Lecture Notes in Fourier Analysis

Mohammad Asadzadeh

1st September 2008

Contents

0 Introduction **7**
0.1 Partial Differential Equations 7
0.1.1 Differential operators , superposition principle 10
0.1.2 Some equations of mathematical physics 13
0.2 Generalized functions 23
0.2.1 The Heaviside step function 24
0.2.2 The Impulse functions 28
0.2.3 Rule of integration for Heaviside θ functions 35
0.2.4 Rules of integration for the Dirac δ functions. 37

1 Laplace Transformation **43**
1.1 The Laplace Transform 43
1.1.1 Definition 43
1.1.2 Existence 45
1.1.3 General Properties of the Laplace Transform 46
1.1.4 Table of Laplace Transforms 51
1.2 The Inverse Laplace Transform 53
1.2.1 Method of Partial Fractions 55
1.3 Applications of Laplace Transforms 58
1.3.1 Initial Value Problems 58
1.3.2 Integral Equations 61

2 Separation of variables **65**
2.1 The heat equation: Dirichlet problem 65
2.2 The heat equation: Neumann problem 71
2.3 The heat conducting problem 73
2.4 The wave equation 75

3 Fourier series for 2π-periodic functions **79**
3.1 Periodic Functions 80
3.2 Fourier series 80
3.3 Even and odd functions 85
3.4 Bessel's Inequalities 88
3.5 Proof of the convergence theorem 91
3.6 Derivatives, primitive functions 98
3.7 Fourier series on the interval $[-\pi,\pi]$ 103
3.8 Fourier series on $2L$-intervals 105

4 The Fourier Transform **111**
4.1 Introduction 111
4.2 Basic properties of the Fourier transform 113
4.3 Convolutions 115
4.4 Some key examples 116
4.5 The Fourier Inversion Theorem 120
4.6 Plancherel Theorem 122
4.7 The symmetry rule 123
4.8 Applications of Fourier transform 127
4.9 Sturm-Liouville problems on $[0,\infty)$. 130
4.10 Generalized functions 134
4.11 Dynamic systems 139
4.12 The discrete Fourier transform 148

5 Orthogonal sets of functions **153**
5.1 Function spaces 153
5.2 Convergence and Completeness 155
5.3 Regular Sturm-Liouville problems 162

6 Boundary value problems **169**
6.1 Fourier techniques for inhomogeneous PDE 169
6.2 Inhomogeneous Sturm-Liouville problems 172
6.3 The Dirichlet problem 175
6.4 Heat flow 178

7 Bessel functions **187**
7.1 Solutions of Bessel's differential equation 188
7.2 The Recurrence Formulas 192

7.3 Orthogonal sets of Bessel functions 197

8 Orthogonal polynomials 205

8.1 Legendre polynomials 205
8.2 Hermite polynomials 216
8.3 Laguerre polynomials 217

9 Distribution theory 219

9.1 Test functions 219
9.2 Delta functions 224
9.3 Distributions 229

10 Derivations of Some PDEs 239

10.1 Some important equations 239
10.2 The Incompressible Navier-Stokes 240
10.2.1 The weak formulation of the Navier-Stokes 247
10.3 Further real world equations 249

Chapter 0

Introduction

This book is an introduction to Fourier analysis and related topics with applications in solving linear partial differential equations (PDEs), integral equations as well as signal problems. In this chapter we introduce some basic PDEs of mathematical physics. We also introduce the step and impulse functions which are crucial in describing the continuous time signals.

0.1 Partial Differential Equations

We shall use the common notation $\mathbb{R}^n$ for the real Euclidean spaces of dimension n with the elements $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. In the most applications n will be $1, 2, 3$ or 4 and the variables x_1, x_2 and x_3 denote coordinates in one, two, or three dimensions, whereas x_4 represents the time variable. In this case we usually replace (x_1, x_2, x_3, x_4) by a most common notation: (x, y, z, t). Further we shall use the common subscript notation for the partial derivatives, viz:

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad \dot{u} = u_t = \frac{\partial u}{\partial t}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad \text{etc.}$$

A more general notation for a partial derivative for a sufficiently smooth function u (see definition below) is given by

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \cdot \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u,$$

where $\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$, $1 \le i \le n$, denotes the partial derivative of order i with respect to the variable x_i, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index of integers $\alpha_i \ge 0$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Definition 1. *A function f of one real variable is said to be of class $\mathcal{C}^{(k)}$ on an interval I if its derivatives $f', \ldots, f^{(k)}$ exist and are continuous on I. A function f of n real variables is said to be of class $\mathcal{C}^{(k)}$ on a set $S \in \mathbb{R}^n$ if all of its partial derivatives of order $\le k$ i.e. $\partial^{|\alpha|} f/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})$ with the multi-index $|\alpha| = \alpha_1 + \ldots + \alpha_n \le k$, exist and are continuous on S.*

A key defining property of a partial differential equation (PDE) is that there is more than one independent variable and a PDE is a relation between an unknown function and its partial derivatives:

$$F(x_1, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, \ldots, \partial^{|\alpha|} u/\partial x_1^{\alpha_1} \ldots \partial x_l^{\alpha_l}, \ldots) = 0. \quad (0.1.1)$$

The order of an equation is defined to be the order of the highest derivative in the equation. The most general PDE of the first order in two independent variables can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \quad (0.1.2)$$

Likewise the most general PDE of the second order in two independent variables is of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \quad (0.1.3)$$

It turns out that, when the equations (0.1.1)-(0.1.3) are considered in bounded domains $\Omega \subset \mathbb{R}^n$, in order to obtain a *unique solution* (see below) one should provide conditions at the boundary of the domain Ω called *boundary conditions*, denoted, e.g. by $B(u) = f$ or $B(u) = 0$ (as well as conditions for $t = 0$, *initial conditions*; denoted, e.g. by $I(u) = g$ or $I(u) = 0$), as we often see in the theory of ordinary differential equations. B and I are expressions on u and its partial derivatives, stated on the whole or a part of the boundary of Ω (or, in case of I, for $t = 0$), and are associated to the underlying PDE. Below we shall discuss the choice of relevant initial and boundary conditions for a PDE.

A *solution* for a PDE of type (0.1.1)-(0.1.3) is a function u that identically satisfies the corresponding PDE, and the associated initial and boundary conditions, in some region of the variables $x_1, x_2, \ldots, x_n$ (or x, y). Note that a solution of an equation of order k has to be k times differentiable. A function in $\mathcal{C}^{(k)}$ that satisfies a PDE of order k is called a classical (or strong) solution of the PDE. We sometimes also have to deal with solutions that are not classical. Such solutions are called weak solutions. We shall discuss the weak solutions in the distribution chapter.

Definition 2. *A problem consisting of a PDE associated with boundary and/or initial conditions is called well-posed if it fulfills the following three criteria:*
1. **Existence** *The problem has a solution.*
2. **Uniqueness** *There is no more than one solution.*
3. **Stability** *A small change in the equation or in the side conditions gives rise to a small change in the solution.*

If one or more of the conditions above does not hold, then we say that the problem is *ill-posed*. The fundamental theoretical question of PDE is whether the problem consisting of the equation and its associated side conditions is well-posed. In this regard, one can fairly say that the fundamental problems of mathematical physics are all well-posed. However, in certain engineering applications we might encounter problems that are ill-posed. In practice, such problems are unsolvable. Therefore, when we face an ill-posed problem, the first step should be to modify it appropriately in order to render it well-posed.

Definition 3. *An equation is called linear if in* (0.1.1)*, F is a linear function of the unknown function u and its derivatives.*

Thus, for example, the equation $e^{x^2y}u_x + x^7u_y + \cos(x^2+y^2)u = y^3$ is a linear equation, while $u_x^2 + u_y^2 = 1$ is nonlinear equation. The nonlinear equations are often further classified into subclasses according to the type of their nonlinearity. Generally, the nonlinearity is more pronounced when it appears in higher order derivatives. For example, the following equations are both nonlinear

$$u_{xx} + u_{yy} = u^3 + u. \tag{0.1.4}$$

$$u_{xx} + u_{yy} = |\nabla u|^2 u. \tag{0.1.5}$$

Here $|\nabla u|$ denotes the norm of the gradient of u. While (0.1.5) is nonlinear, it is still linear as a function of highest-order derivative. Such a nonlinearity is called *quasilinear*. On the other hand in (0.1.4) the nonlinearity is only in the unknown solution u. Such equations are called *semilinear*.

0.1.1 Differential operators , superposition principle

We recall that we denote the set of continuous functions in a domain D (a subset of $\mathbb{R}^n$) by $\mathcal{C}^0(D)$ or $\mathcal{C}(D)$. Further, by $\mathcal{C}^{(k)}(D)$ we mean the set of all functions that are k times continuously differentiable in D. Mappings between function classes as $\mathcal{C}^{(k)}$ are called *differential operators*. We denote by $\mathcal{L}[u]$ the operation of a mapping (operator) $\mathcal{L}$ on a function u.

Definition 4. *An operator $\mathcal{L}$ that satisfies*

$$\mathcal{L}[\beta_1 u_1 + \beta_2 u_2] = \beta_1 \mathcal{L}[u_1] + \beta_2 \mathcal{L}[u_2], \qquad \forall \beta_1, \beta_2 \in \mathbb{R}, \tag{0.1.6}$$

where u_1 and u_2 are arbitrary functions is called a linear operator. We may generalize (0.1.6) *as*

$$\mathcal{L}[\beta_1 u_1 + \ldots + \beta_k u_k] = \beta_1 \mathcal{L}[u_1] + \ldots + \beta_k \mathcal{L}[u_k], \quad \forall \beta_1, \ldots, \beta_k \in \mathbb{R}, \tag{0.1.7}$$

i.e. $\mathcal{L}$ takes any linear combination of u_j's to corresponding linear combination of $\mathcal{L}[u_j]$'s.

For instance the integral operator $\mathcal{L}[f] = \int_a^b f(x)\,dx$ defined on the space of continuous functions on $[a, b]$ defines a linear operator from $\mathcal{C}[a, b]$ into $\mathbb{R}$: satisfies both (0.1.6) and (0.1.7).

A *linear partial differential operator* $\mathcal{L}$ that transforms a function u of the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ into another function $\mathcal{L}$ is given by

$$L[\bullet] = a(\mathbf{x}) + \sum_{i=1}^{n} b_i(\mathbf{x}) \frac{\partial \bullet}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x}) \frac{\partial^2 \bullet}{\partial x_i \partial x_j} + \ldots \tag{0.1.8}$$

where $\bullet$ represents any function u in, say $\mathcal{C}^{(k)}$, and the dots at the end indicate higher-order derivatives, but the sum contains only *finitely* many terms.

The term *linear* in the phrase *linear partial differential operator* refers to the following fundamental property: if $\mathcal{L}$ is given by (0.1.8) and u_j, $1 \leq j \leq k$, are any set of functions possessing the requisite derivatives, and β_j, $1 \leq j \leq k$, are any constants then the relation (0.1.7) is fulfilled. This is an immediate consequence of the fact that (0.1.6) and (0.1.7) are valid for $\mathcal{L}$ replaced with the derivative. A linear differential equation defines a linear differential operator: the equation can be expressed as $\mathcal{L}[u] = F$, where $\mathcal{L}$ is a linear operator and F is a given function. The differential equation

$\mathcal{L}[u] = 0$ is called a *homogeneous equation*. For example, define the operator $\mathcal{L} = \partial^2/\partial x^2 - \partial^2/\partial y^2$. Then

$$\mathcal{L}[u] = u_{xx} - u_{yy} = 0,$$

is a homogeneous equation, while the equation

$$\mathcal{L}[u] = u_{xx} - u_{yy} = x,$$

is an example of a *nonhomogeneous equation*. In a similar way we may define another type of constraint for the PDEs that appears in many applications: *the boundary conditions*. In this regard the linear boundary conditions are defined as operators B satisfying

$$\mathcal{B}(\beta_1 u_1 + \beta_2 u_2) = \beta_1 B(u_1) + \beta_2 B(u_2), \qquad \forall \beta_1, \beta_2 \in \mathbb{R}, \tag{0.1.9}$$

at the boundary of a given domain Ω.

The Superposition principle. An important property of the linear operators is that if the functions u_j, $1 \le j \le k$, satisfy the linear differential equation $\mathcal{L}[u] = F_i$ and the boundary conditions (linear) $B(u_j) = f_j$ for $j = 1, 2, \dots, k$, then the linear combination $v := \sum_{i=1}^n \beta_i u_i$, satisfies the equation $\mathcal{L}[v] = \sum_{i=1}^n \beta_i F_i$ as well as the boundary condition $B(v) = \sum_{i=1}^n \beta_i f_i$. In particular, if each of the functions u_i, $1 \le i \le n$, satisfies the homogeneous equation $\mathcal{L}[u] = 0$ and the homogeneous boundary condition $B(u) = 0$, then every linear combination of them satisfies that equation and boundary condition too. This property is called the *superposition principle*. It allows to construct complex solutions through combining simple solutions: suppose we want to determine all solutions of a differential equation associated with a boundary condition viz,

$$\mathcal{L}[u] = F, \qquad B(u) = f. \tag{0.1.10}$$

We consider the corresponding, *simpler homogeneous problem*:

$$\mathcal{L}[u] = 0, \qquad B(u) = 0. \tag{0.1.11}$$

Now it suffices to find just one solution, say v of the original problem (0.1.10). Then, for any solution u of (0.1.10), $w = u - v$ satisfies (0.1.11). Since $\mathcal{L}[w] = \mathcal{L}[u] - \mathcal{L}[v] = F - F = 0$ and $B(w) = B(u) - B(v) = f - f = 0$. Hence we obtain a general solution of (0.1.10) by adding the general solution w of (0.1.11) to any particular solution of (0.1.10).

Following the same idea one may apply superposition to split a problem involving several inhomogeneous terms into simpler ones each with a single inhomogeneous term. For instance we may split (0.1.10) as

$$\begin{aligned} \mathcal{L}[u_1] &= F, \qquad & B(u_1) &= 0, \\ \mathcal{L}[u_2] &= 0, \qquad & B(u_2) &= f, \end{aligned}$$

and then take $u = u_1 + u_2$.

The most important application of the superposition principle is in the case of linear homogeneous differential equations satisfying homogeneous boundary conditions: *if the functions* u_j, $1 \leq j \leq k$, *satisfy* (0.1.11)*: the linear differential equation* $\mathcal{L}[u] = 0$ *and the boundary conditions (linear)* $B(u_j) = 0$ *for* $j = 1, 2, \ldots, k$, *then the linear combination* $v := \sum_{i=1}^{n} \beta_i u_i$, *satisfies the same equation and boundary condition:* (0.1.11).

Finally, superposition principle is used to prove the uniqueness of solutions to linear PDEs.

Exercises

1. Consider the problem

$$u_{xx} + u = 0, \quad x \in (0, \ell); \qquad u(0) = u(\ell) = 0.$$

Cleasly the function $u(x) \equiv 0$ is a solution. Is this solution unique? Does the answer depend on ℓ?

2. Consider the problem

$$u_{xx} + u_x = f(x), \quad x \in (0, \ell); \qquad u(0) = u'(0) = \frac{1}{2}[u'(\ell) + u(\ell)].$$

a) Is the solution unique? (f is a given function).

b) Under what condition on f a solution exists?

3. Suppose u_i, $i = 1, 2, \ldots, N$ are N solutions of the linear differential equation $\mathcal{L}[u] = F$, where $F \neq 0$. Under what condition on the constant coefficients c_i, $i = 1, 2, \ldots, N$ is the linear combination $\sum_{i=1}^{N} c_i u_i$ also a solution of this equation?

4. Consider the nonlinear ordinary differential equation $u_x = u(1-u)$.

   a) Show that $u_(x) \equiv 1$ and $u_2(x) = 1 - 1/(1+e^x)$ both are solutions, but $u_1 + u_2$ is not a solution.

   b) For which value of c_1 is c_1u_1 a solution? What about c_2u_2?

5. Show that each of the following equations has a solution of the form $u(x,y) = f(ax+by)$ for a proper choice of constants a, b. Fine the constants for each example.

   a) $u_x + 3u_y = 0$. b) $3u_x - \pi u_y = 0$. c) $2u_x + eu_y = 0$.

6. a) Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Write equation in the coordinates $s = x$, $t = x - y$.

   b) Find the general solution of the equation.

   c) Consider the equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$. Write it the coordinates $s = x + y$ and $t = 2x$.

7. a) Show that for $n = 1, 2, 3, \ldots, _n (x, y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0.$$

   b) Find a linear combination of u_n's that satisfies $u(x,1) = \sin 2\pi x - \sin 3\pi x$.

   c) Solve the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} = 0, &\quad u(0,y) = u(1,y) = 0, \\ u(x,0) = 2\sin \pi x, &\quad u(x,1) = \sin 2\pi x - sin 3\pi x. \end{aligned}$$

0.1.2 Some equations of mathematical physics

In this subsection we shall introduce some of the basic partial differential equations of mathematical physics that will be the subject of our studies throughout the book. These equations all involve a fundamental differential operator of order two, called *Laplacian*, acting on $\mathcal{C}^{(2)}(\mathbb{R}^n)$ and defined as follows:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}, \qquad u \in \mathcal{C}^{(2)}(\mathbb{R}^n). \tag{0.1.12}$$

Basically, there are three types of fundamental physical phenomena described by differential equations involving the Laplacian:

$$\begin{aligned} \nabla^2 u &= F(\mathbf{x}, t), && \textit{The Laplace equation} \\ u_t - k\nabla^2 u &= F(\mathbf{x}, t), && \textit{The heat equation} \\ u_{tt} - c^2\nabla^2 u &= F(\mathbf{x}, t), && \textit{The wave equation.} \end{aligned} \tag{0.1.13}$$

Here F is a given function. If $F \neq 0$, then the equations (0.1.13) are *inhomogeneous*. In the special case, when $F \equiv 0$ the equations (0.1.13) are *homogeneous*.

Here the first equation, being time independent, has a particular nature: besides the fact that it describes the steady-state heat transfer and the standing wave equations (loosely speaking, the time independent versions of the other two equations), the Laplace'e equation arises in describing several other physical phenomena such as *electrostatic potential* in regions with no electric charge, *electromagnetic potential*, in the domains lacking gravity as well as problems in *elasticity*, etc.

The heat equation describes the diffusion of thermal energy in a homogeneous material where $u = u(\mathbf{x}, t)$ is the temperature at a position $\mathbf{x}$ at time t and k is a constant called *thermal diffusivity* or *heat conductivity* of the material.

Remark The heat equation can be used to model the heat flow in solids and fluids, in the later case, however, it dose *not* take any account to the convection phenomenon; and provides a reasonable model only if phenomena such as macroscopic currents in the fluid are not present (or negligible). Further, the heat equation is not a fundamental law of physics, and it does not give reliable answers at very low or very high temperatures.

Since temperature is related to heat, which is a form of energy, the basic idea in deriving the heat equation is to use the *law of conservation of energy:*

Fourier's law of heat conduction and the derivation of heat equation
Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a fixed spatial domain with the boundary $\partial\Omega$. The rate of change of thermal energy with respect to time in D is equal to the net flow of energy across the boundary of D plus the rate at which heat is generated within D.

Let now u denote the temperature at the position $\mathbf{x} = (x, y, z) \in D$ and at time t. We assume that the solid is on rest and it is rigid so that the only energy present is thermal energy and the density $\rho(x)$ is independent of the time t and temperature u. Let $\mathcal{E}$ denote the specific internal energy of the solid, that is, the energy per unit mass. Then the amount of thermal energy in Ω is given by

$$\int_\Omega \rho\, \mathcal{E}\, d\mathbf{x},$$

and the time rate (time derivative) of change of thermal energy in Ω is:

$$\frac{d}{dt}\int_\Omega \rho\, \mathcal{E}\, d\mathbf{x} = \int_\Omega \rho\, \mathcal{E}_t\, d\mathbf{x}.$$

Let $q = (q_1, q_2, q_3)$ denote the heat flux vector and $\mathbf{n} = (n_1, n_2, n_3)$ denote the outward unit normal, to the boundary $\partial\Omega$, at the point $\mathbf{x} \in \partial\Omega$. Then $q \cdot \mathbf{n}$ represents the flow of heat per unit cross-sectional area per unit time crossing a surface element. Thus

$$-\int_{\partial\Omega} q \cdot \mathbf{n}\, dS$$

is the amount of heat per unit time flowing across the boundary $\partial\Omega$. Here dS represents the element of surface area. The minus sign reflects the fact that if more heat flows out of the domain D than in, the energy in D decreases. Finally, in general, the heat production is determined by external sources that are independent of the temperature. In some cases (such as an air conditioner controlled by a thermostat) it depends on temperature itself but not on its derivatives. Hence in the presence of a source (or sink) we denote the corresponding rate at which heat is reduced per unit volume by $f = f(\mathbf{x}, t, u)$ so that the source term becomes

$$\int_\Omega f(\mathbf{x}, t, u)\, d\mathbf{x}.$$

Now the law of conservation of energy takes the form

$$\int_\Omega \rho\, \mathcal{E}_t\, d\mathbf{x} - \int_{\partial\Omega} q \cdot \mathbf{n}\, dS = \int_\Omega f(\mathbf{x}, t, u)\, d\mathbf{x}. \tag{0.1.14}$$

Applying the Gauss divergent theorem to the integral over $\partial\Omega$ we get

$$\int_\Omega (\rho\, \mathcal{E}_t + \nabla \cdot q - f)\, d\mathbf{x} = 0, \tag{0.1.15}$$

where $\nabla\cdot$ denotes the divergent operator. In the sequel we shall use the following simple result:

Lemma 1. *Let h be a continuous fiction satisfying $\int_\Omega h(\mathbf{x}\,d\mathbf{x}) = 0$ for every domain $\Omega \subset \mathbb{R}^d$. Then $h \equiv 0$.*

Proof. Let us assume to the contrary that there exists a point $\mathbf{x_0} \in \Omega$ where $h(\mathbf{x_0}) \neq 0$. Assume without loss of generality that $h(\mathbf{x_0}) > 0$. Since h is continuous, there exists a domain (maybe very small) $\Omega_0 \subset \Omega$, containing $\mathbf{x_0}$ and $\varepsilon > 0$, such that $h(\mathbf{x}) > \varepsilon$, for all $\mathbf{x} \in \Omega_0$. Therefore we have $\int_\Omega h(\mathbf{x})\,d\mathbf{x} > \varepsilon \mathrm{Vol}(\Omega_0) > 0$, which contradicts the lemma's assumption. □

From (0.1.15), using the above lemma, we conclude that

$$\rho\,\mathcal{E}_t = -\nabla \cdot q + f. \tag{0.1.16}$$

This is the basic form of our heat conduction law. The functions $\mathcal{E}$ and q are unknown and additional information of an empirical nature is needed to determine the equation for the temperature u. First, for many materials, over fairly wide but not too large temperature range, the function $\mathcal{E} = \mathcal{E}(u)$ depends nearly linearly on u, so that

$$\mathcal{E}_t = \lambda u_t. \tag{0.1.17}$$

Here λ, called the *specific heat*, is assumed to be constant. Next we relate the temperature u to the heat flux q. Here we use *Fourier's law* but, first, to be specific, we describe the simple facts supporting Fourier's law:
(i) the heat flows from regions of high temperature to the regions of low temperature.
(ii) The rate of heat flow is small or large according as temperature changes between neighboring regions are small or large. To describe these quantitative properties of heat flow, we postulate a linear relationship between the rate of heat flow and the rate of temperature change. Recall that if $\mathbf{x}$ is a point in the heat conducting medium and $\mathbf{n}$ is a unit vector specifying a direction at $\mathbf{x}$, then the rate heat flow at $\mathbf{x}$ in the direction $\mathbf{n}$ is $q \cdot \mathbf{n}$ and the rate of change of temperature is $\partial u/\partial \mathbf{n} = \nabla u \cdot \mathbf{n}$, the directional derivative of temperature. Since $q \cdot \mathbf{n} > 0$ requires $\nabla u \cdot \mathbf{n} < 0$, and vice versa, (from the calculus the direction of maximal growth of a function is given by its gradient), our linear relation takes the form $q \cdot \mathbf{n} = -\kappa \nabla u \cdot \mathbf{n}$, with $\kappa = \kappa(\mathbf{x}) > 0$. Since $\mathbf{n}$ specifies any direction from $\mathbf{x}$, this is equivalent to the assumption

$$q = -k\nabla u, \tag{0.1.18}$$

which is concrete statement of the *Fourier's law*. The positive function κ is called the *heat conduction (or Fourier) coefficient*. Let now $\sigma = \kappa/\lambda\rho$ and $F = f/\lambda\rho$ and insert (0.1.16) and (0.1.17) in (0.1.18) to get the final form of the heat equation.

$$u_t = \nabla \cdot (\sigma \nabla u) + F. \tag{0.1.19}$$

The quantity σ is referred to as the *thermal diffusivity (or diffusion) coefficient*. If we assume that σ is constant, then the final form of the heat equation would be

$$u_t = \sigma \nabla^2 u + F, \qquad \text{or} \quad u_t = \sigma\, \Delta u + F. \tag{0.1.20}$$

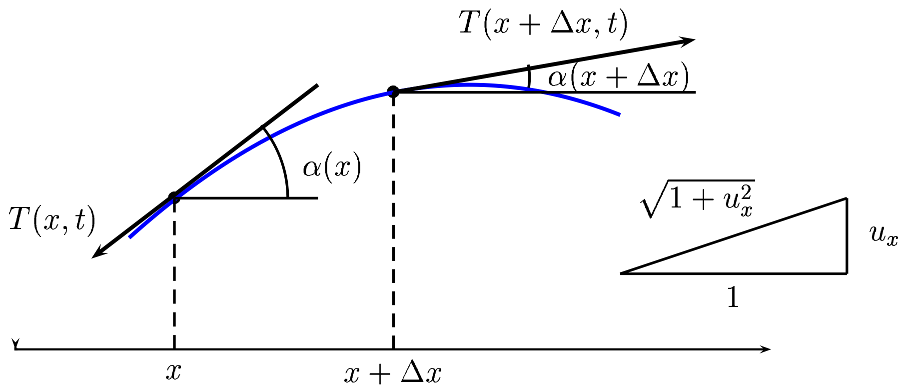
Here $\Delta = \text{div}\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplace's operator in three dimensions (its general form is introduced in the beginning of this subsection).

The third equation in (0.1.13) is the wave equation: $u_{tt} - c\nabla^2 u = F$. Here u represents a wave traveling through an n-dimensional medium; c is the speed of propagation of wave in the medium and $u(\mathbf{x}, t)$ is the amplitude of the wave at position $\mathbf{x}$ and time t. The wave equation provides mathematical model for a number of problems involving different physics processes such as in the following examples:
(i) Vibration of a stretched string, such as guitar string (1-dimensional).
(ii) Vibration of a column of air, such as a clarinet (1-dimensional).
(iii) Vibration of a stretched membrane, such as a drumhead (2-dimensional).
(iv) Waves in an incompressible fluid, such as water (2-dimensional).
(v) Sound waves in air or other elastic media (3-dimensional).
(vi) Electromagnetic, such as light waves and radio waves (3-dimensional).

Note that in (i), (iii) and (iv), u represents the transverse displacement of the string, membrane, or fluid surface; in (ii) and (v), u represents the longitudinal displacement of the air; and in (vi), u is any of the components of the electromagnetic field. For detailed discussions and a derivation of the equations modeling (i)- (vi), see, e.g, Folland [], Guenther and Lee, Gustavsson [], [], Ingard [] Pinchover and Rubinstein [], Strauss [] and Taylor []. We should point out, however, that in most cases the derivation involves making some simplifying assumptions. Hence, the wave equation gives only an approximate description of the actual physical process, and the validity of the approximation will depend on whether certain physical conditions are satisfied. For instance, in example (i) the vibration should be small enough so that the string is not stretched beyond its limits of elasticity. In example (vi), it follows from Maxwell's equations, the fundamental equations of electromagnetism, that the wave equation is satisfied *exactly* in regions containing no electrical charges or current, which of course cannot be guaranteed under normal physical circumstances and can only be approximately justified in the real world. So an attempt to derive the wave equation corresponding to one and each of these examples from the physical principles is far beyond the scope of this book. Nevertheless, to give an idea, below we shall derive the wave equation for a vibrating string which is, by the way, the most considered wave equation model in this book.

The vibrating string, derivation of a wave equation in 1D
Consider a perfectly elastic and flexible string stretched along the segment $[0, L]$ of the x-axis, moving perpendicular to its equilibrium position. Let $\rho_0(x)$ denote the density of the string in the equilibrium position and $\rho(x,t)$ the density at time t. In an arbitrary small interval $[x, x+\Delta x]$ the mass will satisfy

$$\int_x^{x+\Delta x} \rho_0(x)\,dx = m = \int_x^{x+\Delta x} \rho(x,t)\sqrt{1+u_x^2}\,dx. \tag{0.1.21}$$



Thus, using lemma 1, (2.4.1) gives the conservation of mass viz:

$$\rho_0(x) = \rho(x,t)\sqrt{1+u_x^2}. \tag{0.1.22}$$

Now we use the tensions $T(x,t)$ and $T(x+\Delta x, t)$, at the endpoints of an element of the string and determine the force acting on the interval $[x, x+\Delta x]$.

Since we assumed that the string moves only vertically, hence the forces in the horizontal direction should be in balance: i.e.,

$$T(x+\Delta x,t)\cos\alpha(x+\Delta x)-T(x,t)\cos\alpha(x)=0. \tag{0.1.23}$$

Dividing (2.4.3) by Δx and letting $\Delta x \to 0$, we thus obtain

$$\frac{\partial}{\partial x}\Big(T(x,t)\cos\alpha(x)\Big)=0, \tag{0.1.24}$$

hence

$$T(x,t)\cos\alpha(x)=\tau(t), \tag{0.1.25}$$

where $\tau(t)>0$ because it is the magnitude of the horizontal component of the tension.

On the other hand the vertical motion is determined by the fact that the time rate of change of linear momentum is given by the sum of the forces acting in the vertical direction. Hence, using (2.4.2), the momentum of the small element $[x, x+\Delta x]$ is given by

$$\int_x^{x+\Delta x}\rho_0(x)u_t\,dx=\int_x^{x+\Delta x}\rho(x,t)\sqrt{1+u_x^2}\,u_t\,dx, \tag{0.1.26}$$

with the time rate of change:

$$\frac{d}{dt}\int_x^{x+\Delta x}\rho_0u_t\,dx=\int_x^{x+\Delta x}\rho_0u_{tt}\,dx. \tag{0.1.27}$$

There are two kind of forces acting on the segment $[x, x+\Delta x]$ of the string: (i) the forces due to tension that keep the string taut and whose horizontal components are in balance, and (ii) the forces acting a along whole length of the string, such as weight. Thus, using (2.4.5), the net force acting on the ends of the string element $[x, x+\Delta x]$ is

$$\begin{aligned}T(x+\Delta x,t)\sin\alpha(x+\Delta x)-&T(x,t)\sin\alpha(x)=\\&=\tau\Big(\frac{\sin\alpha(x+\Delta x)}{\cos\alpha(x+\Delta x)}-\frac{\sin\alpha(x)}{\cos\alpha(x)}\Big)\\&=\tau\Big(\tan\alpha(x+\Delta x)-\tan\alpha(x)\Big)\\&=\tau\Big(u_x(x+\Delta x,t)-u_x(x,t)\Big).\end{aligned} \tag{0.1.28}$$

Further, the weight of the string acting downward is

$$-\int \rho\, g\, dS = -\int_x^{x+\Delta x} \rho\, g \sqrt{1+u_x^2}\, dx = -\int_x^{x+\Delta x} \rho_0 g\, dx. \qquad (0.1.29)$$

Next, for an external load, with the density $f(x,t)$, acting on the string (e.g., a violin string is bowed), we have

$$\int \rho\, f\, dS = \int_x^{x+\Delta x} \rho_0 f(x,t)\, dx. \qquad (0.1.30)$$

Finally, one should model the friction forces acting on the string segment. We shall assume a linear law of friction of the form:

$$-\int \sigma\, \rho\, u_t\, dS = -\int_x^{x+\Delta x} \sigma\, \rho \sqrt{1+u_x^2}\, u_t\, dx = -\int_x^{x+\Delta x} \sigma\, \rho_0\, u_t\, dx. \quad (0.1.31)$$

Now applying the Newton's second law yields

$$\begin{aligned} -\int_x^{x+\Delta x} \rho_0 u_{tt}\, dx &= \tau[u_x(x+\Delta x,t) - u_x(x,t)] \\ &\quad - \int_x^{x+\Delta x} \sigma\, \rho_0\, u_t\, dx + \int_x^{x+\Delta x} \rho_0 (f-g)\, dx. \end{aligned} \qquad (0.1.32)$$

Dividing (2.4.12) by Δx and letting $\Delta x \to 0$ we obtain the equation

$$\rho_0 u_{tt} = \tau u_{xx} - \sigma\, \rho_0\, u_t + \rho_0 (f-g). \qquad (0.1.33)$$

Letting $c^2 = \tau/\rho_0$ and $F = f - g$, we end up we the following concise form:

$$u_{tt} + \sigma\, u_t = c^2 u_{xx} + F. \qquad (0.1.34)$$

Equation (2.4.14) describes the vibration of the considered string once it is set into motion. The *smallness* assumption here results to a single linear equation for u. Due to the presence of the friction term σu_t, equation (2.4.14) is often referred as the *damped one-dimensional wave equation*. If friction is negligible, then we can let $\sigma = 0$ and get the *inhomogeneous wave equation*

$$u_{tt} = c^2 u_{xx} + F. \qquad (0.1.35)$$

In the absence of the external forces and when the weight of the string is negligible, we may take $F \equiv 0$ to get the *one-dimensional wave equation*:

$$u_{tt} = c^2 u_{xx}. \qquad (0.1.36)$$

Note that since u has the unit of length ℓ, u_{tt} has the unit of acceleration and u_{xx} the unit of ℓ^{-1}, hence c has the unit of velocity.
Remark In Appendix A we include a discussion on derivations of some of the partial differential equations, which are of interest in Fluid and gas dynamics, from the underlying physical laws.

Exercises

1. Show that $u(x,y) = \log(x^2 + y^2)$ satisfies the Laplace's equation $u_{xx} + u_{yy} = 0$ for $(x,y) \neq (0,0)$.

2. $u(x,y,x) = (x^2 + y^2 + x^2)^{-1/2}$ satisfies the Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$.

3. Show that $u(r,\theta) = Br^n \sin(n\theta)$ satisfies the Laplace equation in polar coordinates:
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

4. Verify that
$$u = \frac{-2y}{x^2 + y^2 + 2x + 1}, \qquad v = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 2x + 1}$$
both satisfy the Laplace equation, and sketch the curves $u =$ constant and $v =$ constant. Show that
$$u + iv = \frac{i(z-1)}{z+1}, \qquad \text{where} \quad z = x + iy.$$

5. Show that $u(x,t) = t^{-1/2}\exp(-x^2/4kt)$ satisfies the heat equation $u_t = ku_{xx}$ for $t > 0$.

6. Show that $u(x,y,t) = t^{-1}\exp[-(x^2+y^2)/4kt]$ satisfies the heat equation $u_t = k(u_{xx} + u_{yy})$ for $t > 0$.

7. The spherically symmetric form of the heat conduction equation is:
$$u_{rr} + \frac{2}{r}u_r = \frac{1}{\kappa}u_t.$$
Show that $v = ru$ satisfies the standard one-dimensional heat equation.

8. Show that the equation

$$\theta_t = \kappa\theta_{xx} - h(\theta - \theta_0)$$

can be reduced to the standard heat conduction equation by writing $u = e^{ht}(\theta - \theta_0)$. How do you interpret the term $h(\theta - \theta_0)$?

9. Use the substitution $\xi = x - vt,\ \eta = t$ to transform the one-dimensional convection-diffusion equation

$$u_t = ku_{xx} - vu_x,$$

into a heat equation for $\tilde{u}(\xi, \eta) = u(\xi + v\eta, \eta)$.

