions have the same Laplace transform, even if they differ for $t < 0$. Because of this ambiguity, we shall henceforth always assume that $f(t)$ is *causal*, which is to say, $f(t) = 0$ for all $t < 0$.

If $f(t)$ is not causal to begin with, we can always force it to become so by multiplying it with the *Heaviside step function* $\theta(t)$ (1.1.2). We illustrate such a case below.

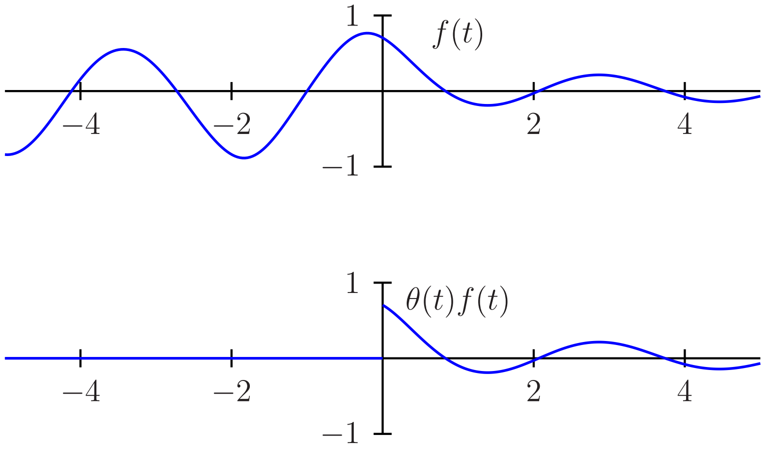

Figure 1.1: A causal restriction of the function $f(t)$.

Example 2. *Find the Laplace transform of $f(t) = e^{ct}$, where c is a constant.*

Solution. Again, by (1.1.1) *we get*

$$F(s) = \int_0^\infty e^{ct}e^{-st}\,dt = \left[-\frac{1}{s-c}\,e^{-(s-c)t}\right]_0^\infty = \frac{1}{s-c}. \tag{1.1.4}$$

Note that, for the above integral to converge, we must assume $s > c$.

1.1.2 Existence

Not any function $f(t)$ have a Laplace transform $\mathcal{L}[f(t)]$. For example, it is easy to see that $\mathcal{L}[e^{t^2}]$ does not exist, since its associated integral diverges as $t \to \infty$. As a rule, $f(t)$ must be of *exponential order* to have a Laplace transform. By this we mean that there must exist a constant, say a, such that

$$\lim_{t\to\infty} |f(t)e^{-at}| = 0. \tag{1.1.5}$$

If this indeed is the case, then by choosing $s > a$, we see that the integrand $f(t)e^{-st}$ of (1.1.1) goes to zero as t tend to infinity and, hence, the integral for $\mathcal{L}[f(t)]$ converges. Let us formalize this result by stating it as a theorem.

Theorem 1. *If $f(t)$ is a piecewise continuous[2] function for all $t \geq 0$, and if*

$$|f(t)| \leq Me^{at}, \tag{1.1.6}$$

for some constants a and C, then the Laplace transform $F(s)$ of $f(t)$ exists.

Proof. If $|f(t)| \leq Me^{at}$ and $s > a$, then

$$|F(s)| \leq \int_0^\infty |f(t)|e^{-st}\,dt \leq \int_0^\infty Me^{-(s-a)t}\,dt = \frac{M}{s-a}. \tag{1.1.7}$$

Hence, if $s > a$ the integral (1.1.1) *converges.*

[2]A function is said to be piecewise continuous if it is discontinuous only at isolated points, and its left and right limits are defined at each discontinuity point.

1.1.3 General Properties of the Laplace Transform

Theorem 2. *Laplace transformation is a linear operation, that is, for any functions $f(t)$ and $g(t)$ whose Laplace transform exist and any constants a and b, we have*

$$\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]. \tag{1.1.8}$$

Proof. By definition, it holds that

$$\begin{aligned}\mathcal{L}[f(t)+g(t)] &= \int_0^\infty (af(t)+bg(t))e^{-st}\,dt \\ &= a\int_0^\infty f(t)e^{-st}\,dt + b\int_0^\infty g(t)e^{-st}\,dt \\ &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].\end{aligned} \tag{1.1.9}$$

Example 3. *Find the Laplace transforms of* $\sinh t = \frac{1}{2}(e^t - e^{-t})$.

Solution. Using the above linearity and (1.1.4) with $c = \pm 1$, we have

$$\mathcal{L}[\sinh t] = \mathcal{L}[\tfrac{1}{2}(e^t - e^{-t})] = \tfrac{1}{2}\mathcal{L}[e^t] - \tfrac{1}{2}\mathcal{L}[e^{-t}] = \tfrac{1}{2}(\tfrac{1}{s-1} - \tfrac{1}{s+1}) = \tfrac{1}{s^2-1}. \tag{1.1.10}$$

Example 4. *Find the Laplace transforms of* $\sin\omega t$ *and* $\cos\omega t$.

Solution. If we set $c = i\omega$ in (1.1.4) then we have

$$\begin{aligned}\mathcal{L}[e^{i\omega t}] &= \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} \\ &= \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}.\end{aligned} \tag{1.1.11}$$

On the other hand we also have

$$\mathcal{L}[e^{i\omega t}] = \mathcal{L}[\cos\omega t + i\sin\omega t] = \mathcal{L}[\cos\omega t] + i\mathcal{L}[\sin\omega t]. \tag{1.1.12}$$

Equating the real and imaginary parts of these two equations, we get

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}, \tag{1.1.13}$$

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}. \tag{1.1.14}$$

As the last examples show, the definition (1.1.1) is rarely the starting point for deriving Laplace transforms. Instead, one usually first consults a table of standard transforms, and then tries to adapt any of these to the problem at hand using a set of general properties, such as the linearity, of the Laplace transform. Below, we derive a number of other such properties and illustrate their use.

Theorem 3 (1^{st} Shifting Rule). *If $f(t)$ has the Laplace transform $F(s)$ then for any constant c, we have*

$$\mathcal{L}[e^{ct}f(t)] = F(s-c). \tag{1.1.15}$$

Proof. Inserting $e^{ct}f(t)$ directly into the definition (1.1.1) *gives, with $s > c$,*

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{ct}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-c)t}\,dt = F(s-c). \tag{1.1.16}$$

Example 5. *Find the Laplace transform of* $3e^{-2t}\cos 5t$.

Solution. By the previous example, we have

$$\mathcal{L}[\cos 5t] = \frac{s}{s^2+25}. \tag{1.1.17}$$

Applying now the linearity, and the 1^{st} Shifting Rule, we get

$$\mathcal{L}[3e^{-2t}\cos 5t] = \frac{3(s+2)}{(s+2)^2+25} = \frac{3s+6}{s^2+4t+29}. \tag{1.1.18}$$

Theorem 4 (2^{nd} Shifting Rule). *Suppose $f(t-T)$ is a function that is zero for $t \le T$, then*

$$\mathcal{L}[f(t-T)] = e^{-Ts}F(s). \tag{1.1.19}$$

Proof. Let $\tau = t - T$, then

$$\begin{aligned}\mathcal{L}[f(t-T)] &= \int_0^\infty f(t-T)e^{-st}\,dt = \int_{-\infty}^\infty f(t-T)e^{-st}\,dt \\ &= \int_{-\infty}^\infty f(\tau)e^{-s(\tau+T)}\,d\tau = e^{-Ts}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau \\ &= e^{-Ts}F(s).\end{aligned} \tag{1.1.20}$$

Introducing a generalized form of the Heaviside step function,

$$\theta(t-T) = \begin{cases} 1, & t > T, \\ 0, & t < T, \end{cases} \tag{1.1.21}$$

we can state the 2^{nd} *Shifting Rule* (1.1.19) formally as

$$\mathcal{L}[\theta(t-T)f(t-T)] = e^{-Ts}F(s). \tag{1.1.22}$$

Theorem 5. *If $f(t)$ satisfies* (1.1.6) *for some constants M and a, then*

$$\mathcal{L}[tf(t)] = -F'(s). \tag{1.1.23}$$

Proof. *Changing the order of differentiation and integration, we have*

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{\partial e^{-st}}{\partial s}\,dt \\ &= \int_0^\infty -tf(t)e^{-st}\,dt = -\mathcal{L}[tf(t)]. \end{aligned} \tag{1.1.24}$$

Example 6. *Find the Laplace transform of $t\sinh t$.*

Solution. *Recall that*

$$\mathcal{L}[\sinh t] = \frac{1}{s^2-1}. \tag{1.1.25}$$

By the last theorem, we get

$$\mathcal{L}[t\sinh t] = -\frac{d}{ds}\frac{1}{s^2-1} = \frac{2s}{(s^2-1)^2}. \tag{1.1.26}$$

Theorem 6. *If $f(t)$ satisfies* (1.1.6) *for some constants M and a, and if $\lim_{t\to 0}\frac{1}{t}f(t)$ exists, then*

$$\mathcal{L}[\tfrac{1}{t}f(t)] = \int_s^\infty F(\omega)\,d\omega. \tag{1.1.27}$$

Proof. *Let $g(t) = \frac{1}{t}f(t)$, i.e., $f(t) = tg(t)$. The previous theorem then gives $F(s) = -G'(s)$. By the fundamental theorem of calculus, and the fact that $G(s) \to 0$ as $s \to \infty$, we have*

$$G(s) = \int_s^\infty F(\omega)\,d\omega. \tag{1.1.28}$$

Example 7. *Find the Laplace transform of* $\dfrac{\sin t}{t}$.

Solution. Recall that $\mathcal{L}[\sin t] = (s^2+1)^{-1}$. *Since*

$$\lim_{t\to 0} \frac{\sin t}{t} = 1, \tag{1.1.29}$$

the assumptions of the last theorem are satisfied and thus we have

$$\mathcal{L}[\tfrac{1}{t}\sin t] = \int_s^\infty \frac{d\omega}{\omega^2+1} = [\arctan\omega]_s^\infty = \frac{\pi}{2} - \arctan s. \tag{1.1.30}$$

A fundamental property of the Laplace transform is the fact that, roughly speaking, taking the derivative of the original function $f(t)$ corresponds to multiplying its transform $F(s)$ by s.

Theorem 7. *Suppose* $f(t)$ *and* $f'(t)$ *are two continuous, piecewise smooth functions satisfying the inequality* (1.1.6) *for the same values of* M *and* a. *Then, it holds*

$$\mathcal{L}[f'(t)] = sF(s) - f(0). \tag{1.1.31}$$

Proof. Integrating by parts, we have

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \big[f(t)e^{-st}\big]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = -f(0) + sF(s). \end{aligned} \tag{1.1.32}$$

Applying this result to $f''(t)$ yields

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2F(s) - sf(0) - f'(0). \tag{1.1.33}$$

Similarly,

$$\mathcal{L}[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0). \tag{1.1.34}$$

By induction, we obtain the transform of the n-th derivative, viz.,

$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \tag{1.1.35}$$

Theorem 8. *The Laplace transform of*

$$\int_0^t f(\tau)\,d\tau, \tag{1.1.36}$$

is given by $\frac{1}{s}F(s)$.

Proof. *Let* $h(t) = \int_0^t f(\tau)\,d\tau$. *By construction we then have* $h'(t) = f(t)$ *and* $h(0) = 0$. *Applying then the result* (1.1.31) *to* $h(t)$ *we immediately get* $F(s) = sH(s) - h(0)$. *Hence,* $H(s) = \frac{1}{s}F(s)$.

Problem 1. *Find the Laplace transform of the following functions.*

a. t	*b.* t^2	*c.* t^3	*d.* t^n
e. $t+1$	*f.* $(t-1)^2$	*g.* $(1+t)^4$	*h.* $\frac{1}{t}$
i. e^{-t}	*j.* e^{3t+4}	*k.* te^t	*l.* e^{t^2}
m. $\cosh t$	*n.* $\cos t$	*o.* $\sin 2t$	*p.* $\sinh^2 t$

Problem 2. *Find the Laplace transform of the following functions.*

a. $e^{at}\cos bt$	*b.* $\theta(t-1)$	*c.* $e^{-t}\theta(t-1)$
d. $t^2\sinh t$	*e.* t^3e^t	*f.* $te^{-t}\cos t$
g. $\sin(\omega t+\alpha)$	*h.* $t\sin\frac{t}{2}$	*i.* $\ln t$
j. $\frac{1}{t}(1-\cos t)$	*k.* $\cosh t\cos t$	*l.* $\cos^2 t$

1.1.4 Table of Laplace Transforms

$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{-at} f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$

Table 1.1: Operational properties of the Laplace transform.

