A Finite Element Crash Course

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1 Continuous Piecewise Linears

Consider a partition,

$$\mathcal{T}_h : 0 = x_0 < x_1 < x_2 < \ldots < x_N = 1, \tag{1.1}$$

of the interval $0 \leq x \leq 1$ into N subintervals of equal length h, see Figure 1 below.

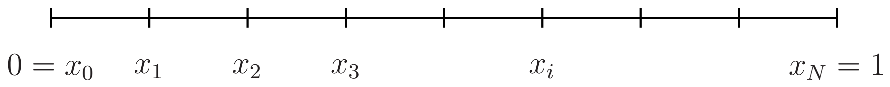

Figure 1: Illustration of a partition $\mathcal{T}_h$.

Let us use this partition to define a function space, V_h, the space of all continuous piecewise linear functions, vanishing at the left end-point $x = 0$ of the interval.

A set of basis functions $\{\varphi_i\}_1^N$ for V_h is defined by

$$\varphi_i(x_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Because of their shape, these functions are commonly called *hat functions*.

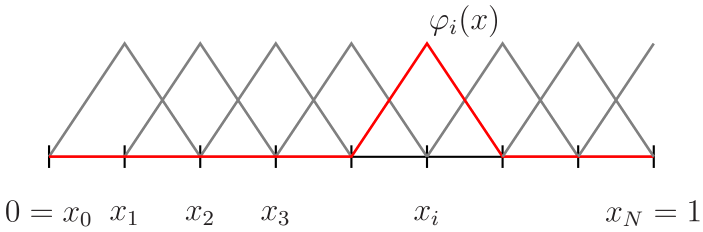

Figure 2: Basis functions, or *hat function*, $\varphi_i(x)$ for V_h.

Note that there is no hat function at $x = 0$ and only a half hat at $x = 1$.

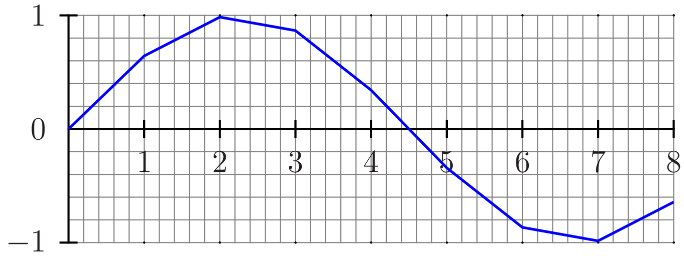

Figure 3: A piecewise linear sine function.

2 Weak and Variational Formulations

Consider the two-point boundary value problem

$$\begin{aligned} -u''(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= 0, \quad u'(1) = 0. \end{aligned} \tag{2.1}$$

A *variational formulation* of (2.16) is obtained by multiplying the *residual* $R(u) = -u''(x) - f(x)$ by any smooth function $v(x)$ satisfying $v(0) = 0$, and integrating over the interval $0 \le x \le 1$. Because $R(u) = 0$, we have trivially

$$\int_0^1 R(u)v(x)\,dx = \int_0^1 -u''(x)v(x) - f(x)v(x)\,dx = 0, \tag{2.2}$$

that is,

$$-\int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx, \tag{2.3}$$

for all functions $v(x)$. Consequently, the variational formulation says that the residual $R(u)$ should be orthogonal to $v(x)$. Usually, $v(x)$ is referred to as a *test function*.

Further, using the regularity (i.e., smoothness) of $v(x)$ we can integrate by parts, to get

$$\begin{aligned} -\int_0^1 u''(x)v(x)\,dx &= -\Big[u'(x)v(x)\Big]_0^1 + \int_0^1 u'(x)v'(x)\,dx \\ &= -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v'(x)\,dx. \end{aligned} \tag{2.4}$$

Since $u'(1) = 0$, and $v(0) = 0$, we have the *weak form*[1] of (2.16)

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \tag{2.5}$$

for all $v(x)$ such that $v(0) = 0$.

2.1 A Finite Element Method

If $U(x)$ is a solution approximation to (2.16) the residual $R(U)$ can no longer be zero for *every* test function $v(x)$. Instead, we replace the condition that the residual should be identically zero by the condition that the residual should be orthogonal to all functions within a suitable so-called *test space* V_h. We choose the space of all continuous piecewise linear functions, which at $x = 0$, as our test space.