10. If $f \in C[0,1]$, let $u(x,t)$ satisfy

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq 1. \end{cases}$$

Derive the identity $2u(u_t - u_{xx}) = (u^2)_t - (2uu_x)_x + 2u_x^2$.

11. Find the possible values of a and b in the expression $u = \cos at \sin bx$, such that it satisfies the wave equation $u_{tt} = c^2 u_{xx}$.

12. Taking $u = f(x + \alpha t)$, where f is any function, find the values of α that will ensure u satisfies the wave equation $u_{tt} = c^2 u_{xx}$.

13. The spherically symmetric version of the wave equation $u_{tt} = c^2 u_{xx}$ takes the form

$$u_{tt} = c^2(u_{rr} + 2u_r/r).$$

Show, by putting $v = ru$ that it has a solution of the form

$$v = f(ct - r) + g(ct + r).$$

14. Let $\xi = x - ct$ and $\eta = x + ct$. Use the chain rule to show that

$$u_{tt} - c^2 u_{xx} = -4u_{\xi\eta}.$$

15. Show that the solution for the initial value problem

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

satisfies the *d'Alembert's formula:*

$$u(x,t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy.$$

0.2 Generalized functions

Here, loosely speaking, by *generalized functions* we shall mean a class of real-valued functions having jump discontinuities at finite or countable number of points, and their derivatives (the concept of derivative is "generalized" to discontinuity points in the context of "weak formulation"). The through study of generalized functions is the subject of the distribution theory (for a brief introduction see Appendix B). Impulse and step functions are examples of most commonly used generalized functions with many applications, e.g. heat conduction, wave propagation, signal analysis and sampling signals, in order to describe the data and solution of the underlying differential equations. Signals are certain type of generalized functions defined over a continuous range of an independent variable such as time. Examples of such signals include voltage, current, power, pressure, flow, volume, angle, displacement, acceleration, and so forth. In this section we derive the step and impulse functions through differentiating either the *ramp function* introduced below, or the *absolute value function*. The Dirac δ-function is derived, e.g. as a "generalized" derivative from the step function. We shall also use, formal, derivatives of δ-functions, which are, rigorously, defined in the distribution sense.

I. The ramp function approach: The ramp function $r(t)$ is defined by

$$r(t) =: \begin{cases} 0, & \text{for} \quad t < 0 \\ t, & \text{for} \quad t \geq 0, \end{cases} \tag{0.2.1}$$

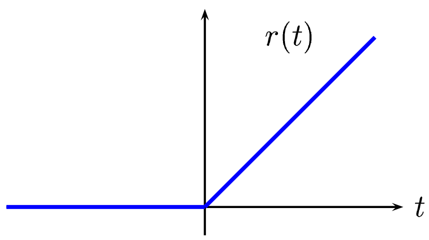


Figure 1: The ramp function $r(t)$.

The function $r(t)$ is everywhere differentiable except at the origin $t = 0$. We define the step function to be the modified derivative $r'(t)$ of $r(t)$ by letting $r'(0) := 1$.

0.2.1 The Heaviside step function

In many engineering applications the underlying differential equation may frequently have a discontinues forcing function, for example a square wave resulting from an on/off switch. In order to accommodate such discontinuities we use the *Heaviside step function* $H(t)$, which we, customary, denote by $\theta(t)$. As for the signal problems: to say that a signal is a continuous-time signal is not the same that it is a continuous function of time, but only the time t is a continuous variable. An important example of this distinction is the *Heaviside step function* $\theta(t)$:

Definition 5. *The Heaviside step function* $\theta(t)$ *is defined by:*

$$\theta(t) =: r'(t) = \begin{cases} 0 & \textit{for} \quad t < 0 \\ 1 & \textit{for} \quad t \geq 0. \end{cases} \tag{0.2.2}$$

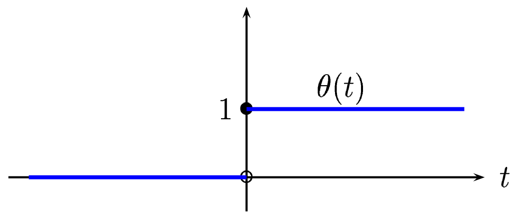


Figure 2: The Heaviside function $\theta(t)$.

The Heaviside step function is also frequently referred to as the *unit step function* or simply the *step function*. As we mentioned above $r'(t)$ is not defined for $t = 0$ and $\theta(t)$ is discontinuous at $t = 0$. Here we define $\theta(0) := 1$, unless otherwise the point $t = 0$ is explicitly excluded from the domain of definition. A function representing a unit step at $t = T$ may be obtained by a horizontal translation of duration T. This shift function is defined by

$$(1) \quad \theta(t-T) = \begin{cases} 0, & \text{for} \quad t < T \\ 1, & \text{for} \quad t \geq T, \end{cases} \qquad (2) \quad r(t) = \int_{-\infty}^{t} \theta(\tau) d\tau. \tag{0.2.3}$$

The product $f(t)\theta(t-T)$ takes values

$$f(t)\theta(t-T) = \begin{cases} 0, & \text{for} \quad t < T \\ f(t), & \text{for} \quad t \geq T. \end{cases} \tag{0.2.4}$$

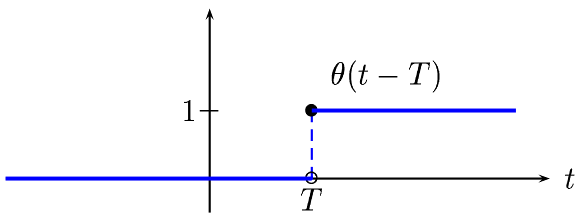


Figure 3: The Heaviside function $\theta(t-T)$.

So the function $\theta(t-T)$ may be interpreted as a device for "switching on"

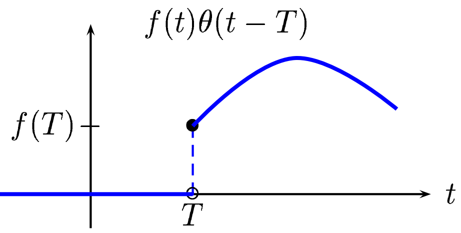


Figure 4: The function $f(t)\theta(t-T)$.

the function $f(t)$ at $t=T$. In this way the unit step function may be used to write a concise formulation of piecewise continuous functions. To illustrate this, consider the piecewise-continuous function $f(t)$ illustrated in the Fig below and defined by

$$(1) \quad f(t) = \begin{cases} f_1(t), & \text{for} \quad 0 \le t < t_1 \\ f_2(t), & \text{for} \quad t_1 \le t < t_2 \\ f_3(t), & \text{for} \quad t \ge t_2. \end{cases} \tag{0.2.5}$$

To construct this function $f(t)$ we use the following switching operations:
(i) switch on the function $f_1(t)$ at $t=0$;
(ii) switch on the function $f_2(t)$ at $t=t_1$ at the same time switch off the function $f_1(t)$;
(iii) switch on the function $f_3(t)$ at $t=t_2$ and at the same time switch off the function $f_2(t)$. In terms of the unit step function the function $f(t)$ may

thus be expressed as

$$f(t) = f_1(t)\theta(t) + [f_2(t) - f_1(t)]\theta(t - t_1) + [f_3(t) - f_2(t)]\theta(t - t_2). \quad (0.2.6)$$

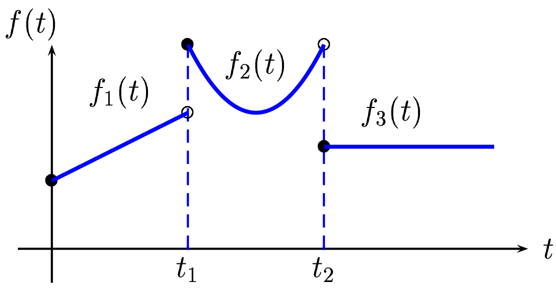


Figure 5: The function $f(t)\theta(t - T)$.

Below are some other illustrative examples:

Exempel 1. *The square pulse function with a given amplitude A:*

$$f(t) = \begin{cases} A, & \text{for} \quad t \in (a, b) \\ 0, & \text{for} \quad t \notin (a, b) \end{cases} \quad (0.2.7)$$

can be representation by the one line expression: $f(t) = A[\theta(t-a) - \theta(t-b)]$.

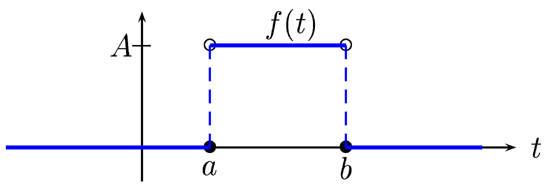


Figure 6: The square pulse function $f(t)$.

Exempel 2. *Using a combination of Heaviside functions The hat function:*

$$\varphi(t) = \begin{cases} 1 + \frac{t}{T} & \text{for} \quad -T \leq t \leq 0 \\ 1 - \frac{t}{T} & \text{for} \quad 0 \leq t \leq T \\ 0 & \text{for} \quad t \notin [-T, T], \end{cases} \quad (0.2.8)$$

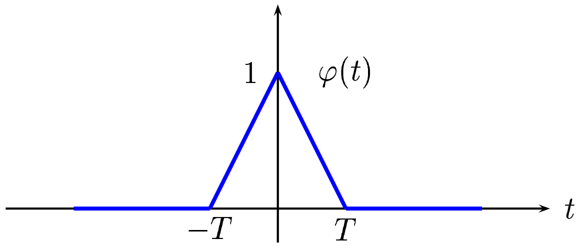


Figure 7: The hat function $\varphi(t)$.

can be expressed in a concise form as

$$g(t) = \left(1 + \frac{t}{T}\right)[\theta(t+T) - \theta(t)] + \left(1 - \frac{t}{T}\right)[\theta(t) - \theta(t-T)]$$
$$= \left(1 + \frac{t}{T}\right)\theta(t+T) - \frac{2t}{T}\theta(t) - \left(1 - \frac{t}{T}\right)\theta(t-T).$$

Exempel 3. *If in the Example 1 we let* $A = \sin(t)$, $a = 0$ *and* $b = \pi$, *i.e.,*

$$f(t) = \begin{cases} \sin t, & \textit{for} \quad t \in [0, \pi] \\ 0, & \textit{for} \quad t \notin (0, \pi) \end{cases} \tag{0.2.9}$$

then, we get the continuous sinus signal: $f(t) = [\theta(t) - \theta(t - \pi)] \sin(t)$. *A cut-off from sinus function as shown in the Fig. below*

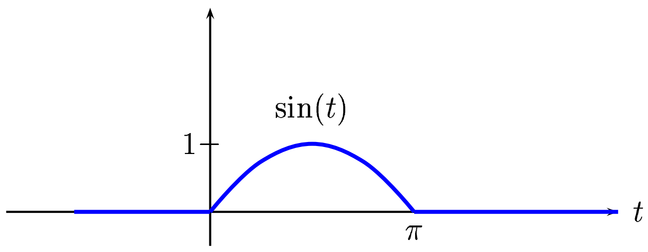


Figure 8: A sinus signal $\sin(t)$.

Exempel 4. *Consider now the, purely, continuous signal* $\theta_\Delta(t)$ *given by*

$$\theta_\Delta(t) = \begin{cases} 0 & \textit{for} \quad t < 0, \\ \frac{t}{\Delta} & \textit{for} \quad 0 \le t \le \Delta, \\ 1 & \textit{for} \quad \Delta \le t. \end{cases} \tag{0.2.10}$$

Then, $\theta_\Delta(t)$ has the following one line expression form:

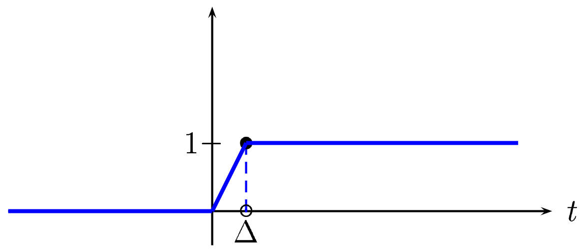


Figure 9: The signal $\theta_\Delta(t)$,

$$\theta_\Delta(t) = \frac{t}{\Delta}\Big[\theta(t) - \theta(t-\Delta)\Big] + \theta(t-\Delta) = \frac{t}{\Delta}\theta(t) + \Big(1 - \frac{t}{\Delta}\Big)\theta(t-\Delta).$$

0.2.2 The Impulse functions

We start with the function for the following *Square pulse*:

$$\delta_\varepsilon(t) := \begin{cases} \frac{1}{\varepsilon} & 0 \le t \le \varepsilon \\ 0 & t \notin [0, \varepsilon] \end{cases} \tag{0.2.11}$$

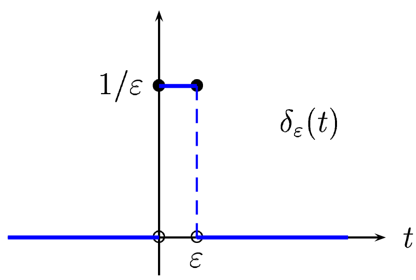


Figure 10: The square pulse function $\delta_\varepsilon(t)$.

Note that δ_ε satisfies

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)dt = 1.$$

Definition 6. *The Dirac's δ function is defined as the limit pulse function:* $\delta(t) = \lim_{\epsilon\to 0}\delta_\varepsilon(t)$. *This function has the following properties:*

(1) $\delta(t) = 0$ *for* $t \neq 0$, (2) $\delta(t)$ *is undefined for* $t = 0$,

(3) $\displaystyle\int_{t_1}^{t_2}\delta(t)dt = \begin{cases} 1 & \text{if } t_1 < 0 < t_2, \\ 0, & \text{otherwise,} \end{cases}$ *in particular* $\displaystyle\int_{-\infty}^{\infty}\delta(t)dt = 1.$

We may use the following geometric approach of the definition for the Dirac delta function: using the, purely, continuous signal of Example 4 we have obviously

$$\theta(t) = \lim_{\Delta\to 0}\theta_\Delta(t), \tag{0.2.12}$$

and the square pulse

$$\delta_\Delta(t) = \frac{d\theta_\Delta(t)}{dt}. \tag{0.2.13}$$

Comparing (0.2.12) and (0.2.13) we get

$$\delta(t) = \lim_{\Delta\to 0}\delta_\Delta(t) = \frac{d\theta(t)}{dt}. \tag{0.2.14}$$

Consequently $\theta(t)$ can be expressed as the *running integral*

$$\theta(t) = \int_{-\infty}^{t}\delta(\tau)\,d\tau, \tag{0.2.15}$$

which is consistent with the original definition of $\delta(t)$, formulated in relation (3) of the definition. The impulse function is depicted graphically as in the Fig. below, where "1" beside the arrow indicates that the *value of the area of impulse* is unity.

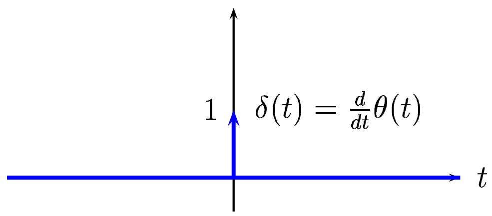


Figure 11: The delta function $\delta(t) = \theta'(t)$.

The scaled impulse function $C\delta(t)$ is simply the derivative of the scaled step function $C\theta(t)$, C is a constant. Hence the value of the scaled impulse is C.

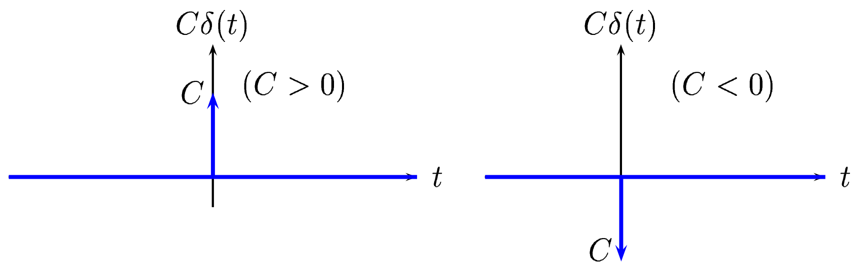


Figure 12: Scaled unit impulse functions for $C > 0$ and $C < 0$.

We will often encounter the product of unit-impulse function with another function $f(t)$, that is

$$g(t) = f(t)\delta(t-T), \tag{0.2.16}$$

where we have included the possibility that $\delta(t)$ may be delayed in time by some amout T (or advanced in time if $T < 0$) to produce $\delta(t-T)$. Another interpretation of $\delta(t-T)$ is simply that

$$\delta(t-T) = \frac{d\theta(t-T)}{dt}. \tag{0.2.17}$$

To interpret the product signal $g(t)$ in (**??**) we again make use of approximate impulse function $\delta_\Delta(t)$ to define the signal

$$g_\Delta(t) = f(t)\delta_\Delta(t-T), \tag{0.2.18}$$

as illustrated in the Fig. below. Assuming that $f(t)$ is continuous over the interval $T \le t \le T+\Delta$, we may approximate $g_\Delta(t)$ for Δ sufficiently small as simply

$$g_\Delta(t) = f(T)\delta_\Delta(t-T), \tag{0.2.19}$$

because $f(t)$ is approximately constant over this interval. Since $\delta(t-T)$ is the limit of $\delta_\Delta(t-T)$ sa $\Delta \to 0$, it follows that

$$g(t) = f(t)\delta(t-T) = f(T)\delta(t-T). \tag{0.2.20}$$

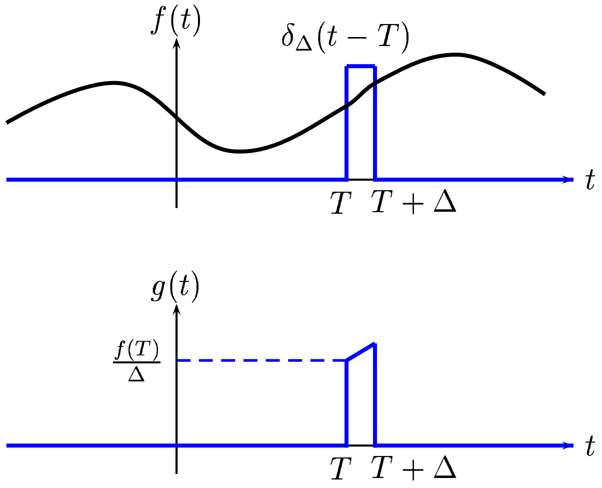


Figure 13: Illustration of approximate product $g_\Delta(t) = f(t)\delta_\Delta(t-T)$.

That is the impulse $\delta(t-T)$ is simply scaled by the value of $f(t)$ at $t=T$ to produce $g(t)$ as illustrated in Fig. above. This is sometimes called the equivalence property of the unit impulse.

A simple, by particularly important extension of this result is that the integral of $g(t)$ over all t equals, from the property (3) of the definition and (0.2.20),

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T), \tag{0.2.21}$$

which is known as *shifting property* of the unit impulse. That is, no matter how complicated the function $f(t)$ may by, the integral of $f(t)\delta(t-T)$ over all t equals simply the value of $f(t)$ at the point $t=T$.

Exempel 5. *We will encounter the integral*

$$F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\delta(t-T)\,dt. \tag{0.2.22}$$

At first glance, this appears to be a complicated integral until we notice that it contains an impulse fiction. The evaluation of the integral is the trivial from (0.2.21) *and is given by*

$$F(j\omega) = e^{-j\omega T} = \cos\omega T + j\sin\omega T.$$

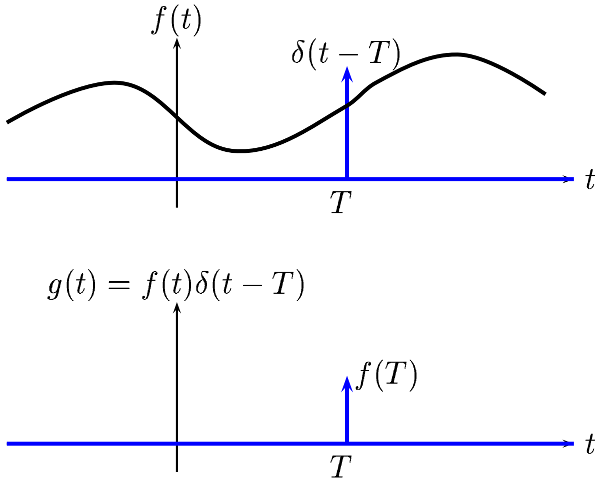


Figure 14: Illustration of product signal $g(t) = f(t)\delta(t - T)$.

The Dirac δ function is the most widely known *generalized function*. Within the traditional realm of functions, the Dirac function does not make sense. Nevertheless, it is one of the most important tools in the study of many problems in pure and applied mathematics, physics, engineering, mathematical statistics, and so forth. One may think of, e.g. $c\delta(x)$ as representing the charge density of a particle of charge c on the x-axis that occupies only the single point $x = 0$: there is no charge except at the origin and the total charge is c. In the sequel, we treat the δ functions as normal functions bearing in mind that their applications require careful interpretation based on properties (1)-(3) above.

Remark. An alternative approach for the definition of the delta function is by replacing the nonsmooth $\delta_\varepsilon(t)$ of the definition above by an infinitely smooth one given by the Gauss-pulse:

$$\delta_\varepsilon(t) = \frac{2}{\sqrt{2\pi}} e^{-t^2/2\varepsilon^2}, \qquad \varepsilon > 0 \tag{0.2.23}$$

where as above

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t).$$

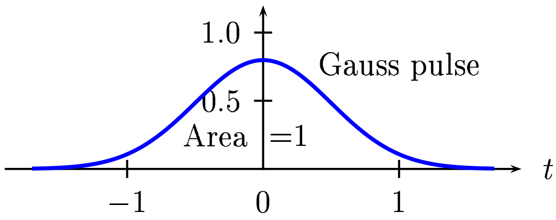


Figure 15: The Gauss pulse $\delta_\varepsilon(t) = \sqrt{2/\pi}\, e^{-t^2/(2\varepsilon^2)}$.

Some properties of the Dirac function

Below we gather some of the most important properties of the Dirac function:

(1) Usually δ is an even function and we have

$$\delta(-t) = \delta(t), \qquad \text{Thus} \qquad \delta(T-t) = \delta(t-T).$$

(2) Recall that $\displaystyle \int_{-\infty}^{t} \delta(\tau)d\tau = \theta(t), \qquad t \neq 0.$

(3) We also recall that using the definition of $\delta_\varepsilon(t)$, we have

$$\delta(t) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t) = \lim_{\varepsilon\to 0} \frac{\theta(t) - \theta(t-\varepsilon)}{\varepsilon} \qquad \Longrightarrow \qquad \delta(t) = \frac{d}{dt}\theta(t).$$

(4) For an arbitrary continuous function f one can prove the following *evaluation formula:*

$$f(t)\delta(t-T) = f(T)\delta(t-T), \tag{0.2.24}$$

$(4)^\circ$ where for $T = 0$ we get $\qquad f(t)\delta(t) = f(0)\delta(t).$

$(4)'$ For continuously differentiable functions f we can, formally, deduce that

$$f(t)\delta'(t-T) = f(T)\delta'(t-T) - f'(T)\delta(t-T). \tag{0.2.25}$$

$(4)'$ is a consequence of applying the product rule in differentiating (0.2.24):

$$[f(t)\delta(t-T)]' = [f(T)\delta(t-T)]' \quad \Longrightarrow \tag{0.2.26}$$

$$f'(t)\delta(t-T) + f(t)\delta'(t-T) = f(T)\delta'(t-T). \tag{0.2.27}$$

■

Remark. By property (3), formally, we have that

$$\delta(t) = \frac{d\theta(t)}{dt} = \frac{d^2 r(t)}{dt^2}, \tag{0.2.28}$$

where $r(t)$ is the *ramp function*.

Below we formulate an alternative approach for deriving the step and impulse functions

II. The absolute value function approach. Recall that

$$|t| = \begin{cases} t, & \text{for } t \geq 0 \\ -t, & \text{for } t < 0, \end{cases}$$

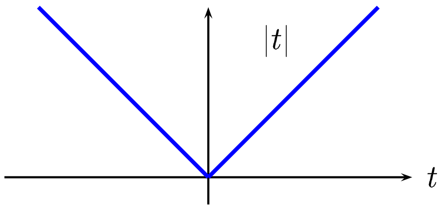


Figure 16: The absolute value function $|t|$.

As the ramp function, the absolute-value function $|t|$ is also everywhere differentiable except at the origin $t = 0$. The *generalized derivative* for $|t|$ is the *signum function*:

$$\text{sign}(t) =: \frac{d(|t|)}{dt} = \begin{cases} -1, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0. \end{cases} \tag{0.2.29}$$

Similarly the signum function is everywhere differentiable except at the origin $t = 0$. A *generalized derivative* of the signum(t) is $2\delta(t)$. The factor 2 is the magnitude of the jump at $t = 0$. That is:

$$\delta(t) = \frac{1}{2}\frac{d\,\text{sign}(t)}{dt} = \frac{1}{2}\frac{d^2(|t|)}{dt}. \tag{0.2.30}$$

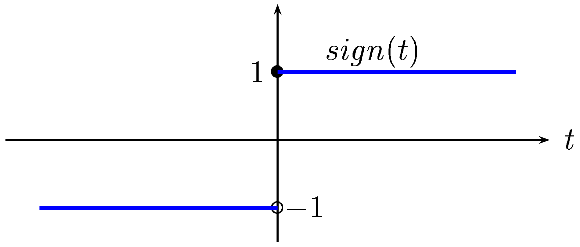


Figure 17: The signum function sign(t).

In solving problems we frequently encounter integrations involving θ and δ functions. Below, we shall formulate some of the most common integration rules:

0.2.3 Rule of integration for Heaviside θ functions

Integrating a causal restriction of $f(t)$, e.g. *the cut off function* $f(t)\theta(t-T)$

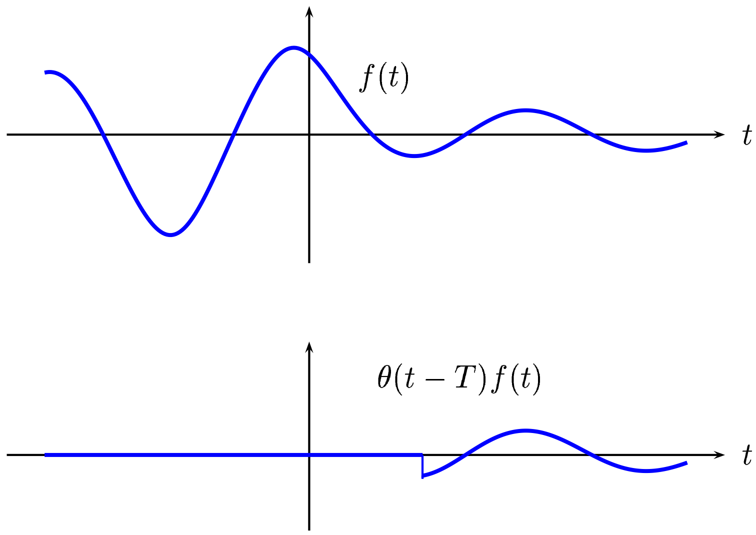


Figure 18: A causal restriction of the function $f(t)$.

we get the rule

$$\int f(t)\theta(t-T)dt = [F(t)-F(T)]\theta(t-T)+C \qquad (0.2.31)$$

where $G(t) = F(t) - F(T)$ is a primitive function to $f(t)$ with $G(T) = 0$ and C is a constant.

Exempel 6. *For example when we compute the integral* $\int (t-T)^p\theta(t-T)dt$ *we have* $f(t) = (t-T)^p$ *and then*

$$\int (t-T)^p\theta(t-T)dt = \frac{(t-T)^{p+1}}{p+1}\theta(t-T)+C, \quad p > -1.$$

Exempel 7. *Let us now evaluate the integral* $\int_{-2}^{2}[\theta(t+3)-2t\theta(t-1)]$ *using the rule of integration* (0.2.31). *The coefficient of* $\theta(t+3)$ *is* 1 *and the coefficient of* $\theta(t-1)$ *is* $2t$. *Let now* $f(t) = 1$ *and* $g(t) = 2t$. *Then we have*

$$\int_{-2}^{2}[\theta(t+3)-2t\theta(t-1)]dt := \int_{-2}^{2} f(t)\theta(t+3)dt - \int_{-2}^{2} g(t)\theta(t-1)dt.$$

To compute the first integral on the right hand side we use the rule of integration

$$\int f(t)\theta(t-T_1)dt = [F(t)-F(T_1)]\theta(t-T_1)+C,$$

where $f(t) = 1 \Rightarrow F(t) = t$ *and* $T_1 = -3 \Rightarrow F(T_1) = F(-3) = -3$, *thus*

$$\int f(t)\theta(t-T_1)dt = [t+3]\theta(t+3)+C.$$

Similarly for the second integral we have $g(t) = 2t \Rightarrow G(t) = t^2$ *and* $T_2 = 1 \Rightarrow G(T_2) = G(1) = 1$. *Hence*

$$\int_{-2}^{2} f(t)\theta(t+3)dt - \int_{-2}^{2} g(t)\theta(t-1)dt = \Big[(t+3)\theta(t+3) - (t^2-1)\theta(t-1)\Big]_{-2}^{2}$$
$$= 5\theta(5) - 3\theta(1) - \theta(+1) + 3\theta(-3) = 5-3-1+0 = 1.$$

Exempel 8. *Compute the following integral*

$$\int \frac{\theta(1-t)}{1+t^2}dt.$$

Here we note that $\theta(1-t) = 1-\theta(t-1)$, *hence we can rewrite the integral as*

$$\int \frac{\theta(1-t)}{1+t^2}dt = \int \frac{1-\theta(t-1)}{1+t^2}dt = \int \frac{dt}{1+t^2} - \int \frac{\theta(t-1)}{1+t^2}dt.$$

We have now

$$f(t) = \frac{1}{1+t^2} \Rightarrow F(t) = \arctan(t)$$

and

$$T = 1 \Rightarrow F(1) = \arctan(1) = \frac{\pi}{4}.$$

Summing up we get

$$\int \frac{dt}{1+t^2} - \int \frac{\theta(t-1)}{1+t^2}dt = \arctan(t) - [\arctan(t) - \frac{\pi}{4}]\theta(t-1) + C.$$

0.2.4 Rules of integration for the Dirac δ functions.

We shall, intuitively, use the following basic rules: An integral of the form

$$\int_T^{T+\beta} \delta(t-T)\, dt \qquad \text{is not well-defined,} \tag{0.2.31}$$

whereas, avoiding the *support point* for δ in integration limits we get the rules:

$$\int_{T-\alpha}^{T+\beta} \delta(t-T)\, dt = 0, \qquad \int_{T+\alpha}^{T+\beta} \delta(t-T)\, dt = 1, \qquad \beta > \alpha > 0. \tag{0.2.32}$$

Exempel 9. *Use property (4) to evaluate the integral* $\int_1^\infty (t^2+2)\delta(t)dt$. *Let* $f(t) = (t^2+2)$, *then we have* $f(0) = 2$. *Thus using the evaluation formula (4)°* *yields*

$$\int_1^\infty (t^2+2)\delta(t)dt = \int_1^\infty 2\delta(t)dt.$$

But since the support of $\delta(t) : 0 \notin [1,\infty]$ *we have* $\int_1^\infty \delta(t)dt = 0$, *and hence*

$$\int_1^\infty 2\delta(t)dt = 0.$$

If instead we evaluate $\int_{-1}^{\infty}(t^2+2)\delta(t)dt$ *then we get the integral*

$$\int_{-1}^{\infty}(t^2+2)\delta(t)dt = \int_{-1}^{\infty} 2\delta(t)dt.$$

Now since $0 \in [-1, \infty]$, *we have* $\int_{-1}^{\infty} \delta(t)dt = 1$ *and thus*

$$\int_{-1}^{\infty}(t^2+2)\delta(t)dt = 2.$$

Remark Note that the integral $\displaystyle\int_{0}^{\infty}(t^2+2)\delta(t)dt$ is not well-defined. In the presence of $\delta(t-T)$ in the integrand, the integrating interval must either contain the point T (0 in this example) as an interior point or start from $t > T$ ($t = a > 0$), alternatively, end at a limit $t < T$ ($t = b < 0$ in the current case). Then $\displaystyle\int_{a}^{\infty}\delta(t)dt = 0$ and we have for instance

$$\int_{0_+}^{\infty}(t^2+2)\delta(t)dt = \lim_{\alpha\to 0^+}\int_{\alpha}^{\infty}(t^2+2)\delta(t)dt = 0.$$

If the integration interval has zero as an interior point, then we have for example

$$\int_{0_-}^{\infty}(t^2+2)\delta(t)dt = \lim_{\alpha\to 0_-}\int_{-\alpha}^{\infty}(t^2+2)\delta(t)dt = 2.$$

Exempel 10. *Evaluate the integral* $\displaystyle\int_{a}^{b}\delta'(t)dt$

After integration we have

$$\int_{a}^{b}\delta'(t)dt = [\delta(t)]_a^b = \delta(b) - \delta(a).$$

But $\delta(t) = 0 \quad if \quad t \neq 0$. *Thus we have*

$$\int_{a}^{b}\delta'(t)dt = [\delta(t)]_a^b = \delta(b) - \delta(a) = 0 - 0 = 0 \textit{ if } a \neq 0 \quad \textit{and } b \neq 0.$$

If either a *or* $b = 0$, *then the integral is not defined.*

Exempel 11. *Use partial integration rule to evaluate the following integral:*

$$\int_{-5}^{\infty} t^2\delta'(t+3)dt$$

Partial integration yields

$$\int_{-5}^{\infty} t^2\delta'(t+3)dt = [t^2\delta(t+3)]_{-5}^{\infty} - \int_{-5}^{\infty} 2t\delta(t+3)dt.$$

Since $\delta(\infty) = \delta(-2) = 0$. *Thus we have*

$$\int_{-5}^{\infty} t^2\delta'(t+3)dt = -\int_{-5}^{\infty} 2t\delta(t+3)dt.$$

Finally, using $f(t)\delta(t-T) = f(T)\delta(t-T)$ *we get* $2t\delta(t+3) = 2(-3)\delta(t+3)$. *Hence using* $\int_{-5}^{\infty} \delta(t+3) = 1$, *we get*

$$\int_{-5}^{\infty} t^2\delta'(t+3)dt = \int_{-5}^{\infty} 2t\delta(t+3)dt = \int_{-5}^{\infty} 2(-3)\delta(t+3) = -6.$$

Exempel 12. *Solve the differential equation:*

$$y'' + y = sign(x); \qquad sign(x) = \begin{Bmatrix} -1 & x < 0 \\ 1 & x \geq 0 \end{Bmatrix} = 2\theta(x) - 1$$

Solution: *The homogen solution* y_h *solves the equation* $y_h'' + y_h = 0$. *This equation has the characteristic equation* $r^2 + 1 = 0$ *with the roots* $r = \pm i$ *and we have*

$$y_h(x) = C_1 e^{ix} + C_2 e^{-ix} \Rightarrow y_h(x) = A\sin(x) + B\cos(x).$$

A particular solution y_p *for the right hand side,* $sign(x) = 2\theta(x) - 1$, *consists of the sum of two particular solutions* y_{p_1} & y_{p_2}, *satisfying* $y_{p_1}'' + y_{p_1} = -1$ *and* $y_{p_2}'' + y_{p_2} = 2\theta(x)$, *respectively. We have that* $y_{p_1} = constant = -1$. *Further,* y_{p_2} *is of the form* $y_{p_2} = u(x)\theta(x)$, *which inserting in the equation for* y_{p_2} *yields*

$$\begin{aligned} y_{p_2}'' + y_{p_2} =& \Big(u'(x)\theta(x) + u(x)\delta(x)\Big)' + u(x)\theta(x) = 2\theta(x) \Leftrightarrow \\ & \Big(u'(x)\theta(x) + u(0)\delta(x)\Big)' + u(x)\theta(x) = 2\theta(x) \Leftrightarrow \\ & u''(x)\theta(x) + u'(0)\delta(x) + u(0)\delta'(x) + u(x)\theta(x) = 2\theta(x) \\ & \Leftrightarrow (u'' + u)\theta(x) + u'(0)\delta(x) + u(0)\delta'(x) = 2\theta(x). \end{aligned}$$

After identifying the coefficients we end up with the initial value problem

$$\begin{cases} u'' + u = 2 \\ u'(0) = u(0) = 0 \end{cases}$$

with the solution $u(x) = 2 - 2\cos(x)$.
Summing all contributions would give the final solution as:

$$y = y_p + y_h = y_{p_1} + y_{p_2} + y_h = -1 + (2 - 2\cos(x))\theta(x) + A\sin(x) + B\cos(x).$$

Exercises.