$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin bt$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin bt - bt\cos bt)$	$\frac{1}{(s^2+b^2)^2}$
$\frac{a}{\sqrt{4\pi t^3}}\, e^{-a^2/4t}$	$e^{-a\sqrt{s}}$

Table 1.2: Standard transform pairs.

1.2 The Inverse Laplace Transform

Finding the *inverse Laplace transform* of a function $f(t)$ is the operation of recovering $f(t)$ from its Laplace transform $F(s)$. One usually denotes this by

$$f(t) = \mathcal{L}^{-1}[F(s)]. \tag{1.2.1}$$

Although there exist a so-called *inversion formula*, which gives a closed form expression for $\mathcal{L}^{-1}[F(s)]$, we shall be content with the simple minded approach of finding inverse Laplace transforms by using a table of standard Laplace transforms. Indeed, it turns out that with the aid of a table and a little algebra, we are able to find $\mathcal{L}^{-1}[F(s)]$ for a large class of functions $f(t)$.

Due to the fact that the Laplace transform is linear it follows that also the inverse Laplace transform is linear. Hence, if a and b are constants, then we have

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)], \tag{1.2.2}$$

Example 8. *Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of*

$$F(s) = \frac{e^{-s}}{s^2} - \frac{e^{-2t}}{s^4}. \tag{1.2.3}$$

Solution. From a table of Laplace transforms, we have

$$\mathcal{L}^{-1}[\tfrac{1}{s^2}] = t, \qquad \mathcal{L}^{-1}[\tfrac{1}{s^4}] = \tfrac{1}{6}\,t^3, \tag{1.2.4}$$

so by the linearity of $\mathcal{L}^{-1}$, we obtain

$$\mathcal{L}^{-1}[\tfrac{e^{-s}}{s^2} - \tfrac{e^{-2s}}{s^4}] = \mathcal{L}^{-1}[\tfrac{e^{-s}}{s^2}] - \mathcal{L}^{-1}[\tfrac{e^{-2s}}{s^4}]. \tag{1.2.5}$$

Using now the 2^{nd} Shifting Rule, we find

$$\mathcal{L}^{-1}[\tfrac{e^{-s}}{s^2}] = \theta(t-1)(t-1), \qquad \mathcal{L}^{-1}[\tfrac{e^{-2s}}{s^4}] = \tfrac{1}{6}\theta(t-2)(t-2)^3. \tag{1.2.6}$$

Hence, the inverse transform of $F(s)$ is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \theta(t-1)(t-1) - \tfrac{1}{6}\theta(t-2)(t-2)^3. \tag{1.2.7}$$

1.2.1 Method of Partial Fractions

A common situation is when $F(s)$ has the form

$$F(s) = \frac{Q(s)}{P(s)}, \tag{1.2.8}$$

where $Q(s)$ and $P(s)$ are real polynomials and the degree of Q is less than the degree of P. It is then necessary to decompose $F(s)$ into *partial fractions* to obtain $\mathcal{L}^{-1}[F(s)]$.

We demonstrate this technique for three cases of denominators $P(s)$.

1. $P(s)$ is a Quadratic with real Roots. Consider, for instance,

$$F(s) = \frac{2s-8}{s^2-5s+6}. \tag{1.2.9}$$

Obviously, $F(s)$ cannot be inverted by inspection and neither do we have it tabulated. However, since the denominator $s^2 - 5s + 6$ has two real roots, $s = 2$ and $s = 3$, it is possible to decompose $F(s)$ into partial fractions, viz.,

$$F(s) = \frac{A}{s-2} - \frac{B}{s-3}. \tag{1.2.10}$$

where A and B are numbers. Our goal is to determine these, because then it is easy to obtain the inverse transform of $F(s)$. By elementary manipulations, we get

$$\frac{A}{s-2} + \frac{B}{s-3} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)} = \frac{(A+B)s+(-3A-2B)}{s^2-5s+6}, \tag{1.2.11}$$

which implies

$$\frac{2s-8}{s^2-5s+6} = \frac{(A+B)s+(-3A-2B)}{s^2-5s+6}. \tag{1.2.12}$$

Comparing the right and left hand side, it is obvious that

$$2 = A + B, \qquad -8 = -3A - 2B, \tag{1.2.13}$$

which is a system of equations for A and B, i.e.,

$$\begin{cases} A + B &= 2, \\ 3A + 2B &= 8. \end{cases} \tag{1.2.14}$$

Solving, we obtain $A = 4$ and $B = -2$. Hence,

$$F(s) = \frac{4}{s-2} - \frac{2}{s-3}. \tag{1.2.15}$$

By recognizing $\frac{1}{s-2}$ as the transform of e^{2t} and $\frac{1}{s-2}$ as that of e^{3t}, we obtain

$$f(t) = \mathcal{L}^{-1}[F(s)] = 4e^{2t} - 2e^{3t}. \tag{1.2.16}$$

2. $P(s)$ is a Quadratic with a Double Root. Let

$$F(s) = \frac{s+1}{(s+2)^2}, \tag{1.2.17}$$

The denominator has a double root -2 and the partial fractions are therefore

$$\frac{s+1}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} = \frac{As + (2A+B)}{(s+2)^2}. \tag{1.2.18}$$

Comparing the left and right hand side of the above expression, we find $A = 1$ and $B = -1$. Recalling that $\mathcal{L}[\frac{1}{s+2}] = e^{-2t}$, we can use $\mathcal{L}[tf(t)] = -F'(s)$ to deduce that the inverse transform of $(s+2)^{-2}$ is te^{-2t}. Hence, the inverse of $F(s)$ is

$$f(t) = e^{-2t} - te^{-2t} = e^{-2t}(1-t). \tag{1.2.19}$$

3. $P(s)$ is a Quadratic and has Complex Conjugated Roots. If

$$F(s) = \frac{s+1}{s^2+4s+5}, \tag{1.2.20}$$

then the denominator has the roots $-2 \pm i$. Completing the square, we get

$$s^2 + 4s + 5 = s^2 + 4s + 4 + 1 = (s+2)^2 + 1, \tag{1.2.21}$$

i.e.,

$$F(s) = \frac{s+1}{(s+2)^2+1}. \tag{1.2.22}$$

By rewriting

$$\frac{s+1}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} + \frac{-1}{(s+2)^2+1}, \tag{1.2.23}$$

and recalling the transforms of $\sin t$ and $\cos t$ it is clear that

$$f(t) = \mathcal{L}^{-1}[F(s)] = e^{-2t}\cos t - e^{-2t}\sin t. \tag{1.2.24}$$

Example 9. *Find the inverse transform of*

$$F(s) = \frac{s+2}{s^3 - s^2 + s - 1}. \tag{1.2.25}$$

Solution. Note that $s^3 - s^2 + s - 1 = s^2(s-1) + (s-1) = (s-1)(s^2+1)$. *Thus,*

$$F(s) = \frac{s+2}{(s-1)(s^2+1)}. \tag{1.2.26}$$

Here, the appropriate decomposition into partial fractions is given by

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{s^2(A+B) + s(C-B) + (A-C)}{s^3 - s^2 + s - 1}. \tag{1.2.27}$$

Identifying coefficients it is clear that

$$A + B = 0, \quad C - B = 1, \quad A - C = 2, \tag{1.2.28}$$

which implies, $A = \frac{3}{2}$, $B = -\frac{3}{2}$, *and* $C = -\frac{1}{2}$. *Hence,*

$$F(s) = \frac{\frac{3}{2}}{s-1} - \frac{\frac{3}{2}s}{s^2+1} - \frac{\frac{1}{2}}{s^2+1}. \tag{1.2.29}$$

Consulting a table of transforms, we recognize $F(s)$ *as the transform of*

$$f(t) = \tfrac{3}{2}e^t - \tfrac{3}{2}\cos t - \tfrac{1}{2}\sin t. \tag{1.2.30}$$

Problem 3. *Find the inverse Laplace transform of the following functions.*

a. $\dfrac{1}{s+1}$	*b.* $\dfrac{1}{s^2+4}$	*c.* $\dfrac{s+1}{s^2+1}$	*d.* $\dfrac{1}{s^2-1}$
e. $\dfrac{s+12}{s^2+4s}$	*f.* $\dfrac{s}{(s+2)^2}$	*g.* $\dfrac{s+1}{(s-3)^4}$	*h.* $\dfrac{e^{-s}}{s}$

Problem 4. *Find the inverse Laplace transform of the following functions.*

a. $\dfrac{s}{s^2-2s-3}$	*b.* $\dfrac{s+2}{s^2+4s+5}$	*c.* $\dfrac{1}{(s-2)^2+9}$
d. $\dfrac{s+1}{s^3+s^2-6s}$	*e.* $\dfrac{3s}{s^2+2s-8}$	*f.* $\dfrac{1}{s(s+1)(s+2)}$

1.3 Applications of Laplace Transforms

1.3.1 Initial Value Problems

Enough with theory, let us find the solution $y(t)$ of the initial value problem

$$y'(t) + 2y(t) = 12e^{3t}, \quad y(0) = 3. \tag{1.3.1}$$

By taking the Laplace transform of every term in the given differential equation, we get

$$\mathcal{L}[y'(t)] + \mathcal{L}[2y(t)] = \mathcal{L}[12e^{3t}]. \tag{1.3.2}$$

Put $Y(s) = \mathcal{L}[y(t)]$. Now,

$$\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s) - 3, \tag{1.3.3}$$

$$\mathcal{L}[2y(t)] = 2Y(s), \tag{1.3.4}$$

$$\mathcal{L}[12e^{3t}] = \frac{12}{s-3}. \tag{1.3.5}$$

Inserting these formulas into (1.3.2) above, we get the *subsidiary equation*

$$sY(s) - 3 + 2Y(s) = \frac{12}{s-3}. \tag{1.3.6}$$

Rearranging, we obtain

$$(s+2)Y(s) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}, \tag{1.3.7}$$

or

$$Y(s) = \frac{3s+3}{(s+2)(s-3)}. \tag{1.3.8}$$

At this point, we decompose $Y(s)$ into partial fractions, viz.,

$$\frac{3s+3}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} = \frac{(A+B)s - 3A + 2B}{(s+2)(s-3)}, \tag{1.3.9}$$

which gives rise to a system of equations for A and B, namely,

$$\left\{ \begin{array}{ll} A + B & = 3, \\ -3A + 2B & = 3. \end{array} \right. \tag{1.3.10}$$

Solving this, we find $A = \frac{3}{5}$ and $B = \frac{12}{5}$. Hence,

$$Y(s) = \frac{\frac{3}{5}}{s+2} + \frac{\frac{12}{5}}{s-3}. \tag{1.3.11}$$

Consulting a table of standard Laplace transforms, we finally have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \tfrac{3}{5}\mathcal{L}^{-1}[\tfrac{1}{s+2}] + \tfrac{12}{5}\mathcal{L}^{-1}[\tfrac{1}{s-3}] \\ &= \frac{3}{5}e^{-2t} + \frac{12}{5}e^{3t}. \end{aligned} \tag{1.3.12}$$

Summary of Solution Process. Note the three steps of the solution process:

1. Take the Laplace transform of both sides of the given hard problem for $y(t)$. As a result a simple algebraic equation for $Y(s) = \mathcal{L}[y(t)]$ is obtained.

2. Solve this so-called *subsidiary equation* for $Y(s)$.

3. Use partial fractions and a table of elementary Laplace transforms to invert $Y(s)$ and so produce the required solution $y(t) = \mathcal{L}^{-1}[Y(s)]$.

