Sample Exam Questions

Applied Mathematics, TMA 682

1. Suppose a is a constant. Verify the following Laplace transforms,

$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}, \qquad \mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}.$$

2. State and prove the 2^{nd} *Shifting Rule* for Laplace transforms, i.e.,

$$\mathcal{L}[f(t - T)] = e^{-Ts}F(s).$$

3. Show, that if the function $f(x)$ is periodic of period p, then the integral

$$\int_a^{a+p} f(x)\, dx,$$

is independent of a.

4. Let $f(x)$ be a given 2π-periodic, integrable function. Show, that if

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx},$$

then the numbers c_n must be given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

5. Prove the *Bessel inequality*, that is,

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx,$$

where c_n are the complex Fourier coefficients of $f(x)$.

6. State and prove the *Riemann-Lebesgue Lemma.*

7. Prove that there exist a constant C, such that,

$$|\pi f(x) - f(x)| \le C(b-a)^2 \max_{a\le x\le b} |f''(x)|,$$

where $\pi f(x)$ is the linear interpolant of $f(x)$ for $a \le x \le b$.

8. Consider the two-point boundary value problem

$$\begin{aligned} -u'' &= f, \quad 0 < x < 1, \\ u(0) &= u'(1) = 0. \end{aligned}$$

Prove that the finite element approximation $U \in V_h$ satisfies

$$\|(u-U)'\| \le \|(u-v)'\|, \text{ for all } v \in V_h,$$

where V_h is the space of all continuous, piecewise linears vanishing at $x = 0$.

9. Verify that the solution $u \in V$ of the *Poisson equation*

$$-u'' = f, \quad 0 < x < 1, \quad u(0) = u'(1) = 0,$$

minimizes the energy functional

$$F(v) = \frac{1}{2}\int_0^1 (v')^2 dx - \int_0^1 fv\, dx,$$

over the space $V = \{v : \|v\| + \|v'\| < \infty,\ v(0) = 0\}$.

10. Consider the *heat equation*

$$\begin{aligned} \dot{u} - u'' &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= u'(1,t) = 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

Show, the stability estimates

a. $\dfrac{d}{dt}\|u\|^2 + 2\|u'\| = 0.$

b. $\|u(\cdot,t)\| \le e^{-t}\|u_0\|.$