1. Draw the graphs for the following functions.

   a) $2\theta(t-1)$ b) $2\theta(1-t)$ c) $t\theta(t-1)$
   d) $t[\theta(t)-\theta(t-1)]$ e) $e^{-t}\theta(t)$ f) $t(t+1)[\theta(t+1)-\theta(2t)]$.

2. Rewrite the following functions using step functions

   a) $f_1(t) = t|t-1|$, b) $f_2(x) = e^{-|x|}$

   c) $f_3(t) = \begin{cases} -1 & t < 0 \\ 1 & t > 0. \end{cases}$ d) $f_4(t) = \begin{cases} A & t < a \\ B & t > a. \end{cases}$

   e) $f_5(t) = \begin{cases} \cos\varphi & |\varphi| < \pi \\ 0 & |\varphi| \geq \pi \end{cases}$ f) $f_6(t) = \begin{cases} 1 & 0 < t < 1 \\ 2 & 1 < t < 2 \\ 0 & \text{else.} \end{cases}$

3. Use the integration rule and solve the following differential equations:

   a) $y' = t\theta(t-1)$ b) $y'' = |t|$ c) $y' + y = |t|$

   d) $\begin{cases} 2xy' + y = 2x\theta(x-4) \\ y(1) = 3. \end{cases}$

4. Calculate $f(t)\delta''(t-T)$,when $f(t)$ has a continuous second derivative.

5. Compute the derivatives, of given order, in the following cases.

   a) Third derivative of $f(t) = (t-1)\theta(t-1)$
   b) Second derivative of $f(t) = \{\ln(t^2+1)\}\theta(t)$
   c) fourth derivative of $f(t) = \begin{cases} t^2 & t \leq 0 \\ 0 & t \geq 0. \end{cases}$

6. Compute n-th derivative of for the following signals:

   a) $f(t) = \begin{cases} kt & 0 \leq t < T \\ 0 & \text{else} \end{cases}$

   b) $f(t) = \theta(t-1)\theta(2-t)$ c) $f(t) = e^t\theta(t)$.

7. Calculate the following integrals

$$a) \int_{-\infty}^{\infty} (e^{-2t} + \sin t)\delta(t)dt, \qquad b) \int_{-\infty}^{\infty} [\delta(t-1) - \delta(t+1)]e^{-i\xi t}dt,$$

$$c) \int_{-\infty}^{\infty} \{\sin\tau + 2e^{\tau}\}\delta(t-\tau)d\tau, d) \int_{a}^{2} e^{-2t}\delta'(t)dt, \;\; a = +1, a = -1,$$

$$e) \int_{0}^{\infty} e^{-st}\delta(t-T)dt,$$

$$f) \int_{a}^{\infty} e^{-st}\delta(t)dt, \;\; a \to 0^-, \;\; a \to 0^+, \;\; a = 0.$$

8. Solve the following differential equations.

a) $y' = 2t + \delta(t), \;\; y(1) = -1,$ b) $y'' = t\delta(t-T),$
c) $y' + ay = \delta(t-T),$
d) $y' + 2y = (t+2)\delta(t+3), \quad y(-1) = 1,$
e) $y'' + y' = \theta(x) + \delta(x),$ f) $y'' - y = \delta''(x).$

9. Determine the general solution of the differential equation

$$y' + 2ty = \delta(t-T).$$

Chapter 1

Laplace Transformation

Laplace transformation is a powerful technique for solving differential equations with constant coefficients. Areas of application are widespread but traditional fields include mechanics, electronics, and automatic control engineering.

Before the advent of computers it was a tedious task to multiply numbers such as 1.4142 and 3.1416. Therefore logarithms were used to transform the complicated operation of multiplication into the simpler operation of addition via the formula

$$\log(1.4142 \cdot 3.1416) = \log(1.4141) + \log(3.1416).$$

By consulting tables of precomputed logarithms and exponentials one obtained the result 4.4429. Roughly speaking, Laplace transformation works analogously and reduces problems of calculus into simple algebraic problems via tables and general properties of the transform.

1.1 The Laplace Transform

1.1.1 Definition

Let $f(t)$ be a function defined for all $t \geq 0$. If the improper integral

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt, \tag{1.1.1}$$

converges for any s, then $F(s)$ is said to be the *Laplace transform*[1] of $f(t)$.

[1]Pierre Simon de Laplace (1749-1827) French mathematician.

Exempel 13. *Find the Laplace transform of the Heaviside step function,*

$$\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \tag{1.1.2}$$

Solution. By the definition (1.1.1) *we get,*

$$F(s) = \int_0^\infty \theta(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s}\,e^{-st}\right]_0^\infty = \frac{1}{s}, \quad s > 0. \tag{1.1.3}$$

The usual way of denoting the Laplace transform of a function $f(t)$ is either $F(s)$ or $\mathcal{L}[f(t)]$. For example, we have $\mathcal{L}[\theta(t)] = \frac{1}{s}$.

Since the integral (1.1.1) has the limits 0 and ∞, it follows that $F(s)$ is not influenced by $f(t)$ when $t < 0$. As a result, if f_1 and f_2 are two functions such that $f_1 = f_2$ for $t \geq 0$, then these functions have the same Laplace transform, even if they differ for $t < 0$. Because of this ambiguity, we shall henceforth always assume that $f(t)$ is *causal*, which is to say, $f(t) = 0$ for all $t < 0$.

If $f(t)$ is not causal to begin with, we can always force it to become so by multiplying it with the *Heaviside step function* $\theta(t)$ (1.1.2). We illustrate such a case below.

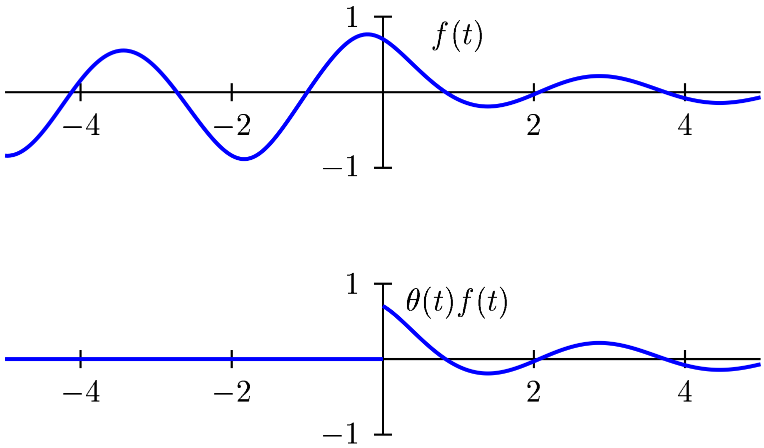


Figure 1.1: A causal restriction of the function $f(t)$.

Exempel 14. *Find the Laplace transform of $f(t) = e^{ct}$, where c is a constant.*

Solution. *Again, by* (1.1.1) *we get*

$$F(s) = \int_0^\infty e^{ct}e^{-st}\,dt = \left[-\frac{1}{s-c}\,e^{-(s-c)t}\right]_0^\infty = \frac{1}{s-c}. \qquad (1.1.4)$$

Note that, for the above integral to converge, we must assume $s > c$.

1.1.2 Existence

Not any function $f(t)$ have a Laplace transform $\mathcal{L}[f(t)]$. For example, it is easy to see that $\mathcal{L}[e^{t^2}]$ does not exist, since its associated integral diverges as $t \to \infty$. As a rule, to have a Laplace transform, it suffices (not necessary) that the function $f(t)$ is of *exponential order*. By this we mean that there must exist a constant, a say, such that

$$\lim_{t\to\infty} |f(t)e^{-at}| = 0. \qquad (1.1.5)$$

If this indeed is the case, then by choosing $s > a$, we see that the integrand $f(t)e^{-st}$ of (1.1.1) goes to zero as t tend to infinity and, hence, the integral for $\mathcal{L}[f(t)]$ converges absolutely for $s > a$. Let us formalize this result by stating it as a theorem.

Theorem 1. *If $f(t)$ is a piecewise continuous*[2] *function for all $t \geq 0$, and if*

$$|f(t)| \leq Me^{at}, \qquad (1.1.6)$$

for some constants a and C, then the Laplace transform $F(s)$ of $f(s)$ exist. We shall also refer to a piecewise continuous function f with the property (1.1.6) *as being in the class $\mathcal{C}$ and simply replace the whole expression by $f \in \mathcal{C}$.*

Proof. *If $|f(t)| \leq Me^{at}$ and $s > a$, then*

$$|F(s)| \leq \int_0^\infty |f(t)|e^{-st}\,dt \leq \int_0^\infty Me^{-(s-a)t}\,dt = \frac{M}{s-a}. \qquad (1.1.7)$$

Hence, if $s > a$ the integral (1.1.1) *converges.*

[2] A function is said to be piecewise continuous if it is discontinuous only at isolated points, and its left and right limits are defined at each discontinuity.

1.1.3 General Properties of the Laplace Transform

Theorem 2. *Laplace transformation is a linear operation, that is, for any functions $f(t)$ and $g(t)$ whose Laplace transform exist and any constants a and b, we have*

$$\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]. \tag{1.1.8}$$

Proof. *By definition, it holds that*

$$\begin{aligned}\mathcal{L}[f(t)+g(t)] &= \int_0^\infty (af(t)+bg(t))e^{-st}\,dt \\ &= a\int_0^\infty f(t)e^{-st}\,dt + b\int_0^\infty g(t)e^{-st}\,dt \\ &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].\end{aligned} \tag{1.1.9}$$

Exempel 15. *Find the Laplace transforms of* $\sinh t = \frac{1}{2}(e^t - e^{-t})$.

Solution. *By the linearity of the Laplace transform, we get*

$$\mathcal{L}[\sinh t] = \mathcal{L}[\tfrac{1}{2}(e^t - e^{-t})] = \tfrac{1}{2}\mathcal{L}[e^t] - \tfrac{1}{2}\mathcal{L}[e^{-t}] = \tfrac{1}{2}(\tfrac{1}{s-1} - \tfrac{1}{s+1}). \tag{1.1.10}$$

Exempel 16. *Find the Laplace transforms of* $\sin\omega t$ *and* $\cos\omega t$.

Solution. *If we set* $c = i\omega$ *in* (1.1.4) *then we have*

$$\begin{aligned}\mathcal{L}[e^{i\omega t}] &= \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} \\ &= \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}.\end{aligned} \tag{1.1.11}$$

On the other hand we also have

$$\mathcal{L}[e^{i\omega t}] = \mathcal{L}[\cos\omega t + i\sin\omega t] = \mathcal{L}[\cos\omega t] + i\mathcal{L}[\sin\omega t]. \tag{1.1.12}$$

Equating the real and imaginary parts of these two equations, we get

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}, \tag{1.1.13}$$

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}. \tag{1.1.14}$$

As the last examples show, the definition (1.1.1) is rarely the starting point for deriving Laplace transforms. Instead, one usually first consults a table of standard transforms, and then tries to adapt any of these to the problem at hand using a set of general properties, such as the linearity, of the Laplace transform. Below, we derive a number of other such properties and illustrate their use.

Theorem 3 (1^{st} Shifting Rule). *If $f(t)$ has the transform $F(s)$ then for any constant c, we have*

$$\mathcal{L}[e^{ct}f(t)] = F(s-c). \tag{1.1.15}$$

Proof. *Inserting $e^{ct}f(t)$ directly into the definition* (1.1.1) *gives, with $s > c$,*

$$\mathcal{L}[e^{ct}f(t)] = \int_0^\infty e^{ct}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-c)t}\,dt = F(s-c). \tag{1.1.16}$$

Exempel 17. *Find the Laplace transform of $3e^{-2t}\cos 5t$.*

Solution. By the previous example, we have

$$\mathcal{L}[\cos 5t] = \frac{s}{s^2+25}. \tag{1.1.17}$$

Applying now the 1^{st} Shifting Rule, we get

$$\mathcal{L}[3e^{-2t}\cos 5t] = \frac{3(s+2)}{(s+2)^2+25} = \frac{3s+6}{s^2+4t+29}. \tag{1.1.18}$$

Theorem 4 (2^{nd} Shifting Rule). *Assume that $T > 0$ and $f(t-T)$ is a function that is zero until $t = T$, then*

$$\mathcal{L}[f(t-T)] = e^{-Ts}F(s). \tag{1.1.19}$$

Proof. *Let $\tau = t - T$, then*

$$\begin{aligned}\mathcal{L}[f(t-T)] &= \int_0^\infty f(t-T)e^{-st}\,dt = \int_{-\infty}^\infty f(t-T)e^{-st}\,dt \\ &= \int_{-\infty}^\infty f(\tau)e^{-s(\tau+T)}\,d\tau = e^{-Ts}\int_0^\infty f(\tau)e^{-s(\tau)}\,d\tau \\ &= e^{-Ts}F(s).\end{aligned} \tag{1.1.20}$$

Introducing a generalized form of the Heaviside step function,

$$\theta(t-T) = \begin{cases} 1, & t > T, \\ 0, & t < T, \end{cases} \tag{1.1.21}$$

we can state the 2^{nd} *Shifting Rule* (1.1.19) formally as

$$\mathcal{L}[\theta(t-T)f(t-T)] = e^{-Ts}F(s). \tag{1.1.22}$$

Theorem 5. *If $f(t)$ satisfies* (1.1.6) *for some constants M and a, then*

$$\mathcal{L}[tf(t)] = -F'(s). \tag{1.1.23}$$

Proof. Differentiating under the integral sign we get

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{\partial e^{-st}}{\partial s}\,dt \\ &= \int_0^\infty -tf(t)e^{-st}\,dt = -\mathcal{L}[tf(t)]. \end{aligned} \tag{1.1.24}$$

Exempel 18. *Find the Laplace transform of $t\sinh t$.*

Solution. Recall that

$$\mathcal{L}[\sinh t] = \frac{1}{s^2-1}. \tag{1.1.25}$$

By the last theorem, we get

$$\mathcal{L}[t\sinh t] = \frac{d}{ds}\frac{1}{s^2-1} = -\frac{2s}{(s^2-1)^2}. \tag{1.1.26}$$

Theorem 6. *If $f(t)$ satisfies* (1.1.6) *for some constants M and a, and if $\lim_{t\to 0}\frac{1}{t}f(t)$ exists, then*

$$\mathcal{L}[\tfrac{1}{t}f(t)] = \int_s^\infty F(\omega)\,\omega. \tag{1.1.27}$$

Proof. Put $g(t) = \frac{1}{t}f(t)$, i.e., $f(t) = tg(t)$. The previous theorem then gives $F(s) = -G'(s)$. By the fundamental theorem of calculus, and the fact that $G(s) \to 0$ as $s \to \infty$, we have

$$G(s) = \int_s^\infty F(\omega)\,d\omega. \tag{1.1.28}$$

Exempel 19. *Find the Laplace transform of* $\dfrac{\sin t}{t}$.

Solution. Recall that $\mathcal{L}[\sin t] = (s^2+1)^{-1}$. *Since*

$$\lim_{t\to 0} \frac{\sin t}{t} = 1, \tag{1.1.29}$$

the assumptions of the last theorem are satisfied and thus we have

$$\mathcal{L}[\tfrac{1}{t}\sin t] = \int_s^\infty \frac{d\omega}{\omega^2+1} = [\arctan\omega]_s^\infty = \frac{\pi}{2} - \arctan s. \tag{1.1.30}$$

A fundamental property of the Laplace transform is the fact that, roughly speaking, taking the derivative of the original function $f(t)$ corresponds to multiplying its transform $F(s)$ by s.

Theorem 7. *Suppose* $f(t)$ *and* $f'(t)$ *are continuous and piecewise smooth for* $t > 0$ *and* a *is sufficiently large so that* $|f(t)| \le Me^{at}$ *and* $|f'(t)| \le Me^{at}$. *Then, it follows that*

$$\mathcal{L}[f'(t)] = sF(s) - f(0). \tag{1.1.31}$$

Proof. Integrating by parts, we have

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = -f(0) + sF(s).\end{aligned} \tag{1.1.32}$$

Applying this result to $f''(t)$ yields

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2F(s) - sf(0) - f'(0). \tag{1.1.33}$$

Similarly,

$$\mathcal{L}[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0). \tag{1.1.34}$$

By induction, we obtain the transform of the n-th derivative, viz.,

$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \tag{1.1.35}$$

Theorem 8. *The Laplace transform of*

$$\int_0^t f(\tau)\, d\tau, \tag{1.1.36}$$

is given by $\frac{1}{s}F(s)$.

Proof. Let $h(t) = \int_0^t f(\tau)\, d\tau$. *By construction we then have* $h'(t) = f(t)$ *and* $h(0) = 0$. *Using now that* $\mathcal{L}[f'(t)] = s\mathcal{L}[f'(t)] - f(0)$ *we immediately get* $F(s) = sH(s) - h(0)$. *Hence,* $H(s) = \frac{1}{s}F(s)$.

Problem 1. *Find the Laplace transform of the following functions.*

a. t	*b.* t^2	*c.* t^3	*d.* t^n
e. $t+1$	*f.* $(t-1)^2$	*g.* $(1+t)^4$	*h.* $\frac{1}{t}$
i. e^{-t}	*j.* e^{3t+4}	*k.* te^t	*l.* e^{t^2}
m. $\cosh t$	*n.* $\cos t$	*o.* $\sin 2t$	*p.* $\sinh^2 t$

Problem 2. *Find the Laplace transform of the following functions.*

a. $e^{at}\cos bt$	*b.* $\theta(t-1)$	*c.* $e^{-t}\theta(t-1)$
d. $t^2 \sinh t$	*e.* $t^3 e^t$	*f.* $te^{-t}\cos t$
g. $\sin(\omega t + \alpha)$	*h.* $t \sin \frac{t}{2}$	*i.* $\ln t$
j. $\frac{1}{t}(1 - \cos t)$	*k.* $\cosh t \cos t$	*l.* $\cos^2 t$

1.1.4 Table of Laplace Transforms

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$

Table 1.1: Operational properties of the Laplace transform.

$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin bt$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin bt - bt\cos bt)$	$\frac{1}{(s^2+b^2)^2}$
$\frac{a}{\sqrt{4\pi t^3}}\,e^{-a^2/4t}$	$e^{-a\sqrt{s}}$

Table 1.2: Standard transform pairs.

1.2 The Inverse Laplace Transform

Finding the *inverse Laplace transform* of a function $f(t)$ is the operation of recovering $f(t)$ from its Laplace transform $F(s)$. One usually denotes this by

$$f(t) = \mathcal{L}^{-1}[F(s)]. \tag{1.2.1}$$

Remark. The Laplace transform of a function $f \in \mathcal{C}$ is defined for complex s for $Res > a$, viz

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt. \tag{1.2.2}$$

In this way one may use the inversion Formula for Fourier transforms to obtain the so-called *inversion formula*, which gives a closed form expression for $\mathcal{L}^{-1}[F(s)]$. This however, needs the Fourier transform formalism which we shall introduce in a later chapter. Nevertheless, we can formulate a criterion (without proof) as follows:

Lemma 2. *If f and g are in $\mathcal{C}$ and $\mathcal{L}f = \mathcal{L}g$, the $f = g$. (More specifically, $f(t) = g(t)$ at all points t where both f and g are continuous*

By this Lemma, a function is $f \in \mathcal{C}$ is uniquely determined (up to modifications at its discontinuities) by its Laplace transform F, and we shall say that f is the *inverse Laplace transform* of F and write $f(t) = \mathcal{L}^{-1}[F(s)]$:

$$f = \mathcal{L}^{-1}F \Longleftrightarrow F = \mathcal{L}f.$$

Having in mind the above criterion we shall be content with the simple minded approach of finding inverse Laplace transforms by using a table of standard Laplace transforms.

Indeed, it turns out that with the aid of a table and a little algebra, we are able to find $\mathcal{L}^{-1}[F(s)]$ for a large number of functions $f(t)$.

Due to the fact that the Laplace transform is linear it follows that also the inverse Laplace transform is linear. Hence, if a and b are constants, then we have

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)], \tag{1.2.3}$$

Exempel 20. *Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of*

$$F(s) = \frac{e^{-s}}{s^2} - \frac{e^{-2t}}{s^4}. \tag{1.2.4}$$

Solution. From a table of standard transforms, we have

$$\mathcal{L}^{-1}[\tfrac{1}{s^2}] = t, \qquad \mathcal{L}^{-1}[\tfrac{1}{s^4}] = \tfrac{1}{6}\,t^3, \tag{1.2.5}$$

so by the linearity of $\mathcal{L}^{-1}$, we obtain

$$\mathcal{L}^{-1}[\tfrac{e^{-s}}{s^2} - \tfrac{e^{-2s}}{s^4}] = \mathcal{L}^{-1}[\tfrac{e^{-s}}{s^2}] - \mathcal{L}^{-1}[\tfrac{e^{-2s}}{s^4}]. \tag{1.2.6}$$

Using now the 2^{nd} Shifting Rule, we find

$$\mathcal{L}^{-1}[\tfrac{e^{-s}}{s^2}] = \theta(t-1)(t-1), \qquad \mathcal{L}^{-1}[\tfrac{e^{-2s}}{s^4}] = \tfrac{1}{6}\theta(t-2)(t-1)^3. \tag{1.2.7}$$

Hence, the inverse transform of $F(s)$ is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \theta(t-1)(t-1) - \tfrac{1}{6}\theta(t-2)(t-1)^3. \tag{1.2.8}$$

1.2.1 Method of Partial Fractions

A common situation is when $F(s)$ has the form

$$F(s) = \frac{Q(s)}{P(s)}, \tag{1.2.9}$$

where $Q(s)$ and $P(s)$ are real polynomials and the degree of Q is less than the degree of P. It is then necessary to decompose $F(s)$ into *partial fractions* to obtain $\mathcal{L}^{-1}[F(s)]$.

We demonstrate this technique for three cases of denominators $P(s)$.

1. $P(s)$ is a Quadratic with real Roots. Consider, for instance,

$$F(s) = \frac{2s-8}{s^2-5s+6}, \tag{1.2.10}$$

Obviously, $F(s)$ cannot be inverted by inspection and neither do we have it tabulated. However, since the denominator $s^2 - 5s + 6$ has two real roots, $s = 2$ and $s = 3$, it is possible to decompose $F(s)$ into partial fractions, viz.,

$$F(s) = \frac{A}{s-2} - \frac{B}{s-3}. \tag{1.2.11}$$

where A and B are numbers. Our goal is to determine these, because then it is easy to obtain the inverse transform of $F(s)$. By elementary manipulations, we get

$$\frac{A}{s-2} + \frac{B}{s-3} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)} = \frac{(A+B)s+(-3A-2B)}{s^2-5s+6}, \tag{1.2.12}$$

which implies

$$\frac{2s-8}{s^2-5s+6} = \frac{(A+B)s+(-3A-2B)}{s^2-5s+6}. \tag{1.2.13}$$

Comparing the right and left hand side, it is obvious that

$$2 = A + B, \qquad -8 = -3A - 2B, \tag{1.2.14}$$

which is a system of equations for A and B, i.e.,

$$\begin{cases} A + B &= 2, \\ 3A + 2B &= 8. \end{cases} \tag{1.2.15}$$

Solving, we obtain $A = 4$ and $B = -2$. Hence,

$$F(s) = \frac{4}{s-2} - \frac{2}{s-3}. \tag{1.2.16}$$

By recognizing $\frac{1}{s-2}$ as the transform of e^{2t} and $\frac{1}{s-2}$ as that of e^{3t}, we obtain

$$f(t) = \mathcal{L}^{-1}[F(s)] = 4e^{2t} - 2e^{3t}. \tag{1.2.17}$$

2. $P(s)$ is a Quadratic with a Double Root. Let

$$F(s) = \frac{s+1}{(s+2)^2}, \tag{1.2.18}$$

The denominator has a double root -2 and the partial fractions are therefore

$$\frac{s+1}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} = \frac{As + (2A+B)}{(s+2)^2}. \tag{1.2.19}$$

Comparing the left and right hand side of the above expression, we find $A = 1$ and $B = -2$. Recalling that $\mathcal{L}[\frac{1}{s+2}] = e^{-2t}$, we can use $\mathcal{L}[tf(t)] = -F'(s)$ to deduce that the inverse transform of $(s+2)^{-2}$ is te^{2t}. Hence, the inverse of $F(s)$ is

$$f(t) = e^{2t} - 2te^{2t}. \tag{1.2.20}$$

3. $P(s)$ is a Quadratic and has Complex Conjugated Roots. If

$$F(s) = \frac{s+1}{s^2+4s+5}, \tag{1.2.21}$$

then the denominator has the roots $-2 \pm i$. Completing the square, we get

$$s^2 + 4s + 5 = s^2 + 4s + 4 + 1 = (s+2)^2 + 1, \tag{1.2.22}$$

i.e.,

$$F(s) = \frac{s+1}{(s+2)^2+1}. \tag{1.2.23}$$

By rewriting

$$\frac{s+1}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} + \frac{-1}{(s+2)^2+1}, \tag{1.2.24}$$

and recalling the transforms of $\sin t$ and $\cos t$ it is clear that

$$f(t) = \mathcal{L}^{-1}[F(s)] = e^{-2t}\cos t - e^{-2t}\sin t. \tag{1.2.25}$$

Exempel 21. *Find the inverse transform of*

$$F(s) = \frac{s+2}{s^3 - s^2 + s - 1}. \tag{1.2.26}$$

Solution. Calculating the roots of the denominator $s^3 - s^2 + s - 1$*, we find* $s_1 = 1$*,* $s_2 = i$*, and* s_3*, i.e.,*

$$F(s) = \frac{s+2}{(s-1)(s+i)(s-i)} = \frac{s+2}{(s-1)(s^2+1)}. \tag{1.2.27}$$

Here, the appropriate decomposition into partial fractions is given by

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{s^2(A+B) + s(C-B) + (A-C)}{s^3 - s^2 + s - 1}. \tag{1.2.28}$$

Identifying coefficients it is clear that

$$s^2(A+B) = 0, \quad s(C-B) = s, \quad A - C = 2, \tag{1.2.29}$$

which implies, $A = \frac{3}{2}$*,* $B = -\frac{3}{2}$*, and* $C = -\frac{1}{2}$*. Hence,*

$$F(s) = \frac{\frac{3}{2}}{s-1} - \frac{\frac{3}{2}s}{s^2+1} - \frac{\frac{1}{2}}{s^2+1}. \tag{1.2.30}$$

Consulting a table of transforms, we recognize $F(s)$ *as the transform of*

$$f(t) = \tfrac{3}{2}e^t - \tfrac{3}{2}\cos t - \tfrac{1}{2}\sin t. \tag{1.2.31}$$

Problem 3. *Find the inverse Laplace transform of the following functions.*

a. $\frac{1}{s+1}$	*b.* $\frac{1}{s^2+4}$	*c.* $\frac{s+1}{s^2+1}$	*d.* $\frac{1}{s^2-1}$
e. $\frac{s+12}{s^2+4s}$	*f.* $\frac{s}{(s+2)^2}$	*g.* $\frac{s+1}{(s-3)^4}$	*h.* $\frac{e^{-s}}{s}$

Problem 4. *Find the inverse Laplace transform of the following functions.*

a. $\frac{s}{s^2-2s-3}$	*b.* $\frac{s+2}{s^2+4s+5}$	*c.* $\frac{1}{(s-2)^2+9}$
d. $\frac{s+1}{s^3+s^2-6s}$	*e.* $\frac{3s}{s^2+2s-8}$	*f.* $\frac{1}{s(s+1)(s+2)}$

1.3 Applications of Laplace Transforms

1.3.1 Initial Value Problems

Enough with theory, let us find the solution $y(t)$ of the initial value problem

$$y'(t) + 2y(t) = 12e^{3t}, \quad y(0) = 3. \tag{1.3.1}$$

By taking the Laplace transform of every term in the given differential equation, we get

$$\mathcal{L}[y'(t)] + \mathcal{L}[2y(t)] = \mathcal{L}[12e^{3t}]. \tag{1.3.2}$$

Put $Y(s) = \mathcal{L}[y(t)]$. Now,

$$\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s) - 3, \tag{1.3.3}$$

$$\mathcal{L}[2y(t)] = 2Y(s), \tag{1.3.4}$$

$$\mathcal{L}[12e^{3t}] = \frac{12}{s-3}. \tag{1.3.5}$$

Inserting these formulas into (1.3.2) above, we get the *subsidiary equation*

$$sY(s) - 3 + 2Y(s) = \frac{12}{s-3}. \tag{1.3.6}$$

Rearranging, we obtain

$$(s+2)Y(s) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}, \tag{1.3.7}$$

or

$$Y(s) = \frac{3s+3}{(s+2)(s-3)}. \tag{1.3.8}$$

At this point, we decompose $Y(s)$ into partial fractions, viz.,

$$\frac{3s+3}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} = \frac{(A+B)s - 3A + 2B}{(s+2)(s-3)}, \tag{1.3.9}$$

which gives rise to a system of equations for A and B, namely,

$$\left\{ \begin{array}{ll} A + B & = 3, \\ -3A + 2B & = 3. \end{array} \right. \tag{1.3.10}$$

Solving this, we find $A = \frac{3}{5}$ and $B = \frac{12}{5}$. Hence,

$$Y(s) = \frac{\frac{3}{5}}{s+2} + \frac{\frac{12}{5}}{s-3}. \tag{1.3.11}$$

Consulting a table of standard Laplace transforms, we finally have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \tfrac{3}{5}\mathcal{L}^{-1}[\tfrac{1}{s+2}] + \tfrac{12}{5}\mathcal{L}^{-1}[\tfrac{1}{s-3}] \\ &= \frac{3}{5}e^{-2t} + \frac{12}{5}e^{3t}. \end{aligned} \tag{1.3.12}$$

Summary of Solution Process. Note the three steps of the solution process:

1. Take the Laplace transform of both sides of the given hard problem for $y(t)$. As a result a simple algebraic equation for $Y(s) = \mathcal{L}[y(t)]$ is obtained.

2. Solve this so-called *subsidiary equation* for $Y(s)$.

3. Use partial fractions and a table of elementary Laplace transforms to invert $Y(s)$ and so produce the required solution $y(t) = \mathcal{L}^{-1}[Y(s)]$.

Exempel 22. *Solve the following initial value problem for $t > 0$*

$$y''(t) + 4y'(t) + 3y(t) = 0, \tag{1.3.13}$$
$$y(0) = 3, \quad y'(0) = 1. \tag{1.3.14}$$

Solution. We have

$$\mathcal{L}[y'(t)] = sY(s) - 3, \qquad \mathcal{L}[y''(t)] = s^2Y(s) - 3s - 1, \tag{1.3.15}$$

Laplace transformation of (1.3.13) *yields the subsidiary equation*

$$s^2Y(s) + 4sY(s) + 3Y(s) = 3s + 13, \tag{1.3.16}$$

or

$$(s+3)(s+1)Y(s) = 3s + 13. \tag{1.3.17}$$

Solving for $Y(s)$ and using a decomposition into partial fractions, we get

$$\begin{aligned} Y(s) &= \frac{3s+13}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} \\ &= \frac{A(s+1)+B(s+3)}{(s+3)(s+1)} = \frac{(A+B)s+A+3B}{(s+3)(s+1)}, \end{aligned} \tag{1.3.18}$$

from which we obtain $A = -2$, and $B = 5$. Thus,

$$Y(s) = -\frac{2}{s+3} + \frac{5}{s+1}. \tag{1.3.19}$$

Recalling (1.1.4) *it is obvious that*

$$\mathcal{L}^{-1}\left[\tfrac{1}{s+3}\right] = e^{-3t}, \qquad \mathcal{L}^{-1}\left[\tfrac{1}{s+1}\right] = e^{t}. \tag{1.3.20}$$

Hence, the solution is given by

$$y(t) = -2e^{-3t} + 5e^{-t}. \tag{1.3.21}$$

A simple way to check whether the correct solution has been obtained is to see if the initial condition is satisfied by the found function $y(t)$. Here we have $y(0) = \frac{3}{5} + \frac{12}{5} = 3$ and, since $y'(t) = 6e^{-3t} - 5e^{-t}$, we also have $y'(0) = 6 - 5 = 1$. Hence, the requirements $y(0) = 3$ and $y'(0) = 1$ are indeed satisfied by (1.3.21).

Problem 5. *Solve the following differential equations for $t > 0$*

a. $y' + 2y = e^{-3t}, \quad y(0) = 4.$

b. $y' - y = e^{2t}, \quad y(0) = -1.$

c. $y'' + 2y' + y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$

d. $y'' + 4y' + 13y = 2e^{-t}, \quad y(0) = 0, \quad y'(0) = -1.$

e. $y'' + 4y = 8e^{2t}, \quad y(0) = 0, \quad y'(0) = 3.$

f. $y'' - 2y' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0.$

g. $y'' + 4y' = 3e^{t}, \quad y(0) = 2, \quad y'(0) = 1.$

h. $y'' + 2y' + 2y = 2, \quad y(0) = 0$ *for* $t < 0.$

1.3.2 Integral Equations

Apart from solving differential equations, the Laplace transform technique may also be used to solve integral equations. For instance, consider the flow of electric current around a circuit consisting of a resistor, a capacitance, and a battery.

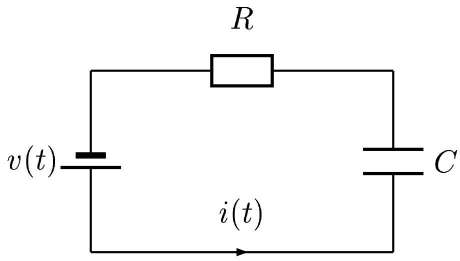


Figure 1.2: Electric RC-circuit.

It follows[3] that the current $i(t)$ satisfies the integral equation

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t), \tag{1.3.22}$$

where R and C are respectively the resistance and capacity of the circuit, and $v(t)$ is the electromotive force of the battery. For simplicity, let us assume that $C = R = 1$, and that $v(t)$ has the form of a square pulse of amplitude 1, applied between $t = 1$ and $t = 2$, i.e.,

$$v(t) = \theta(t-1) - \theta(t-2) = \begin{cases} 0, & t < 1, \\ 1, & 1 < t < 2, \\ 0, & t > 2. \end{cases} \tag{1.3.23}$$

The Laplace transform $V(s)$ of $v(t)$ is given by

$$V(s) = \int_0^\infty v(t)e^{-st}\,dt = \int_1^2 e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_1^2 = -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s}. \tag{1.3.24}$$

Assuming that $i(0) = 0$, we may transform (1.3.22) to obtain

$$RI(s) + \frac{I(s)}{Cs} = V(s), \tag{1.3.25}$$

[3]From the Kirchoff voltage law.

or, since $R = C = 1$,

$$I(s) + \frac{I(s)}{s} = \frac{1}{s}(e^{-s} - e^{-2s}). \tag{1.3.26}$$

Solving for $I(s)$ we obtain, after elementary manipulations,

$$I(s) = \frac{1}{s+1}(e^{-s} + e^{-2s}). \tag{1.3.27}$$

Noting that $\mathcal{L}^{-1}[\frac{1}{s+1}] = e^{-t}$, we then use the 2^{nd} *Shifting Rule*, to obtain

$$i(t) = e^{-(t-1)}\theta(t-1) - e^{-(t-2)}\theta(t-2). \tag{1.3.28}$$

Hence,

$$i(t) = \begin{cases} 0, & t < 1, \\ e^1 e^{-t}, & 1 < t < 2, \\ (e^1 - e^2)e^{-t}, & t > 2. \end{cases} \tag{1.3.29}$$

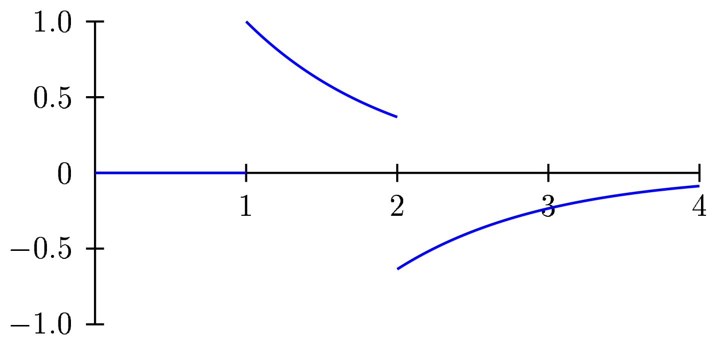


Figure 1.3: Graph of $i(t)$ for $0 < t < 4$.