Example 10. *Solve the following initial value problem for* $t > 0$

$$y''(t) + 4y'(t) + 3y(t) = 0, \tag{1.3.13}$$
$$y(0) = 3, \quad y'(0) = 1. \tag{1.3.14}$$

Solution. We have

$$\mathcal{L}[y'(t)] = sY(s) - 3, \qquad \mathcal{L}[y''(t)] = s^2Y(s) - 3s - 1, \tag{1.3.15}$$

Laplace transformation of (1.3.13) *yields the subsidiary equation*

$$s^2Y(s) + 4sY(s) + 3Y(s) = 3s + 13, \tag{1.3.16}$$

or

$$(s+3)(s+1)Y(s) = 3s + 13. \tag{1.3.17}$$

Solving for $Y(s)$ *and using a decomposition into partial fractions, we get*

$$\begin{aligned} Y(s) &= \frac{3s+13}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} \\ &= \frac{A(s+1)+B(s+3)}{(s+3)(s+1)} = \frac{(A+B)s+A+3B}{(s+3)(s+1)}, \end{aligned} \tag{1.3.18}$$

from which we obtain $A = -2$, *and* $B = 5$. *Thus,*

$$Y(s) = -\frac{2}{s+3} + \frac{5}{s+1}. \tag{1.3.19}$$

Recalling (1.1.4) *it is obvious that*

$$\mathcal{L}^{-1}\left[\tfrac{1}{s+3}\right] = e^{-3t}, \qquad \mathcal{L}^{-1}\left[\tfrac{1}{s+1}\right] = e^{-t}. \tag{1.3.20}$$

Hence, the solution is given by

$$y(t) = -2e^{-3t} + 5e^{-t}. \tag{1.3.21}$$

A simple way to check whether the correct solution has been obtained is to see if the initial condition is satisfied by the found function $y(t)$. Here we have $y(0) = -2 + 5 = 3$ and, since $y'(t) = 6e^{-3t} - 5e^{-t}$, we also have $y'(0) = 6 - 5 = 1$. Hence, the given initial conditions are indeed satisfied by (1.3.21).

Problem 5. *Solve the following differential equations for* $t > 0$

a. $y' + 2y = e^{-3t}, \quad y(0) = 4.$

b. $y' - y = e^{2t}, \quad y(0) = -1.$

c. $y'' + 2y' + y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$

d. $y'' + 4y' + 13y = 2e^{-t}, \quad y(0) = 0, \quad y'(0) = -1.$

e. $y'' + 4y = 8e^{2t}, \quad y(0) = 0, \quad y'(0) = 3.$

f. $y'' - 2y' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0.$

g. $y'' + 4y' = 3e^{t}, \quad y(0) = 2, \quad y'(0) = 1.$

h. $y'' + 2y' + 2y = 2, \quad y(0) = 0, \quad y'(0) = 0.$

1.3.2 Integral Equations

Apart from solving differential equations, the Laplace transform technique may also be used to solve integral equations. For instance, consider the flow of electric current around a circuit consisting of a resistor, a capacitance, and a battery.

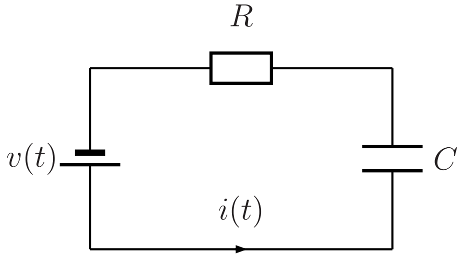

Figure 1.2: Electric RC-circuit.

It follows[3] that the current $i(t)$ satisfies the integral equation

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t), \tag{1.3.22}$$

where R and C are respectively the resistance and capacity of the circuit, and $v(t)$ is the electromotive force of the battery. For simplicity, let us assume that $C = R = 1$, and that $v(t)$ has the form of a square puls of amplitude 1, applied between $t = 1$ and $t = 2$, i.e.,

$$v(t) = \theta(t-1) - \theta(t-2) = \begin{cases} 0, & t < 1, \\ 1, & 1 < t < 2, \\ 0, & t > 2. \end{cases} \tag{1.3.23}$$

The Laplace transform $V(s)$ of $v(t)$ is given by

$$V(s) = \int_0^\infty v(t)e^{-st}\,dt = \int_1^2 e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_1^2 = -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s}. \tag{1.3.24}$$

Assuming that $i(0) = 0$, we may transform (1.3.22) to obtain

$$RI(s) + \frac{I(s)}{Cs} = V(s), \tag{1.3.25}$$

[3]From the Kirchoff voltage law.

or, since $R = C = 1$,

$$I(s) + \frac{I(s)}{s} = \frac{1}{s}(e^{-s} - e^{-2s}). \tag{1.3.26}$$

Solving for $I(s)$ we obtain, after elementary manipulations,

$$I(s) = \frac{1}{s+1}(e^{-s} + e^{-2s}). \tag{1.3.27}$$

Noting that $\mathcal{L}^{-1}[\frac{1}{s+1}] = e^{-t}$, we then use the 2^{nd} *Shifting Rule*, to obtain

$$i(t) = e^{-(t-1)}\theta(t-1) - e^{-(t-2)}\theta(t-2). \tag{1.3.28}$$

Hence,

$$i(t) = \begin{cases} 0, & t < 1, \\ e^1 e^{-t}, & 1 < t < 2, \\ (e^1 - e^2)e^{-t}, & t > 2. \end{cases} \tag{1.3.29}$$

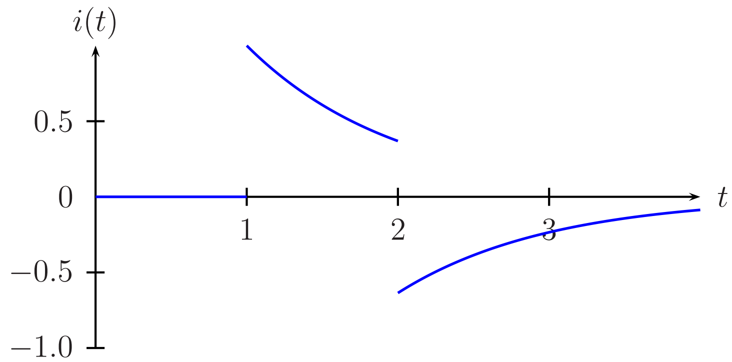

Figure 1.3: Graph of $i(t)$ for $0 < t < 4$.

Problem 6. *Solve the integral equation for* $t > 0$

$$2y(t) + \int_0^t y(\tau)\, d\tau = 4.$$

Problem 7. *Solve the following integral equations for $t > 0$*

a. $y'(t) + 2y(t) + \int_0^t y(\tau)\, d\tau = \cos t, \quad y(0) = 1.$

b. $y'(t) + 2y(t) + 2\int_0^t y(\tau)\, d\tau = 1 + e^{-t}, \quad y(0) = 1.$

c. $y''(t) - 7y(t) + 6\int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 7, \quad y'(0) = -12.$

Chapter 2

Fourier Analysis

Periodic phenomenon occur frequently throughout nature and their study is of the utmost importance for our understanding of many real-world systems. For example, the signals from radio pulsars allow astronomers to study space, the seasonal periodicity of the weather governs the crop of corn, and the regular beats of a heart is necessary for the survival of every mammal. Periodicity can be found everywhere and concern any absolute variable, i.e., time, space, velocity, etc. In this chapter we shall begin to study periodic functions, and especially, their representation as sums of sine and cosine functions, Fourier series.

Fourier series has long provided one of the principal tools of analysis for mathematical physics, engineering, and signal processing. It has spurred many generalizations, and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions describing wave motion, such as with light and sound, its generalizations often relate to wider settings, for example, the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis. We shall, however, be content with presenting the basic theory and its application to the solution of partial differential equations.

2.1 Periodic Functions

A function $f(x)$ is said to be periodic if there is a constant $p > 0$, such that

$$f(x+p) = f(x), \tag{2.1.1}$$

for all x. Any positive number p with this property is called a period of $f(x)$. For example, $f(x) = \cos x$ has periods 2π, 4π, etc. However, the smallest number $p > 0$ with the property (2.1.1) is called the *prime period*, and it is generally this value that is meant when a function is referred to as being p-periodic, or, of period p.

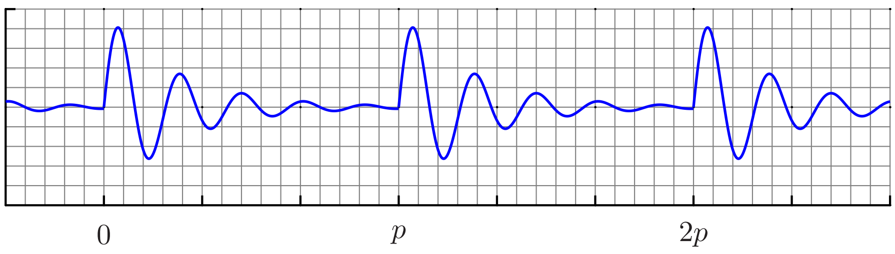

Figure 2.1: Illustration of a p-periodic function.

Lemma 1. *Suppose $f(x)$ is periodic with period p, then the integral*

$$\int_a^{a+p} f(x)\,dx, \tag{2.1.2}$$

is independent of starting point a.

Proof: Let

$$g(a) = \int_a^{a+p} f(x)\,dx = \int_0^{a+p} f(x) - \int_0^a f(x)\,dx. \tag{2.1.3}$$

By the fundamental theorem of calculus, we have $g'(a) = f(a+p) - f(a)$ but since $f(p+a) = f(a)$ we also have $g'(a) = 0$. Hence, $g(a)$ is constant and thus independent of a.

2.2 Fourier Series

2.2.1 Definition

From about 1800 onwards, the French scientist Joseph Fourier[1] was lead by problems of heat conduction to consider the possibility of representing a more or less arbitrary 2π-periodic function $f(x)$ as a linear combination[2] of the functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \sin 3x,\ \ldots \tag{2.2.2}$$

Fourier conjectured that any integrable periodic function $f(x)$ of period 2π can be written, at almost every point x, as the sum of a trigonometric series of the form

$$f(x) = \tfrac{1}{2}\,a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.2.3}$$

where a_n and b_n, $n = 0, 1, 2, \ldots$, are real numbers, defined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \tag{2.2.4}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{2.2.5}$$

Here the term $\frac{1}{2}\,a_0$ is due to the constant function $\cos 0 = 1$, the factor $\frac{1}{2}$ being included for reasons of later convenience. Further, b_0 does not exist, since $\sin 0 = 0$.

Fourier managed to solve several problems of heat flow using such series representations, and, as a result, (2.2.3) is today called the *Fourier series* of $f(x)$. Similarly, the corresponding numbers a_n and b_n are called the *Fourier coefficients* of $f(x)$.

---

[1] Jean Baptiste Fourier (1768-1830) French physicist and mathematician.

[2] Recall that if $\{\phi_1, \phi_2, \phi_3, \ldots\}$ is a collections of things (e.g., numbers, vectors, or functions) that can be multiplied by scalars and added together, then a linear combination is any expression of the form

$$c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \ldots, \tag{2.2.1}$$

where c_j, $j = 1, 2, 3, \ldots$ are constants.

Example 11. *Find the Fourier series of the 2π-periodic ramp function*

$$f(x) = x, \quad 0 \le x \le 2\pi. \tag{2.2.6}$$

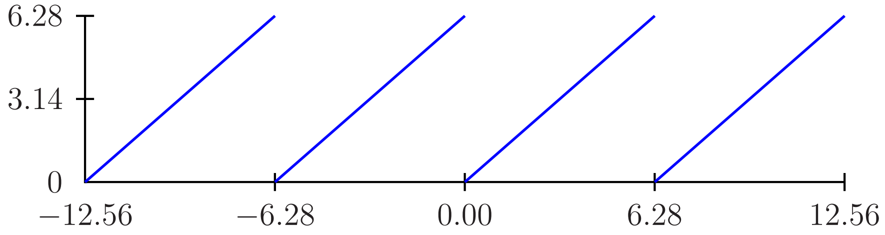

Figure 2.2: Graph of the ramp function.