A *Finite Element Method* for (2.16) reads: find $U(x) \in V_h$, such that

$$\int_0^1 U'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \tag{2.6}$$

for all $v(x) \in V_h$.

[1] Note that the weak form only involves one derivative of u whereas (2.16) involves two.

Note that since we also choose $U(x) \in V_h$ we have the *ansatz*,

$$U(x) = \xi_1\varphi_1(x) + \xi_2\varphi_2(x) + \ldots + \xi_N\varphi_N(x). \tag{2.7}$$

Here, ξ_i, $i = 1, 2, \ldots, N$ are unknown constants that must be determined. We do this by evaluating (2.6) for N different choices of test functions $v(x)$. As a result, we get a system of equations from which the constants ξ_i can be computed.

Hence, substituting (2.7) into (2.6) and choosing the basis functions as test functions[2], i.e., $v(x) = \varphi_j(x)$ for $j = 1, 2, \ldots, N$, we get a system of equations,

$$\begin{cases} \int_0^1 (\xi_1\varphi_1' + \xi_2\varphi_2' + \ldots + \xi_N\varphi_N')\varphi_1'\,dx = \int_0^1 f\varphi_1 dx \\ \int_0^1 (\xi_1\varphi_1' + \xi_2\varphi_2' + \ldots + \xi_N\varphi_N')\varphi_2'\,dx = \int_0^1 f\varphi_2 dx \\ \vdots \\ \int_0^1 (\xi_1\varphi_1' + \xi_2\varphi_2' + \ldots + \xi_N\varphi_N')\varphi_N'\,dx = \int_0^1 f\varphi_N dx \end{cases} . \tag{2.8}$$

Using matrix notation, we write this as

$$\boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{b}, \tag{2.9}$$

where the entries of the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ are given by

$$a_{ij} = \int_0^1 \varphi_i'(x)\,\varphi_j'(x)\,dx, \quad b_j = \int_0^1 f(x)\varphi_j(x)\,dx, \quad i,j = 1,2,\ldots,N. \tag{2.10}$$

Assembly Process. To compute the integrals of (2.10) we note that

$$\varphi_i(x) = \frac{1}{h}\begin{cases} x - x_{i-1}, & \text{if } x_{i-1} \le x \le x_i, \\ x_{i+1} - x, & \text{if } x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

[2] It is easy to verify that testing (2.6) against all $v \in V_h$ reduces to testing against the basis functions φ_j. This is a consequence of the fact that these bases are linearly independent. In other words, if (2.6) is satisfied for each φ_j separately, then it must also be satisfied for an arbitrary combination of them.

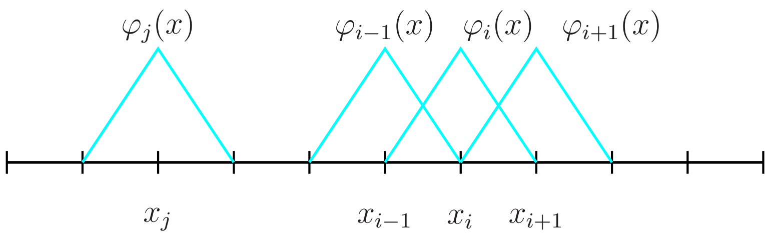

Figure 4: Overlapping and non-overlapping hat functions.

For $|i-j|>1$, we have $a_{ij}=0$, since the bases $\varphi_i(x)$ and $\varphi_j(x)$ lack any overlapping support, that is, one of then is always zero. However, if $i=j$, then

$$a_{ii}=\int_0^1 \varphi_i(x)\varphi_i(x)\,dx=\int_{x_{i-1}}^{x_i}\frac{1}{h^2}\,dx+\int_{x_i}^{x_{i+1}}\frac{(-1)(-1)}{h^2}\,dx=\frac{2}{h}. \tag{2.12}$$

where we have used that $x_i-x_{i-1}=x_{i+1}-x_i=h$. Further, if $j=i+1$, then

$$a_{i,i+1}=\int_0^1 \varphi_i(x)\varphi_{i+1}(x)\,dx=\int_{x_i}^{x_{i+1}}\frac{(-1)}{h}\frac{1}{h}\,dx=-\frac{1}{h}. \tag{2.13}$$

Changing i to $i-1$, we also get $a_{i-1,i}=-1/h$. Hence, by symmetry, we have

$$\boldsymbol{A}=\frac{1}{h}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}. \tag{2.14}$$

Note the last entry which is just $1/h$, because the basis function $\varphi_N(x)$ is half.