Problem 6. *Solve the integral equation for* $t > 0$

$$\int_0^t y(\tau)\, d\tau + 2y(t) = 4.$$

Problem 7. *Solve the following integral equations for $t > 0$*

a. $y'(t) + 2y(t) + \int_0^t y(\tau)\, d\tau = \cos t, \quad y(0) = 1.$

b. $y'(t) + 2y(t) + 2\int_0^t y(\tau)\, d\tau = 1 + e^{-t}, \quad y(0) = 1.$

c. $y''(t) - 7y(t) + 6\int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 7, \quad y'(0) = -12.$

Chapter 2

Separation of variables

The separation of variables is a widely used technique that transforms linear partial differential equations to ordinary differential equations in the variables of the pde.

We illustrate the method through some examples.

2.1 The heat equation: Dirichlet problem

Consider the 1-dimensional heat equation for a rod $[0, L]$, with homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t = ku_{xx}, & x \in [0, L], & t > 0, & \text{(PDE)} \\ u(0,t) = u(L,t) = 0, & & t > 0, & \text{(BC)} \\ u(x,0) = f(x), & x \in [0, L]. & & \text{(IC)} \end{cases} \tag{2.1.1}$$

We recall the general form of a standard pde in two variables x and t:

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + F = 0, \tag{2.1.2}$$

with the *discriminant* defined by

$$d := AC - B^2. \tag{2.1.3}$$

In (2.1.1), since $A = 1,\ B = 0$ and $C = 0.$ we have $d = 0$ and hence (2.1.1) is a *parabolic equation*. Further (2.1.1) is described by totally three partial derivatives of the function u, one with respect to t in u_t and two with respect to x in u_{xx}.

To solve a differential equation, e.g., (2.1.1) corresponds to finding an analytic expression for the function u, i.e., loosely speaking, through integrating with respect to the differentiated variable. Since there are three differentiations involved in the heat equation (2.1.1), to regain u one needs to perform 3 integrations each creating a degree of freedom. Therefore it is necessary to supply totally 3 (initial and boundary) conditions to determine these degrees of freedom.

To convert the above pde to odes, in x and t, we let $u(x,t) = X(x)T(t) \neq 0$. (**Note!** that $u \equiv 0$ does not work if $f \neq 0$. For $f \equiv 0$, $u(x,t) \equiv 0$ is a solution. Here we seek *non-trivial solutions*).

Inserting $u(x,t) = X(x)T(t)$ in the pde in (2.1.1) we get

$$X(x)T'(t) = kX''(x)T(t).$$

Dividing both sides by $kX(x)T(t) \neq 0$ we thus obtain

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = \lambda, \tag{2.1.4}$$

where the left hand side depends on only t whereas the right hand side depends on only x. This indicates that λ must be an absolute constant independent of x and t.

The differential equation for the function T(t) is now:

$$\frac{T'(t)}{kT(t)} = \lambda \quad \text{or} \quad T'(t) - \lambda k T(t) = 0. \tag{2.1.5}$$

Here the integrating factor is $e^{-\int_0^t \lambda k ds} = e^{-\lambda k t}$. Thus multiplying both side by $e^{-\lambda k t}$ we have,

$$T'(t)e^{-\lambda kt} - \lambda k T(t) e^{-\lambda kt} = 0 \quad \text{or} \quad \frac{d}{dt}\Big(T(t)e^{-\lambda kt}\Big) = 0,$$

Integrating over $(0,t)$ we get

$$\int_0^t \frac{d}{ds}\Big(T(s)e^{-\lambda ks}\Big)dt = T(t)e^{-\lambda kt} - T(0)e^{-\lambda k0} = 0.$$

which gives

$$T(t) = T(0)e^{k\lambda t}. \tag{2.1.6}$$

The differential equation for the function X(x) is:

$$\begin{cases} \frac{X''(t)}{X(x)} = \lambda, & \text{i.e.,} \qquad X''(x) = \lambda X(x), \\ \text{boundary condition (BC)} \Longrightarrow & X(0) = X(L) = 0. \end{cases} \tag{2.1.7}$$

The characteristic equation is now $r^2 = \lambda$. Thus if $\lambda \geq 0$, then $r = \pm\sqrt{\lambda}$ and for $\lambda < 0$ we have $r = \pm i\sqrt{-\lambda}$. We therefore need to consider *three cases*.

I. $\underline{\lambda = 0}$, then $X''(x) = 0$, which gives $X(x) = Ax + B$. But $X(0) = 0$ gives $B = 0$ and thus $X(x) = Ax$. Further $X(L) = 0$ gives $AL = 0$, where $L \neq 0$ and we get $A = 0$. Consequently, (since $A = B = 0$), $u(x,t) \equiv 0$ and we do not get a non-trivial solution in this case. Therefore $\lambda = 0$ is not acceptable.

II. $\underline{\lambda > 0}$, then we have a solution of the form

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x},$$

where $X(0) = 0$ gives $A + B = 0$, i.e., $A = -B$. Further $X(L) = 0$ together with $A = -B$ gives that

$$X(L) = A\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) = 0.$$

But since $L \neq 0$ and $\lambda \neq 0$, hence we have $\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) \neq 0$ and consequently $A = 0$. Thus $B = -A = 0$ gives finally $u(x,t) \equiv 0$. This is also a trivial solution, i.e., $\lambda > 0$ is not acceptable.

III. $\underline{\lambda < 0}$, yields a solution of the form

$$X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x,$$

where $X(0) = 0$ gives $A = 0$ and thus $X(x) = B\sin\sqrt{-\lambda}x$. Now $X(L) = 0$ gives $B\sin\sqrt{-\lambda}L = 0$. If $B = 0$ we have $X(x) \equiv 0$, leading again to the trivial, and thus not acceptable, solution $u(x,t) \equiv 0$. However, in contrary to the cases **I.** and **II.** , here we have other choices than $B = 0$. If $B \neq 0$

then $\sin\sqrt{-\lambda}L = 0$, i.e., $\sqrt{-\lambda}L = n\pi, \quad n \geq 1$, or $\lambda = -\left(\frac{n\pi}{L}\right)^2$. Thus for each $n = 1, 2, ...$, we have the following *eigenvalues and eigenfunctions:*

$$\lambda_n = -\frac{n^2\pi^2}{L^2}, \quad \text{and} \quad X_n(x) = \sin\frac{n\pi}{L}x. \tag{2.1.8}$$

Since $u(x,t) = X(x)T(t) \neq 0$ and $T(t) = T(0)\cdot\ e^{k\lambda t} = T(0)e^{-k\frac{n^2\pi^2}{L^2}t}$ we have for each n a solution $u_n(x,t)$ given by

$$u_n(x,t) = X_n(x)T_n(t) = e^{-k\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi}{L}x. \tag{2.1.9}$$

Since our pde is linear and homogeneous, thus by the superposition principle below, the finite linear combination of all solutions $u_n(x,t), \quad n = 1, 2, ...$ is a solution as well, and then passing to *infinite* linear combination we have

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2}{L^2}kt}\sin\frac{n\pi}{L}x. \tag{2.1.10}$$

There are some obvious questions concerning the convergence of such series as well as their termvis differentiability. We postpone investigating these phenomena and for the moment shall not worry about them. Below we recall the *Superposition Principle* which is also stated in introduction.

Lemma 3 (The Superposition Principle). *If $u_1, u_2, ..., u_m$ satisfy the linear differential equations $L(u_j) = F_j$ and the boundary conditions $B(u_j) = f_j$ for $j = 1, ..., m$ and $c_1, ..., c_m$ are arbitrary constants, then $u = c_1u_1 + ... + c_mu_m$ satisfies the partial differential equation*

$$L(u) = c_1F_1 + ... + c_kF_k \quad \textit{where} \quad B(u) = c_1f_1 + ... + c_mf_m.$$

Our problem corresponds to $F_j = 0$, for $j = 1, ..., m$.

Now it remains to determine the coefficients C_n. To this approach we use the initial condition (IC): $u(x,0) = f(x)$, which gives

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}x = f(x). \tag{2.1.11}$$

Note! C_n is the so called *Fourier sinus coefficients* for f in the interval $[0, L]$. Now we multiply (2.1.11) by $\sin\left(\frac{m\pi}{L}x\right)$, integrate over $[0, L]$, and change the order of integration and summation to obtain

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = \sum_{n=1}^{\infty} C_n \int_0^L \sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{L}x\,dx. \tag{2.1.12}$$

Let now $X_n = \sin\frac{n\pi}{L}x$ and $X_m = \sin\frac{m\pi}{L}x$ and define the scalar product of two functions f and g by

$$\langle f, g\rangle = \int_0^L f(x)g(x)dx.$$

Thus

$$\begin{cases} \langle f, X_m\rangle = \displaystyle\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx \\ \langle X_n, X_m\rangle = \displaystyle\int_0^L \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx. \end{cases} \tag{2.1.13}$$

Using simple trigonometric formulas we have

$$\int_0^L \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx = \frac{1}{2}\int_0^L \Big[\cos(n-m)\frac{\pi}{L}x - \cos(n+m)\frac{\pi}{L}x\Big]dx.$$

Thus if $n \neq m$ we have

$$\langle X_n, X_m\rangle = \frac{L}{2\pi}\Big[\frac{1}{n-m}\sin(n-m)\frac{\pi}{L}x - \frac{1}{n+m}\sin(n+m)\frac{\pi}{L}x\Big]_0^L = 0,$$

i.e., X_n and X_m are orthogonal functions, while for $n = m$ we have

$$\begin{aligned} \int_0^L \sin\frac{m\pi}{L}x\sin\frac{m\pi}{L}x\,dx &= \frac{1}{2}\int_0^L \Big[1 - \cos\frac{2m\pi}{L}x\Big]dx \\ &= \frac{L}{2} - \Big[\frac{L}{2\pi m}\sin\frac{2m\pi}{L}x\Big]_0^L = \frac{L}{2}. \end{aligned} \tag{2.1.14}$$

Hence we have

$$\sum_{n=1}^{\infty} C_n \int_0^L \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx = C_m\int_0^L \sin\frac{m\pi}{L}x\sin\frac{m\pi}{L}x\,dx = C_m\cdot\frac{L}{2}$$

so that (2.1.12) is written as

$$\int_0^L f(x) \sin \frac{m\pi}{L} x \, dx = C_m \cdot \frac{L}{2}$$

and finally changing m to n

$$C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx. \tag{2.1.15}$$

Let us now give an abstract form for $f(x)$ using scalar products. From the above notation we can write $\langle f, X_m \langle = C_m \langle X_m, X_m \langle$ and thus

$$C_m = \frac{\langle f, X_m \rangle}{\langle X_m, X_m \rangle} = \frac{\langle f, X_m \rangle}{|X_m|^2}. \tag{2.1.16}$$

Let now $f_m := C_m X_m$, then

$$f_n = C_n X_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} X_n, \tag{2.1.17}$$

and (2.1.11) gives that

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{L} x = \sum_{n=1}^{\infty} C_n X_n = \sum_{n=1}^{\infty} f_n. \tag{2.1.18}$$

Note that $\Big\{ X_n \Big\}_1^\infty$ are orthogonal basis functions and f_n, the "n-th" component of f is the orthogonal projection of $f = (f_1, f_2, ..., f_n, ...)$ on X_n. Thus denoting the unit vector parallel to X_n by $u_n := \frac{X_n}{|X_n|}$, it follows that

$$f_n = |f_n| u_n = |f_n| \frac{X_n}{|X_n|}. \tag{2.1.19}$$

and since $|f_n| = |f| cos\theta$, we get using the definition of the scalar product

$$\langle f, X_n \rangle = |X_n||f| cos\theta = |X_n||f_n|. \tag{2.1.20}$$

Hence

$$|f_n| = \frac{\langle f, X_n \rangle}{|X_n|}, \tag{2.1.21}$$

Thus we have shown once again that

$$f_n = |f_n| u_n = \frac{\langle f, X_n \rangle}{|X_n|} \cdot \frac{X_n}{|X_n|} = \frac{\langle f, X_n \rangle}{|X_n|^2} X_n = C_n X_n, \tag{2.1.22}$$

where we have used (2.1.16) and (2.1.18) is justified geometrically as well.

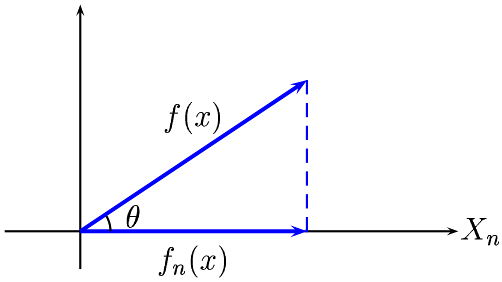


Figure 2.1: The orthogonal projection of $f(x)$.

2.2 The heat equation: Neumann problem

In this part we consider the heat flow with Neumann boundary conditions:

$$\begin{cases} u_t = ku_{xx}, & x \in [0, L], \quad t > 0, & \text{(PDE)} \\ u_x(0, t) = u_x(L, t) = 0, & \qquad\qquad\quad t > 0, & \text{(BC)} \\ u(x, 0) = f(x), & x \in [0, L], & \text{(IC)} \end{cases} \tag{2.2.1}$$

The same procedure as in the Dirichlet case yields

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{constant},$$

with the boundary conditions $X'(0) = X'(L) = 0$, and the same

differential equation for the function T(t): $T'(t) = \lambda kT(t)$.

The differential equation for the function X(x): $\frac{X''(x)}{X(x)} = \lambda$ is studied in a similar way as in the previous section:

I'. $\underline{\lambda = 0}$, i.e., $X(x) = Ax + B$ and thus $X'(x) = A$.

The boundary conditions $X'(0) = X'(L) = 0$ imply that $A = 0$. Hence we have $X(x) = B$ and it follows that $u(x, t) = B \cdot T(0)e^{k\lambda t}$ is the solution for this case.

II'. $\underline{\lambda > 0}$ yields

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}. \tag{2.2.2}$$

Thus $X'(x) = A\sqrt{\lambda} \cdot e^{\sqrt{\lambda}x} - B\sqrt{\lambda} \cdot e^{-\sqrt{\lambda}x}$ and hence $X'(0) = \sqrt{\lambda}(A - B) = 0$, which implies that $A = B$. Inserting $B = A$ in the other boundary condition: $X'(L) = 0$, we get $A\sqrt{\lambda}\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) = 0$, which together with $\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) \neq 0$ gives $A = 0$. Thus $A = B = 0$ and consequently $X(x) \equiv 0$ and we get the trivial, i.e., zero solution in this case.

III'. $\underline{\lambda < 0}$ gives the well-known solution

$$X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x. \tag{2.2.3}$$

with the derivative $X'(x) = -A\sqrt{-\lambda}\sin\sqrt{-\lambda}x + B\sqrt{-\lambda}\cos\sqrt{-\lambda}x$, which associated with the boundary data $X'(0) = 0$ gives $B = 0$ and thus $X(x) = A\cos(\sqrt{-\lambda}x)$.
The second boundary condition: $X'(L) = 0$ yields $A\sqrt{-\lambda}\cdot\sin(\sqrt{-\lambda}L) = 0$, which assuming $A \neq 0$, yields $\sin\sqrt{-\lambda}L = 0$, i.e., $\sqrt{-\lambda}L = n\pi, n \geq 1$, so that as above we once again the eigenvalues $\lambda = -\left(\frac{n\pi}{L}\right)^2$. Thus for the Neumann problem we have the following *eigenvalues and eigenfunctions*:

$$\lambda_n = -\frac{n^2\pi^2}{L^2}, \quad \text{and} \quad X_n(x) = \cos\frac{n\pi}{L}x, \quad n = 0, 1, 2, \ldots \tag{2.2.4}$$

Note that in this case for $n = 0$, we get the eigenvalue $\lambda = 0$ with the corresponding, non-trivial, eigenfunction $X_0(x) = 1$. This means that the case **III'.** contains **I'.** as well.
In summary we get

$$u_n(x,t) = X_n(x)T(t) = C_n e^{-\frac{n^2\pi^2}{L^2}kt}\cos\frac{n\pi}{L}x, \quad n \geq 0. \tag{2.2.5}$$

Using superposition we get the solution

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-\frac{n^2\pi^2}{L^2}kt}\cos\frac{n\pi}{L}x, \tag{2.2.6}$$

where by a similar argument as in the Dirichlet case we have

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \quad n \geq 0. \tag{2.2.7}$$

2.3 The heat conducting problem

To proceed below we study the heat conducting in a circular ring. This model has many important features, e.g., as a by product, we can derive the basic formulas for the Fourier series expansions.

For the heat conducting in a circular ring, we do not have a natural boundary condition, however, we can use the *periodicity* and write the equation as follows:

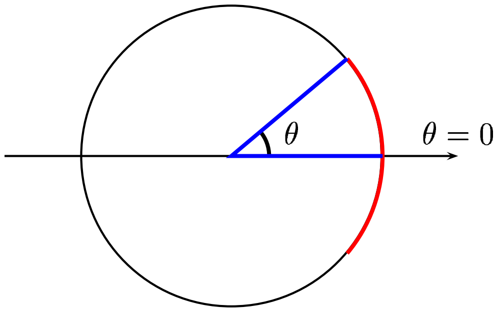


Figure 2.2: The circular ring

$$\begin{cases} u_t = ku_{\theta\theta}, \\ u(0,\theta) = f(\theta), \\ u(t,0) = u(t,2\pi), \end{cases} \tag{2.3.1}$$

Applying the principle of the separation of variables we let now the solution $u(t,\theta) = T(t)\Theta(\theta) \neq 0$ and get the following eigenvalue problems:

$$\frac{T'(t)}{kT(t)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda. \tag{2.3.2}$$

The general solution for the eigenvalue problem $T'(t) = \lambda kT(t)$ is given by

$$T(t) = C_0 e^{\lambda kt}, \tag{2.3.3}$$

whereas for the general solution of the eigenvalue problem $\Theta''(\theta) = \lambda\Theta(\theta)$ we have

$$\Theta(\theta) = A\cos\sqrt{-\lambda}\,\theta + B\sin\sqrt{-\lambda}\,\theta. \tag{2.3.4}$$

For the equation (2.3.4) we have no natural boundary conditions, which means that we cannot find values for A and B as in the previous procedure. However, the 2π periodicity in (2.3.4) would yield for $\Theta(\theta)$: i.e., we have $\Theta(0) = \Theta(2\pi)$ and hence since

$$\begin{cases} \Theta(0) = A, \\ \Theta(2\pi) = A\cos\sqrt{-\lambda}\,2\pi + B\sin\sqrt{-\lambda}\,2\pi, \end{cases} \tag{2.3.5}$$

$$\Theta(0) = \Theta(2\pi) \Longrightarrow A = A\cos\sqrt{-\lambda}\,2\pi + B\sin\sqrt{-\lambda}\,2\pi. \tag{2.3.6}$$

Now identifying the coefficients (Note! that B need not be zero) we get

$$\cos\sqrt{-\lambda}\,2\pi = 1 \quad \text{and} \quad \sin\sqrt{-\lambda}\,2\pi = 0, \quad B \neq 0. \tag{2.3.7}$$

Thus we have $\sqrt{-\lambda} = n$, where n is a integer, i.e., $\lambda = -n^2, \quad n = 0, 1, 2, ...$ Summing up we have

$$\begin{cases} T_n(t) = C_0 e^{-n^2 kt}, \\ \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \end{cases} \tag{2.3.8}$$

Let now $a_n = C_0 A_n$, $b_n = C_0 B_n$ and use superposition principle to write

$$u(t,\theta) = \sum_{n=0}^{\infty} e^{-n^2 kt}\Big(a_n\cos n\theta + b_n\sin n\theta\Big). \tag{2.3.9}$$

It remains to determine a_n and b_n, where, as before, we use the initial condition $u(0,\theta) = f(\theta)$ to get

$$f(\theta) = \sum_{n=0}^{\infty}(a_n\cos n\theta + b_n\sin n\theta). \tag{2.3.10}$$

This is the well-known Fourier series expansion for f, which we shall study in details in chapter 4.

2.4 The wave equation

In this section we want to solve the initial boundary value problem for the wave equation using separation of variables. We illustrate the procedure studying the following example:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in [0, L], \quad t > 0, & \text{(PDE)} \\ u(0,t) = u(L,t) = 0, & \qquad\qquad\quad t > 0, & \text{(BC)} \\ u(x,0) = f(x), & x \in [0, L], & \text{(IC1)} \\ u_t(x,0) = g(x) & x \in [0, L], & \text{(IC2)}. \end{cases} \tag{2.4.1}$$

Let $u(x,t) = X(x)T(t) \neq 0$. Our (PDE) can we now be written as $T''(t)X(x) = c^2T(t)X''(x)$, which yields

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \lambda, \tag{2.4.2}$$

where using the same argument as for the Dirichlet problem for the heat equation we have $\lambda < 0$. Further, due to that fact that the ordinary differential equation $T''(t) = \lambda T(t)$ has the characteristic equation $r^2 = c^2\lambda$ with the rots $r = \pm ic\sqrt{-\lambda}$. It follows using the preceding examples that

$$T(t) = A\cos\sqrt{-\lambda}\,ct + B\sin\sqrt{-\lambda}\,ct, \tag{2.4.3}$$

Similarly for $X(x)$ we get

$$X(x) = C\cos\sqrt{-\lambda}\,x + D\sin\sqrt{-\lambda}\,x. \tag{2.4.4}$$

Now the boundary condition give $X(0) = C = 0$, hence $X(x) = D\sin(\sqrt{-\lambda}\,x)$. Further $X(L) = 0$ together with $D \neq 0$ yields $\sqrt{-\lambda}\,L = n\pi$, $n = 1, 2, \ldots$. Thus we have eigenvalues and eigenfunctions

$$\lambda = -\frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi}{L}, \qquad n = 1, 2, \ldots \tag{2.4.5}$$

Hence,

$$u_n(x,t) = \Big(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\Big)\sin\frac{n\pi}{L}x \tag{2.4.6}$$

where we may interpret $X_n(x) = D_n\sin\frac{n\pi}{L}$ (not! normalized), $a_n = A_nD_n$ and $b_n = B_nD_n$ with obvious notations for A_n and B_n. Finally using superposition we get

$$u(x,t) = \sum_{n=1}^{\infty}\Big(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\Big)\sin\frac{n\pi}{L}x. \tag{2.4.7}$$

Now it remain to determine a_n and b_n. Since $u(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x. \tag{2.4.8}$$

Multiplying (2.4.8) by $\sin \frac{m\pi}{L} x$ and integrating over $[0, L]$ yields

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx, \quad n \geq 1. \tag{2.4.9}$$

Similarly

$$u_t(x,t) = \sum_{n=1}^{\infty} \Big[\frac{a_n n\pi c}{L} \sin \frac{n\pi ct}{L} + \frac{b_n n\pi c}{L} \cos \frac{n\pi ct}{L} \Big] \sin \frac{n\pi}{L} x \tag{2.4.10}$$

together with $u_t(x,0) = g(x)$ yields

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{b_n n\pi c}{L} \sin \frac{n\pi}{L} x. \tag{2.4.11}$$

Multiplying (2.4.11) by $\sin \frac{m\pi}{L} x$ and integrating over $[0, L]$ it follows that

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx, \quad n \geq 1. \tag{2.4.12}$$

In this way we have an expression for $u(x,t)$ as a function of $f(x)$ and $g(x)$:

$$u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \Big) \sin \frac{n\pi}{L} x. \tag{2.4.13}$$

Now we want to present $u(x,t)$, as a function of $f(x)$ and $g(x)$, in the form of *Hadamard's formula*. To this approach we write

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}. \tag{2.4.14}$$

Using the elementary trigonometric relations: $\sin(x \pm t) = \sin x \cdot \cos t \pm \sin t \cdot \cos x$ and $\cos(x \pm t) = \cos x \cdot \cos t \mp \sin x \cdot \sin t$ we rewrite (2.4.13) as

$$\begin{aligned} u(x,t) = & \sum_{n=1}^{\infty} a_n \frac{1}{2} \Big[\sin \frac{n\pi}{L}(x + ct) + \sin \frac{n\pi}{L}(x - ct) \Big] \\ & + \sum_{n=1}^{\infty} b_n \frac{1}{2} \Big[\cos \frac{n\pi}{L}(x - ct) - \cos \frac{n\pi}{L}(x + ct) \Big]. \end{aligned} \tag{2.4.15}$$

Now we using (2.4.8), that

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n \frac{1}{2}\Big[\sin\frac{n\pi}{L}(x+ct)+\sin\frac{n\pi}{L}(x-ct)\Big] \\
&=\frac{1}{2}\sum_{n=1}^{\infty} a_n\sin\frac{n\pi}{L}(x+ct)+\frac{1}{2}\sum_{n=1}^{\infty} a_n\sin\frac{n\pi}{L}(x-ct) \qquad (2.4.16)\\
&=\frac{1}{2}\Big[f(x+ct)+f(x-ct)\Big].
\end{aligned}$$

Further using the identity

$$\Big[\cos\frac{n\pi}{L}(x-ct)-cos\frac{n\pi}{L}(x+ct)\Big]=\frac{n\pi}{L}\int_{x-ct}^{x+ct}\sin\frac{n\pi}{L}y\,dy, \qquad (2.4.17)$$

we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} b_n\frac{1}{2}\Big[\cos\frac{n\pi}{L}(x-ct)-\cos\frac{n\pi}{L}(x+ct)\Big]\\
&=\frac{1}{2}\sum_{n=1}^{\infty} b_n\frac{n\pi}{L}\int_{x-ct}^{x+ct}\sin\frac{n\pi}{L}y\,dy \qquad (2.4.18)\\
&=\frac{1}{2}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} b_n\frac{n\pi}{L}\sin\frac{n\pi}{L}y\,dy,
\end{aligned}$$

where, in the last step, we have changed the order of the summation and integration. Now using (2.4.11) it follows that

$$\frac{1}{2}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty}\frac{1}{c}\cdot\frac{n\pi c}{L}b_n\sin\frac{n\pi}{L}y\,dy=\frac{1}{2c}\int_{x-ct}^{x+ct}g(y)\,dy. \qquad (2.4.19)$$

Thus we conclude the *Hadamard's formula:*

$$u(x,t)=\frac{1}{2}\Big[f(x+ct)+f(x-ct)\Big]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(y)dy. \qquad (2.4.20)$$

Chapter 3

Fourier series for 2π-periodic functions

Periodic phenomenon occur frequently throughout nature and their study is of the utmost importance for our understanding of many real-world systems. For example, the signals from radio pulsars allow astronomers to study space, the seasonal periodicity of the weather governs the crop of corn, and the regular beats of a heart is necessary for the survival of every mammal. Periodicity can be found everywhere and concern any absolute variable, i.e., time, space, velocity, etc. In this chapter we shall begin to study periodic functions, and especially, their representation as sums of sine and cosine functions, Fourier series. Fourier series has long provided one of the principal tools of analysis for mathematical physics, engineering, and signal processing. It has spurred many generalizations, and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions describing wave motion, such as with light and sound, its generalizations often relate to wider settings, for example, the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis. We shall, however, be content with presenting the basic theory and its application to the solution of partial differential equations.

3.1 Periodic Functions

A function $f(x)$ is said to be periodic in there is a constant p, such that

$$f(x+p) = f(x), \tag{3.1.1}$$

for all x. Any positive number with this property is called a period of $f(x)$. For example, $f(x) = \sin x$ has periods 2π, 4π, etc. However, the smallest number $p > 0$ with the property (3.1.1) is called the *prime, period*, and it is generally this value that is meant when a function is referred to as being p-periodic, or, of period p.

First we state and prove a frequently used result, viz:

Lemma 4. *Suppose $f(x)$ is periodic with period p, then the integral*

$$\int_a^{a+P} F(x)dx \tag{3.1.2}$$

is independent of the starting point a.

Proof. Let

$$g(a) := \int_a^{a+p} f(x)dx = \int_0^{a+P} f(x) - \int_0^a f(x)dx.$$

By the fundamental theorem of calculus $g'(a) = f(a+P) - f(a)$ but since $f(p+a) = f(a)$ we have that $g'(a) = 0$. Hence, $g(a)$ is constant and independent of a. ☐

3.2 Fourier series

From about 1800 onwards, the French scientist Joseph Fourier [1] was lead by problems of heat conduction to consider the possibility of representing a more or less arbitrary 2π-periodic function as a linear combination of the functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots \tag{3.2.1}$$

[1]Jean Baptiste Fourier (1768-1830) French physicist and mathematician.

Fourier conjectured that any integrable periodic function $f(x)$ of period 2π can be written, *at almost every point* x (which we specify later), as the sum of a trigonometric series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos nx + b_n \sin nx\Big). \tag{3.2.2}$$

where a_n and b_n, $n = 0, 1, 2, \dots$, are real numbers defied by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \tag{3.2.3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \tag{3.2.4}$$

Here the term $\frac{1}{2}a_0$ is due to the constant function $\cos 0 = 1$, the factor $\frac{1}{2}$ being included for reasons of later convenience. Further, b_0 does not exist, since $\sin 0 = 0$.

Fourier manged to solve several problems of heat flow using such series representations, and, as a result, (3.2.2) is today called the *Fourier series* of $f(x)$. Similarly, the corresponding coefficients a_n and b_n are called *Fourier coefficients* of $f(x)$.

As we stated above the *equality* (3.2.2) is not always true, and therefore we replace this equality by a "$\sim$" sign indicating that the right hand side is the Fourier series of the function f.

To be more specific, suppose that $f(\theta)$ is a 2π-periodic Riemann integrable function, e.g., $f(\theta)$ is piecewise continuous and 2π-periodic. We summarize the above discussion in the following definition which also reformulates (2.3.10) in a slightly modified form

Definition 7. *The real Fourier series expansion of a 2π periodic Riemann integrable function $f(\theta)$ is given by:*

$$f(\theta) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos n\theta + b_n \sin n\theta\Big). \tag{3.2.5}$$

Here a_0, a_n and b_n are called the Fourier coefficients for f. We shall return to the reason why a_0 is isolated.

Inserting

$$\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \quad \text{and} \quad \sin n\theta = \frac{1}{2i}(e^{in\theta} - e^{-in\theta})$$

in (3.2.5) we get

$$\begin{aligned} f(\theta) &\sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(\frac{1}{2}a_n(e^{in\theta} + e^{-in\theta}) - \frac{i}{2}b_n(e^{in\theta} - e^{-in\theta})\Big) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\theta} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n + ib_n)e^{-in\theta} \end{aligned} \tag{3.2.6}$$

We rewrite (3.2.6) as the following complex Fourier series expansion of f:

$$f(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta} = C_0 + \sum_{n=1}^{\infty} C_n e^{in\theta} + \sum_{n=1}^{\infty} C_{-n} e^{-in\theta}. \tag{3.2.7}$$

We identify the coefficients on the right hand sides of (3.2.6) and (3.2.7) to obtain

$$C_0 = \frac{1}{2}a_0, \quad C_n = \frac{1}{2}(a_n - ib_n), \quad C_{-n} = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots, \tag{3.2.8}$$

or equivalently

$$a_0 = 2C_0, \quad a_n = C_n + C_{-n}, \qquad b_n = i(C_n - C_{-n}), \qquad n = 1, 2, \ldots, \tag{3.2.9}$$

Obviously we can calculate complex Fourier coefficients C_n in terms of $f(\theta)$ using (3.2.8) and (3.2.2) and (3.2.3). Now, *when* (3.2.7) *is an equality*, we may use this equality and first calculate the complex Fourier coefficients C_n in terms of $f(\theta)$ and the using (3.2.9) we recompute a_n and b_n. The idea is to use the orthogonality of the set $\{e^{in\theta}\}$ (as that of $\sin n\theta$ and $\cos n\theta$ in the previous chapter). To this approach we multiply both sides in (3.2.7) by $e^{-ik\theta}$ and integrate over $[\pi, -\pi]$ to obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta = \sum_{-\infty}^{\infty} C_n \int_{-\pi}^{\pi} e^{i(n-k)\theta}d\theta, \tag{3.2.10}$$

where we have changed the order of integration and summation. We compute for $n \neq k$ that

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta = \Big[\frac{1}{i(n-k)} e^{i(n-k)\theta}\Big]_{-\pi}^{\pi} = \frac{(-1)^{(n-k)} - (-1)^{(n-k)}}{i(n-k)} = 0, \tag{3.2.11}$$

whereas $n = k$ yields

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta = \int_{-\pi}^{\pi} d\theta = 2\pi. \tag{3.2.12}$$

Thus in concise form we have the orthogonality relation viz,

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta = \begin{cases} 0, & \text{for } n \neq k, \\ 2\pi, & \text{for } n = k. \end{cases} \tag{3.2.13}$$

Hence only for $n = k$ we can get a contribution from the sum on the left hand side in (3.2.10), i.e.

$$\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta = 2\pi C_k. \tag{3.2.14}$$

Relabeling the integers k and n, we have the following formula for the complex Fourier coefficients C_n:

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta. \tag{3.2.15}$$

We note that by (3.2.15),

$$C_0 = \frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta, \tag{3.2.16}$$

is average or mean value of f on *any* interval of length 2π. The real coefficient a_0 is given by

$$a_0 = 2C_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta. \tag{3.2.17}$$

Further for $n = 1, 2, \ldots$, we have using (3.2.8) and (3.2.15) that

$$a_n = C_n + C_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} + e^{in\theta}) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) 2 \cos n\theta d\theta.$$

Thus

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(\theta)\cos n\theta d\theta. \tag{3.2.18}$$

Note that the formula (3.2.18) holds for $n = 0$, as well. This is the reason why we use the factor $\frac{1}{2}$ in the formula (3.2.5). Analogously the real Fourier coefficients b_n are then given by

$$b_n = i(C_n - C_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} - e^{in\theta})d\theta = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-2i)\sin n\theta d\theta,$$

and hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta. \tag{3.2.19}$$

We summarize the result of this section in the following formal definition:

Definition 8. *Suppose $f(\theta)$ is a 2π periodic Riemann integrable function. Then $f(\theta)$ has a Complex Fourier series expansion as*

$$f(\theta) \sim \sum_{-\infty}^{\infty} C_n e^{in\theta}. \tag{3.2.20}$$

The corresponding real Fourier series representation of $f(\theta)$ is given by

$$f(\theta) \sim \frac{1}{2}a_0 + \sum_{1}^{\infty}\big(a_n\cos n\theta + b_n \sin n\theta\big). \tag{3.2.21}$$

The complex Fourier coefficients C_n of f are given by

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta. \tag{3.2.22}$$

Equivalently the real Fourier coefficients a_n and b_n, for f are defined as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \quad (n \geq 0), \tag{3.2.23}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta, \quad (n \geq 1). \tag{3.2.24}$$

So far a crucial question is when we can write equality in (3.2.5) or (3.2.21)? And defining a_n and b_n as in (3.2.6) and (3.2.7), what is the relation between the right hand side in, e.g., (3.2.5) and $f(x)$? We put the whole answer in the *Fourier convergence theorem* below:

Theorem 9 (Convergence Theorem). *If f is 2π-periodic and piecewise smooth on $\mathbb{R}$, and*

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_{1}^{N}\Big(a_n \cos n\theta + b_n \sin n\theta\Big) = \sum_{-N}^{N} C_n e^{in\theta} \tag{3.2.25}$$

then

$$\lim_{N\to\infty} S_N^f(\theta) = \frac{1}{2}\Big[f(\theta-) + f(\theta+)\Big]$$

for every θ and in particular,

$$\lim_{N\to\infty} S_N^f(\theta) = f(\theta)$$

for every θ at which f is continuous.