Solution. By definition we have for $n = 0$,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi, \tag{2.2.7}$$

and for $n = 1, 2, 3, \ldots$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx \\ &= \left[\frac{x}{\pi}\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi}\sin nx\,dx \\ &= \left[\frac{1}{\pi n^2}\ \cos nx\right]_0^{2\pi} = 0. \end{aligned} \tag{2.2.8}$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx \\ &= \left[-\frac{x}{\pi}\frac{\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi}\cos nx\,dx = -\frac{2}{n}. \end{aligned} \tag{2.2.9}$$

Hence, the Fourier series of $f(x)$ is given by

$$f(x) = \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$
$$= \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \dots\right). \tag{2.2.10}$$

To get a feeling for Fourier series, it is helpful to add up and graph the first few terms of this series. Below, we show the sum (2.2.10) terminated after 5 terms.

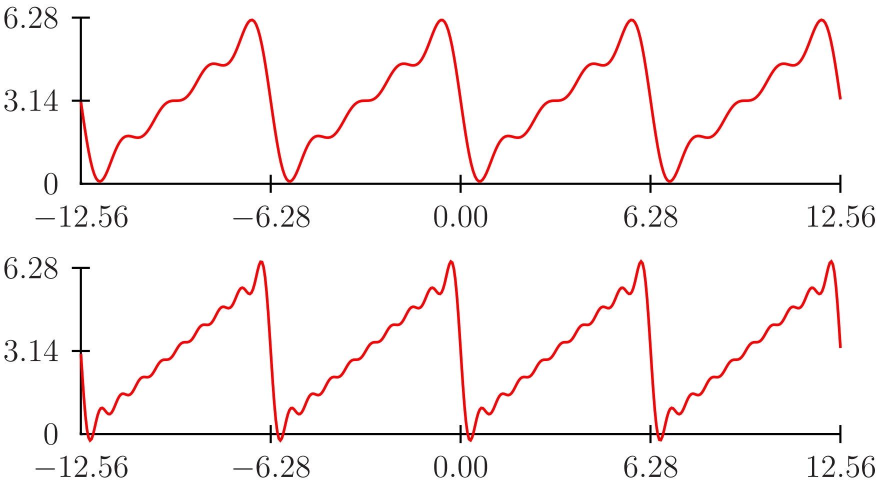

Figure 2.3: Partial sums of the Fourier series for the ramp function.

As seen, the sum is indeed attempting to reproduce the ramp function rather successfully. Adding more terms will further reduce the remaining wiggles.

Problem 8. *Find the Fourier series of the 2π-periodic function*

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ 0, & -\pi < x < 0. \end{cases}$$

Problem 9. *Find the Fourier series of the 2π-periodic function*

$$f(x) = \begin{cases} x, & 0 \le x < \pi, \\ 0, & -\pi < x \le 0. \end{cases}$$

2.2.2 Complex Representation of Fourier Series

We can greatly simplify the expression for the Fourier series (2.2.3) by using the relations

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \qquad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}, \tag{2.2.11}$$

where $i = \sqrt{-1}$ is the imaginary unit. Hence,

$$\begin{aligned} f(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \tfrac{1}{2}a_n(e^{inx} + e^{-inx}) - \tfrac{i}{2}b_n(e^{inx} - e^{-inx}) \\ &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \tfrac{1}{2}(a_n - ib_n)e^{inx} + \sum_{n=1}^{\infty} \tfrac{1}{2}(a_n + ib_n)e^{-inx}. \end{aligned} \tag{2.2.12}$$

Defining $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ for $n = 1, 2, 3, \ldots$, we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} = \sum_{-\infty}^{\infty} c_n e^{inx}, \tag{2.2.13}$$

which is called the *complex Fourier series* of $f(x)$.

2.2.3 Derivation of the Euler Formulas

Although the complex Fourier series and the sine-cosine series are identical, some manipulations and calculations become more streamlined when using the complex representation. For instance, it is easy to justify the definition of the numbers a_n and b_n, the so-called *Euler formulas* (2.2.4) and (2.2.5). To do so, we need the next lemma.

Lemma 2. *$\{e^{inx}\}_{-\infty}^{\infty}$ forms a so-called orthogonal set on $-\pi \leq x \leq \pi$, in the sense that*

$$\int_{-\pi}^{\pi} e^{i(n-k)x}\, dx = \begin{cases} 0, & n \neq k, \\ 2\pi, & n = k. \end{cases} \tag{2.2.14}$$

Proof.

$$\int_{-\pi}^{\pi} e^{i(n-k)x}\, dx = \left[\frac{e^{i(n-k)x}}{i(n-k)}\right]_{-\pi}^{\pi} = \frac{(-1)^{n-k} - (-1)^{n-k}}{i(n-k)} = 0, \quad n \neq k, \tag{2.2.15}$$

whilst

$$\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx = \int_{-\pi}^{\pi} dx = 2\pi, \quad \textit{for} \quad n = k. \tag{2.2.16}$$

Now, multiplying (2.2.13) by e^{-ikx}, and integrating term-by-term from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)x}\,dx. \tag{2.2.17}$$

By the orthogonality property above, this leads to

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = 2\pi c_k \quad \text{for} \quad n = k. \tag{2.2.18}$$

Relabeling the integer k by n we thus have a formula for c_n, namely,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx. \tag{2.2.19}$$

Finally, using the fact that $a_n = c_n + c_{-n}$ and that $b_n = i(c_n - c_{-n})$ we find for $n = 0$,

$$a_0 = 2c_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx, \tag{2.2.20}$$

and for $n = 1, 2, 3\ldots$,

$$\begin{aligned} a_n &= c_n + c_{-n} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{inx} + e^{-inx})\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, &(2.2.21)\\ b_n &= i(c_n - c_{-n}) \\ &= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx})\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. &(2.2.22)\end{aligned}$$

2.2.4 Even and Odd Functions

Much unnecessary work and corresponding sources of error can be avoided when calculating a_n and b_n, if $f(x)$ is either *even* or *odd*.

Recall that a function $g(x)$ is said to be *even* if $g(x) = g(-x)$ for all x. Similarly, a function $h(x)$ is said to be *odd* if $h(x) = -h(-x)$ for all x.

From the graphs of $g(x)$ and $h(x)$ it is obvious that the integrals of these functions over any symmetric interval, e.g., $-a \leq x \leq a$, may be simplified, viz.,

$$\int_{-a}^{a} g(x)\, dx = 2\int_{0}^{a} g(x)\, dx, \qquad \int_{-a}^{a} h(x)\, dx = 0. \tag{2.2.23}$$

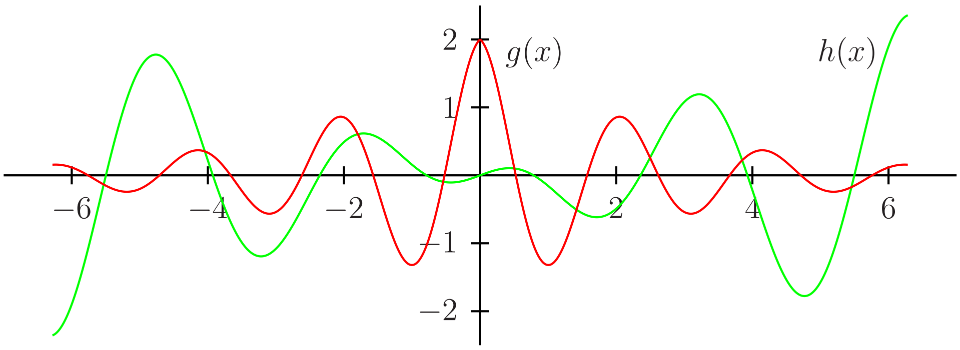

Figure 2.4: Illustration of even and odd functions.

Moreover, the product $q(x) = g(x)h(x)$ is odd, since

$$q(-x) = g(-x)h(-x) = g(-x)[-h(x)] = -g(x)h(x) = -q(x). \tag{2.2.24}$$

Thus, if $f(x)$ is even, then $f(x)\sin nx$ is odd, since $\sin nx$ is odd. Similarly, if $f(x)$ is odd, then $f(x)\cos nx$ is odd, since $\cos nx$ is even. Hence, from the definitions (2.2.4) and (2.2.5) of a_n and b_n we get the results of the next lemma.

Lemma 3. *If $f(x)$ is even, then*

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx, \quad b_n = 0. \tag{2.2.25}$$

whereas if $f(x)$ is odd, then

$$a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx. \tag{2.2.26}$$

Example 12. *Find the Fourier series of the square wave, defined by*

$$f(x) = \begin{cases} +1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \end{cases} \tag{2.2.27}$$

$$f(x) = f(x + 2\pi). \tag{2.2.28}$$

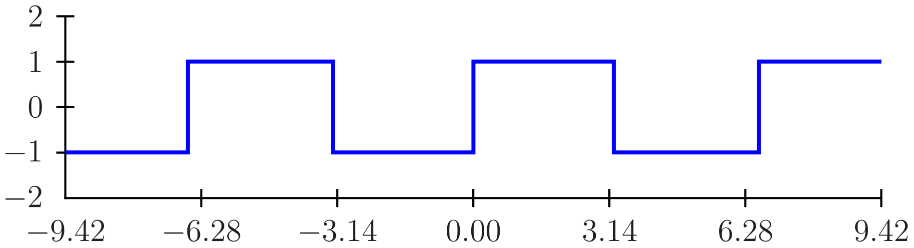

Figure 2.5: Graph of the square wave.

Solution. Since $f(x)$ is odd we have $a_n = 0$ for all n, and

$$b_n = \frac{2}{\pi}\int_0^\pi \sin nx\, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^\pi = -\frac{2}{\pi}\frac{(-1)^n - 1}{n}, \tag{2.2.29}$$

since $\cos n\pi = (-1)^n$. Hence, the Fourier series of $f(x)$ is given by

$$\begin{aligned} f(x) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx \\ &= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \dots\right). \end{aligned} \tag{2.2.30}$$

Below, we graph a few partial sums of this Fourier series. As is evident, they provide increasingly better approximations to the square wave as the number of terms increase, except possibly near points of discontinuity, i.e., $x = 0$, $\pm\pi$, $\pm 2\pi$, etc. Intuitively, this is what is to be expected, since it is hard to express a jump discontinuity using the perfectly smooth functions $\sin nx$ and $\cos nx$.

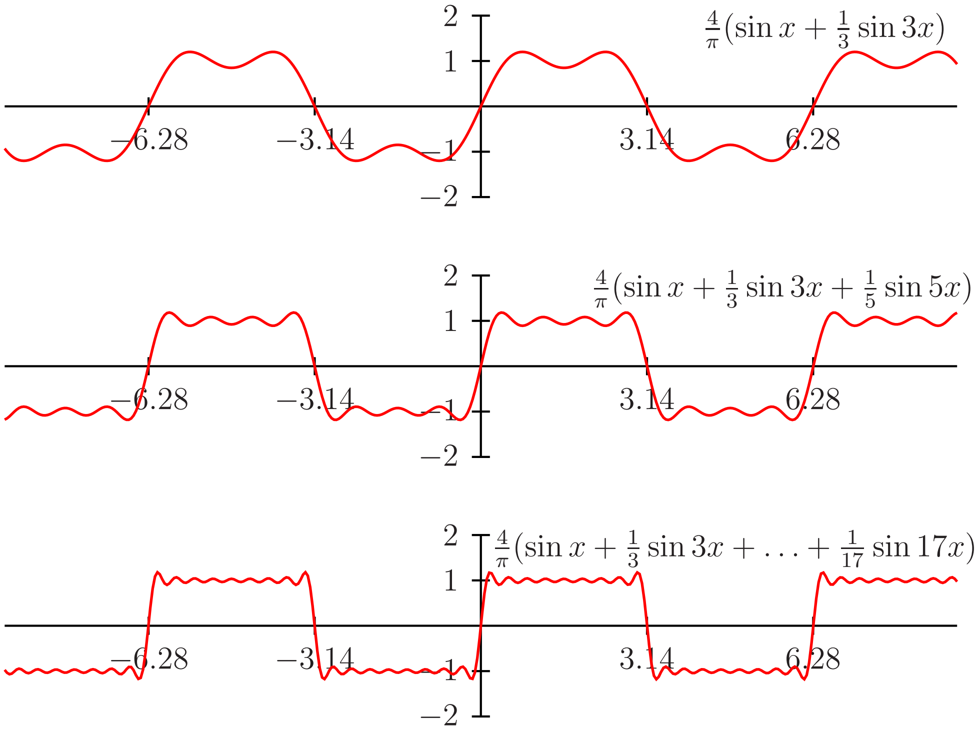

Figure 2.6: Partial sums of the Fourier series for the square wave.