Generally, the entries b_i of the vector $\boldsymbol{b}$ must be evaluated by quadrature, since they involve the function $f(x)$. However, if $f(x)=1$, then we can compute b_i by noting that it is the area under $\varphi_i(x)$. We thus get

$$b_i = \int_0^1 f(x)\varphi_i(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)\,dx = h. \tag{2.15}$$

Solving (2.9) using $f(x) = 1$ and $N = 30$, gives $U(x)$ as shown below.

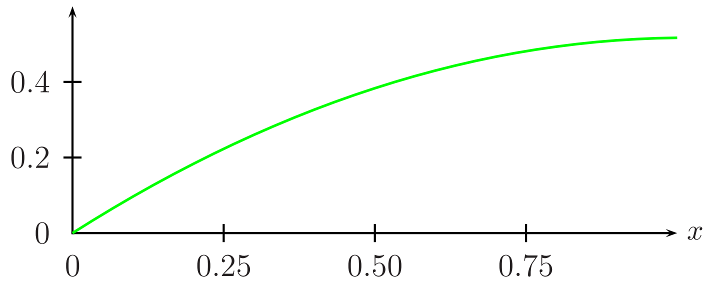

Figure 5: Finite element solution $U(x)$.

2.2 Three Equivalent Problems

Let us turn to consider the more general boundary value problem

$$-(au')' = f, \quad 0 < x < 1, \quad u(0) = u(1) = 0. \tag{2.16}$$

where $a(x) > 0$ is a positive, piecewise continuous, and bounded function for $0 \leq x \leq 1$.

Let us also introduce the function space V, defined by

$$V = \{v : \|v\| + \|v'\| < \infty, \quad v(0) = v(1) = 0\}. \tag{2.17}$$

As before, the variational formulation of the problem (2.16) is obtained by multiplying the equation by a function $v \in V$ and integrating over the interval $0 \leq x \leq 1$, i.e.,

$$-\int_0^1 (au')'v\,dx = \int_0^1 fv\,dx. \tag{2.18}$$

By partial integration, we get

$$-[au'v]_0^1 + \int_0^1 au'v'\,dx = \int_0^1 fv\,dx. \tag{2.19}$$

but, since $v(0) = v(1) = 0$, we have

$$\int_0^1 au'v'\,dx = \int_0^1 fv\,dx. \tag{2.20}$$

So, that the *variational formulation* reads: find $u \in V$, such that

$$\int_0^1 au'v'\,dx = \int_0^1 fv\,dx, \tag{2.21}$$

for all $v \in V$.

Theorem 1. *A solution u to* (2.16) *also satisfies the variational formulation* (2.21)*.*

Proof. Consider (2.21) *written as* $-\int_0^1 (au')'v\,dx = \int_0^1 fv\,dx$*, which can also be written as*

$$\int_0^1 (-(au')' - f)v\,dx = 0, \tag{2.22}$$

for all $v \in V$. We claim that this also implies $-(au')' = f$. To see this, suppose that our claim is not true. Then, there exists a point ξ within the interval $0 \le x \le 1$, such that $-(au')' - f > 0$ (or, < 0) at this point. In this case, let us take a positive (or, negative) function v and insert it into (2.22)*. We get*

$$\int_0^1 (-(au')' - f)v\,dx > 0, \tag{2.23}$$

which contradicts (2.22). Thus, our claim is true and the proof is complete.

Equivalent Minimization Problem. Our boundary value problem (2.16) can also be formulated as a minimization problem, namely, find $u \in V$, such that

$$F(u) = \min_{w \in V} F(w). \tag{2.24}$$

Here, $F(w)$ is the total *energy* of $w(x)$ given by

$$F(w) = \tfrac{1}{2}\int_0^1 a(w')^2\,dx - \int_0^1 fw\,dx. \tag{2.25}$$

Theorem 2. *The minimization problem* (2.24) *above is equivalent with the variational formulation* (2.21).