We postpone the proof of Fourier convergence theorem. However, from now on as soon as we can justify that a function f is continuous at a point θ, we use equality between the Fourier series expansion of f and $f(\theta)$.

3.3 Even and odd functions

Below we examine some useful properties of even and off functions:

- For an even function $F(\theta)$ we have $F(\theta) = F(-\theta)$, hence

$$\int_{-a}^{a} F(x)dx = 2\int_{0}^{a} F(x)dx. \tag{3.3.1}$$

- For an odd function $F(\theta)$ we have $F(-\theta) = -F(\theta)$, and hence

$$\int_{-a}^{a} F(x)dx = -\int_{0}^{a} F(x)dx + \int_{0}^{a} F(x)dx = 0. \tag{3.3.2}$$

Now since $\cos(n\theta)$ is an even function and $\sin(n\theta)$ is an odd function, we get

$$\text{For even } f \qquad a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta \qquad \text{and} \qquad b_n = 0. \tag{3.3.3}$$

$$\text{For odd } f \qquad a_n = 0 \qquad \text{and} \qquad b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin n\theta\, d\theta. \tag{3.3.4}$$

Exempel 1. *The function $f(\theta) = |\theta|$, $-\pi < \theta < \pi$, is 2π periodic. Express f in Fourier series.*

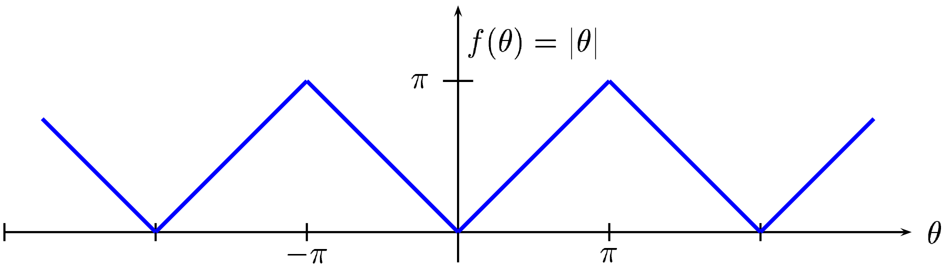


Figure 3.1: The 2π periodic function $f(\theta) = |\theta|$.

The function $f(\theta)$ is 2π-periodic and even, thus we have according to (3.3.3) $b_n = 0$ *and*

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta$$

Since $f(\theta) = |\theta|$ and $\theta > 0$ on the interval $[0, \pi]$ we get first for $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi \theta\cos 0\, d\theta = \frac{2}{\pi}\int_0^\pi \theta\, d\theta = \frac{2}{\pi}\left[\frac{\theta^2}{2}\right]_0^\pi = \pi,$$

and then for $n > 0$ we have using partial integration that

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi |\theta|\cos n\theta\, d\theta = \frac{2}{\pi}\int_0^\pi \theta\cos n\theta\, d\theta \\ &= \frac{2}{\pi}\left[\theta\cdot\frac{1}{n}\sin n\theta\right]_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{1}{n}\sin n\theta\, d\theta = \frac{2}{\pi n}\left[\frac{\cos n\theta}{n}\right]_0^\pi \\ &= \frac{2}{\pi n^2}\Big[(-1)^n - 1\Big] = \begin{cases} \frac{-4}{\pi(2k-1)^2}, & \textit{for odd } n := 2k-1 \\ 0, & \textit{for even } n := 2k. \end{cases} \end{aligned} \tag{3.3.5}$$

Thus the Fourier series expansion formula

$$f(\theta) = \frac{1}{2}a_0 + \sum_1^\infty \Big(a_n \cos n\theta + b_n \sin n\theta\Big)$$

yields

$$|\theta| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^\infty \frac{1}{(2k-1)^2}\cos(2k-1)\theta. \tag{3.3.6}$$

Note that

$$\sum_{k=1}^\infty \frac{1}{(2k-1)^2}\cos(2k-1)\theta, \tag{3.3.7}$$

convergence quadratically!

Exempel 2. *The function $g(\theta) = \theta$, $-\pi < \theta < \pi$, is 2π-periodic. Express g in Fourier series.*

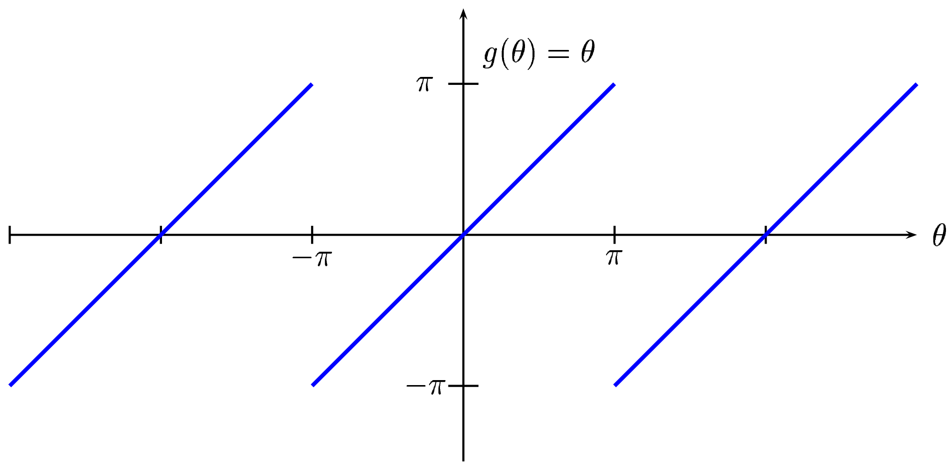


Figure 3.2: The 2π periodic function $g(\theta) = \theta$.

Here $g(\theta) = \theta$ is an odd function, thus by (3.3.4) *we have $a_n = 0$ for $n = 0, 1, \ldots$ and*

$$b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin n\theta\, d\theta = \frac{2}{\pi}\int_0^\pi \theta \cdot \sin n\theta\, d\theta$$

Now using partial integration we get

$$\begin{aligned} b_n &= \frac{2}{\pi}\Big[\theta\cdot\frac{-1}{n}\cos n\theta\Big]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{-1}{n}\cos n\theta\, d\theta \\ &= \frac{2}{\pi}\cdot\pi\cdot\frac{-1}{n}(-1)^n = 2\cdot\frac{(-1)^{n+1}}{n}. \end{aligned} \tag{3.3.8}$$

and hence

$$g(\theta) \sim \frac{1}{2}a_0 + \sum_1^{\infty}\Big(a_n\cos n\theta + b_n\sin n\theta\Big) \tag{3.3.9}$$

yields

$$g(\theta) = \theta = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\theta. \tag{3.3.10}$$

Note that

$$\sum_1^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\theta) \sim \sum_1^{\infty}\frac{1}{n}, \tag{3.3.11}$$

which means that $g(\theta) = \theta$ *has poor convergence properties than the previous example:* $f(\theta) = |\theta|$. *Below we comment this phenomenon:*

Remark. Note that, comparing the graphs of f and g we see that f is a continuous, linear, oscillating functions, whereas g is discontinuous (which is also the reason behind the $\sim$ sign in (3.3.9) rather than an equality, we shall discuss this later on in this chapter). Now if in (3.3.6) and (3.3.11) we take the summation only up to n and denoting the resulting finite sums by $f_n(\theta)$ and $g_n(\theta)$, then $f_n(\theta)$ goes faster towards $f(\theta)$ than $g_n(\theta)$ goes towards $g(\theta)$. This is due to stronger continuity property in f than in g. Thus, if f is regular the convergence of the Fourier series for f is faster towards f .

3.4 Bessel's Inequalities

Theorem 10 (The Bessel's Inequality I). *If* $f(\theta)$ *is a* 2π- *periodic Riemann integrable function on* $[-\pi,\pi]$ *and* C_n *are the Fourier coefficients of* f, *then*

$$\sum_{-\infty}^{\infty}|C_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta. \tag{3.4.1}$$

Proof. We use the partial sum, of order N, of the complex Fourier series expansion for $f(\theta)$, i.e.,

$$f_N(\theta) = \sum_{-N}^{N} C_n e^{in\theta}, \tag{3.4.2}$$

and compute

$$\begin{aligned} \Big| f(\theta) - \sum_{-N}^{N} C_n e^{in\theta} \Big|^2 &= \Big(f(\theta) - \sum_{-N}^{N} C_n e^{in\theta} \Big) \Big(\overline{f(\theta)} - \sum_{-N}^{N} \overline{C}_n e^{-in\theta} \Big) \\ &= |f(\theta)|^2 - \sum_{-N}^{N} \Big[C_n \overline{f(\theta)} e^{in\theta} + \overline{C_n} f(\theta) e^{-in\theta} \Big] \qquad (3.4.3) \\ &+ \sum_{m,n=-N}^{N} C_m \overline{C_n} e^{i(m-n)\theta}. \end{aligned}$$

Now integrating (3.4.3) over $[-\pi, \pi]$, and changing the order of summation and integration, it follows that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| f(\theta) - \sum_{-N}^{N} C_n e^{in\theta} \Big|^2 d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \\ &- \frac{1}{2\pi} \sum_{-N}^{N} \Big[C_n \int_{-\pi}^{\pi} \overline{f(\theta)} e^{in\theta} d(\theta) + \overline{C_n} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \Big] \qquad (3.4.4) \\ &+ \frac{1}{2\pi} \sum_{m,n=-N}^{N} \int_{-\pi}^{\pi} C_m \overline{C_n} e^{i(m-n)\theta} d\theta. \end{aligned}$$

Recall that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta = \begin{cases} 0, & \textit{for} \quad m \neq n \\ 1, & \textit{for} \quad m = n. \end{cases}$$

Further

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \quad \textit{implies} \quad \overline{C_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\theta)} e^{in\theta} d\theta.$$

Hence we can rewrite (3.4.4) *as*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\theta)-\sum_{-N}^{N}C_ne^{in\theta}\right|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta \\ -\sum_{-N}^{N}(C_n\overline{C_n}+\overline{C_n}C_n)+\sum_{-N}^{N}C_n\overline{C_n}. \tag{3.4.5}$$

Thus

$$0\le\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\theta)-\sum_{-N}^{N}C_ne^{in\theta}\right|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta-\sum_{-N}^{N}|C_n|^2,$$

and letting $N\to\infty$ *we obtain the first Bessel's Inequality:*

$$\sum_{-\infty}^{\infty}|C_n|^2\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta.$$

Below using the relations between real and complex Fourier coefficients we derive the real version of (3.4.1):

Theorem 11 (Bessel's Inequality II). *If* $f(\theta)$ *is a* 2π*- periodic Riemann integrable function on* $[-\pi,\pi]$ *and* a_n, $n=0,1,\dots,$ *and* b_n, $n=1,2,\dots,$ *are the real Fourier coefficients of* f, *then*

$$\frac{1}{4}|a_0|^2+\frac{1}{2}\sum_{1}^{\infty}(|a_n|^2+|b_n|^2)\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta \tag{3.4.6}$$

Proof. As we have seen earlier we have for $n\ge 1$ *that* $a_n=C_n+C_{-n}$ *and* $b_n=i(C_n-C_{-n})$*. Hence*

$$|a_n|^2+|b_n|^2=(C_n+C_{-n})(\overline{C_n}+\overline{C_{-n}})+i(C_n-C_{-n})(-i)(\overline{C_n}-\overline{C_{-n}}) \\ =2C_n\overline{C_n}+2C_{-n}\overline{C_{-n}}=2|C_n|^2+2|C_{-n}|^2.$$

Thus, for $n\ge 1$, *we have*

$$\frac{1}{2}(|a_n|^2+|b_n|^2)=|C_n|^2+|C_{-n}|^2, \tag{3.4.7}$$

whereas for $n=0$ *we get*

$$|a_0|^2=2C_{-0}2\overline{C_{-0}}=4|C_0|^2. \tag{3.4.8}$$

Combining (3.4.7) *and* (3.4.8) *and the Bessel's inequality I, we obtain*

$$\begin{aligned} \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_1^\infty(|a_n|^2 + |b_n|^2) &= |C_0|^2 + \sum_1^\infty \left(|C_n|^2 + |C_{-n}|^2\right) \\ &= \sum_{-\infty}^{\infty} |C_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \end{aligned} \tag{3.4.9}$$

which is the Bessel's Inequality (II) and the proof is complete.

As a consequence of these two theorems we conclude that for 2π Riemann integrable functions, the series obtained from the Fourier coefficients:

$$\sum_1^\infty |a_n|^2, \quad \sum_1^\infty |b_n|^2, \quad \sum_1^\infty |C_n|^2 \quad \text{and} \quad \sum_{-1}^{-\infty} |C_n|^2$$

are all convergent. Thus, $|a_n|^2$, $|b_n|^2$ and $|C_n|^2$, being the *n-th* terms of convergent series, tend to zero as $n \to \infty$ (also as $n \to -\infty$ in the case of C_n) and hence so do a_n, b_n and C_n. We summarize these properties, viz,

Lemma 5. *For a 2π periodic, Riemann integrable function f, the Fourier coefficients a_n, b_n and C_n all tend to zero as as $n \to \infty$ (also as $n \to -\infty$ in the case of C_n).*

3.5 Proof of the convergence theorem

To prove the convergence theorem for the Fourier series we need to define the concepts as *piecewise continuous* and *piecewise smooth* functions.

Definition 9. *A function f on the closed interval $[a, b]$ is said to be* **piecewise continuous** *on $[a, b]$ if*
(i) f is continuous on $[a, b]$ except perhaps at finitely many points $x_1, x_2, ..., x_k$.
(ii) at each of the points $x_1, x_2, ..., x_k$, f has both the left-hand and the Right-hand limits, i.e., $f(x_j-)$ and $f(x_j+)$, $j = 1, \ldots, k$, exist.

Definition 10. *A function f, defined on the closed interval $[a, b]$ is* **piecewise smooth** *on $[a, b]$ if f and its first derivative f' are both piecewise continuous on $[a, b]$.*

Definition 11. *Let f be a 2π-periodic piecewise smooth function then, for a fixed θ, the Nth partial sum for f is defined by*

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_1^N \Big(a_n \cos n\theta + b_n \sin n\theta\Big) = \sum_{-N}^N C_n e^{in\theta} \tag{3.5.1}$$

with a_n, b_n and C_n being the usual Fourier coefficients, viz;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos(n\psi)d\psi, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin(n\psi)d\psi, \quad \textit{and}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)e^{-in\psi}d\psi.$$

To prepare for the proof of the convergence theorem we note that

$$S_N^f(\theta) = \sum_{-N}^N C_n e^{in\theta} = \frac{1}{2\pi}\sum_{-N}^N \int_{-\pi}^{\pi} f(\psi)e^{in(\theta-\psi)}d\psi.$$

is symmetric on n, since n ranges from $-N$ to N. Hence replacing n by $-n$, does not affect the above sum and thus we can write

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^N \int_{-\pi}^{\pi} f(\psi)e^{in(\psi-\theta)}d\psi \tag{3.5.2}$$

Let now $\phi = \psi - \theta$, then $\psi = \theta + \phi$ and $d\psi = d\phi$, so that

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^N \int_{-\pi+\theta}^{\pi+\theta} f(\theta+\phi)e^{in\phi}d\phi. \tag{3.5.3}$$

Further since both f and $e^{in\psi}$ are 2π-periodic, using Lemma 2, it follows that

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^N \int_{-\pi}^{\pi} f(\theta+\phi)e^{in\phi}d\phi. \tag{3.5.4}$$

Now we define the *Nth Dirichlet kernel* by

$$D_N(\phi) = \frac{1}{2\pi}\sum_{-N}^N e^{in\phi} \tag{3.5.5}$$

and rewrite $S_N^f(\theta)$, in (3.5.4) as

$$S_N^f(\theta) = \int_{-\pi}^{\pi} f(\theta+\phi) D_N(\phi) d\phi. \tag{3.5.6}$$

Some properties of the Dirichlet kernel:
The function $D_N(\phi)$ is the sum of a finite geometric progression, which for $\phi \neq 0$ can be written as

$$\begin{aligned} D_N(\phi) &= \frac{1}{2\pi}\Big(e^{-iN\phi} + e^{-i(N+1)\phi} + \ldots + e^0 + e^{i2\phi} + \ldots + e^{iN\phi}\Big) \\ &= \frac{1}{2\pi} e^{-iN\phi}\Big(1 + e^{i\phi} + \ldots + e^{i2N\phi}\Big) = \frac{1}{2\pi} e^{-iN\phi} \sum_0^{2N} e^{in\phi} \qquad (3.5.7) \\ &= \frac{1}{2\pi} \frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}. \end{aligned}$$

Multiply both the numerator and denominator by $e^{-i\frac{\phi}{2}}$ we get

$$D_N(\phi) = \frac{1}{2\pi} \frac{e^{i(N+\frac{1}{2})\phi} - e^{-i(N+\frac{1}{2})\phi}}{e^{i\frac{1}{2}\phi} - e^{-i\frac{1}{2}\phi}} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi} \tag{3.5.8}$$

From this formula we can sketch the graph of $D_N(\phi)$. It is rapidly oscillating to zero viz.

We conclude this part by the following lemma:

Lemma 6. *For any N, we have that*

$$\int_{-\pi}^{0} D_N(\phi) d\phi = \int_0^{\pi} D_N(\phi) d\phi = \frac{1}{2}$$

Proof: We give the proof for $\phi \in (0,\pi)$. The case $\phi \in (-\pi, 0)$ is proved in the same way. We rewrite the Nth Dirichlet kernel as

$$D_N(\phi) = \frac{1}{2\pi} \sum_{-N}^{N} e^{in\phi} = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_1^N \Big(e^{in\phi} + e^{-in\phi}\Big) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_1^N \cos n\phi.$$

Integrating over $\phi \in (0,\pi)$ yields

$$\int_0^{\pi} D_N(\phi) d\phi = \int_0^{\pi} \Big(\frac{1}{2\pi} + \frac{1}{\pi}\sum_1^N \cos n\phi\Big) d\phi = \Big[\frac{\theta}{2\pi} + \frac{1}{\pi}\sum_1^N \frac{\sin n\phi}{n}\Big]_0^{\pi} = \frac{1}{2},$$

and the proof is complete.

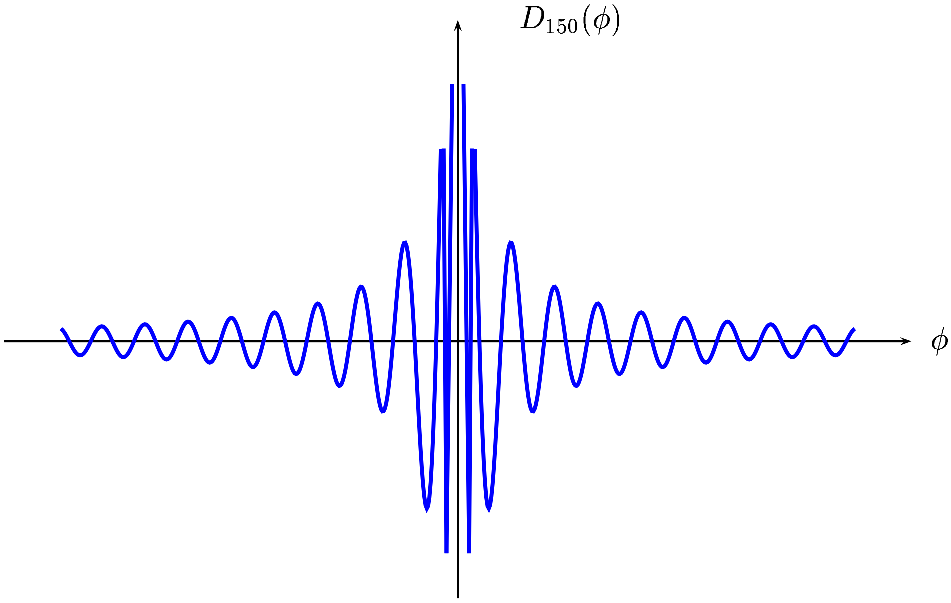


Figure 3.3: The Dirichlet kernel $D_N(\phi)$ for $N = 150$.

Now we return to the proof of our main result:
Proof of the Convergence Thereon: Since f is piecewise smooth we have that $f(\theta) = f(\theta-)$ for $\theta < 0$ and $f(\theta) = f(\theta+)$ for $\theta > 0$. Further using lemma 4 we can write

$$\frac{1}{2}f(\theta-) = f(\theta-)\int_{-\pi}^{0} D_N(\phi)d\phi, \qquad \frac{1}{2}f(\theta+) = f(\theta+)\int_{0}^{\pi} D_N(\phi)d\phi. \quad (3.5.9)$$

By (3.5.6)

$$S_N^f(\theta) = \int_{-\pi}^{0} f(\theta+\phi)D_N(\phi)d\phi + \int_{0}^{\pi} f(\theta+\phi)D_N(\phi)d\phi. \quad (3.5.10)$$

Subtracting (3.5.9) from (3.5.10) we get

$$\begin{aligned} S_N^f(\theta) - \frac{1}{2}\Big[f(\theta-) + f(\theta+)\Big] &= \int_{-\pi}^{0} \Big[f(\theta+\phi) - f(\theta-)\Big] D_N(\phi)d\phi \\ &+ \int_{0}^{\pi} \Big[f(\theta+\phi) - f(\theta+)\Big] D_N(\phi)d\phi \end{aligned} \quad (3.5.11)$$

We now wish to show that for each **fixed** θ this approaches zero as $N \to \infty$. Recalling (3.5.7) we have

$$D_N(\phi) = \frac{1}{2\pi}\frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}$$

We now define $g(\phi)$ by

$$g(\phi) = \begin{cases} \frac{f(\theta+\phi)-f(\theta-)}{e^{i\phi}-1} & \text{for} \quad -\pi < \phi < 0 \\ \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} & \text{for} \quad 0 < \phi < \pi, \end{cases} \tag{3.5.12}$$

and rewrite (3.5.11) as

$$S_N^f(\theta) - \frac{1}{2}\Big[f(\theta-) + f(\theta+)\Big] = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\Big(e^{i(N+1)\phi} - e^{-iN\phi}\Big)d\phi. \tag{3.5.13}$$

We can easily see that $g(\phi)$ is a well-behaved function on $[-\pi, \pi]$. In fact $g(\phi)$ is as smooth as $f(\theta)$, except near $\phi = 0$, where $e^{i\phi} - 1$ vanishes. Using l'Hôpital's rule it follows that

$$\lim_{\phi\to 0+} g(\phi) = \lim_{\phi\to 0+} \frac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1} = \lim_{\phi\to 0+} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta+)}{i}.$$

Similarly $g(\phi)$ approaches the finite limit $i^{-1}f'(\theta-)$ as $\phi \to 0-$. Hence $g(\phi)$ is actually piecewise continuous on $[-\pi, \pi]$, and as a consequence of Bessel's inequality (Lemma 3) its Fourier coefficients

$$C_n(g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{-in\phi}d\phi \longrightarrow 0 \qquad \text{as} \quad n \to \pm\infty. \tag{3.5.14}$$

Now since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{i(N+1)\phi}d\phi = C_{-(N+1)}(g) \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{-iN\phi}d\phi = C_N(g),$$

using (3.5.13) it follows that

$$\lim_{N\to\infty} S_N^f(\theta) - \frac{1}{2}\Big[f(\theta-) + f(\theta+)\Big] = 0.$$

and the proof is complete.

Below we give an alternative approach to the proof of the Convergence Theorem. This approach is based on the following properties of the Fourier coefficients:

Lemma 7. *Let f and g be 2π periodic Riemann integrable (piecewise smooth) function. Then the Fourier coefficient*

$$C_n(f) = C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta,$$

as an operator, is linear, i.e.,

$$C_n(\alpha f + \beta g) = \alpha C_n(f) + \beta C_n(g), \quad \forall\alpha,\, \beta \in \mathbb{R}. \tag{3.5.15}$$

Further we have that

$$C_n(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}d\theta = \delta_{n0}. \tag{3.5.16}$$

This is justified by the fact that for $n = 0$ we have

$$C_0(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta = \frac{1}{2\pi}\Big[\theta\Big]_{-\pi}^{\pi} = 1, \tag{3.5.17}$$

while $n \neq 0$ yields

$$C_n(1) = \frac{1}{2\pi}\Big[-\frac{1}{in}e^{-in\theta}\Big]_{-\pi}^{\pi} = -\frac{1}{2\pi in}\Big[\cos n\theta - i\sin n\theta\Big]_{-\pi}^{\pi} = 0. \tag{3.5.18}$$

Finally

$$C_n(e^{ik\theta}f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}f(\theta)e^{-in\theta}d\theta = c_{n-k}(f) \tag{3.5.19}$$

Proof of the Convergence Theorem, Method II: Suppose that f is continuous at θ_0. Let

$$\textbf{(i)} \quad g(\theta) = \frac{f(\theta) - f(\theta_0)}{e^{i\theta} - e^{i\theta_0}} = \frac{\Big(f(\theta) - f(\theta_0)\Big)/(\theta - \theta_0)}{\Big(e^{i\theta} - e^{i\theta_0}\Big)/(\theta - \theta_0)} \tag{3.5.20}$$

We know that

$$\lim_{\theta\to\theta_0} \frac{f(\theta) - f(\theta_0)}{\theta - \theta_0} = f'(\theta_0) \tag{3.5.21}$$

and

$$\lim_{\theta\to\theta_0} \frac{e^{i\theta} - e^{i\theta_0}}{\theta - \theta_0} = \Big(e^{i\theta}\Big)'\Big|_{\theta=\theta_0} = ie^{i\theta_0} \tag{3.5.22}$$

Thus it follows from (3.5.20)- (3.5.22) that

$$\lim_{\theta\to\theta_0\pm} g(\theta) = \lim_{\theta\to\theta_0\pm} \frac{f'(\theta)}{\left(e^{i\theta}\right)'} = \frac{f'(\theta_0\pm)}{ie^{i\theta_0}}. \tag{3.5.23}$$

Since $f'(\theta)$ is piecewise continuous, $g(\theta)$ is piecewise continuous. Further using the (3.5.20) we get

$$f(\theta) = f(\theta_0) + e^{i\theta}g(\theta) - e^{i\theta_0}g(\theta)$$

Thus by linearity of $C_n(f)$, and the relations (3.5.16) and (3.5.19) (see Lemma 5) we have

$$\begin{aligned} C_n(f(\theta)) &= C_n\Big[f(\theta_0) + e^{i\theta}g(\theta) - e^{i\theta_0}g(\theta)\Big] \\ &= f(\theta_0)C_n(1) + = C_{n-1}(g(\theta)) - e^{i\theta_0}C_n(g(\theta)) \\ &= f(\theta_0)\delta_{n0} + C_{n-1}(g(\theta)) - e^{i\theta_0}C_n(g(\theta)). \end{aligned} \tag{3.5.24}$$

Hence

$$\begin{aligned} S_N^f(\theta_0) &= \sum_{-N}^{N} C_n(f(\theta))e^{in\theta_0} \\ &= \sum_{-N}^{N} \Big[f(\theta_0)\delta_{n0} + C_{n-1}(g(\theta)) - e^{i\theta_0}C_n(g(\theta))\Big]e^{in\theta_0}. \end{aligned} \tag{3.5.25}$$

Evidently we have

$$\sum_{-N}^{N} f(\theta_0)\delta_{n0} = f(\theta_0) \tag{3.5.26}$$

Now to calculate the reaming part of the sum in (3.5.25) , we define

$$r_n = C_{n-1}(g(\theta))e^{in\theta_0}, \tag{3.5.27}$$

and write

$$\begin{aligned} \Big[C_{n-1}(g(\theta)) - e^{i\theta_0}C_n(g(\theta))\Big]e^{in\theta_0} &= C_{n-1}(g(\theta))e^{in\theta_0} - e^{i(n+1)\theta_0}C_n(g(\theta)) \\ &= r_n - r_{n+1} \end{aligned} \tag{3.5.28}$$

Thus we have

$$
\begin{aligned}
S_N^f(\theta) &= f(\theta_0) + \sum_{-N}^{N}[r_n - r_{n+1}] \\
&= f(\theta_0) + (r_{-N} - r_{-N+1}) + \dots + (r_N - r_{N+1}) \\
&= f(\theta_0) + r_{-N} - r_{N+1}
\end{aligned}
\tag{3.5.29}
$$

Finally since g is piecewise continuous, then using Lemma 3,

$$|r_n| = |C_{n-1}(g(\theta))e^{in\theta_0}| = |C_{n-1}(g(\theta)| \to 0, \qquad \text{as} \quad |n| \to \infty. \tag{3.5.30}$$

$$S_N^f(\theta) \to f(\theta_0), \quad \text{as} \quad N \to \infty.$$

and the proof is complete.

3.6 Derivatives, primitive functions

The fundamental theorem of calculus:

$$\int_a^b f'(\theta)d\theta = f(b) - f(a) \tag{3.6.1}$$

applies to functions f that are continuous and piecewise smooth, even though f' is undefined at the "corners". For example, if f is differentiable except at the point $c \in (a, b)$, we have

$$
\begin{aligned}
\int_a^b f'(\theta)d\theta &= \int_a^c f'(\theta)d\theta + \int_c^b f'(\theta)d\theta \\
&= \Big[f(c) - f(a)\Big] + \Big[f(b) - f(c)\Big] = f(b) - f(a).
\end{aligned}
$$

Now we shall show how to relate the Fourier coefficients of a function to those of its derivatives.

Theorem 12. *Suppose f is $2\pi-$periodic, continuous, and piecewise smooth. Let a_n, b_n and C_n be the Fourier coefficients of the function f, and let a'_n, b'_n and C'_n be the corresponding Fourier coefficients of the derivative f'. Then*

$$a'_n = nb_n, \qquad b'_n = -na_n, \quad \text{and} \quad C'_n = inC_n \tag{3.6.2}$$

Note! *a'_n, b'_n and C'_n are not derivatives of a_n, b_n and C_n.*

Proof: Using integration by parts it follows that:

$$\begin{aligned} a'_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\Big[f(\theta)\cos n\theta\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)(-n\sin n\theta) \\ &= n\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = nb_n \end{aligned}$$

where we use the fact that, because of 2π*-periodicity* $f(\pi) = f(-\pi)$. *Similarly*

$$\begin{aligned} b'_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(\theta)\sin n\theta\, d\theta = \frac{1}{\pi}\Big[f(\theta)\sin n\theta\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) n\cos n\theta\, d\theta \\ &= -n\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = -na_n. \end{aligned}$$

As for C'_n *we use once again* $f(-\pi) = f(\pi)$ *and also* $e^{in\pi} = e^{-in\pi} = (-1)^n$ *to obtain*

$$\begin{aligned} C'_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}d\theta = \frac{1}{2\pi}\Big[f(\theta)e^{-in\theta}\Big]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-ine^{-in\theta})d\theta \\ &= 0 + in\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = inC_n \end{aligned}$$

and the proof is complete.

In the next theorem we derive the Fourier series expansion of $f'(\theta)$ in terms of the Fourier coefficients of f and also give the convergence of the f's Fourier series expansion (convergence theorem for f').

Theorem 13. *Suppose that* $f(\theta)$ *is a* $2\pi-$*periodic, continuous, and piecewise smooth function, and that* $f'(\theta)$ *is piecewise smooth. Then the Fourier series expansion of* f' *is obtained by the termwise derivation of the Fourier series expansion for* f, *i.e., if*

$$f(\theta) = \sum_{-\infty}^{\infty} C_n e^{in\theta} = \frac{1}{2}a_0 + \sum_{1}^{\infty}\big(a_n\cos n\theta + b_n\sin n\theta\big),$$

then

$$f'(\theta) = \sum_{-\infty}^{\infty} inC_n e^{in\theta} = \sum_{1}^{\infty}\big(nb_n\cos n\theta - na_n\sin n\theta\big) \qquad (3.6.3)$$

for all θ at which $f'(\theta)$ exists. At the exceptional points where $f'(\theta)$ has jumps, the series (3.6.3) *converges to*

$$\frac{1}{2}\Big[f'(\theta-)+f'(\theta+)\Big]. \tag{3.6.4}$$

Proof: We know that $f(\theta)$,$f'(\theta)$ and $f''(\theta)$ are continuous and piecewise smooth. Using theorem 11 we have that $f'(\theta)$ is the sum of its Fourier series at every point (with appropriate modifications at the jumps). By theorem 12 the coefficients of $e^{in\theta}, \cos n\theta$ and $\sin n\theta$ in this series are inC_n ,nb_n and a_n, and we get the desired result.

Now we link the Fourier coefficients of f and that of its primitive functions F. Note that a periodic function $f(\theta)$ has a periodic integral $F(\theta)$ if and only if the constant a_0 in the Fourier series expansion for $f(\theta)$ is identically 0, i.e., if and only if f's average vanishes:

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta = 0.$$

Exempel 3. *We know that $f(\theta)=1$ is periodic but $F(\theta)=\int f(\theta)d\theta$ is not periodic. However, except the constant term a_0, the integral of every term in a Fourier series expansion is periodic. From this we see that a periodic function has a periodic integral precisely when the constant term in its Fourier series vanishes. We therefore arrive at the following result.*

Theorem 14. *Suppose $f(\theta)$ is $2\pi-$periodic and piecewise continuous, with Fourier coefficients a_n, b_n and C_n, and let $F(\theta)=\int_0^\theta f(\phi)d\phi$. If $C_0=\frac{1}{2}a_0=0$, then by termwise integration of the Fourier expansion for f we have that for all θ*

$$F(\theta) = C_0(F) + \sum_{-\infty, n\neq 0}^{\infty}\frac{C_n}{in}e^{in\theta} = \frac{1}{2}A_0 + \sum_{1}^{\infty}\Big(\frac{a_n}{n}\sin n\theta - \frac{b_n}{n}\cos n\theta\Big) \tag{3.6.5}$$

where the constant term

$$C_0(F) = \frac{1}{2}A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)d\theta$$

is the mean value of F on $[-\pi,\pi]$.

Proof: Since $f(\theta)$ is piecewise continuous, it follows that $F = \int f$ is continuous. $C_0 = \frac{1}{2}a_0 = 0$ gives that F is $2\pi-$periodic since by lemma 2

$$F(\theta + 2\pi) - F(\theta) = \int_{\theta}^{\theta+2\pi} f(\phi)d\phi - \int_{-\pi}^{\pi} f(\phi)d\phi = 2\pi C_0 = 0.$$

Hence by the convergence theorem 11 $F(\theta)$ is the sum of its Fourier series at every θ. But theorem 12 applied to F, yields

$$A_n = -\frac{b_n}{n}, \quad B_n = \frac{a_n}{n} \quad \textit{and} \quad \hat{C}_n := C_n(F) = \frac{C_n}{in} \quad (n \neq 0) \tag{3.6.6}$$

and the proof is complete.