Notice that if we let $x = \frac{\pi}{2}$, then the Fourier series (2.2.30) yields

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \ldots\right), \tag{2.2.31}$$

since $f(\frac{\pi}{2}) = 1$. But this implies that we can find the sum of a non-trivial, alternating series, namely,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}. \tag{2.2.32}$$

Problem 10. *Calculate the sum*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$$

by using the Fourier series expansion of the 2π*-periodic function*

$$f(x) = x^2, \quad -\pi \le x \le \pi.$$

2.2.5 Bessel Inequality and Riemann-Lebesgue Lemma

Theorem 9 (Bessel Inequality). *If $\int_{-\pi}^{\pi} |f(x)|^2\,dx$ is finite, then*

$$\sum_{-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx. \tag{2.2.33}$$

Proof. Let

$$S_N^f(x) = \sum_{n=-N}^{N} c_n e^{-inx}, \tag{2.2.34}$$

denote the N-th partial sum of the Fourier series for $f(x)$. We then have

$$\begin{aligned}
|f(x) - S_N^f(x)|^2 &= |f(x) - \sum_{n=-N}^{N} c_n e^{inx}|^2 \\
&= \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right)\left(\bar{f}(x) - \sum_{n=-N}^{N} \bar{c}_n e^{-inx}\right) \\
&= |f(x)|^2 - \sum_{n=-N}^{N} \left(c_n \bar{f}(x) e^{inx} + \bar{c}_n f(x) e^{-inx}\right) \\
&\quad + \sum_{n=-N}^{N}\sum_{m=-N}^{N} c_m \bar{c}_n e^{i(m-n)x},
\end{aligned} \tag{2.2.35}$$

where bars denote complex conjugates. Dividing by 2π and integrating from $-\pi$ to π, we get

$$\begin{aligned}
0 &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - \sum_{n=-N}^{N} c_n e^{inx}|^2\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx - \frac{1}{2\pi}\sum_{n=-N}^{N}\left(c_n \int_{-\pi}^{\pi} \bar{f}(x) e^{inx}\,dx + \bar{c}_n \int_{-\pi}^{\pi} \bar{f}(x) e^{inx}\,dx\right) \\
&\quad + \frac{1}{2\pi}\sum_{n=-N}^{N}\sum_{m=-N}^{N} c_m \bar{c}_n \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx.
\end{aligned} \tag{2.2.36}$$

Now, using the orthogonality (2.2.14) *and the formula* (2.2.19) *for* c_n*, this simplifies, viz.,*

$$\begin{aligned} 0 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx - \sum_{n=-N}^{N} (c_n\bar{c}_n + \bar{c}_n c_n) + \sum_{n=-N}^{N} |c_n|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx - \sum_{n=-N}^{N} |c_n|^2. \end{aligned} \tag{2.2.37}$$

Finally, let $N \to \infty$ *and the proof is complete.*

Because of the Bessel inequality, it follows that

$$\lim_{|n|\to\infty} c_n = 0, \tag{2.2.38}$$

which is known as the *Riemann-Lebesgue Lemma*.

Lemma 4 (Riemann-Lebesgue Lemma). *If* $\int_{-\pi}^{\pi} |f(x)|^2\,dx$ *is finite, then it holds*

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{n\to\infty} b_n = 0, \quad \lim_{|n|\to\infty} c_n = 0. \tag{2.2.39}$$

Proof. From (2.2.33) *we know that the sum*

$$\sum_{-\infty}^{\infty} |c_n|^2, \tag{2.2.40}$$

converges, so it follows that $c_n \to 0$ *as* $|n| \to \infty$*. But since* $a_n = c_n + c_{-n}$ *and* $b_n = i(c_n - c_{-n})$ *it also follows that* $a_n \to 0$ *and that* $b_n \to 0$ *as* $n \to \infty$*.*

Hence, according to the Riemann-Lebesgue lemma the terms of a Fourier series decrease as n tend to infinity, which is necessary if the series is to converge.

Problem 11. *Is there a function* $f(x)$ *with a finite* $\int_{-\pi}^{\pi} |f(x)|^2\,dx$*, such that*

$$f(x) = \sum_{n=1}^{\infty} \sin nx. \tag{2.2.41}$$

2.2.6 Convergence of Fourier Series

All convergence theorems are concerned with how the partial sums

$$S_N^f(x) = \sum_{n=-N}^{N} c_n e^{inx}, \tag{2.2.42}$$

converge to $f(x)$.

The question of *pointwise convergence*, for example, concerns whether

$$\lim_{N\to\infty} S_N^f(x_0) = f(x_0), \tag{2.2.43}$$

holds for each fixed value of x_0.

Below, we state a fundamental convergence theorem for Fourier series, which says that $S_N^f(x_0)$ converges pointwise to the average value of the left and right hand limits of $f(x)$ at x_0, as N tend to infinity. Consequently, if $f(x)$ is continuous at x_0, then its Fourier series will eventually converge to $f(x_0)$. On the other hand, if $f(x)$ is discontinuous at x_0, then we must redefine $f(x_0)$ as $\frac{1}{2}(f(x_{0-}) + f(x_{0+}))$ to be able to claim the convergence of its Fourier series at this point.

Theorem 10 (Dirichlet Theorem). *If $f(x)$ is a 2π-periodic function, which is piecewise smooth for all x, and if $S_N^f(x)$ is the N-th partial sum of its Fourier series, then*

$$\lim_{N\to\infty} S_N^f(x_0) = \frac{f(x_{0-}) + f(x_{0+})}{2}, \tag{2.2.44}$$

for all points x_0. In particular, $\lim_{N\to\infty} S_N^f(x) = f(x_0)$ for every point x_0 at which $f(x)$ is continuous.

A trivial consequence of the last theorem is that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - S_N^f(x)|^2 \, dx = 0, \tag{2.2.45}$$

i.e., the average error between $f(x)$ and its Fourier series tends to zero as N tend to infinity. However, this means that the Bessel inequality (2.2.33) is actually a genuine equality. Usually, it is then called the *Parseval Indentity.*

Theorem 11 (Parseval Identity). *With reference to the last theorem, it holds that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\sum_{-\infty}^{\infty}|c_n|^2, \tag{2.2.46}$$

or, equivalently,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\tfrac{1}{2}|a_0|^2+\sum_{n=1}^{\infty}|a_n|^2+|b_n|^2. \tag{2.2.47}$$

Example 13. *Compute the sum*

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{4}{7^4}+\dots, \tag{2.2.48}$$

using the Fourier series expansion of $f(x)=|x|$, $-\pi\le x\le\pi$.

Solution. We notice that $f(x)$ is even, so $b_n=0$ and

$$a_n=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx. \tag{2.2.49}$$

Thus, for $n=0$,

$$a_0=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi, \tag{2.2.50}$$

and for $n>0$,

$$\begin{aligned}a_n&=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx\\&=\frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}=\frac{2}{\pi}\frac{\cos n\pi-1}{n^2}=\frac{2}{\pi}\frac{(-1)^n-1}{n^2}.\end{aligned} \tag{2.2.51}$$

since $\sin n\pi=0$ and $\cos n\pi=(-1)^n$. Consequently,

$$f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx. \tag{2.2.52}$$

Furthermore,

$$\int_{-\pi}^{\pi}|f(x)|^2\,dx=2\int_0^{\pi}x^2\,dx=\frac{2}{3}\pi^3. \tag{2.2.53}$$

Thus, using the Parseval identity (2.2.47) *we obtain*

$$\frac{1}{\pi}\frac{2}{3}\pi^3 = \frac{1}{2}\pi^2 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)^2}{n^4}, \tag{2.2.54}$$

i.e.,

$$\frac{2}{3}\pi^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots\right). \tag{2.2.55}$$

Hence,

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{4}{7^4} + \dots = \frac{\pi^4}{96}. \tag{2.2.56}$$

Problem 12. *Compute the sum*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}, \tag{2.2.57}$$

using the Fourier series expansion of $f(x) = x$, $-\pi \le x \le \pi$.

2.2.7 Functions of Arbitrary Period

Fourier series expansions are not limited to periodic functions of period 2π only. For a periodic function $f(x)$ of arbitrary period, $2L$ say, a simple change of variables can be used to transform the period from $-L \le x \le L$ to $-\pi \le x \le \pi$. Therefore, let $g(t)$ be periodic of period 2π and let $t = \pi x/L$. We have $f(x+2L) = f(x)$ and $g(t) = f(\frac{L}{\pi}t) = f(x)$. Thus, by the periodicity, we get

$$g(t + 2\pi) = f(\tfrac{L}{\pi}t + 2L) = f(\tfrac{L}{\pi}t) = g(t). \tag{2.2.58}$$

It is now clear that the Fourier series for g,

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt, \tag{2.2.59}$$

can be written as

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right). \tag{2.2.60}$$

Its Fourier coefficients are given by

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\,dt \\
&= \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\frac{\pi}{L}\,dx \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.
\end{aligned} \tag{2.2.61}$$

Example 14. *Find the Fourier series of the rectified sine wave,*

$$f(x) = |\sin\pi x|, \quad 0 \le x \le 1, \quad f(x+1) = f(x). \tag{2.2.62}$$

Solution. Since $f(x)$ is even $b_n = 0$, and

$$a_0 = 2\int_{-1/2}^{1/2} f(x)\,dx = 4\int_0^{1/2}\sin\pi x\,dx = -\tfrac{4}{\pi}[\cos\pi x]_0^{1/2} = \frac{4}{\pi}. \tag{2.2.63}$$

Further, using that $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$, we get

$$\begin{aligned}
a_n &= 4\int_0^{1/2}\sin\pi x\cos 2n\pi x\,dx \\
&= 2\int_0^{1/2}\sin(1+2n)\pi x\,dx + 2\int_0^{1/2}\sin(1-2n)\pi x\,dx \\
&= -\frac{2}{(1+2n)\pi}[\cos(2n+1)\pi x]_0^{1/2} - \frac{2}{(1-2n)\pi}[\cos(1-2n)\pi x]_0^{1/2} \\
&= -\frac{2}{(1+2n)\pi}(-1) - \frac{2}{(1-2n)\pi}(-1) = \frac{4}{(1-4n^2)\pi}.
\end{aligned} \tag{2.2.64}$$

Hence, we have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos 2n\pi x. \tag{2.2.65}$$

Problem 13. *Let the function $f(x)$ be periodic of period 4 and let*

$$f(x) = |x|, \quad |x| < 2.$$

a. Graph $f(x)$ for $|x| < 6$.

b. Find the Fourier series of $f(x)$.

c. Use the series obtained to compute the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

d. Show, using the Parseval identity, that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{\pi^4}{96}.$$

2.2.8 Sine and Cosine Series

In various physical and engineering problems there is a practical need to apply trigonometric series expansions to functions $f(x)$ only defined for a finite interval, say $0 \le x \le L$. This can be accomplished by first defining $f(x)$ for $-L \le x \le 0$ and then extending it to a periodic function of period $2L$. In doing so, we can use the freedom of definition to make the extension either even or odd as is convenient.