Proof. *($\Leftarrow$) For $w \in V$, let $v = w - u$, then $v \in V$ and*

$$
\begin{aligned}
F(w) = F(u+v) &= \tfrac{1}{2}\int_0^1 a(u'+v')^2\,dx - \int_0^1 f(u+v)\,dx \\
&= \underbrace{\tfrac{1}{2}\int_0^1 au'^2\,dx}_{A} + \underbrace{\int_0^1 au'v'dx}_{B} + \tfrac{1}{2}\int_0^1 av'^2\,dx - \underbrace{\int_0^1 fu\,dx}_{C} - \underbrace{\int_0^1 fv\,dx}_{D},
\end{aligned}
\tag{2.26}
$$

but $B + D = 0$ by virtue of (2.21)*. Further, by the definition of F we have $A + C = F(u)$. Thus,*

$$
F(w) = F(u) + \tfrac{1}{2}\int_0^1 a(v')^2\,dx, \tag{2.27}
$$

and since $a(x) > 0$ we must have $F(w) > F(u)$.

($\Rightarrow$) Let us now assume that $F(u) \leq F(w)$ and set $g(\varepsilon) = F(u + \varepsilon v)$. We know then that g has a minimum at $\varepsilon = 0$, that is to say, $g'(0) = 0$. However,

$$
\begin{aligned}
g(\varepsilon) = F(u+\varepsilon v) &= \tfrac{1}{2}\int_0^1 a(u'+\varepsilon v')^2\,dx - \int_0^1 f(u+\varepsilon v)\,dx \\
&= \tfrac{1}{2}\int_0^1 a(u')^2 + 2a\varepsilon u'v' + a\varepsilon^2(v')^2\,dx - \int_0^1 fu\,dx - \varepsilon\int_0^1 fv\,dx,
\end{aligned}
\tag{2.28}
$$

so $g'(\varepsilon)$ is given by

$$
g'(\varepsilon) = \tfrac{1}{2}\int_0^1 2a\varepsilon(v')^2 + 2au'v'\,dx - \int_0^1 fv\,dx. \tag{2.29}
$$

Evaluating this derivative at $\varepsilon = 0$ implies that

$$
\int_0^1 au'v'dx - \int_0^1 fv\,dx = 0, \tag{2.30}
$$

which is precisely the variational formulation (2.21).

2.3 A Basic Error Estimate

Numerical techniques, such as the Finite Element Method, are worthless unless they are supplemented by some form of error analysis. *A priori* error analyses are performed theoretically and before any computations has been done. Here, we show a basic error estimate.

Theorem 3. *Let $u \in V$ be a solution to*

$$-u'' = f, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0. \tag{2.31}$$

Also, let $U \in V_h$ be a solution approximation satisfying

$$\int_0^1 U'v'\,dx = \int_0^1 fv\,dx, \tag{2.32}$$

for all $v \in V_h$. Here, $V_h \subset V = \{v : \|v\| + \|v\| < \infty,\ v(0) = v(1) = 0\}$ is the usual space of piecewise linears vanishing at $x = 0$ and $x = 1$. It holds that the error $e = u - U$ obeys

$$\|(u - U)'\| \leq \|(u - v)'\|, \tag{2.33}$$

for all $v \in V_h$.

Proof: Pick a function $v \in V_h$. We then have

$$\begin{aligned}
\|(u - U)'\|^2 &= \int_0^1 (u' - U')^2\,dx \\
&= \int_0^1 (u' - U')(u' - v' + v' - U')\,dx \\
&= \int_0^1 (u' - U')(u' - v')\,dx + \int_0^1 (u' - U')(v' - U')\,dx.
\end{aligned} \tag{2.34}$$

Further, it also holds that

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx, \tag{2.35}$$

for all $v \in V$, and that

$$\int_0^1 U'v'\,dx = \int_0^1 fv\,dx, \tag{2.36}$$

for all $v \in V_h \subset V$*. Subtracting these two relations we have*

$$\int_0^1 (u' - U')v'\,dx = 0, \tag{2.37}$$

for all $v \in V_h$*. As a consequence, the last integral of the estimate vanish, i.e.,*

$$\|(u-U)'\|^2 = \int_0^1 (u' - U')(u' - v')\,dx \le \|(u-U)'\|\|(u-v)'\|, \tag{2.38}$$

where we used the Cauchy-Schwarz inequality. Hence,

$$\|(u-U)'\| \le \|(u-v)'\|, \tag{2.39}$$

and the proof is complete.