Note! If $C_0 \neq 0$, the above argument can be applied to the function $F(\theta) - C_0\theta$, since then for the function

$$G(\theta) := \int_0^{\theta} \Big(f(\phi) - C_0\Big)d\phi = F(\theta) - C_0\theta,$$

the derivative $G'(\theta) = g(\theta) - C_0$ has, in its Fourier series expansion, the constant term $C_0(g) = 0$. Then

$$G(\theta) = C_0(G) + \sum_{-\infty, n\neq 0}^{\infty} \frac{C_n}{in}e^{in\theta} = \frac{1}{2}A_0 + \sum_{1}^{\infty}\Big(\frac{a_n}{n}\sin n\theta - \frac{b_n}{n}\cos n\theta\Big)$$

where

$$C_0(G) = \frac{1}{2}A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\theta)d\theta.$$

Exempel 4. *$f(\theta)$ is $2\pi-$periodic and piecewise continuous. Give the Fourier series expansion for $F(\theta)$!*

$$f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & -\pi < \theta < 0, \end{cases}$$

Clearly $F(\theta) = |\theta|$ for $|\theta| \leq \pi$. Since f is odd we have $a_n = 0$. Consequently $C_0 = \frac{1}{2}a_0 = 0$. We also have

$$f(0) = \frac{1}{2}\Big[f(0-) + f(0+)\Big] = \frac{1}{2}\Big[-1 + 1\Big] = 0,$$

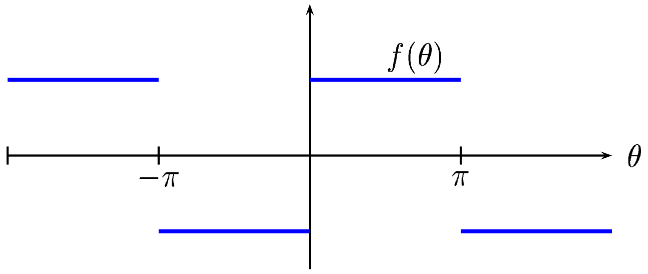


Figure 3.4: The signum function $f(\theta)$.

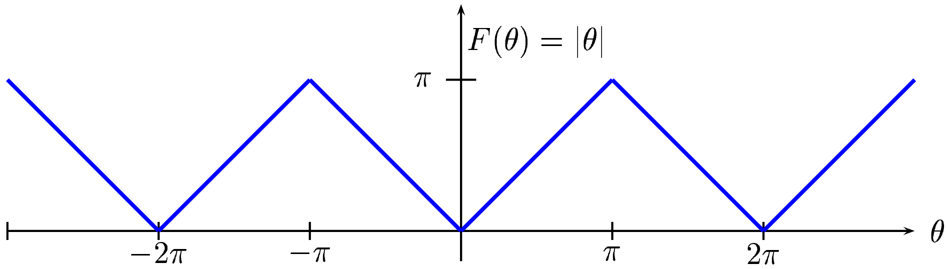


Figure 3.5: The (periodic) primitive function $F(\theta)$.

We compute b_n, $n \geq 1$ *viz,*

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\sin n\theta d\theta.$$

But since $f(\theta) = 1$ *for* $0 < \theta < \pi$ *and we get*

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin n\theta d\theta = -\frac{2}{\pi}\Big[\frac{1}{n}\cos n\theta\Big]_0^{\pi} = -\frac{2}{n\pi}\Big((-1)^n - 1\Big)$$

and

$$b_n = \begin{cases} \frac{4}{(2k-1)\pi} & n = 2k-1 \\ 0 & n = 2k \end{cases}$$

Thus we have

$$f(\theta) \sim \sum_{n=1}^{\infty} b_n \sin n\theta = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin(2n-1)\theta.$$

Since $F(\theta)$ is continuous we now have that the Fourier series for $F(\theta) = |\theta|$, $|\theta| \le \pi$, is

$$\begin{aligned} F(\theta) &= \int_0^\theta f(\phi)d\phi = \frac{4}{\pi}\int_0^\theta \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)\phi d\phi \\ &= C_0 - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} \end{aligned} \tag{3.6.7}$$

where

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| d\theta = \frac{1}{\pi}\int_0^\pi \theta d\theta = \frac{1}{\pi}\left[\frac{\theta^2}{2}\right]_0^\pi = \frac{\pi}{2}.$$

Note that this C_0 is the lower bound of the integral in (3.6.7)*, i.e.,*

$$C_0 = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2},$$

which, as a by-product, gives the sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

3.7 Fourier series on the interval $[-\pi, \pi]$

Fourier series give expansions of periodic functions on the whole real line in terms of trigonometric functions. They can also be used to give expansions of functions defined on a finite interval in terms of trigonometric functions on that interval. We start with the simplest case:

Suppose that $f(\theta)$ is defined on $[0, \pi]$, as in the Figure below.

We want to extend $f(\theta)$ to the whole real line by requiring it to be $2\pi-$ periodic. We have the *even extension*: $f_{even}(\theta)$, of f to $[-\pi, \pi]$ is defined by

$$f_{even}(-\theta) = f(\theta), \qquad \theta \in [0, \pi]$$

and the *odd extension*: $f_{odd}(\theta)$, of f to $[-\pi, \pi]$: defined by

$$f_{odd}(-\theta) = -f(\theta), \qquad \theta \in (0, \pi], \qquad f_{odd}(0) = 0.$$

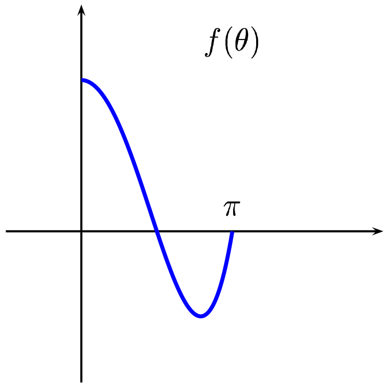


Figure 3.6: A function $f(\theta),\ \theta \in [0, \pi]$.

From lemma 2.2 it follows that for even extensions we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(\theta)\cos n\theta\, d\theta = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\cos n\theta\, d\theta$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(\theta)\sin n\theta\, d\theta = 0.$$

These give the following *Fourier cosine series* of f, if f is an even integrable function on $[0, \pi]$.

$$f_{even}(\theta) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta, \quad \text{where} \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\cos n\theta\, d\theta. \quad (3.7.1)$$

In the same way we have for the odd extension functions

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(\theta)\cos n\theta\, d\theta = 0$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(\theta)\sin n\theta\, d\theta = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\sin n\theta\, d\theta$$

These give the following *Fourier sine series* of f, if f is an odd integrable function on $[0, \pi]$.

$$f_{odd}(\theta) \sim \sum_{n=1}^{\infty} b_n \sin n\theta, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\sin n\theta\, d\theta. \quad (3.7.2)$$

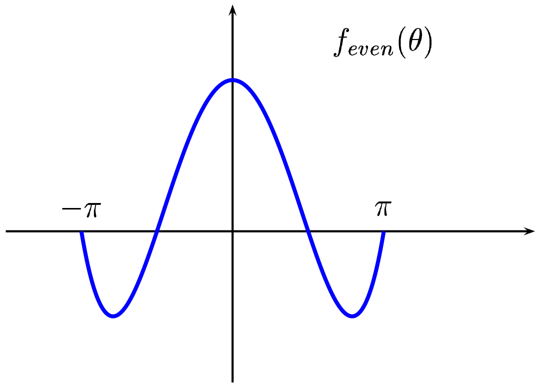


Figure 3.7: The even extension of $f(\theta)$: $f_{even}(\theta),\ \theta \in [-\pi, \pi]$.

3.8 Fourier series on $2L$-intervals

Suppose that $f(x)$ is $2L$-periodic. Making the change of variables:

$$x = \frac{L\theta}{\pi},$$

we get

$$f(x) = f\Big(\frac{L\theta}{\pi}\Big) = g(\theta).$$

Obviously $g(\theta)$ is $2\pi-$periodic, since

$$g(\theta + 2\pi) = f\Big(\frac{L(\theta + 2\pi)}{\pi}\Big) = f\Big(\frac{L\theta}{\pi} + 2L\Big) = f\Big(\frac{L\theta}{\pi}\Big) = g(\theta).$$

If $g(\theta)$ is piecewise smooth we can expand it in 2π-periodic Fourier series:

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \quad \text{where} \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta.$$

Now the substitution

$$\theta = \frac{\pi x}{L}, \quad d\theta = \frac{\pi}{L} dx, \quad \text{and} \quad \theta = \pm\pi \leftrightarrow x = \pm L.$$

yields

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x},$$

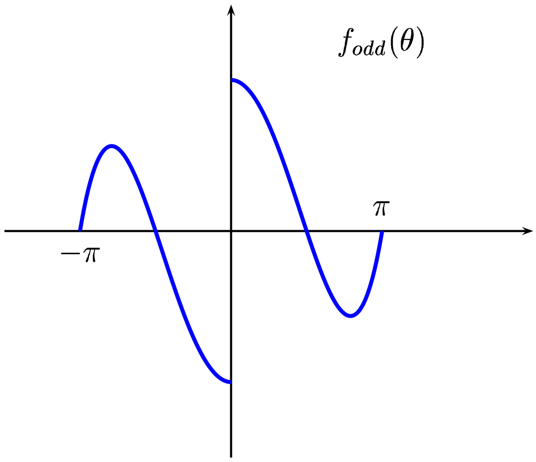


Figure 3.8: The odd extension of $f(\theta)$: $f_{odd}(\theta)$, $\theta \in [-\pi, \pi]$.

where

$$C_n = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{-\frac{in\pi}{L}x}\frac{\pi}{L}dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi}{L}x}dx.$$

Thus we obtain with $\theta = \frac{\pi x}{L}$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \tag{3.8.1}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx. \tag{3.8.2}$$

Thus it follows that the *Fourier cosine expansion* of an even piecewise smooth function f on the interval $[0, L]$ is

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \tag{3.8.3}$$

where

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\,dx \quad \text{and} \quad b_n = 0. \tag{3.8.4}$$

Analogously, the *Fourier sine expansion* of an odd piecewise smooth function f on the interval $[0, L]$ is

$$f(x) = x \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}x, \tag{3.8.5}$$

where

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x\, dx. \tag{3.8.6}$$

Exempel 5. *Find the Fourier cosine expansion of* $f(x) = x$ *on* $[0, L]$.

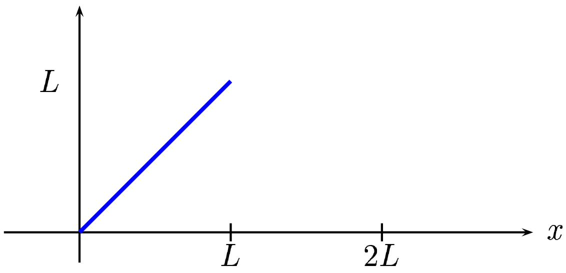


Figure 3.9: The function $f(x) = x,\ x \in [0, L]$.

We expand the function $f(x) = x$ *to an even 2L-periodic function ,* $f_{even} = |x|$*, for* $x \in [-L, L]$*:*

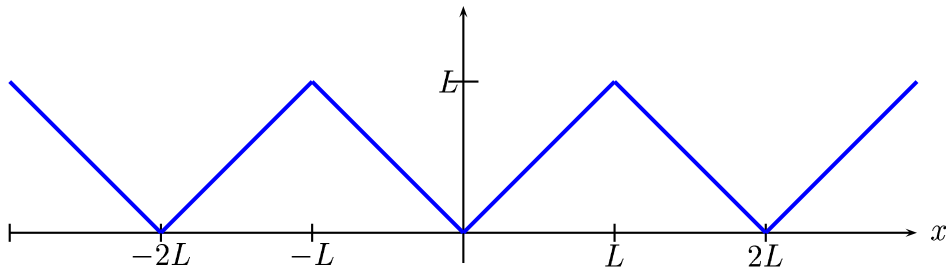


Figure 3.10: The $2L$ periodic even function $f_{even}(x) = |x|,\ x \in [-L, L]$.

Since the expanded function is even, we have that $b_n = 0$*. As for* a_n *we have*

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx,$$

and with $f(x) = x$ *on* $[0, L]$, *we get*

$$a_0 = \frac{2}{L}\int_0^L x\cos 0\,dx = \frac{2}{L}\int_0^L x\,dx = \frac{2}{L}\Big[\frac{x^2}{2}\Big]_0^L = L, \tag{3.8.7}$$

and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = \frac{2}{L}\Big[x\frac{L}{n\pi}\sin\frac{n\pi}{L}x\Big]_0^L \\ &- \frac{2}{L}\int_0^L \frac{L}{n\pi}\sin\frac{n\pi x}{L}dx = \frac{2}{L}\Big(\frac{L}{n\pi}\Big)^2\Big[\cos\frac{n\pi}{L}x\Big]_0^L \\ &= \frac{2L}{n^2\pi^2}\Big((-1)^n - 1\Big) \end{aligned} \tag{3.8.8}$$

Thus we have

$$a_n = \begin{cases} \frac{-4L}{(2k-1)^2\pi^2} & n = 2k-1 \\ 0 & n = 2k. \end{cases} \tag{3.8.9}$$

Note that since $f_{even} = |x|$ *is continuous and piecewise differentiable on* $\mathbb{R}$, *it follows that*

$$f(x) = f_{even}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}x \tag{3.8.10}$$

and thus

$$f(x) = |x| = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{L}x, \qquad x \in \mathbb{R}. \tag{3.8.11}$$

In particular

$$f(x) = x = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{L}x, \qquad x \in [0, L]. \tag{3.8.12}$$

Exempel 6. *Expand the function $f(x) = x$ to a 2L-periodic odd function on $[-L, L]$.*

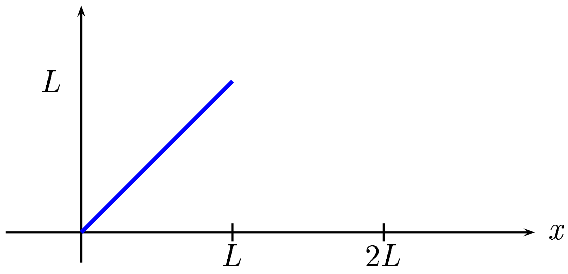


Figure 3.11: The function $f(x) = x, \ x \in [0, L]$.

We expand the function $f(x) = x$ to an odd 2L-periodic function , $f_{odd} = x$, for $x \in [-L, L]$:

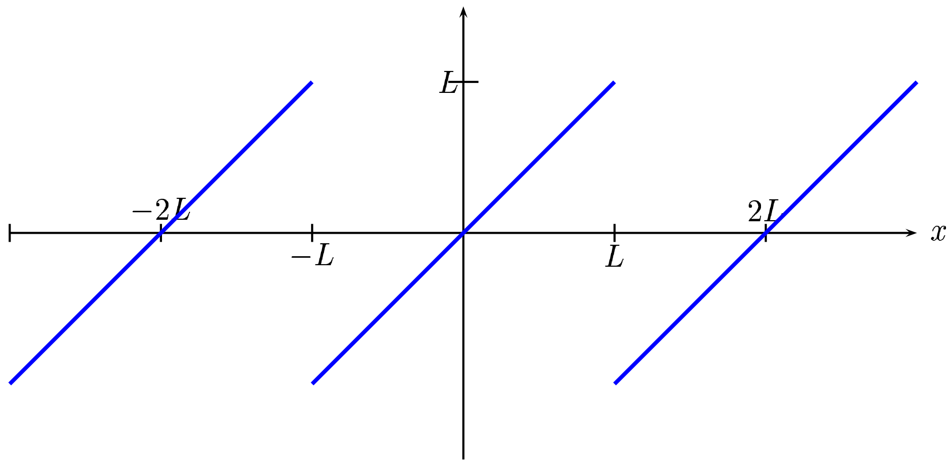


Figure 3.12: The $2L$ periodic odd function $f_{odd}(x) = x, \ x \in [-L, L]$.

Since the expand function is odd, we have $a_n = 0$, and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx,$$

Inserting $f(x) = x$ *on* $[0, L]$ *and using partial integration it follows that*

$$b_n = \frac{2}{L}\int_0^L x \sin\frac{n\pi}{L}x\,dx = \frac{2}{L}\Big[x\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\Big]_0^L - \frac{2}{L}\int_0^L \frac{-L}{n\pi}\cos\frac{n\pi}{L}x\,dx$$
$$= \frac{2}{L}L\frac{-L}{n\pi}\cos(n\pi) + \frac{2}{L}\Big(\frac{L}{n\pi}\Big)^2\Big[sin\frac{n\pi}{L}x\Big]_0^L = \frac{2L}{n\pi}(-1)^{n+1}.$$

Since f *is continuous on* $(0, L)$ *we get*

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}x. \qquad (3.8.13)$$

Thus for $x \in (0, L)$ *we have using* (3.8.13) *and* (3.8.11) *that*

$$x = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}x = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi}{L}x.$$

This equality is valid for $x = 0$ *as well. Thus we have*

$$\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin 0 = 0 \Longleftrightarrow \frac{L}{2} = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos 0.$$

which gives

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

which we recognize also from previous computations.

Chapter 4

The Fourier Transform

Fourier transformation is the most powerful technique for solving differential equations of different type arising in science and engineering. There are a variety of both analytical and numerical approaches rely on Fourier transforms. FFT (Fast Fourier Transform) is , e.g., the backbone of numerical approaches for problems in signal analysis. Besides all the traditional applications the modern technique of wavelet transform is based on (actually is an special version of) the Fourier transform.

.....

4.1 Introduction

We now turn to the study of Fourier transform which is an integral transform, as Laplace transform, defined on the whole real line $\mathbb{R}$, and focused on analyzing functions and deriving relevant techniques to solve differential equations. We start with an analogy with Fourier series viz:

Suppose that f is a function on $\mathbb{R}$. For any $L > 0$ we can expand f on the interval $[-L, L]$ in a Fourier series,

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty} c_{n,L}e^{i\frac{\pi n}{L}x}, \quad \text{where} \quad C_{n,L} = \int_{-L}^{L} f(y)e^{-i\frac{\pi n}{L}y}dy. \qquad (4.1.1)$$

Let

$$\frac{\pi}{L} = \Delta\xi \quad \text{and define} \quad \xi_n := \frac{\pi n}{L} = n\Delta\xi.$$

Then the formulas in (4.1.1) become

$$f(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} c_{n,L}e^{i\xi_n x}\Delta\xi, \quad \text{where} \quad C_{n,L} = \int_{-L}^{L} f(y)e^{-i\xi_n y}dy. \tag{4.1.2}$$

Suppose that $f(x)$ vanishes rapidly as $x \to \pm\infty$, then for sufficiently large L we get

$$C_{n,L} = \int_{-L}^{L} f(y)e^{-i\xi_n y}dy \approx \int_{-\infty}^{\infty} f(y)e^{-i\xi_n y}dy. \tag{4.1.3}$$

Introducing the notation

$$\hat{f}(\xi_n) := \int_{-\infty}^{\infty} f(y)e^{-i\xi_n y}dy, \tag{4.1.4}$$

we have

$$f(x) \approx \frac{1}{2\pi}\sum_{-\infty}^{\infty} \hat{f}(\xi_n)e^{i\xi_n x}\Delta\xi, \quad \text{where} \quad |x| < L. \tag{4.1.5}$$

Let $L \to \infty$, so that $\Delta\xi \to 0$ and the sum in (4.1.5) should turn into an integral, thus:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi, \quad \text{where} \quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx, \tag{4.1.6}$$

$\hat{f}$ is called the *Fourier transform* of f and the formula (4.1.6) is the *Fourier inversion theorem.*

Definition 12. *If f is an integrable function on $\mathbb{R}$, i.e., $f \in L^1(\mathbb{R})$, its Fourier transform is the function $\hat{f}$ on $\mathbb{R}$, defined by*

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx := \mathcal{F}\Big[f(x)\Big](\xi) := \mathcal{F}\Big[f(x)\Big]. \tag{4.1.7}$$

Lemma 8. *The Fourier transform $\hat{f}(\xi)$ is (i) bounded, and (ii) continuous.*

Proof. (i) Since $\hat{f}(\xi)$ is defined for $f \in L^1(\mathbb{R})$, and $|e^{-i\xi x}| = 1$, the integral converges absolutely for all ξ,

$$|\hat{f}(\xi)| = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx \le \int_{-\infty}^{\infty} f(x)dx < \infty \quad \text{where} \quad f \in L^1(\mathbb{R}).$$

(ii) Let $\xi \to \xi_0$. We want to show that $\hat{f}(\xi) \to \hat{f}(\xi_0)$. Since

$$|f(x)e^{i\xi x}| = |f(x)| \quad \forall\ \xi \quad \text{and} \quad f \in L_1(\mathbb{R}),\ \textit{i.e.,}\ \int_{-\infty}^{\infty} f(x)dx < \infty,$$

the *dominating convergence theorem* give us

$$\lim_{\xi \to \xi_0} \hat{f}(\xi) = \int_{-\infty}^{\infty} \lim_{\xi \to \xi_0} f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)e^{-i\xi_0 x}dx = \hat{f}(\xi_0),$$

and the proof is complete. □

4.2 Basic properties of the Fourier transform

Some of the basic properties of the Fourier transform are given in the following theorem.

Theorem 15. *Suppose $f \in L^1$, then*
(a) *For any $a \in \mathbb{R}$, we have*

$$(\mathbf{a1}) \quad \mathcal{F}\Big[(x-a)\Big] = e^{-ia\xi}\hat{f}(\xi) \quad \textit{and} \quad (\mathbf{a2}) \quad \mathcal{F}\Big[e^{ia\xi}f(x)\Big] = \hat{f}(\xi - a).$$

(b) *If $\delta > 0$, then we have the scaling formula:*

$$\mathcal{F}\Big[f(\delta x)\Big](\xi) = \frac{1}{\delta}\hat{f}\Big(\frac{\xi}{\delta}\Big).$$

(c) *If f is continuous and piecewise smooth and $f' \in L^1$, then*

$$(\mathbf{c1}) \qquad \mathcal{F}[f'(x)](\xi) = i\xi\hat{f}(\xi).$$

On the other hand, if $xf(x)$ is integrable, then

$$(\mathbf{c2}) \qquad \mathcal{F}\Big[xf(x)\Big] = i\hat{f}'(\xi).$$

Proof. **(a1)** From the definition we have

$$\mathcal{F}\Big[(x-a)\Big] = \int_{-\infty}^{\infty} f(x-a)e^{-i\xi x}dx.$$

Substituting $x - a = y$ and thus $dx = dy$ we get

$$\mathcal{F}\Big[(x-a)\Big] = \int_{-\infty}^{\infty} f(y)e^{-i\xi(y+a)}dy = e^{-i\xi a}\int_{-\infty}^{\infty} f(y)e^{-i\xi y}dy = e^{-i\xi a}\hat{f}(\xi).$$

(a2) Using the definition it follows that

$$\mathcal{F}\Big[e^{iax}f(x)\Big] = \int_{-\infty}^{\infty} e^{iax}f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)e^{-i(\xi-a)x}dx = \hat{f}(\xi - a).$$

(b) The Fourier transform formula gives

$$\mathcal{F}\Big[f(\delta x)\Big] = \int_{-\infty}^{\infty} f(\delta x)e^{-i\xi x}dx.$$

Substituting $\delta x = y$ and thus $dx = \delta^{-1}dy$ we obtain

$$\mathcal{F}\Big[f(\delta x)\Big] = \int_{-\infty}^{\infty} f(y)e^{-i\frac{\xi}{\delta}y}\frac{1}{\delta}dy = \frac{1}{\delta}\hat{f}\Big(\frac{\xi}{\delta}\Big).$$

(c1) Using partial integration it follows that

$$\mathcal{F}[f'(x)](\xi) = \int_{-\infty}^{\infty} f'(x)e^{-i\xi x}dx = \Big[f(x)e^{-i\xi x}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\xi)e^{-i\xi x}dx.$$

But since $f \in L^1$, the limit

$$\lim_{x\to\infty} f(x) = f(0) + \int_0^{\infty} f'(x)dx$$

exists and since $f' \in L^1$ this limit must be zero. Likewise $\lim_{x\to-\infty} f(x) = 0$. Thus we have

$$\Big[f(x)e^{-i\xi x}\Big]_{-\infty}^{\infty} = 0 \quad \text{and} \quad \mathcal{F}[f'(x)](\xi) = (i\xi)\hat{f}(\xi).$$

(c2) Since $\frac{d}{d\xi}e^{-i\xi x} = (-ix)e^{-i\xi x}$, we may write $xe^{-i\xi x} = i\frac{d}{d\xi}e^{-i\xi x}$. Then we have

$$\mathcal{F}\Big[xf(x)\Big] = \int_{-\infty}^{\infty} xf(x)e^{-i\xi x}dx = i\frac{d}{d\xi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = i\hat{f}'(\xi).$$

□

4.3 Convolutions

In this part we derive one of the most powerfully Fourier transform formulas:f he Fourier transform of the *convolution product* of two functions.

Definition 13. *If f and g are functions on $\mathbb{R}$, their convolution is the function $f * g$ defined by*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \qquad \forall\, x \in \mathbb{R}. \tag{4.3.1}$$

With a change of variables we have evidently

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy. \tag{4.3.2}$$

We can think of the convolution integral as a limit of the Riemann sum:

$$\int_{-\infty}^{\infty} f(x-y)g(y)dy \approx \sum_{j=-\infty}^{\infty} f(x-y_j)g(y_j)\Delta y_j.$$

The function $f_j(x) := f(x-y_j)$ is a translation of f along the x-axis by the amount y_j, so the sum on the Right is a linear combination of translates of f with coefficients $g(y_j)\Delta y_j$. We can therefore think of $f * g$ as a continuous superposition of translates of f.

The weighted average of f on $[a, b]$ with respect to a nonnegative weight function g is

$$\frac{\int_a^b f(y)g(y)dy}{\int_a^b g(y)dy}.$$

Suppose now that $\int_a^b g(y)dy = 1$. If we now use the identity (4.3.2) and write $f * g(x)$ as $\int_{-\infty}^{\infty} f(y)g(x-y)dy$, we see that $f * g(x)$ is the weighted average of f with respect to the weight function $g(x-y)$.

In the next two theorems we state (without proof) some basic algebraic and analytic properties of convolutions.

Theorem 16. *Convolution obeys the same algebraic laws as ordinary multiplication:*
(i) The associative law: $f*(ag+bh) = a(f*g)+b(f*h)$, *for a, b constants.*
(ii) The commutative law: $f * g = g * f$.
(iii) The distributive law: $f * (g * h) = (f * g) * h$.

Theorem 17. *Suppose that f and g are differentiable and the convolutions $f * g$, $f' * g$ and $f * g'$ are well-defined. Then $f * g$ is differentiable and*

$$(f * g)'(x) = (f' * g)(x) = (f * g')(x).$$

Now we can give the proof for the convolution theorem:

Theorem 18 (The convolution theorem). *Suppose that $f,\ g \in L^1$, then*

$$\mathcal{F}[f * g] = (f * g)\hat{} = \hat{f}\hat{g}.$$

Proof. By the definition

$$(f * g)\hat{}(\xi) = \int_{-\infty}^{\infty} (f * g)(x) e^{-i\xi x} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-i\xi x} dy dx.$$

Since $f,\ g \in L^1$ we can use Fubini's theorem to change the order of integration. Substituting also $x - y = z$, it follow that

$$\begin{aligned}
(f * g)\hat{}(\xi) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-i\xi x} dx dy \\
&= \int_{-\infty}^{\infty} g(y) \Big\{ \int_{-\infty}^{\infty} f(z) e^{-i\xi(y+z)}\, dz \Big\} dy \\
&= \Big(\int_{-\infty}^{\infty} g(y)) e^{-i\xi y} dy \Big) \Big(\int_{-\infty}^{\infty} f(z) e^{-i\xi z} dz \Big) = \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

and thus we have

$$(f * g)\hat{}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

and the proof is complete. □

4.4 Some key examples

Exempel 7. *Determine the Fourier transform for the function $f(x) = e^{-|x|}$.*

Solution: Using the definition of the Fourier transform it follows that

$$\begin{aligned}
\mathcal{F}\Big[e^{-|x|}\Big](\xi) &= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\xi x} dx = \int_{-\infty}^{0} e^{(1-i\xi)x} dx + \int_{0}^{\infty} e^{-(1+i\xi)x} dx \\
&= \Big[\frac{e^{(1-i\xi)x}}{1-i\xi}\Big]_{-\infty}^{0} + \Big[\frac{e^{-(1+i\xi)x}}{-(1+i\xi)}\Big]_{0}^{\infty} = \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}.
\end{aligned}$$

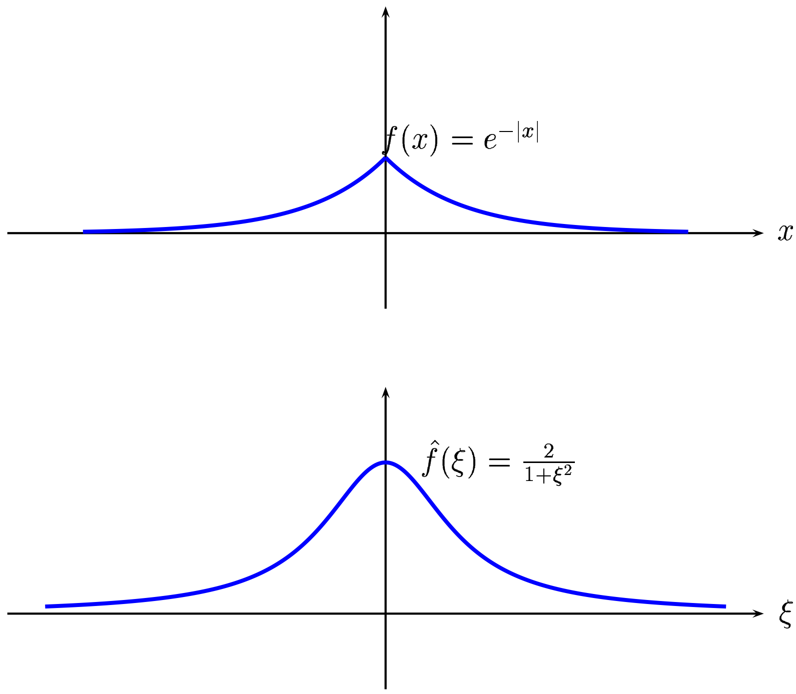


Figure 4.1: The function $f(t) = e^{-|x|}$ and its Fourier transform $\hat{f}(\xi) = \frac{2}{1+\xi^2}$.

Note that, although the graphs for f and $\hat{f}$ have similar profiles, unlike f, $\hat{f}$ is differentiable at zero.

Now using the *scaling formula* (theorem 15b):

$$\mathcal{F}\Big[f(\delta x)\Big](\xi) = \frac{1}{\delta}\hat{f}\Big(\frac{\xi}{\delta}\Big),$$

with $\delta = a$ we get

$$\mathcal{F}\Big[e^{-a|x|}\Big] = \frac{1}{a}\cdot\frac{2}{1+(\xi/a)^2} = \frac{2a}{\xi^2+a^2}, \quad \text{for} \quad a > 0.$$

Next Fourier transform is used deriving several key formulas and deserves a special attention:

Lemma 9. *Let* $f(x) = sign(x) \cdot e^{-a|x|}$*, then* $\hat{f}(\xi) = \frac{-2i\xi}{a^2+\xi^2}$.

Proof. A straightforward calculation yields

$$\begin{aligned}\mathcal{F}\Big[sign(x) \cdot e^{-a|x|}\Big] &= \int_{-\infty}^{\infty} sign(x) \cdot e^{-a|x|} e^{-i\xi x} dx \\ &= \int_{-\infty}^{0} -e^{(a-i\xi)x} dx + \int_{0}^{\infty} e^{-(a+i\xi)x} dx \\ &= \Big[\frac{e^{(a-i\xi)x}}{a-i\xi}\Big]_{-\infty}^{0} + \Big[\frac{e^{-(a+i\xi)x}}{-(a+i\xi)}\Big]_{0}^{\infty} \\ &= \frac{-1}{a-i\xi} + \frac{1}{a+i\xi} = \frac{-2i\xi}{a^2+\xi^2}.\end{aligned} \tag{4.4.1}$$

□

Exempel 8. *Find the Fourier transform for the function* $f(x) = e^{-x^2}$.

Solution: By the definition we have that the Fourier transform for $f(x) = e^{-x^2}$ *is*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-x^2} e^{-i\xi x} dx.$$

It will be easier if we first compute $(\hat{f})'(\xi)$. *Then* $\hat{f}(\xi)$ *will follow easily using theorem 15(c):*

$$\begin{aligned}(\hat{f})'(\xi) &= \int_{-\infty}^{\infty} (-ix) e^{-x^2} e^{-i\xi x} dx \\ &= \Big[\frac{i}{2} e^{-x^2} e^{-i\xi x}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{i}{2} e^{-x^2} (-i\xi) e^{-i\xi x} dx \\ &= -\frac{\xi}{2} \int_{-\infty}^{\infty} e^{-x^2} e^{-i\xi x} dx = -\frac{\xi}{2} \hat{f}(\xi),\end{aligned} \tag{4.4.2}$$

where we used partial integration and the fact that $\Big[\frac{i}{2} e^{-x^2} e^{-i\xi x}\Big]_{-\infty}^{\infty} = 0$. *Consequently we have the differential equation* $\hat{f}'(\xi) + \frac{\xi}{2}\hat{f}(\xi) = 0$*, where solution is* $\hat{f}(\xi) = Ce^{-\frac{\xi^2}{4}}$*, with* $C = \hat{f}(0)$.

Note that for $\xi = 0$,

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad thus \quad C = \sqrt{\pi},$$

and hence

$$\hat{f}(\xi) = \mathcal{F}\left[e^{-x^2}\right](\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}. \tag{4.4.3}$$

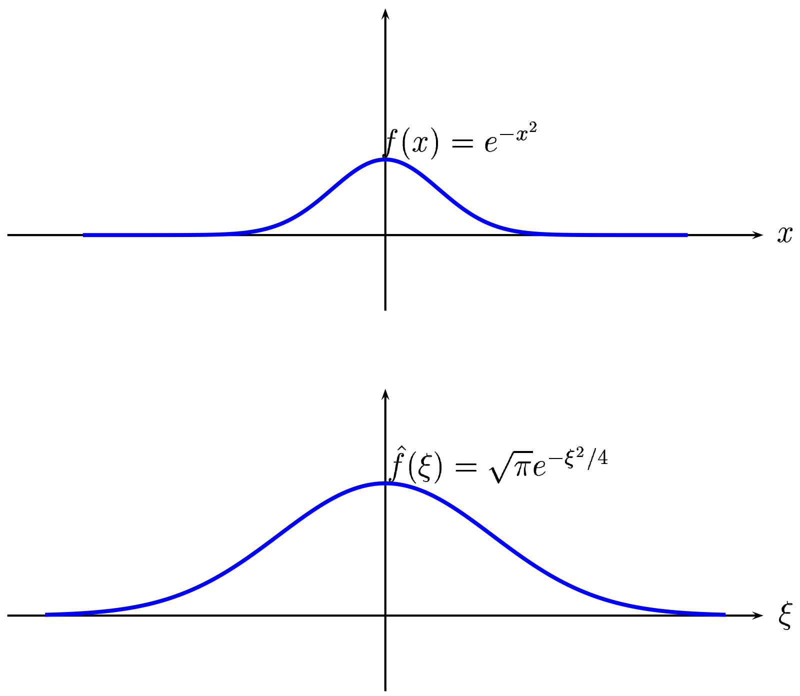


Figure 4.2: $f(x) = e^{-x^2}$ and its Fourier transform $\hat{f}(\xi) = \sqrt{\pi}e^{-\xi^2/4}$.

This means that for a Gaussian distribution f its Fourier transform $\hat{f}$ is equivalent to a scaling of f preserving both its shape and regularity. In particular, as we shall see below, the Fourier transform of $e^{-x^2/2}$ is the same function multiplied by $\sqrt{2\pi}$.