Fourier Sine Series. We define the *odd extension* $F(x)$ of $f(x)$ by

$$F(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L, \\ 0 & \text{if } x = 0, \\ -f(-x) & \text{if } -L \le x \le 0, \end{cases} \tag{2.2.66}$$

$$F(x + 2L) = F(x). \tag{2.2.67}$$

By construction, $F(x)$ is odd and periodic, i.e., its series expansion is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{2.2.68}$$

where b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{2.2.69}$$

Fourier Cosine Series. Similarly, we define the *even extension* $F(x)$ of $f(x)$ by

$$F(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L, \\ f(-x) & \text{if } -L \leq x \leq 0, \end{cases} \tag{2.2.70}$$

$$F(x+2L) = F(x). \tag{2.2.71}$$

In this case, we thus have

$$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx. \tag{2.2.72}$$

Example 15. *Develop a sine series for the function $f(x) = \cos x$, $0 \leq x \leq \pi$.*

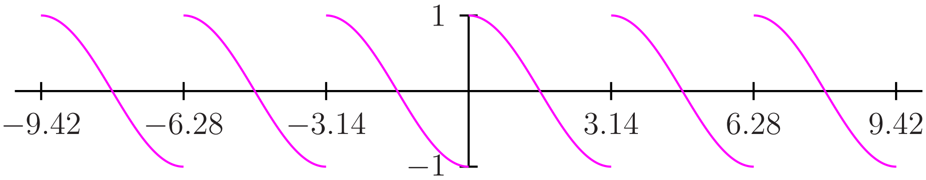

Figure 2.7: Odd extension of $\cos x$.

Solution. We have $b_1 = 0$ and

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx \\ &= \frac{1}{\pi}\int_0^\pi \sin(n-1)x\,dx + \frac{1}{\pi}\int_0^\pi \sin(n+1)x\,dx \\ &= \frac{1}{\pi}\left[-\frac{\cos(n-1)x}{n-1}\right]_0^\pi + \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n-1}\right]_0^\pi \\ &= \frac{1}{\pi}\frac{(-1)^n-1}{n-1} + \frac{1}{\pi}\frac{(-1)^n-1}{n+1} = \frac{2n}{\pi}\frac{(-1)^n-1}{n^2-1}. \end{aligned} \tag{2.2.73}$$

Hence, the sine series of $\cos x$ is given by

$$2\sum_{n=1}^{\infty}\frac{n}{\pi}\frac{(-1)^n-1}{n^2-1}\sin nx. \tag{2.2.74}$$

Problem 14. *Classify the following functions as even, odd, or neither even nor odd.*

a. e^x *b.* $x \sin x$ *c.* $\ln x$

d. $\sin^2 x$ *e.* e^{-x^2} *f.* $1 + x + x^5 + x^7$

Problem 15. *Develop a cosine series for the function*

$$f(x) = x, \quad 0 \leq x \leq 1.$$

Problem 16. *Develop a cosine series for the function*

$$f(x) = x^2, \quad 0 \leq x \leq \pi.$$

Problem 17. *Let the function $f(x)$ be odd and periodic of period 2π, where*

$$f(x) = x - \pi, \quad 0 < x < \pi.$$

a. Graph $f(x)$ for $-3\pi \leq x \leq 3\pi$.

b. Find the Fourier series of $f(x)$.

c. Use the series obtained to compute the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Chapter 3

Partial Differential Equations

Partial differential equations are of vast importance in science and engineering, since they provide mathematical models to a wide variety of real-world systems. Indeed almost all natural laws of physics can be formulated in terms of partial differential equations. For example, the Ampère law of electromagnetics, which relates the magnetic field $\boldsymbol{H}$ around a conducting electric wire of current density $\boldsymbol{J}$, can be stated simply as $\nabla \times \boldsymbol{H} = \boldsymbol{J}$, which is a partial differential equation for $\boldsymbol{H}$.

Although mathematical physics is the traditional area of application, partial differential equations are today found in such diverse areas as biology, chemistry, economics, and medicine.

Our goal here is to develop a few of the most basic ideas of the theory of partial differential equations, and apply them to the simplest models arising from physics.

3.1 Basic Concepts

3.1.1 Definition

Definition 1. *A partial differential equation* [1] *for a function $u(x, y, \ldots)$ with partial derivatives u_x, u_y, u_{xx}, u_{xy}, …, is a mathematical relation of the form*

$$F(x, y, \ldots, u, u_x, u_y, u_{xy}, \ldots) = 0, \tag{3.1.1}$$

where F is a given function of the variables $x, y, \ldots, u, u_x, u_y, u_{xx}, \ldots$

[1] Sometimes abbreviated PDE.

Generally, the variable u which we differentiate is called the dependent variable, whereas those which we differentiate with respect to (e.g., x and y) are called the independent variables.

By itself, the equation (3.1.1) is too general to allow for any systematic study. Instead, each of the major types of equations that commonly arise is studied individually. However, there are a few basic concepts which are essential for studying all types of equations. One such concept is the *order* of a PDE.

Definition 2. *The order of a partial differential equation is the order of the highest partial derivative present within the equation.*

So, for example,

$$\begin{aligned} u_t + uu_x &= 0, && \text{is of first order,} \\ u_t - 6uu_x + u_{xxx} &= 0, && \text{is of third order, etc.} \end{aligned}$$

Because the dynamics of most real-world systems usually involve at most two derivatives[2], second-order equations are the most frequently occurring and thus the most important.

3.1.2 Linearity and Superposition

One should recall from elementary calculus that the operation of taking a derivative is a linear operation, that is, for any functions $u(x)$ and $v(x)$ it holds that $(u+v)_x = u_x + v_x$. Therefore, (3.1.1) is said to be *linear*, if it can be rewritten as

$$L(u) = f, \tag{3.1.2}$$

where the *operator* $L(u)$ involves only sums and compositions of derivatives without higher power terms (e.g., u^2 or uu_x). More specific, L has to satisfy

$$L(av + bw) = aL(v) + bL(w), \tag{3.1.3}$$

for all functions v and w, and all numbers a and b. Thus,

$$e^{\sin y} u_{xx} + \log(1 + x^3) u_{yy} = 0, \tag{3.1.4}$$

[2]Recall that, e.g., the equation of motion $F = ma$ involves two time derivatives.

is linear in u, even though its so-called *coefficients* $e^{\sin y}$ and $\log(1 + x^3)$ are highly non-linear functions of x and y. However, the much simpler-looking *minimal surface equation*

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \tag{3.1.5}$$

is *non-linear* since the derivatives of u are multiplied together and squared.

Finally, if the right hand side f is identically zero, then (3.1.2) is said to be *homogeneous*; otherwise it is *inhomogeneous*.

Theorem 12 (Superposition Principle). *If u_1 and u_2 are two individual solutions to the linear homogeneous equation $L(u) = 0$, then so is any linear combination of them, i.e.,*

$$u = c_1 u_1 + c_2 u_2, \tag{3.1.6}$$

where c_1 and c_2 are any constants.

Proof. *Using the linearity of $L(u)$ we have*

$$L(u) = L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2). \tag{3.1.7}$$

But as $L(u_1) = 0$ and $L(u_2) = 0$, it immediately follows that also $L(u) = 0$.

For example, since $u_1 = x^2 - y^2$ and $u_2 = e^x \cos y$, both satisfy *Laplace' equation* $L(u) = u_{xx} + u_{yy} = 0$, so does any linear combination of them (i.e., $u = c_1 u_1 + c_2 u_2 = c_1(x^2 - y^2) + c_2 e^x \cos y$). As we shall see, this property is extremely useful for constructing solutions that must obey certain auxiliary constraints.

3.1.3 Classification

Let it suffice to classify linear second-order equations of the type

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = G, \tag{3.1.8}$$

where $A, \ldots, G$ can be constants, or given functions of x and y.

By analogy with the conic sections, ellipse, hyperbola, and parabola, all linear partial differential equations can be classified as of either elliptic, hyperbolic, or parabolic type.

Definition 3. *Depending on if the quantity $4AC - B^2$, is positive, negative or zero, the general linear second-order equation* (3.1.8) *is said to be either elliptic, hyperbolic, or parabolic, respectively.*

Example 16. *Classify*

$$u_{xx} + uu_{xy} + (1 + u^2)u_{yy} = 0, \tag{3.1.9}$$

as either hyperbolic, parabolic, or elliptic.

Here, $A = 1$, $B = u$, and $C = 1 + u^2$, so

$$4AC - B^2 = 4(1 + u^2) - u^2 = 3u^2 + 4 > 0. \tag{3.1.10}$$

Hence, the equation is always elliptic.

Problem 18. *Classify the following equations as linear or non-linear.*

a. $u_{xx} + yu_x + xu_y = x^3y$.

b. $u_{xyx} + 2x^2u_{xx}y_{yy} = \sin xy$.

c. $u_{xy} + \frac{1}{x}u_y = 0$.

Problem 19. *Classify the following into elliptic, hyperbolic, or parabolic types*

a. $u_{xx} + 2u_{xy} + (1 - x^2 - y^2)u_{yy} = x^2 + y^2$.

b. $u_{xx} + x^3u_{yy} = 0$.

3.1.4 Physical Derivation of the Heat Equation

One of the classical[3] partial differential equations of mathematical physics is the *heat equation*, describing the evolution of temperature within a heat conducting solid.

Consider a long thin rod of length L, oriented along the x-axis. We wish to develop a model of heat flow through the rod. For simplicity, we assume that the rod is laterally insulated so that any flow of heat is essentially directed lengthwise.

According to the first law of thermodynamics, which states conservation of energy, the rate of change of internal energy equals at all times the net inflow of heat. Thus, if $j(x, t)$ and $e(x, t)$ are the heat flux and the thermal energy per unit length of the rod at a point x and time t, respectively, we have

$$\int_0^L e_t \, dx = j(0, t) - j(L, t). \tag{3.1.11}$$

By the fundamental theorem of calculus, it follows that

$$j(L, t) - j(0, t) = \int_0^L j_x \, dx, \tag{3.1.12}$$

so (3.1.11) can also be written as

$$\int_0^L (e_t + j_x) \, dx = 0. \tag{3.1.13}$$

However, since the length of the rod L is arbitrary, we must conclude that

$$e_t + j_x = 0. \tag{3.1.14}$$

Experimental studies show that the relation between the total thermal energy H of a body of mass m and its temperature u is roughly linear and given by

$$H = cmu, \tag{3.1.15}$$

where c is known as the *specific heat capacity* of the material. Because our rod is long and thin, it is reasonable to believe that its temperature u only varies with longitudinal position x and time t, i.e., $u = u(x, t)$. Moreover,

[3] It originates back to Fourier and his thesis *Theorie Analytique de la Chaleur* (1822).

since we have defined the total energy of the rod by $H = \int_0^L e\,dx$, it follows that

$$e = \varrho c u, \tag{3.1.16}$$

where ϱ is the mass density of the rod.

Heat always flow from warm regions to cooler ones, but at rate which is dependent upon the conducting medium. For example, if a copper rod and an iron rod are joined together end to end, and the ends are subsequently heated, the heat will conduct through the copper end more quickly than the iron end because copper has a higher *thermal conductivity*.

Fourier considered the above properties and summarized them elegantly, viz.,

$$j = -\kappa u_x, \tag{3.1.17}$$

a relation which determines the heat flux j for a given temperature profile u and thermal conductivity κ.

Substituting (3.1.16) and (3.1.17) into (3.1.14) and assuming κ constant, we finally arrive at

$$\varrho c u_t = \kappa u_{xx}, \tag{3.1.18}$$

which is generally know as the *heat equation*.

3.1.5 Boundary and Initial Conditions

Motivation for Boundary Conditions. Consider the simple equation

$$y'(t) = 4t. \tag{3.1.19}$$

By integration we can easily obtain its general solution $y(t) = 2t^2 + C$, where C is a constant of integration. However, because C is arbitrary, we cannot determine a unique solution. To do so, we must have another piece of information about $y(t)$.

For example, if we were asked for the solution of (3.1.19) that satisfies $y(0) = 0$, then there would be only one possible answer, namely $y(t) = 2t^2$. If we try any of the other solutions to the equation, e.g., $y(t) = 2t^2 + 1$ or $y(t) = 2t^2 + 2$, say, we find that $y(0)$ is not zero, but 1 or 2, respectively. A constraint, such as $y(0) = 0$, which is given together with a differential equation to fix a constant of integration, and so give a unique solution, is called a *boundary condition*.

With second-order differential equations, we need to integrate twice to obtain its general solution. As a result, we end up with two constants of integration and need two boundary conditions to specify which solution we want.

Boundary and Initial Conditions for the Heat Equation. Since the heat equation contains a total of three derivatives, two spatial ones and one with respect to time, we expect its solution to contain three degrees of freedom, that is, constants of integration. Consequently, it is necessary to supply three auxiliary constraints along with this equation to get a unique solution.