As a consequence of this example we have the following important formula for the Fourier transform of a general Gaussian function:

Lemma 10.

$$\mathcal{F}\left[e^{-\frac{ax^2}{2}}\right](\xi) = \sqrt{\frac{2\pi}{a}}e^{-\frac{\xi^2}{2a}}. \tag{4.4.4}$$

Proof. The proof is straightforward using the scaling formula with $\delta = \sqrt{\frac{a}{2}}$, viz,

$$\mathcal{F}\Big[e^{-\frac{ax^2}{2}}\Big](\xi) = \sqrt{\frac{2}{a}}\sqrt{\pi}e^{-\frac{\left(\xi\sqrt{\frac{2}{a}}\right)^2}{4}} = \sqrt{\frac{2\pi}{a}}e^{-\frac{\xi^2}{2a}}.$$

□

Later on we shall use the above formula with the substituting: $x = \xi$ and $\xi = (x - y)$:

$$\mathcal{F}\Big[e^{-\frac{a\xi^2}{2}}\Big](x-y) = \sqrt{\frac{2\pi}{a}}e^{-\frac{(x-y)^2}{2a}}. \tag{4.4.5}$$

4.5 The Fourier Inversion Theorem

By the Fourier inversion theorem we mean a procedure that justifies recovering f from $\hat{f}$.

Theorem 19 (The Fourier Inversion Theorem). *Suppose $f \in L^1(\mathbb{R})$, f, piecewise continuous, and defined at its points of discontinuity so as to satisfy $f(x) = \frac{1}{2}\Big[f(x-) + f(x+)\Big]$ for all $x \in \mathbb{R}$. Then*

$$f(x) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi. \tag{4.5.1}$$

Moreover, since $\hat{f} \in L^1(\mathbb{R})$, the f is continuous and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}d\xi. \tag{4.5.2}$$

Proof. Note that the *cutoff function* $e^{-\varepsilon^2\xi^2/2}$ in (4.5.1) is just to make the integrals converge, then passing to the limit the cutoff is removed. A straightforward calculation yields

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(y)e^{-i\xi y}e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}dyd\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\Big\{\int_{-\infty}^{\infty}e^{-\frac{\varepsilon^2\xi^2}{2}}e^{-i\xi(y-x)}d\xi\Big\}dy \quad (4.5.3) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\mathcal{F}\Big[e^{-\frac{\varepsilon^2\xi^2}{2}}\Big](y-x)dy.\end{aligned}$$

Now we apply (4.4.5) above with $a = \varepsilon^2$ to get

$$\mathcal{F}\Big[e^{-\frac{\varepsilon^2\xi^2}{2}}\Big](y-x) = \frac{\sqrt{2\pi}}{\varepsilon}e^{-\frac{(x-y)^2}{2\varepsilon^2}}.$$

Replacing in (4.5.3) it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\frac{1}{\varepsilon}e^{-\left(\frac{y-x}{\sqrt{2}\varepsilon}\right)^2}dy. \tag{4.5.4}$$

Substituting $\frac{y-x}{\sqrt{2}\varepsilon} = z$ gives $y = x + \sqrt{2}\varepsilon z$ and $dy = \sqrt{2}\varepsilon dz$. Thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}f(x+\sqrt{2}\varepsilon z)e^{-z^2}dz. \tag{4.5.5}$$

Now since f is bounded we have

$$\Big|f(x+\sqrt{2}\varepsilon z)e^{-z^2}\Big| \le Me^{-z^2} \quad \text{and} \quad \Big|\hat{f}(\xi)e^{i\xi x}e^{-\frac{\varepsilon^2\xi^2}{2}}\Big| \le \Big|\hat{f}(\xi)\Big| \in L^1.$$

Taking limit in both sides of (4.5.5), by Lebesgue dominated convergence theorem, we can pass the limits inside integrals to get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}\Big\{\lim_{\varepsilon\to 0}e^{-\frac{\varepsilon^2\xi^2}{2}}\Big\}\,d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{\varepsilon\to 0}f(x+\sqrt{2}\varepsilon z)e^{-z^2}dz.$$

Hence by the continuity of f it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{i\xi x}d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}f(x)e^{-z^2}dz = f(x)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}dz = f(x),$$

and the proof is complete. □

The Fourier inversion formula can simply be interpreted as a improper integral if f is integrable and piecewise smooth on $\mathbb{R}$. Below, we state this as a theorem (without proof!):

Theorem 20. *If f is integrable and piecewise smooth on $\mathbb{R}$, then*

$$\lim_{r\to\infty}\int_{-r}^{r}e^{i\xi x}\hat{f}(\xi)d\xi = \frac{1}{2}\Big[f(x-)+f(x+)\Big], \tag{4.5.6}$$

for every $x \in \mathbb{R}$.

4.6 Plancherel Theorem

Below we show that the Fourier transform preserves the inner products up to a factor of 2π.

Theorem 21. *Suppose that f,$\hat{f}$,g and $\hat{g}$ are in L^1. Then*

$$2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle. \tag{4.6.1}$$

Proof. Using the Fourier inversion theorem for g:

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\xi)e^{i\xi x}d\xi,$$

and the definition of the inner product yields

$$\begin{aligned}
2\pi\langle f, g\rangle &= 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} \overline{\hat{g}(\xi)}e^{-i\xi x}d\xi \\
&= \int_{-\infty}^{\infty} \overline{\hat{g}(\xi)}\Big\{\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\Big\}d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \langle \hat{f}, \hat{g}\rangle,
\end{aligned}$$

where we used the fact that since $f, \hat{g} \in L^1$, and the proof is complete. □

Remark. The definition of the Fourier transform can be developed to arbitrary L^2-functions. If $f, g, \hat{f}$ and $\hat{g}$ are in L^1, then $f, g, \hat{f}$ and $\hat{g}$ are also in L^2.

Because of our interest in L_2 spaces we formulate the following result:

Theorem 22 (The Plancherel Theorem). *The Fourier transform, defined originally on $L^1 \cap L^2$, extends uniquely to a map on L^2 satisfying*

$$2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \quad \textit{for all} \quad f, g \in L^2.$$

As a consequence of the Plancherel theorem we have
The Parsevals formula: For $f = g \in L^2$ we have that

$$2\pi\int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2d\xi,$$

or

$$2\pi\|f(x)\|_{L^2}^2 = \|\hat{f}(\xi)\|_{L^2}^2. \tag{4.6.2}$$

4.7 The symmetry rule

In this part we derive the symmetry rule which links, in a most simple way, a function f with its Fourier transform $\hat{f}$ so that knowing one of f or $\hat{f}$ the other one will follow using this rule:

Theorem 23. *The Fourier inversion theorem can be formulated as a symmetry in the following way: If* $\varphi(\xi) = \hat{f}(\xi)$, *where* $\hat{\varphi}(x) = \hat{\hat{f}}(x)$, *we have*

$$\hat{\varphi}(-x) = 2\pi f(x) \quad or \quad \hat{\varphi}(x) = 2\pi f(-x). \tag{4.7.1}$$

Proof. In the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}d\xi.$$

we substitute $x = -t$. Then we get

$$f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-i\xi t}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\xi)e^{-i\xi t}d\xi = \hat{\varphi}(t).$$

Thus, with$t = x$, we have

$$f(-x) = \frac{1}{2\pi}\hat{\varphi}(x) \quad \text{i.e.,} \quad \hat{\varphi}(x) = 2\pi f(-x).$$

The formulas in (4.7.1) are often given in transform variable ξ, viz

$$\hat{\varphi}(\xi) = 2\pi f(-\xi).$$

□

Remark. All the formulas and rules, including *the symmetry rule* are also valid for the L^2 case.

Applications of the symmetry rule

Exempel 9. *In this example we derive the following formula for the Fourier transform of the cut-off function:*

$$\chi_a(x) = \theta(x+a) - \theta(x-a) \Longleftrightarrow \hat{\chi}_a(\xi) = \frac{2\sin(a\xi)}{\xi}. \tag{4.7.2}$$

Solution: We have that

$$\hat{\chi}_a(\xi) = \int_{-\infty}^{\infty} \chi_a(x))e^{-i\xi x} d\xi.$$

Since $\chi_a(x) = 0$ for $|x| > a$ and $\chi_a(x) = 1$ for $|x| < a$ we get

$$\hat{\chi}_a(\xi) = \int_{-a}^{a} e^{-i\xi x} d\xi = \Big[\frac{1}{-i\xi} e^{-i\xi x}\Big]_{-a}^{a} = \frac{1}{i\xi}\Big(e^{i\xi x} - e^{-i\xi x}\Big) = \frac{2\sin(a\xi)}{\xi}.$$

Using the symmetry rule we get the following formulas:

$$\mathcal{F}\Big[\frac{2\sin(ax)}{x}\Big](\xi) = 2\pi\chi_a(-\xi)$$

and since $\chi_a(-\xi) = \chi_a(\xi)$ thus

$$\mathcal{F}\Big[\frac{\sin(ax)}{x}\Big](\xi) = \pi\chi_a(\xi). \tag{4.7.3}$$

Exempel 10. *Recalling som of our key examples:*

$$\mathcal{F}\Big[e^{-|x|}\Big] = \frac{2}{1+\xi^2} \quad \text{and} \quad \mathcal{F}\Big[e^{-a|x|}\Big] = \frac{2a}{\xi^2 + a^2}.$$

The symmetry rule give us

$$\mathcal{F}\Big[\frac{2}{1+x^2}\Big] = 2\pi e^{-|-\xi|} = 2\pi e^{-|\xi|} \Longrightarrow \mathcal{F}\Big[\frac{1}{1+x^2}\Big] = \pi e^{-|\xi|}. \tag{4.7.4}$$

Similarly, by the symmetry rule

$$\mathcal{F}\Big[\frac{2a}{x^2+a^2}\Big] = 2\pi e^{-a|\xi|} \Longrightarrow \mathcal{F}\Big[\frac{1}{x^2+a^2}\Big] = \Big(\frac{\pi}{a}\Big)e^{-a|\xi|}. \tag{4.7.5}$$

Exempel 11. *Since*

$$\mathcal{F}\Big[e^{-\frac{ax^2}{2}}\Big](\xi) = \sqrt{\frac{2\pi}{a}} e^{-\frac{\xi^2}{2a}}.$$

by the symmetry rule

$$\mathcal{F}\Big[\sqrt{\frac{2\pi}{a}} e^{-\frac{x^2}{2a}}\Big](\xi) = 2\pi e^{-\frac{a\xi^2}{2}} \tag{4.7.6}$$

The symmetry rule in general

If $\hat{f}(\xi)$ is a Fourier transform of $f(x)$, $f(x) \supset^{\mathcal{F}} \hat{f}(\xi) := g(\xi)$, then

$$g(\xi) \supset^{\mathcal{F}} 2\pi f(-x) = \hat{g}(x) \quad \text{i.e.} \quad g(x) \supset^{\mathcal{F}} 2\pi f(-\xi) = \hat{g}(\xi).$$

Exempel 12. *Recalling the formula*

$$\chi_a(x) \supset^{\mathcal{F}} \frac{2\sin(a\xi)}{\xi},$$

by the Fourier inversion theorem we have

$$\chi_a(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\xi)}{\xi} e^{i\xi x} d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\xi)}{\xi} e^{i\xi x} d\xi.$$

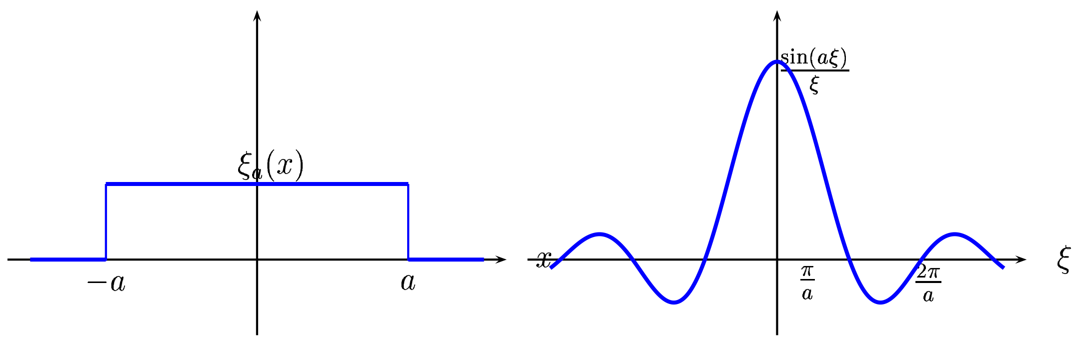


Figure 4.3: $f(x) = \chi_a(x)$ and its Fourier transform $\hat{f}(\xi) = \frac{\sin(a\xi)}{\xi}$.

Let $a = 1$ and $x = 0$. Then we get the following important result:

$$\int_{-\infty}^{\infty} \frac{\sin(\xi)}{\xi} d\xi = \pi. \tag{4.7.7}$$

The Fourier inversion formula can be interpreted as a (principal value) improper integral, that is:

$$f(x) = \lim_{r\to\infty} \int_{-r}^{r} e^{i\xi x} \hat{f}(\xi) d\xi,$$

which give us the following theorem.

Theorem 24. *Suppose $f \in L^1(\mathbb{R})$, is continuous at x_0, and has both right and left derivatives, $f'(x_0^+)$ and $f'(x_0^-)$. Then*

$$f(x_0) = \lim_{r\to\infty} \int_{-r}^{r} \hat{f}(\xi) e^{i\xi x_0} d\xi.$$

Proof. Let

$$g(x) = \frac{f(x) - f(x_0)e^{-\frac{1}{2}(x-x_0)^2}}{x - x_0}. \tag{4.7.8}$$

By the L'Hôpital's rule

$$\lim_{x\to x_0^+} g(x) = f'(x_0^+) + \lim_{x\to x_0^+} f(x_0)(x-x_0)e^{-\frac{1}{2}(x-x_0)^2} = f'(x_0^+).$$

Similarly $\lim_{x\to x_0^-} g(x) = f'(x_0^-)$. Thus $g(x)$ is bounded in a neighborhood of x_0 and both $g(x)$ and $xg(x) = x_0 g(x) + f(x) - f(x_0)(x-x_0)e^{-\frac{1}{2}(x-x_0)^2}) \in L^1(\mathbb{R})$. Further, by (4.7.8)

$$f(x) = f(x_0)e^{-\frac{1}{2}(x-x_0)^2} + xg(x) - x_0 g(x). \tag{4.7.9}$$

Let $h(x) = e^{-\frac{1}{2}(x-x_0)^2}$ then by the Fourier transform rules: $\hat{h}(\xi - a) = \hat{h}(\xi)e^{-ix_0\xi}$. Further $\mathcal{F}\Big[xg(x)\Big] = i\hat{g}'(\xi)$. Now recalling $e^{-\frac{x^2}{2}} \supset^{\mathcal{F}} \sqrt{2\pi}e^{-\frac{\xi^2}{2}}$, the Fourier transform of $f(x)$, see (4.7.9), is

$$\hat{f}(\xi) = f(x_0)\sqrt{2\pi}e^{-\frac{\xi^2}{2}}e^{-ix_0\xi} + i\hat{g}'(\xi) - x_0\hat{g}(\xi). \tag{4.7.10}$$

Multiplying (4.7.10) by $e^{i\xi x_0}$ and integrating over $[-r, r]$ we get

$$\begin{aligned} \frac{1}{2\pi}\int_{-r}^{r} \hat{f}(\xi)e^{i\xi x_0}d\xi &= f(x_0)\frac{1}{\sqrt{2\pi}}\int_{-r}^{r} e^{-\frac{\xi^2}{2}}d\xi \\ &\quad + \frac{i}{2\pi}\int_{-r}^{r} \hat{g}'(\xi)e^{i\xi x_0}d\xi - \frac{x_0}{2\pi}\int_{-r}^{r} \hat{g}(\xi)e^{i\xi x_0}d\xi. \end{aligned} \tag{4.7.11}$$

Partial integration in the second term on the right hand side yields

$$\begin{aligned} \frac{1}{2\pi}\int_{-r}^{r} \hat{f}(\xi)e^{i\xi x_0}d\xi &= f(x_0)\frac{1}{\sqrt{2\pi}}\int_{-r}^{r} e^{-\frac{\xi^2}{2}}d\xi + \frac{i}{2\pi}\Big[\hat{g}(\xi)e^{i\xi x_0}\Big]_{-r}^{r} \\ &\quad - \frac{i}{2\pi}\int_{-r}^{r} \hat{g}(\xi)(ix_0)e^{i\xi x_0}d\xi - \frac{x_0}{2\pi}\int_{-r}^{r} \hat{g}(\xi)e^{i\xi x_0}d\xi, \end{aligned} \tag{4.7.12}$$

where the last two terms are identical, and hence

$$\frac{1}{2\pi}\int_{-r}^{r}\hat{f}(\xi)e^{i\xi x_0}d\xi = f(x_0)\frac{1}{\sqrt{2\pi}}\int_{-r}^{r}e^{-\frac{\xi^2}{2}}d\xi + \frac{1}{2\pi}\Big[\hat{g}(r)e^{irx_0} - \hat{g}(-r)e^{-irx_0}\Big].$$

Riemann-Lebesgue's lemma give that for $g \in L^1$, $\hat{g}(\xi) \to 0$, as $\xi \to \pm\infty$ and since $\int_{-r}^{r} e^{-\frac{\xi^2}{2}}d\xi = \sqrt{2\pi}$, we finally get

$$\lim_{r\to\infty}\frac{1}{2\pi}\int_{-r}^{r}\hat{f}(\xi)e^{i\xi x_0}d\xi = f(x_0),$$

and the proof is complete. □

4.8 Applications of Fourier transform

Partial differential equations

We now use the Fourier transform to solve problems on unbounded regions. The Fourier transform converts differentiation into a simple algebraic operation and we can reduce partial differential equations to easily solvable ordinary differential equations.

Exempel 13. *Consider the heat flow in an infinitely long road, given the initial temperature $u(x,0) = f(x)$:*

$$u_t = ku_{xx}, \quad t > 0, \quad -\infty < x < \infty. \tag{4.8.1}$$

Solution: To find the temperature $u(x,t)$, let $\hat{u}(\xi,t) = \mathcal{F}_x\Big[u(x,t)\Big](\xi)$. Then

$$\mathcal{F}[u_t](\xi) = \int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{-i\xi x}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}u(x,t)e^{-i\xi x}dx = \frac{\partial \hat{u}}{\partial t}.$$

Further $\mathcal{F}[u_x](\xi) = i\xi\hat{u}(\xi)$ gives that $\mathcal{F}[u_{xx}](\xi) = (i\xi)^2\hat{u}(\xi) = -\xi^2\hat{u}(\xi)$. Hence the Fourier transform of (4.8.1) *yields*

$$\frac{\partial \hat{u}}{\partial t} = -k\xi^2\hat{u}(\xi), \tag{4.8.2}$$

with the general solution

$$\hat{u}(\xi,t) = Ce^{-k\xi^2 t}. \tag{4.8.3}$$

Fourier transform of the the initial data $u(\xi, 0) = f(\xi)$*:* $\hat{u}(\xi, 0) = \hat{f}(\xi)$*, inserted in* (4.8.3) *give* $\hat{u}(\xi, 0) = C = \hat{f}(\xi)$*. Thus we have*

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-k\xi^2 t}. \tag{4.8.4}$$

To recover the solution u *we recall that* $\mathcal{F}\Big[e^{-\frac{ax^2}{2}}\Big](\xi) = \sqrt{\frac{2\pi}{a}}e^{-\frac{\xi^2}{2a}}$*. Letting* $\frac{1}{2a} = kt$ *thus* $a = \frac{1}{2kt}$*, we then have*

$$\mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi) = \sqrt{4\pi kt} \cdot e^{-k\xi^2 t}, \quad \textit{hence} \quad e^{-k\xi^2 t} = \frac{1}{\sqrt{4\pi kt}}\mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi).$$

Inserting in (4.8.4) *we get*

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{4\pi kt}}\hat{f}(\xi)\mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi) := \frac{1}{\sqrt{4\pi kt}}\hat{g}(\xi)\hat{f}(\xi), \tag{4.8.5}$$

where $\hat{g}(\xi) := \mathcal{F}\Big[e^{-\frac{x^2}{4kt}}\Big](\xi)$*. Using the convolution theorem:* $\hat{f}\hat{g} = (f * g)\hat{}$ *it follows that*

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}}(f * g)(x) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)dy. \tag{4.8.6}$$

Exempel 14. *Solve the Poisson's equation,*

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \tag{4.8.7}$$

where the boundary condition, $u(x, 0) = f(x)$*, is bounded.*

Solution: As in the previous example the Fourier transform of the equation and the boundary, with respect to x*, yields to the following ordinary differential equation in* y*;*

$$-\xi^2\hat{u}(\xi, y) + \frac{\partial^2\hat{u}(\xi, y)}{\partial y^2} = 0 \quad \textit{and} \quad \hat{u}(\xi, 0) = \hat{f}(\xi), \tag{4.8.8}$$

with the general solution given by

$$\hat{u}(\xi, y) = C_1(\xi)e^{|\xi|y} + C_2(\xi)e^{-|\xi|y}. \tag{4.8.9}$$

By the boundedness requirement we have that $C_1(\xi) = 0$*. Moreover using the Fourier transform of the boundary data from* (4.8.8) *we get* $\hat{u}(\xi, 0) = C_2(\xi) = \hat{f}(\xi)$*. Thus*

$$\hat{u}(\xi, y) = \hat{f}(\xi)e^{-|\xi|y}. \tag{4.8.10}$$

To take the inverse transform, in this case, the appropriate Fourier transform formula is:

$$\mathcal{F}\Big[\frac{1}{x^2+a^2}\Big] = \frac{\pi}{a}e^{-a|\xi|}, \quad \textit{where} \quad a > 0. \tag{4.8.11}$$

Choosing $a = y$ *in* (4.8.11) *we get*

$$\mathcal{F}\Big[\frac{y}{\pi}\cdot\frac{1}{x^2+y^2}\Big] = \frac{\pi}{y}\cdot e^{-|\xi|y}. \tag{4.8.12}$$

Thus the inverse transform of (4.8.10) *is*

$$u(x,y) = f(x) * \frac{y}{\pi}\cdot\frac{1}{x^2+y^2} = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-s)}{s^2+y^2}\,ds, \tag{4.8.13}$$

which is the Poisson integral formula for the solution the given problem.

Remark. This solution make sense since the Poisson kernel $\frac{y}{\pi(x^2+y^2)} \in L^1$ and $f(x)$ is bounded, $|f(x)| \leq M$. Thus we have

$$|u(x,y)| \leq M\cdot\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{s^2+y^2}ds = \frac{M}{\pi}\arctan\Big(\frac{s}{y}\Big)_{-\infty}^{\infty} = M.$$

Exempel 15. *Solve the Dirichlet problem*

$$u_{xx} + u_{yy} = 0, \quad x > 0 \qquad y > 0, \quad \textit{where} \tag{4.8.14}$$

$$u(0,y) = 0, \quad u(x,0) = \frac{x}{x^2+1} \quad \textit{and} \quad u(x,y) \quad \textit{is bounded.} \tag{4.8.15}$$

Solution: First we solve the following full range (in x) problem:

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \qquad y > 0, \quad \textit{where} \tag{4.8.16}$$

$$u(x,0) = \frac{x}{x^2+1} \quad \textit{and} \quad u(x,y) \quad \textit{is bounded.} \tag{4.8.17}$$

In this case since $\frac{x}{x^2+1}$ *is odd then* $u(x,y)$ *is odd in* x *and we have automatically the condition* $u(0,y) = 0$. *Now we recall the formula*

$$signx\cdot e^{-a|x|} \supset^{\mathcal{F}} \frac{-2i\xi}{a^2+\xi^2}.$$

By the symmetry rule we get

$$\frac{x}{a^2+x^2} \supset^{\mathcal{F}} -\pi i \cdot sign(\xi) \cdot e^{-a|\xi|}. \tag{4.8.18}$$

Thus for $a = 1$ we have

$$\frac{x}{1+x^2} \supset^{\mathcal{F}} -\pi i \cdot sign(\xi) \cdot e^{-|\xi|}.$$

hence

$$u(x,0) := f(x) \supset^{\mathcal{F}} -\pi i \cdot sign(\xi) \cdot e^{-a|\xi|}.$$

Now the Fourier transform of the solution $\hat{u}(\xi, y) = \hat{f}(\xi)e^{-|\xi|y}$, (see previous example), can be written as

$$\hat{u}(\xi, y) = -\pi i \cdot sign(\xi) \cdot e^{-|\xi|} e^{-y|\xi|} = -\pi i \cdot sign(\xi) \cdot e^{-(1+y)|\xi|}.$$

Thus with $a = 1 + y$ in (4.8.18) we finally get

$$u(x,y) = \frac{x}{x^2+(1+y)^2}. \tag{4.8.19}$$

4.9 Sturm-Liouville problems on $[0, \infty)$.

Solving PDEs by the separation of variables technique we encountered the Sturm-Liouville problems. Here we study *a singular* Sturm-Liouville problem in $\mathbb{R}$:

$$X''(x) + \xi^2 X(x) = 0, \qquad -\infty < x < \infty, \quad \text{where } \xi \in \mathbb{R}, \tag{4.9.1}$$

where the general solution $X(x) = c_1 e^{i\xi x} + c_2 e^{-i\xi x} \notin L_2(\mathbb{R})$ and therefore we do not have an orthogonal basis of eigenfunctions. Instead a function $f \in L_2(\mathbb{R})$ can be expanded, in terms of these eigenfunctions, by the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\,d\xi = \frac{1}{2\pi}\left[\hat{f}(\xi)e^{i\xi x} + \hat{f}(-\xi)e^{-i\xi x}\right]d\xi. \tag{4.9.2}$$

Similarly for the half-line problems:

$$X''(x) + \xi^2 X(x) = 0, \qquad 0 < x < \infty, \quad X'(0) = 0; \tag{4.9.3}$$

and

$$X''(x) + \xi^2 X(x) = 0, \qquad 0 < x < \infty, \quad X(0) = 0, \tag{4.9.4}$$

the corresponding multipliers $\cos \xi x$ and $\sin \xi x$, respectively, are not forming orthogonal basis in $L_2(0, \infty)$. So again, for an arbitrary function $f \in L_2(0, \infty)$, we seek Fourier type expansion formulas viz,

$$f(x) = \int_0^\infty a(\xi) \cos \xi x \, d\xi, \qquad f(x) = \int_0^\infty b(\xi) \sin \xi x \, d\xi.$$

Here the idea is to employ even and odd extensions of f to $\mathbb{R}$.

Indeed, if $f \in L^1(\mathbb{R}$ and f is *even*, then

$$\hat{f}(\xi) = \int_{-\infty}^\infty f(x)(\cos \xi x - i \sin \xi x) \, dx = 2 \int_0^\infty f(x) \cos \xi x \, dx.$$

Clearly $\hat{f}$ is even and the inversion formula give us

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(\xi))(\cos \xi x + i \sin \xi x) \, d\xi = \frac{1}{\pi} \int_0^\infty \hat{f}(\xi) \cos \xi x \, d\xi.$$

Similarly if f is *odd* so is $\hat{f}$, and hence

$$\hat{f}(\xi) = -2i \int_0^\infty f(x) \sin \xi x \, dx, \qquad f(x) = \frac{i}{\pi} \int_0^\infty \hat{f}(\xi)) \sin \xi x \, d\xi.$$

Definition 14. *Let $f \in L^1(0, \infty)$. Then the Fourier cosine transform and Fourier sine transform of f are the functions $\mathcal{F}_c[f](\xi)$ and $\mathcal{F}_s[f](\xi)$ on $[0, \infty)$ defined by*

$$\mathcal{F}_c[f](\xi) = \int_0^\infty f(x) \cos \xi x \, dx \quad \text{and} \quad \mathcal{F}_s[f](\xi) = \int_0^\infty f(x) \sin \xi x \, dx. \tag{4.9.5}$$

Thus, if f_{even} and f_{odd} are the even and odd extensions of f to $\mathbb{R}$, then $\mathcal{F}_c[f](\xi)$ and $\mathcal{F}_s[f](\xi)$ are restrictions to $[0, \infty)$ of $\frac{1}{2}\hat{f}_{even}$ and $\frac{i}{2}\hat{f}_{odd}$, since

$$\hat{f}_{even}(\xi) = 2 \int_0^\infty f_{even}(x) \cos \xi x \, dx = 2\mathcal{F}_c[f](\xi),$$

$$\hat{f}_{odd}(\xi) = -2i \int_0^\infty f_{odd}(x) \sin \xi x \, dx = -2i\mathcal{F}_s[f](\xi) = \frac{2}{i}\mathcal{F}_s[f](\xi).$$

The *inversion formulas* therefore become

$$f(x) = \frac{2}{\pi}\int_0^\infty \mathcal{F}_c[f](\xi)\cos\xi x\,d\xi = \frac{2}{\pi}\int_0^\infty \mathcal{F}_s[f](\xi)\sin\xi x\,d\xi.$$

Plancherel Theorem for $\mathcal{F}_c[f]$ and $\mathcal{F}_s[f]$.

Using the above relations it follows that the norm of $\mathcal{F}_c[f](\xi)$ on $[0,\infty)$ is given by

$$\|\mathcal{F}_c[f]\|^2_{L^2([0,\infty))} = \int_0^\infty \left|\frac{1}{2}\hat{f}_{even}(\xi)\right|^2 d\xi = \frac{1}{4}\cdot\frac{1}{2}\int_{-\infty}^\infty |\hat{f}_{even}(\xi)|^2 d\xi,$$

i.e.,

$$\|\mathcal{F}_c[f]\|^2_{L^2[0,\infty)} = \frac{1}{8}\|\hat{f}_{even}(\xi)\|^2_{L^2(-\infty,\infty)}. \tag{4.9.6}$$

Recalling the Parsevals formula: $\|\hat{f}(\xi)\|^2_{L^2(-\infty,\infty)} = 2\pi\|f(x)\|^2_{L^2(-\infty,\infty)}$, the relation (4.9.6) is written as

$$\|\mathcal{F}_c[f]\|^2_{L^2[0,\infty)} = \frac{\pi}{4}\int_{-\infty}^\infty |f_{even}(x)|^2\,dx = \frac{\pi}{2}\|f_{even}\|^2. \tag{4.9.7}$$

Similarly,

$$\|\mathcal{F}_s[f]\|^2 = \frac{\pi}{2}\int_0^\infty |f_{odd}(x)|^2\,dx = \frac{\pi}{2}\|f_{odd}\|^2. \tag{4.9.8}$$

We summarize the relation (4.9.7) and (4.9.8) in the:

Theorem 25 (Plancherel Theorem for cos and sin transforms). *$\mathcal{F}_c[f]$ and $\mathcal{F}_s[f]$ extend to maps from $L^2(0,\infty)$ onto itself that satisfy*

$$\|\mathcal{F}_c[f]\|^2 = \|\mathcal{F}_s[f]\|^2 = \frac{\pi}{2}\|f\|^2.$$

Exempel 16. *Use the Fourier sine transform to find a bounded solution $u(x,y)$ for the problem:*

$$u_{xx} + u_{yy} = 0, \qquad x > 0, \qquad y > 0, \tag{4.9.9}$$

with the boundary conditions

$$u(0,y) = 0, \quad \text{and} \quad u(x,0) = \frac{x}{x^2+1}.$$

Solution: The Fourier sine expansion of $u(x, y)$, *in* x *is:*

$$u(x, y) = \frac{2}{\pi}\int_0^\infty v(\xi, y)\sin\xi x \, d\xi, \quad \textit{where} \tag{4.9.10}$$

$$v(\xi, y) = \mathcal{F}_s[u(x, y)](\xi) = \int_0^\infty u(x, y)\sin\xi x \, dx.$$

Differentiating (4.9.10) *with respect to* x *yields*

$$u_x = \frac{d}{dx}\Big(\frac{2}{\pi}\int_0^\infty v(\xi, y)\sin\xi x \, d\xi\Big) = \frac{2}{\pi}\int_0^\infty v(\xi, y)\xi\cos\xi x \, d\xi$$

and hence

$$u_{xx} = \frac{2}{\pi}\int_0^\infty v(\xi, y)(-\xi^2)\sin\xi x \, d\xi. \tag{4.9.11}$$

Thus the equation (4.9.9) *can be written as*

$$u_{xx} + u_{yy} = \frac{2}{\pi}\int_0^\infty \Big[v_{yy}(\xi, y) - \xi^2 v(\xi, y)\Big]\sin\xi x \, d\xi = 0.$$

So that we get the differential equation $v_{yy} - \xi^2 v = 0$ *with the general solution*

$$v(\xi, y) = C_1(\xi)e^{|\xi|y} + C_2(\xi)e^{-|\xi|y}.$$

Since we seek a bounded solution $u(x, y)$ *thus* $v(\xi, y)$ *must be bounded and hence* $C_1 = 0$. *Further,*

$$v(\xi, 0) = \mathcal{F}_s[u(x, 0)](\xi) = \mathcal{F}_s\Big[\frac{x}{x^2+1}\Big](\xi) = C_2(\xi).$$

Now since $\hat{f}_s = \frac{1}{2}i\hat{f}$, *it follows that*

$$\mathcal{F}_s\Big[\frac{x}{x^2+1}\Big](\xi) = \frac{i}{2}\mathcal{F}\Big[\frac{x}{x^2+1}\Big](\xi). \tag{4.9.12}$$

Moreover, using our previously known transforms:

$$\frac{i}{2}\mathcal{F}\Big[\frac{x}{x^2+a^2}\Big](\xi) = \frac{i}{2}(-i\pi)sign(\xi)e^{-a|\xi|} = \frac{\pi}{2}sign(\xi)e^{-a|\xi|}, \tag{4.9.13}$$

for $a = 1$ *we get, using* (4.9.12) *and* (4.9.13), *that* $C_2(\xi) = \frac{\pi}{2}sign(\xi)e^{-|\xi|}$, *and consequently*

$$v(\xi, y) = \frac{\pi}{2}sign(\xi)e^{-|\xi|}e^{-|\xi|y} = \frac{\pi}{2}sign(\xi)e^{-(1+y)|\xi|} = \mathcal{F}_s\Big[u(x, y)\Big](\xi).$$

Finally, by (4.9.13) *with* $a = y + 1$, *we have the solution*

$$u(x, y) = \frac{x}{x^2 + (1 + y)^2}.$$

This is the restriction on $[0, \infty)$ *of the odd expansion of the solution* $u(x, y)$ *on* $\mathbb{R} \times \{y > 0\}$.

4.10 Generalized functions

Consider the generalized function

$$f(x) := sign(x) = \begin{cases} 1, & \text{for} \quad x > 0, \\ -1, & \text{for} \quad x < 0. \end{cases}$$

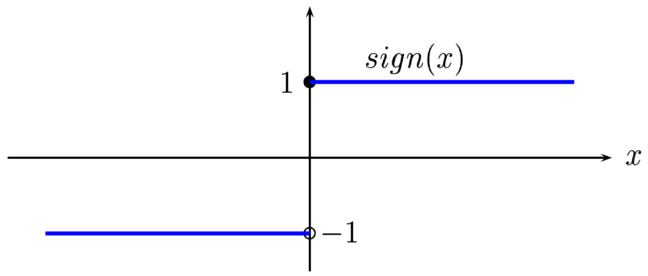


Figure 4.4: The signum function sign(x).