Regarding our heat conducting rod, we must thus know how heat flows through the ends of the rod, and we must know the temperature distribution within the rod at some initial time.

If the ends of the rod are perfectly insulated, so that the heat flux across these at $x = 0$ and $x = L$ are zero, then by (3.1.17) we have the boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0, \qquad t > 0. \tag{3.1.20}$$

On the other hand, if the ends of the rod are held at the fixed temperature zero, e.g., by submerging them into ice water, then we have the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0. \tag{3.1.21}$$

Either of these boundary conditions can be inhomogeneous, i.e., have a non-zero right hand side, and we could of course also have mixed conditions, where one end of the rod is insulated and the other end is held at a constant temperature.

A boundary condition that specifies the value of the solution is called a *Dirichlet condition*, while a condition that specifies the value of a derivative is called a *Neumann condition*. If the boundary condition is of mixed type, then it is called a *Robin condition*.

To completely determine the temperature as a function of time t and space x, we must also specify the temperature profile at some initial time t_0 via a so-called *initial condition*

$$u(x,t_0) = f(x). \tag{3.1.22}$$

3.2 Separation of Variables

3.2.1 Overview of Method

Separation of variables is one of the oldest[4] analytical methods for solving PDE-problems. In its traditional form, however, it can only be applied to linear homogeneous equations subject to homogeneous boundary conditions.

The basic assumption of separation of variables is that the solution u of a linear homogeneous equation involving the variables x and t, say, can be written as a product of two functions X and T, each of which is only dependent upon a single independent variable, i.e.,

$$u(x,t) = X(x)T(t). \tag{3.2.1}$$

Separation of variables is really best grasped by example, so instead of discussing the general ideas of this technique any further, let us apply it to a specific problem.

3.2.2 The Heat Equation

Recall that the temperature u within a laterally insulated rod of length L, which has its ends kept a zero temperature, is given by the *initial-boundary value problem*

$$u_t = ku_{xx}, \quad 0 \le x \le L, \quad t \ge 0, \tag{3.2.2}$$
$$u(0,t) = u(L,0) = 0, \tag{3.2.3}$$
$$u(x,0) = f(x). \tag{3.2.4}$$

Substituting the *ansatz* $u(x,t) = X(x)T(t)$ into (3.2.2) we get

$$X(x)T'(t) = kX''(x)T(t). \tag{3.2.5}$$

Dividing both sides by $kX(x)T(x)$ we then obtain

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{3.2.6}$$

Here, the left hand side is a function of t alone, whereas the right hand side is a function of x alone. But since x and t are independent variables, this is

[4] Usually credited to J. Bernoulli who developed it around 1755.

impossible, unless both sides are equal to a constant, λ say. Hence, we arrive at

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda, \tag{3.2.7}$$

where λ is called the *separation constant*. Rearranging, we have

$$T' = k\lambda T, \qquad X'' = \lambda X, \tag{3.2.8}$$

which is two simple *ordinary* differential equations for X and T that can be solved by any elementary method. Solving the ODE for T by inspection, we have

$$T(t) = Ce^{k\lambda t}, \tag{3.2.9}$$

where C is arbitrary. Contemplating this result, we see that $T(t)$ increases exponentially with time, if $\lambda \geq 0$. However, without any additional heating mechanism, such a behavior is obviously unphysical for the temperature u of the rod. Therefore, we choose the separation constant negative. To force this we let

$$\lambda = -\mu^2. \tag{3.2.10}$$

As a result, the ODE for X is then

$$X''(x) + \mu^2 X(x) = 0, \tag{3.2.11}$$

Thus,

$$X(x) = A\cos\mu x + B\sin\mu x, \tag{3.2.12}$$

where A and B are constants of integration. Multiplying X and T, we get

$$u(x,t) = X(x)T(t) = e^{-\mu^2 t}(A\cos\mu x + B\sin\mu x). \tag{3.2.13}$$

Although these functions do satisfy the heat equation $u_t = ku_{xx}$, they do not satisfy the prescribed boundary conditions $u(0,t) = u(L,t) = 0$. So, the next step is to enforce these by specifying A, B, and μ suitably. In doing, so we find

$$u(0,t) = e^{-\mu^2 t}(A\cos 0 + B\sin 0) = e^{-\mu^2 t}A = 0, \tag{3.2.14}$$

i.e., $A = 0$, and

$$u(L,t) = Be^{-\mu^2 t}\sin\mu L = 0, \tag{3.2.15}$$

which implies $\sin \mu L = 0$. Consequently, we pick

$$\mu L = \pm\pi,\ \pm 2\pi,\ \pm 3\pi, \ldots, \tag{3.2.16}$$

or, equivalently

$$\lambda = -\frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots \tag{3.2.17}$$

We have thus obtained a sequence of solutions of the form

$$u_n(x,t) = B_n e^{-k\pi^2 n^2 t/L^2} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3 \ldots, \tag{3.2.18}$$

but, since (3.2.2) obeys the superposition principle, its general solution is given by a linear combination of these individual solutions u_n, that is to say,

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k\pi^2 n^2 t/L^2} \sin \frac{n\pi x}{L}. \tag{3.2.19}$$

Finally, the initial condition implies

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x), \tag{3.2.20}$$

and the problem is to choose the numbers B_n, so that the series for $u(x,0)$ equals $f(x)$. However, recognizing (3.2.20) as a Fourier sine series, we know that this can be done by choosing B_n as

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{3.2.21}$$

As a particular example, let us consider the case $L = 1$, $k = \frac{1}{10}$, and

$f(x) = x(1-x)$. We get

$$\begin{aligned}
B_n &= 2\int_0^1 x(1-x)\sin n\pi x\,dx \\
&= \frac{2}{n\pi}\big[-x(1-x)\cos n\pi x\big]_0^1 + \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x\,dx \\
&= \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x\,dx \\
&= \frac{2}{(n\pi)^2}\big[(1-2x)\sin n\pi x\big]_0^1 + \frac{2}{(n\pi)^2}\int_0^1 (-2)\sin n\pi x\,dx \\
&= -\frac{4}{(n\pi)^2}\int_0^1 \sin n\pi x\,dx \\
&= -\frac{4}{(n\pi)^3}\big[-\cos n\pi x\big]_0^1 = 4\,\frac{(-1)^n-1}{(n\pi)^3}.
\end{aligned} \tag{3.2.22}$$

Hence, the final solution is given by

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}\,e^{-n^2\pi^2 t/10}\sin n\pi x. \tag{3.2.23}$$

Problem 20. *Solve formally by separation of variables the heat equation*

$$\begin{gathered}
u_t = u_{xx}, \quad 0<x<1, \quad t>0, \\
u(0,t) = u(1,t) = 0, \\
u(x,0) = 1.
\end{gathered}$$

Problem 21. *Solve the homogeneous heat equation*

$$\begin{gathered}
u_t = 7u_{xx}, \quad 0<x<\pi, \quad t>0, \\
u(0,t) = u(\pi,t) = 0, \\
u(x,0) = 3\sin 2x - 6\sin 5x.
\end{gathered}$$

Problem 22. *Solve the homogeneous heat equation*

$$\begin{gathered}
u_{xx} = 4u_t, \quad 0<x<1, \quad t>0, \\
u(0,t) = u(1,t) = 0, \\
u(x,0) = \min\{x, 1-x\}.
\end{gathered}$$

Problem 23. *Solve the homogeneous heat equation*

$$\begin{gathered} u_{xx} = u_t + u, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \sin \pi x. \end{gathered}$$

Problem 24. *Solve the heat equation*

$$\begin{gathered} u_{xx} = u_t + u, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = 1. \end{gathered}$$

Example 17. *Solve the heat equation*

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t \geq 0, \tag{3.2.24}$$

$$u_x(0,t) = u_x(1,0) = 0, \tag{3.2.25}$$

$$u(x,0) = f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} < x < 1. \end{cases} \tag{3.2.26}$$

Solution. Put $u(x,t) = X(x)T(t)$ *and plug it into* (3.2.24)*. We get*

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda, \tag{3.2.27}$$

where we choose the separation constant $\lambda = -\mu^2$*. Solving the differential equations* $T'(t) = -\mu^2 T(t)$ *and* $X''(x) + \mu^2 X(x) = 0$*, for* $X(x)$ *and* $T(t)$ *yields* $T(t) = Ce^{-\mu^2 t}$ *and* $X(x) = A\cos\mu x + B\sin\mu x$*. Hence, we have solutions of the form*

$$u(x,t) = X(x)T(t) = e^{-t\mu^2}(A\cos\mu x + B\sin\mu x). \tag{3.2.28}$$

Further, the boundary conditions gives us

$$u_x(0,t) = \mu B e^{-t\mu^2} = 0, \tag{3.2.29}$$

i.e., $B = 0$*, and*

$$u_x(1,t) = -Ae^{-t\mu^2}\sin\mu = 0, \tag{3.2.30}$$

which implies $\sin\mu = 0$. *Thus, we must choose* $\mu = \pm n\pi$, $n = 0, 1, 2, \ldots$ *to get a non-trivial solution. Moreover, by the superposition principle we have*

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\pi^2 n^2 t} \cos n\pi x.$$

This last result suggests that we should expand the initial condition $f(x)$ *using a cosine series. In doing so, it is better to write the solution* $u(x,t)$, *viz.,*

$$u(x,t) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\pi^2 n^2 t} \cos n\pi x.$$

that is, to redefine A_0. *We then have, for* $n = 0$

$$A_0 = 2\int_0^1 f(x)\,dx = 2\int_{1/2}^1 dx = 1, \tag{3.2.31}$$

and for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} A_n &= 2\int_0^1 f(x)\cos n\pi x\,dx = 2\int_{1/2}^1 \cos n\pi x\,dx \\ &= 2\left[\frac{\sin n\pi x}{n\pi}\right]_{1/2}^1 = \frac{2}{n\pi}\begin{cases} -1, & \text{if } n = 1, 5, 9, \ldots \\ 0, & \text{if } n = 2, 4, 6, \ldots \\ 1, & \text{if } n = 3, 7, 11, \ldots \end{cases} \end{aligned} \tag{3.2.32}$$

Hence, the final solution is given by

$$\begin{aligned} u(x,t) = \frac{1}{2} &- \frac{2}{\pi}e^{-\pi^2 t}\cos \pi x + \frac{2}{3\pi}e^{-9\pi^2 t}\cos 3\pi x \\ &- \frac{2}{5\pi}e^{-25\pi^2 t}\cos 5\pi x + \frac{2}{7\pi}e^{-49\pi^2 t}\cos 7\pi x - \ldots. \end{aligned} \tag{3.2.33}$$

Problem 25. *Solve the heat equation*

$$\begin{gathered} u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0 \\ u_x(0,t) = u_x(1,t) = 0, \\ u(x,0) = x. \end{gathered}$$

3.2.3 The Wave Equation

Let us illustrate separation of variables for something other than the heat equation.

The one-dimensional *wave equation* is a model of the small transverse oscillations of a string stretched between two points, e.g., a vibrating violin string. We want to find a function $u(x,t)$ that gives the deflection of the string at any point x, $0 \le x \le L$, where L is the string length, and at any time $t > 0$.

For simplicity, we assume that the string is perfectly flexible, made of a single homogeneous material, and of constant tension, so that the effects of gravity can be ignored. Taking the above assumptions into consideration, the motion of the string can be modeled by the following initial-boundary value problem.

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{3.2.34}$$

$$u(0,t) = u(L,t) = 0, \tag{3.2.35}$$

$$u(x,0) = f(x), \tag{3.2.36}$$

$$u_t(x,0) = g(x). \tag{3.2.37}$$

Here, the constant $c > 0$ is the wave speed of the string and depends upon the linear density and tension of the string. The fact that the string is fixed at the endpoints $x = 0$ and $x = L$ is reflected by the boundary conditions (3.2.35). Furthermore, the initial displacement $u(x,0)$ of the string and the initial velocity $u_t(x,0)$ at each point along it are specified by the functions $f(x)$ and $g(x)$, respectively.

Now applying separation of variables $u(x,t) = X(x)T(t)$ to (3.2.34) we get

$$T''(t)X(x) = c^2 T(t)X''(x). \tag{3.2.38}$$

Dividing through by $c^2 X(x)T(t)$ we have

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}, \tag{3.2.39}$$

but, since a function of x alone that equals a function of t alone must be a constant, we find

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{3.2.40}$$

We then put $\lambda = -\mu^2 < 0$, to obtain

$$T'' + (c\mu)^2 T = 0, \qquad X'' + \mu^2 X = 0, \tag{3.2.41}$$

and thus

$$T(t) = A\cos\mu ct + B\sin\mu ct, \quad X(x) = C\cos\mu x + D\sin\mu x, \tag{3.2.42}$$

where A, B, C, and D are arbitrary constants. Hence,

$$u(x,t) = (A\cos\mu ct + B\sin\mu ct)(C\cos\mu x + D\sin\mu x).$$

As with the heat equation, the boundary conditions $u(0,t) = u(L,t) = 0$, imply that $C = 0$, and that $\mu = n\pi/L$. Consequently, we obtain the series solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L} \right) \sin\frac{n\pi x}{L}. \tag{3.2.43}$$

Finally, substituting the series for $u(x,t)$ into the initial conditions gives

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x), \tag{3.2.44}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\frac{n\pi x}{L} = g(x). \tag{3.2.45}$$

For A_n and B_n, the following formulas apply

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{3.2.46}$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx. \tag{3.2.47}$$

Problem 26. *Solve, by separation of variables, the wave equation*

$$\begin{gathered} u_{tt} = 4u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = \sin 3x, \quad u_t(x,0) = 2\sin 4x. \end{gathered}$$

Problem 27. *Solve the homogeneous wave equation*

$$\begin{gathered} u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = u_x(1,t) = 0, \\ u_t(x,0) = 0, \quad u(x,0) = x^2 - 2x. \end{gathered}$$

3.2.4 Inhomogeneous Equations

If either the PDE or the boundary conditions are inhomogeneous, then the classical separation of variables method will not work. In such cases, one technique that is often successful is to break the desired solution into two parts, namely, one that is a function of both independent variables and one that is a function of only one of these. More specific, one can often convert the original problem into a homogeneous PDE with homogeneous boundary conditions that can be solved by separation of variables, and a common ODE that can be solved by traditional methods.

As a particular example, consider heat flow within a laterally insulated rod of length L. Let the left and right ends of the rod be attached to a thermostat, so that the temperature at $x = 0$ and $x = L$ is fixed at α and β degrees, respectively. Moreover, let also a heater be attached to the rod, so that a constant heat q is added to the rod per unit length. If the initial temperature is zero, then the situation is governed by the initial-boundary value problem

$$u_t = ku_{xx} + q, \quad 0 < x < L, \quad t > 0, \tag{3.2.48}$$

$$u(0,t) = \alpha, \quad u(L,t) = \beta, \tag{3.2.49}$$

$$u(x,0) = 0. \tag{3.2.50}$$

Following the above guidelines, we try to solve this problem by writing the temperature $u(x,t)$ as the sum of a *transient solution* $v(x,t)$ and a *steady state solution* $s(x)$

$$u(x,t) = v(x,t) + s(x). \tag{3.2.51}$$

Substituting this expression into (3.2.48) yields

$$v_t = kv_{xx} + ks'' + q, \tag{3.2.52}$$

so if we let

$$s''(x) = -\frac{q}{k}, \tag{3.2.53}$$

then $v(x,t)$ satisfies the homogeneous equation $v_t = kv_{xx}$. However, we also desire homogeneous boundary conditions for $v(x,t)$. To accomplish this, we note that

$$u(0,t) = v(0,t) + s(0) = \alpha, \tag{3.2.54}$$

$$u(L,t) = v(L,t) + s(L) = \beta, \tag{3.2.55}$$

which suggest that we put $s(0) = \alpha$ and $s(L) = \beta$, to get $v(0,t) = 0$ and $v(L,t) = 0$, as desired.

As a last step, the ansatz (3.2.51) is inserted into the initial condition, i.e.,

$$u(x,0) = v(x,0) + s(x) = 0. \tag{3.2.56}$$

yielding the initial condition for the $v(x,t)$ problem as

$$v(0,t) = -s(x). \tag{3.2.57}$$

Hence, the original inhomogeneous problem has been converted into two problems, each of which is straightforward to solve. First, $s(x)$ is found by solving

$$s''(x) = -\frac{q}{k}, \quad s(0) = \alpha, \quad s(L) = \beta. \tag{3.2.58}$$

Next, $v(x,t)$ is determined by applying separation of variables to

$$v_t = v_{xx}, \quad 0 < x < L, \quad t > 0, \tag{3.2.59}$$

$$v(0,t) = 0, \quad v(L,t) = 0, \tag{3.2.60}$$

$$v(x,0) = -s(x). \tag{3.2.61}$$

Finally, the solutions to the two separate problems given by (3.2.58) and (3.2.59) are added to give the desired space-time temperature distribution, $u(x,t)$.

Problem 28. *Solve the heat equation*

$$u_t = u_{xx} + \sin 3\pi x, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = 0, \quad u(1,t) = 0,$$
$$u(x,0) = \sin \pi x.$$

Problem 29. *Solve the problem*

$$u_t = u_{xx} + \sin \pi x, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = 0, \quad u(1,t) = 0,$$
$$u(x,0) = 1.$$

Problem 30. *Solve the wave equation*

$$u_{xx} = u_{tt} - 1, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = 0, \quad u(1,t) = 1,$$
$$u(x,0) = x, \quad u_t(x,0) = 0.$$

Appendix A

Answers to Exercises

Problem 1

a. $1/s^2$

b. $2/s^3$

c. $6/s^4$

d. $n!/s^{n+1}$

e. $1/s^2 + 1/s$

f. $2/s^3 - 2/s^2 + 1/s$

g. $24/s^5 + 24/s^4 + 12/s^3 + 4/s^2 + 1/s$

h. do not exist

i. $1/(s+1)$

j. $e^4/(s-3)$

k. $1/(s-1)^2$

l. do not exist

m. $s/(s^2-1)$

n. $s/(s^2+1)$

o. $2/(s^2+4)$

p. $2/(s(s^2-4))$

Problem 2

a. $(s-a)/((s-a)^2+b^2)$

b. e^{-s}/s

c. $e^{-(s+1)}/(s+1)$

d. $2(3s^2+1)/(s^2-1)^3$

e. $6/(s-1)^4$

f. $s(2+s)/(s^2+2s+2)^2$

g. $(\omega\cos\alpha + s\sin\alpha)/(s^2+\omega^2)$

h. $s/(s^2+1/4)^2$

i. do not exist

j. $\frac{1}{2}\ln(1+1/s^2)$

k. $s^3/(s^4+4)$

l. $(2+s^2)/(s(s^2+4))$

Problem 3

a. e^{-t} *b.* $\frac{1}{2}\sin 2t$ *c.* $\cos t + \sin t$ *d.* $\sinh t$

e. $3 - 2e^{-4t}$ *f.* $e^{-2t}(1-2t)$ *g.* $\frac{1}{6}t^2e^{3t}(4t+3)$ *h.* $\theta(t-1)$

Problem 4

a. $\frac{1}{4}e^{-t} + \frac{3}{4}e^{3t}$ *b.* $e^{-2t}\cos t$ *c.* $\frac{1}{3}e^{2t}\sin 3t$

d. $-\frac{1}{6} - \frac{2}{15}e^{-3t} + \frac{3}{10}e^{2t}$ *e.* $2e^{-4t} + e^{2t}$ *f.* $\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$

Problem 5

a. $y(t) = 5e^{-2t} - e^{-3t}$

b. $y = e^{2t} - 2e^t$

c. $y(t) = \frac{1}{2}e^{-t}t(t+2)$

d. $y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}(\cos 3t + 2\sin 3t)$

e. $e^{2t} - \cos 2t + \frac{1}{2}\sin 2t$

f. $y = \frac{1}{5}(\cos t - 2\sin t + 4e^t\cos t - 2e^t\sin t)$

g. $y = \frac{3}{2} - \frac{1}{10}e^{-4t} + \frac{3}{5}e^t$

h. $1 - e^{-t}(\cos t + \sin t)$

Problem 6 $y(t) = 2e^{-t/2}$

Problem 7

a. $y(t) = \frac{1}{2}(e^{-t} - te^{-t} + \cos t)$

b. $y(t) = (-1 + 2\cos t + \sin t)e^{-t}$

c. $y(t) = e^t + e^{2t} + 5e^{-3t}$

Problem 8

$$\begin{aligned} f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin nx \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right) \end{aligned}$$

Problem 9

$$\begin{aligned} f(x) &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \\ &= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right) \\ &\quad + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right) \end{aligned}$$

Problem 10

$$\begin{aligned} x^2 &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \\ &= \frac{\pi^2}{3} + 4\left(-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \dots\right) \end{aligned}$$

Let $x = 0$ to get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots = \frac{\pi^2}{12}.$$

Problem 11 No, it contradicts the Riemann-Lebesgue lemma.

Problem 12

$$\begin{aligned} x &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \\ &= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right) \end{aligned}$$

Use the Parseval formula to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}.$$

Problem 13

a. -

b.

$$|x| = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \frac{1}{49}\cos\frac{7\pi x}{2} + \dots\right)$$

c. $\pi^2/8$ (put $x = 0$)

d. -

Problem 14

a. Neither odd nor even	*b.* Even	*c.* Neither odd nor even
d. Even	*e.* Even	*f.* Neither odd nor even

Problem 15

$$f(x) = \frac{1}{2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x$$

Problem 16

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Problem 17

a. -

b. $f(x) = -2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

c. $\pi/4$

Problem 18 a. Linear, b. Non-linear, c. Linear.

Problem 19

a. Hyperbolic, except at the point $(0,0)$ where it is parabolic.

b. Hyperbolic if $x<0$, parabolic if $x=0$, and elliptic if $x>0$.

Problem 20

$$u(x,t)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\,e^{-n^2\pi^2 t}\sin n\pi x$$

Problem 21

$$u(x,t)=3e^{-28t}\sin 2x-6e^{-175t}\sin 5x$$

Problem 22

$$\begin{aligned}u(x,t)&=\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}e^{-\frac{1}{4}(2k+1)^2\pi^2 t}\sin(2k+1)\pi x\\&=\frac{4}{\pi^2}\left(e^{-\frac{1}{4}\pi^2 t}\sin\pi x-\frac{1}{9}e^{-\frac{9}{4}\pi^2 t}\sin 3\pi x+\frac{1}{25}e^{-\frac{25}{4}\pi^2 t}\sin 5\pi x-\ldots\right)\end{aligned}$$

Problem 23

$$u(x,t)=e^{-(\pi^2+1)\,t}\sin\pi x$$

Problem 24

$$u(x,t)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\,e^{-(n^2\pi^2+1)t}\sin n\pi x$$

Problem 25

$$u(x,t)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2 t}\cos n\pi x$$

Problem 26

$$u(x,t)=\cos 6t\sin 3x+\frac{1}{4}\sin 8t\sin 4x$$

Problem 27

$$u(x,t) = -\frac{16}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}(\sin(k+\tfrac{1}{2})\pi(x+t) + \sin(k+\tfrac{1}{2})\pi(x-t))$$

Problem 28

$$u(x,t) = e^{-\pi^2 t}\sin \pi x + \frac{1-e^{-9\pi^2 t}}{9\pi^2}\sin 3\pi x$$

Problem 29

$$\begin{aligned} u(x,t) &= \left(\frac{4}{\pi}e^{-\pi^2 t} + \frac{1-e^{-\pi^2 t}}{\pi^2}\right)\sin \pi x \\ &\quad + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\pi^2 t}\sin(2k+1)\pi x \end{aligned}$$

Problem 30

$$u(x,t) = -\frac{x^2}{2} + \frac{3x}{2} - \sum_{n=1,3,5,\dots}^{\infty}\frac{4}{\pi^3 n^3}\sin n\pi x\cos n\pi t.$$