Formally since

$$\int_{-\infty}^{\infty} |sign(x)| dx = \int_{-\infty}^{\infty} 1\, dx = \infty,$$

thus $sign(x) \notin L^1$ and hence, a priori, the Fourier transform is not defined for $f(x) := sign(x)$. However, to define a *generalized Fourier transform*, we multiply $sign(x)$ by the convergence factor $e^{-\varepsilon|x|}$ where $\varepsilon > 0$. Now $g(x) := e^{-\varepsilon|x|} sign(x) \in L_1$, and we have

$$\begin{aligned} \hat{g}(x) \supset^{\mathcal{F}} \int_{-\infty}^{\infty} e^{-\varepsilon|x|} sign(x) e^{-i\xi x} dx &= \int_0^{\infty} e^{-\varepsilon x} e^{-i\xi x} dx - \int_{-\infty}^{0} e^{\varepsilon x} e^{-i\xi x} dx \\ &= \Big[- \frac{1}{\varepsilon + i\xi} e^{-(\varepsilon + i\xi)x} \Big]_0^{\infty} - \Big[\frac{1}{\varepsilon - i\xi} e^{(\varepsilon - i\xi)x} \Big]_{-\infty}^{0} = \frac{-2i\xi}{\varepsilon^2 + \xi^2}. \end{aligned}$$

Thus as $\varepsilon \to 0$ we get

$$sign(x) \supset^{\mathcal{F}} = \frac{-2i\xi}{\xi^2} = \frac{2}{i\xi}. \tag{4.10.1}$$

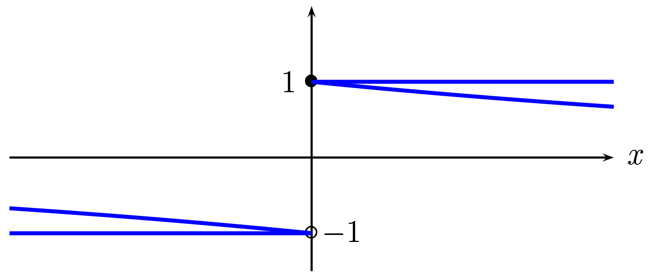


Figure 4.5: The signum function $e^{-\varepsilon|x|}sign(x)$.

Remark By the symmetry rule it follows that

$$\frac{-2ix}{\varepsilon^2 + x^2} \supset^{\mathcal{F}} 2\pi e^{-\varepsilon|-\xi|}sign(-\xi) = -2\pi e^{-\varepsilon|\xi|}sign(\xi), \tag{4.10.2}$$

or, $\varepsilon = a$,

$$\frac{i}{2}\frac{x}{a^2 + x^2} \supset^{\mathcal{F}} \frac{\pi}{2}e^{-a|\xi|}sign(\xi). \tag{4.10.3}$$

Alternatively, to show $\mathcal{F}[sign(x)] = \frac{2}{i\xi}$, using Heaviside's function $\theta(x)$ we may write

$$sign(x) = \theta(x) + (-1)(1 - \theta(x)) = 2\theta(x) - 1.$$

Now we state, without proof, the Fourier transforms of two important generalized functions:

$$\mathcal{F}[1] = 2\pi\delta(\xi) \tag{4.10.4}$$

and

$$\mathcal{F}[\theta(x)] = \pi\delta(\xi) + \frac{1}{i\xi}. \tag{4.10.5}$$

By (4.10.5) and (**??**) we have that

$$\mathcal{F}[sign(x)] = 2\pi\delta(\xi) + \frac{2}{i\xi} - 2\pi\delta(\xi) = \frac{2}{i\xi}.$$

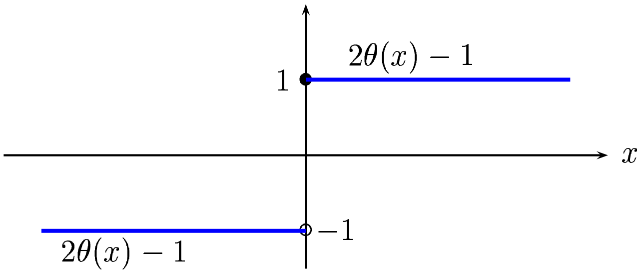


Figure 4.6: The signum function $\text{sign}(x)$.

Fourier Transforms of impulse functions

The Dirac's delta function is an even function defined by

$$\delta(x) = 0, \qquad \text{for} \quad x \neq 0, \tag{4.10.6}$$

and

$$\int_{-a}^{a} \delta(x)dx = 1 \qquad \text{for all} \quad a > 0. \tag{4.10.7}$$

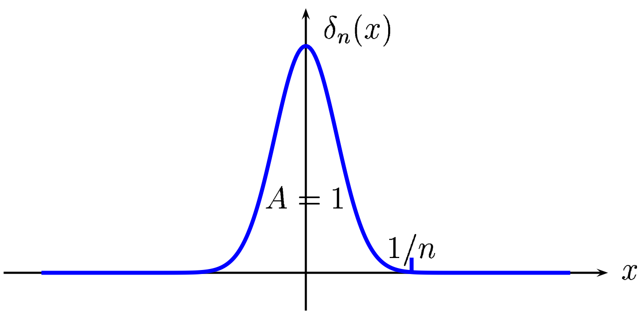


Figure 4.7: The Dirac function $\delta_n(x)$.

For $x = t - T$ this definition give

$$\delta(t-T) = \int_{-\infty}^{\infty} \delta(t-T)dx = 1. \tag{4.10.8}$$

To derive the Fourier transform of $\delta(t-T)$, we recall that by the evaluation formula:

$$f(t)\delta(t-T) = f(T)\delta(t-T) \quad \text{we have} \quad e^{-j\omega t}\delta(t-T) = e^{-j\omega T}\delta(t-T)$$

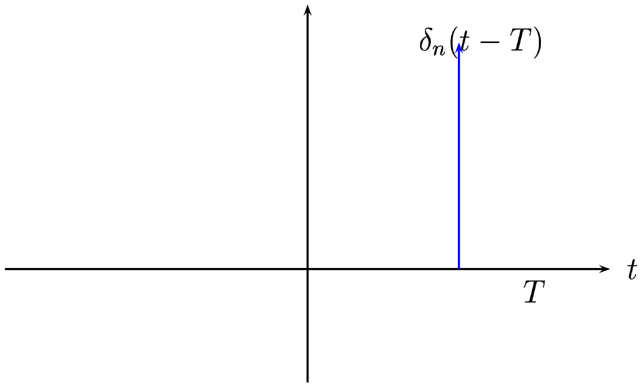


Figure 4.8: The Dirac function $\delta_{(t-T)}$.

and thus we have

$$\delta(t-T) \supset^{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t-T)e^{-j\omega t}dt = e^{-jwT} \int_{-\infty}^{\infty} \delta(t-T)dt = e^{-j\omega T}. \quad (4.10.9)$$

Then for $T = 0$: $\delta(t) \supset^{\mathcal{F}} = e^0 = 1$. Using symmetry rule and the fact that δ is an even function we have the following "formal relations": $1 \supset^{\mathcal{F}} 2\pi\delta(-\omega) = 2\pi\delta(\omega)$, i.e., we have

$$\delta(t) \supset^{\mathcal{F}} = 1, \qquad \text{and} \qquad 1 \supset^{\mathcal{F}} 2\pi\delta(\omega). \quad (4.10.10)$$

Remark Note that $1 \notin L^1$. Therefore the formulas in (4.10.10) are only formal (they are valid in distribution sense).

Signal analysis

Let $f(t)$ represent the amplitude of a signal at time t. The Fourier representation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega, \qquad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t))e^{-i\omega t}dt,$$

exhibits f as a continuous superposition of the simple periodic waves $e^{i\omega t}$ as ω ranges over all possible frequencies. This representation is the *basic* one in the analysis of signals in electrical engineering:

Let $f(t) = \delta(t-T)$ be *the time impulse function*. Then its Fourier transform: $\hat{f}(\omega) = e^{-j\omega T}$, is the *the frequency function*.

The power of a signal $f(t)$ is proportional to the square of the amplitude, $|f(t)|^2$, so the total energy of the signal is proportional to $\int_{-\infty}^{\infty} |f(t)|^2 dt$. Thus the condition for the finiteness of the total energy corresponds to $f(t) \in L^2$.

The impulse energy in the frequency band , $[\omega, \omega + d\omega]$, is given by

$$|\hat{f}(t)|^2 \frac{d\omega}{2\pi} = |e^{-j\omega T}|^2 \frac{d\omega}{2\pi} = d\nu.$$

Thus the energy is uniformly distributed over the whole frequency band, $-\infty < \omega < \infty$, which means that:

a. The total energy is ∞.

b. The exact impulse $\delta(t-T)$ is unphysical.

c.

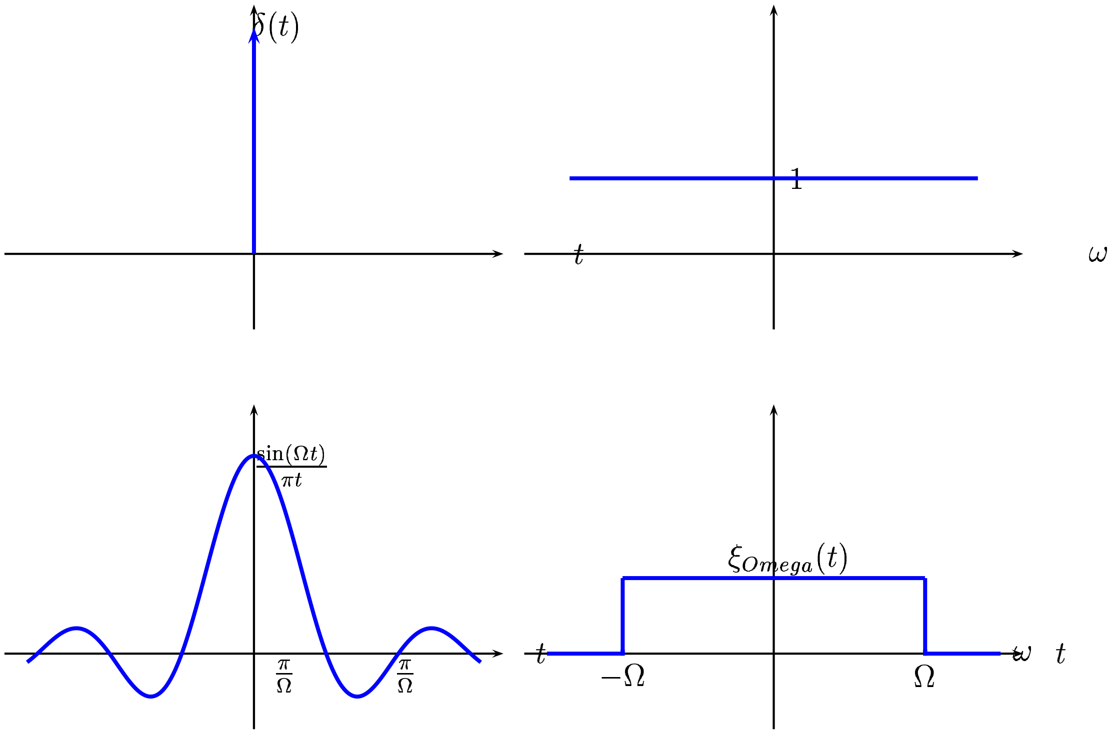
$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\hat{\delta}(\omega)}_{=1} e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega \qquad \text{is divergent.}$$

Let us look at the frequencies $|\omega| < \Omega$, where Ω is large. Then we may approximate the delta function as:

$$\delta(t) \cong \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{j\omega t} d\omega = \frac{1}{2\pi} \left[\frac{1}{jt} e^{j\omega t} \right]_{-\Omega}^{\Omega} = \frac{1}{\pi t} \sin(\Omega t).$$

Hence we have

$$\delta(t) = \lim_{\Omega \to \infty} \frac{\sin(\Omega t)}{\pi t}. \tag{4.10.11}$$



4.11 Dynamic systems

We begin by giving the definition of a linear space of functions for a dynamic system and the definition for the impulse response.

Definition 15. *A linear system L is a linear map from a linear space of input signals, $x_i(t)$, to a linear space of output signals, $L[x_i]$, where*

$$L[c_1 x_1(t) + c_2 x_2(t)] = c_1 L[x_1(t)] + c_2 L[x_2(t)] \tag{4.11.1}$$

$$L\Big[\sum_{n=1}^{\infty} c_n x_n\Big] = \sum_{n=1}^{\infty} c_n L[x_n] \tag{4.11.2}$$

$$L\Big[\int_a^b x(\alpha, t) d\alpha\Big] = \int_a^b L\Big[x(\alpha, t)\Big] d\alpha. \tag{4.11.3}$$

Let $y(t) = L[x(t)]$. Then the linear system L is time invariant if

$$L[x(t-T)] = y(t-T) \quad \textit{for all} \quad T \in \mathbf{R}. \tag{4.11.4}$$

Definition 16. *The impulse response, $H(t)$, is the output signal for the Dirac (δ-function) input signal.*

Theorem 26. *If L is a linear time invariant system with the impulse response H, then*

$$L[x(t)] = (x * H)(t) \qquad \text{or} \quad (L[x(t)])\hat{} = h\hat{f}, \tag{4.11.5}$$

where h is called the system function.

$$L[e^{i\omega t}] = \hat{H}(\omega)e^{i\omega t} \quad \text{for every} \quad \omega \in \mathbb{R} \tag{4.11.6}$$

If $H(t)$ is real, then we have for $\omega > 0$ that

$$L[\cos(\omega t)] = Re[\hat{H}(\omega)e^{i\omega t}]. \tag{4.11.7}$$

Proof. We know that

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(\tau)\delta(t-\tau)d\tau.$$

By (4.11.3) we have

$$L[x(t)] = L\Big[\int_{-\infty}^{\infty} \hat{x}(\tau)\delta(t-\tau)d\tau\Big] = \int_{-\infty}^{\infty} L[\hat{x}(\tau)\delta(t-\tau)]d\tau,$$

where using (4.11.1) yields

$$L[x(t)] = \int_{-\infty}^{\infty} \hat{x}(\tau)L[\delta(t-\tau)]d\tau,$$

and finally (4.11.1) gives the first assertion of the theorem:

$$L[x(t)] = \int_{-\infty}^{\infty} \hat{x}(\tau)H(t-\tau)]d\tau = (x * H)(t).$$

To prove (4.11.6) we let $x(t) = e^{i\omega t}$ in (4.11.5), then

$$L[x(t)] = \int_{-\infty}^{\infty} H(\tau)e^{i\omega(t-\tau)}d\tau = e^{i\omega t}\int_{-\infty}^{\infty} H(\tau)e^{-i\omega\tau}d\tau = e^{i\omega t}\hat{H}(\omega).$$

Finally we let $x(t) = \cos(\omega t)$, in (4.11.5), then

$$L[x(t)] = L[\cos(\omega t)] = \int_{-\infty}^{\infty} H(\tau)\cos[\omega(t-\tau)]d\tau = Re\int_{-\infty}^{\infty} H(\tau)e^{i\omega(t-\tau)}d\tau.$$

Thus

$$L[\cos(\omega t)] = Re\Big[\hat{H}(\omega)e^{i\omega t}\Big].$$

□

Definition 17. *The linear system L is called* causal *if the output signal value at the time t only depends on the input signal value at the times $\leq t$.*

Theorem 27. *Suppose L is a linear and time invariant system. Then L is causal if and only if $H(t) = 0$ for $t \leq 0$.*

Definition 18. *A linear time invariant system with the impulse response $H(t)$ is called stable if*

$$\int_{-\infty}^{\infty} |H(t)|dt < \infty.$$

Exempel 17. *For a linear time invariant system we have that the input signal, $\theta(t)e^{-2t}$, give us the output signal, $t^2\theta(t)e^{-3t}$. What will the output signal, $y(t)$, be if the input signal, $x(t)$, is 2π-periodic and $x(t) = t$ for $0 < t < 2\pi$ Give the answer in the form of complex trigonometric Fourier series.*

Solution: Let us only study the interval $[0,\infty)$ since $\theta(t) = 0$ if $t < 0$. Then the Fourier transform of $\theta(t)e^{-\varepsilon t}$ is

$$\int_0^{\infty} e^{-\varepsilon t}e^{-i\xi t}dt = \int_0^{\infty} e^{-(\varepsilon+i\xi)t}dt = \left[\frac{-e^{-(\varepsilon+i\xi)t}}{\varepsilon+i\xi}\right]_0^{\infty} = \frac{1}{\varepsilon+i\xi}.$$

Then we calculate the limit value, when $\varepsilon \to \infty$.

$$\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon+i\xi} = \lim_{\varepsilon\to\infty}\frac{\varepsilon-i\xi}{\varepsilon^2+\xi^2} = \lim_{\varepsilon\to\infty}\frac{\varepsilon}{\varepsilon^2+\xi^2} + \lim_{\varepsilon\to\infty}\frac{\xi}{i(\varepsilon^2+\xi^2)} = \pi\delta(\xi) + \frac{1}{i\xi}.$$

Let now $\varepsilon = 2$, then we have that the Fourier transform of $x_1(t) = \theta(t)e^{-2t}$ is $\hat{x}_1(\xi) = \frac{1}{2+i\xi}$ and the Fourier transform of $y_1(t) = t^2\theta(t)e^{-3t}$ is

$$\hat{y}_1(\xi) = -\frac{d^2}{d\xi^2}\Big(\frac{1}{3+i\xi}\Big) = \frac{2}{(3+i\xi)^3}.$$

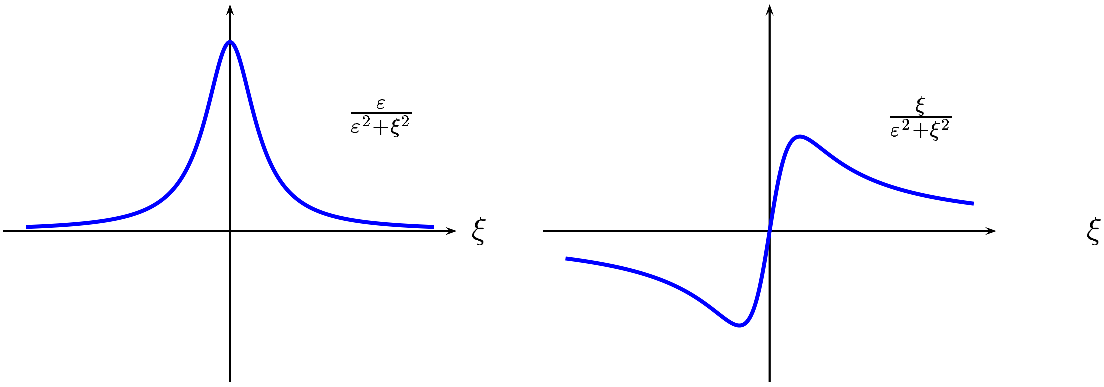


Figure 4.9: $f(\xi) = \frac{\varepsilon}{\varepsilon^2+\xi^2}$ and $g(\xi) = \frac{\xi}{\varepsilon^2+\xi^2}$.

Then the Fourier transform of the impulse response $H(t)$ will be

$$\hat{H}(\xi) = \frac{\hat{y}_1(\xi)}{\hat{x}_1(\xi)} = \frac{2(2+i\xi)}{(3+i\xi)^3}.$$

We get immediately that for $\xi = 0$ we get

$$\hat{H}(0) = \frac{4}{27}.$$

Further, the complex Fourier series for 2π periodic ($T = 2\pi$), $x(t)$ is

$$x(t) = t = \sum_{n=-\infty}^{\infty} C_n e^{in\Omega t}, \quad \text{with} \quad \Omega = \frac{2\pi}{T} = 1.$$

Hence, the Fourier coefficients for $n \neq 0$ are

$$\begin{aligned} C_n &= \frac{1}{2\pi}\int_0^{2\pi} te^{-int}dt = \frac{1}{2\pi}\left[\frac{te^{-int}}{-in}\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{-in}e^{-int}dt \\ &= \frac{1}{2\pi}\cdot\frac{2\pi e^{-i2\pi n}}{-in} = -\frac{1}{in}. \end{aligned} \tag{4.11.8}$$

For $n = 0$ we have

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} te^0 dt = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = \pi.$$

For $\xi = n$ and with $e^{i\xi t}$ mapped on $\hat{H}(\xi)e^{i\xi t}$ we have that

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{int} \quad \textit{will be mapped on} \quad \sum_{n=-\infty}^{\infty} C_n \hat{H}(n) e^{int} = y(t).$$

Thus we get the out signal viz,

$$y(t) = C_0 \hat{H}(0) + \sum_{n=-\infty, n\neq 0}^{\infty} C_n \hat{H}(n) e^{int} = \frac{4}{27}\pi + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2(2+in)}{(3+in)^3} e^{int}.$$

The sampling Theorem

We start with the definition of a band-limited signal $f(t)$.

Definition 19. *The signal $f(t)$ is called band-limited if it involves only frequencies smaller than some constant Ω. That is, $\hat{f}$ vanishes outside the finite interval $[-\Omega, \Omega]$.*

Theorem 28 (The Sampling Theorem). *Suppose that f is a band-limited function, $f \in L^2$ and $\hat{f}(\omega) = 0$ for $|\omega| \geq \Omega$. Then f is completely determined by its values at the points $t_n = n\pi/\Omega$, where $n = 0, \pm 1, \pm 2, \ldots$. In fact*

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}$$

Proof. Expand $\hat{f}$ in a Fourier series on the interval $[-\Omega, \Omega]$. Since $\hat{f}$ is even in n we have that

$$\hat{f}(\omega) = \sum_{-\infty}^{\infty} C_n e^{\frac{in\pi\omega}{\Omega}} = \sum_{-\infty}^{\infty} C_{-n} e^{\frac{-in\pi\omega}{\Omega}}, \quad \text{where} \quad |\omega| \leq \Omega. \tag{4.11.9}$$

The Fourier coefficients C_{-n} are given by

$$C_{-n} = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{\frac{in\pi\omega}{\Omega}} d\omega \tag{4.11.10}$$

and since $\hat{f}(\omega) = 0$ for $|\omega| \geq \Omega$ hence using the Fourier inversion formula it follows that

$$C_{-n} = \frac{1}{2\Omega} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{\frac{in\pi\omega}{\Omega}} d\omega = \frac{\pi}{\Omega} f\left(\frac{n\pi}{\Omega}\right). \tag{4.11.11}$$

By the bound-limit condition $\hat{f} = 0$ for $|\omega| > 0$, i.e., we may write

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)e^{i\omega t}d\omega. \tag{4.11.12}$$

Further by (4.11.11)

$$\hat{f}(\omega) = \sum_{-\infty}^{\infty} C_{-n}e^{\frac{-in\pi\omega}{\Omega}} = \sum_{-\infty}^{\infty} \frac{\pi}{\Omega} f\left(\frac{n\pi}{\Omega}\right)e^{\frac{-in\pi\omega}{\Omega}}, \tag{4.11.13}$$

which, inserting in (4.11.13) yields

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\sum_{-\infty}^{\infty}\frac{\pi}{\Omega}f\left(\frac{n\pi}{\Omega}\right)e^{\frac{-in\pi\omega}{\Omega}}e^{i\omega t}d\omega = \frac{1}{2\Omega}\sum_{-\infty}^{\infty}\left[f\left(\frac{n\pi}{\Omega}\right)\int_{-\Omega}^{\Omega} e^{-i(n\pi-\Omega t)\frac{\omega}{\Omega}}d\omega\right].$$

Evaluating the integral:

$$\int_{-\Omega}^{\Omega} e^{-i(n\pi-\Omega t)\frac{\omega}{\Omega}}d\omega = \frac{e^{i(\Omega t-n\pi)} - e^{-i(\Omega t-n\pi)}}{i(\Omega t-n\pi)\frac{1}{\Omega}} = \frac{2\sin(n\pi-\Omega t)}{(\Omega t-n\pi)\frac{1}{\Omega}},$$

we get the desired result

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{\sin(n\pi-\Omega t)}{(\Omega t-n\pi)}$$

and the proof is complete. □

Theorem 29 (The LP_α-sampling Theorem). *Suppose the band-limited signal $f(t)$ is a continuous function and $\hat{f}(\omega) = 0$ for $|\omega| \geq 0$. If we sample this signal by the frequencies $\frac{1}{T} \geq \frac{\alpha}{\pi}$ (i.e., with the angular frequencies $\Omega = \frac{2\pi}{T} \geq 2\alpha$), then we can regain $f(t)$ from the sampled signal through a low-pass filtering by the cutoff angel frequency α: LP_α-filtering, multiplied by T.*

Proof. For the time discrete signal $\{f(nT)\}_{n=-\infty}^{\infty}$ the sampled signal, in the time continuous form, is:

$$\sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) = f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = f(t)S_T(t). \tag{4.11.14}$$

The Fourier series expansion of $S_T(t)$ is

$$S_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\Omega t}, \quad \text{where} \quad C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-in\Omega t} dt = \frac{1}{T}. \tag{4.11.15}$$

Then we have the sampled signal

$$f(t)S_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{in\Omega t} f(t) \supset^{\mathcal{F}} \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}(\omega - n\Omega). \tag{4.11.16}$$

LP_α-filtering means a convolution of the signal by the function $h_\alpha(t) = \sin(\alpha t)/\pi t$. Consequently we multiply the Fourier transform of $f(t)S_T(t)$ by $\hat{h}_\alpha(\omega)$, where

$$\hat{h}_\alpha(\omega) = \begin{cases} 1, & \text{for} \quad |\omega| \le \alpha \\ 0, & \text{for} \quad |\omega| > \alpha. \end{cases} \tag{4.11.17}$$

Thus we have

$$LP_\alpha\Big(f(t)S_T(t)\Big) \supset^{\mathcal{F}} \frac{1}{T}\hat{h}_\alpha(\omega) \cdot \sum_{n=-\infty}^{\infty} \hat{f}(\omega - n\Omega). \tag{4.11.18}$$

The only non-vanishing term of the sum above, corresponds to $n = 0$, i.e., for $n \neq 0$ we have $\hat{f}(\omega - n\Omega) = 0$. Thus

$$\hat{h}_\alpha(\omega) = \sum_{n=-\infty}^{\infty} \hat{f}(\omega - n\Omega) = \hat{f}(\omega) \tag{4.11.19}$$

and hence

$$TLP_\alpha\Big(f(t)S_T(t)\Big) = f(t). \tag{4.11.20}$$

□

Exempel 18 (Sampling example). *Let* $\hat{h}_\alpha(\omega)$ *be given as in* (4.11.17) *We know that, see* (4.7.3),

$$\hat{h}_\alpha(\omega) \subset^{\mathcal{F}} \frac{1}{\pi t}\sin(\alpha t). \tag{4.11.21}$$

Thus we have

$$\hat{f}(\omega) = \hat{h}_\alpha(\omega) \cdot \sum_{n=-\infty}^{\infty} \hat{f}(\omega - n\Omega), \tag{4.11.22}$$

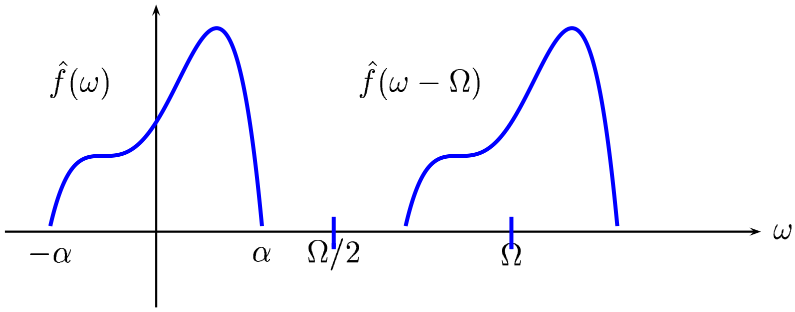


Figure 4.10: $\hat{f}(\omega)$ and $\hat{f}(\omega-\Omega)$.

which give us with $\Omega = 2\pi/T$,

$$f(t) = h_\alpha(t) * Tf(t)\delta_T(t). \tag{4.11.23}$$

Performing this convolution with respect to t *it follows that*

$$\begin{aligned} f(t) &= T\frac{\sin(\alpha t)}{\pi t} * \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} f(nT)\frac{T\sin[\alpha(t-nT)]}{\pi(t-nT)} * \delta(t) = \sum_{n=-\infty}^{\infty} f(nT)\frac{T\sin[\alpha(t-nT)]}{\pi(t-nT)}. \end{aligned}$$

Exempel 19 (The T-Sampled signal). *Let*

$$f_s(t) = T\sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) = Tf(t)\sum_{n=-\infty}^{\infty}\delta(t-nT). \tag{4.11.24}$$

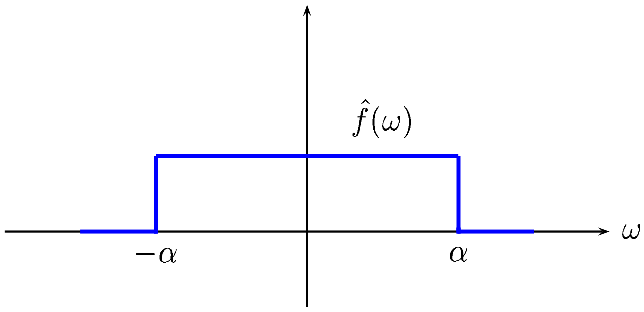
Then we have by (**??**)

$$Tf(t)S_T(t) \supset^{\mathcal{F}} \sum_{n=-\infty}^{\infty}\hat{f}(\omega-n\Omega).$$

The low pas filtering is convolution by h_α, *where* $\hat{h}_\alpha(\omega) = \theta(\omega+\alpha)-\theta(\omega-\alpha)$, *which,here, coincides with* $\hat{f}(\omega)$.

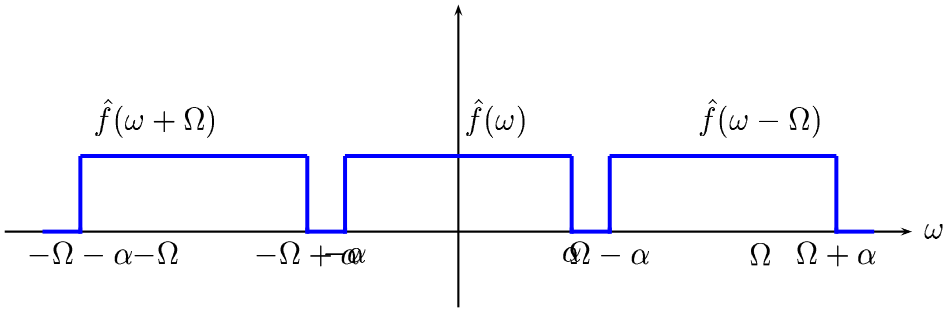
It follows that

$$g(t) = h_\alpha(t) * f_s(t) \quad \textit{and thus} \quad \hat{g}(\omega) = \hat{h}_\alpha(\omega)\sum_{n=-\infty}^{\infty}\hat{f}(\omega-n\Omega).$$



Compare by the proof of the sampling theorem. Now we can have one of the following cases:

Case 1. *If $\Omega \leq 2\alpha$ we only have $\hat{f}(\omega - n\Omega) \neq 0$ for $n = 0$, which means that $\Omega - \alpha \geq \alpha$.*



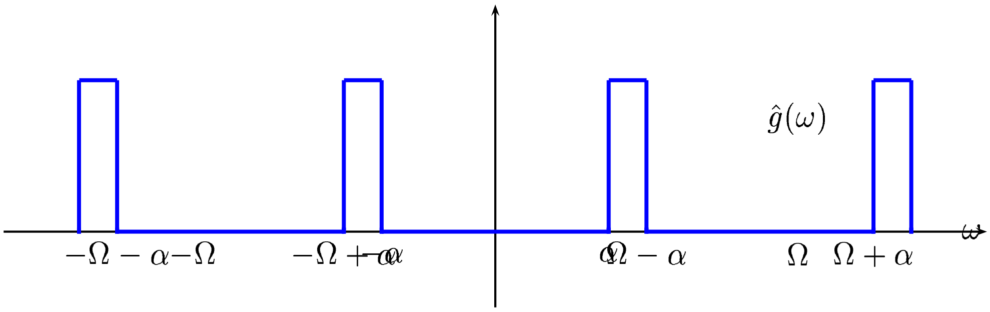
Since $\hat{h}(\omega) = 1$ for $|\omega| \leq \alpha$ we get $\hat{g}(\omega) = \hat{g}(\omega)$ and thus $g(t) = f(t)$. That we could have seen directly from the sampling theorem.

Case 2. *If $\alpha < \Omega \leq 2\alpha$ it occurs overlapping. From above we have*

$$\hat{g}(\omega) = \hat{h}_\alpha(\omega) \sum_{n=-\infty}^{\infty} \hat{f}(\omega - n\Omega). \tag{4.11.25}$$

In general the Plancheral theorem would give

$$\int_{-\infty}^{\infty} |f(t) - g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega) - \hat{g}(\omega)|^2 d\omega = \frac{1}{\pi} \int_{0}^{\infty} |\hat{f}(\omega) - \hat{g}(\omega)|^2 d\omega$$
$$= \frac{1}{\pi} \int_0^\alpha |\hat{f}(\omega) - \hat{g}(\omega)|^2 d\omega = \begin{cases} 0, & \Omega \geq 2\alpha \\ \frac{1}{\pi} \int_{\Omega-\alpha}^{\alpha} 1 d\omega = \frac{2\alpha - \Omega}{\pi}, & \alpha < \Omega \leq 2\alpha. \end{cases}$$



4.12 The discrete Fourier transform

The discrete Fourier transform is a linear mapping that operates on complex N-dimensional vectors in the same way that the Fourier transform operates on functions on $\mathbb{R}$. Let us consider the problem of numerical approximation of Fourier transforms, thus we must approximate the Fourier transform by something that involves only a finite number of algebraic calculations performed on a finite set of data. First we replace the integral over $(-\infty, \infty)$ by the integral over a finite interval $[0, \Omega]$: We may assume that f vanishes outside the bounded interval $[0, \Omega]$. Thus

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = \int_{0}^{\Omega} f(x) e^{-ix\xi} dx.$$

Using the sampling points $x = \Omega/N$ we approximate $\hat{f}$ by the Riemann sum

$$\hat{f}(\xi) \approx \sum_{n=0}^{N-1} f\Big(\frac{n\Omega}{N}\Big) e^{-in\frac{\Omega}{N}\xi} \cdot \frac{\Omega}{N}. \tag{4.12.1}$$

The sum is periodic in ξ with the period $\frac{2\pi N}{\Omega}$.

Now we calculate $\hat{f}(\xi)$ in the points $\xi = \frac{2\pi m}{\Omega}$, $m = 0, 1, ..., N-1$:

$$\hat{f}\Big(\frac{2\pi m}{\Omega}\Big) \cong \frac{\Omega}{N} \sum_{n=0}^{N-1} e^{\frac{-2\pi i n m}{N}} f\Big(\frac{n\Omega}{N}\Big), \tag{4.12.2}$$

and let $a_n = f\left(\frac{n\Omega}{N}\right)$, then we get

$$\hat{f}\Big(\frac{2\pi m}{\Omega}\Big) \cong \frac{\Omega}{N} \hat{a}_m, \quad \text{where} \quad |m| << N \quad \text{and} \quad \hat{a}_m = \sum_{n=0}^{N-1} e^{-i\frac{2\pi n m}{N}} a_n.$$

We have therefore arrived at a mapping that transforms a given $\mathbf{C}^N$ vector $\mathbf{a} = (a_0, a_1, ..., a_{N-1})$ into another $\mathbf{C}^N$ vector $\hat{\mathbf{a}} = (\hat{a}_0, \hat{a}_1, ..., \hat{a}_{N-1})$, which gives rise to the following definition:

Definition 20. *The "N-point discrete Fourier transform" $\mathcal{F}_N : \mathbf{C}^N \to \mathbf{C}^N$, that maps $\mathbf{a} = (a_0, a_1, ..., a_{N-1}) \in \mathbf{C}^N$ to $\hat{\mathbf{a}}$, is the linear defined by*

$$\mathcal{F}_N \mathbf{a} = \hat{\mathbf{a}}, \quad \text{and} \quad \hat{a}_m = \sum_{n=0}^{N-1} e^{-i\frac{2\pi nm}{N}} a_n \quad \text{where} \quad 0 \leq m < N. \tag{4.12.3}$$

The discrete Fourier transform have the nice property that it converts discrete convolution into ordinary multiplication:

$$\mathcal{F}_N(\mathbf{a} * \mathbf{b}) = (\hat{a}_0 \hat{b}_0, ..., \hat{a}_{N-1} \hat{b}_{N-1}), \tag{4.12.4}$$

where the discrete convolution $\mathbf{a} * \mathbf{b}$ is defined by

$$(\mathbf{a} * \mathbf{b})_n = \sum_{k=0}^{N-1} a_k b_{[n-k]}, \tag{4.12.5}$$

where $[n-k] = n-k$ if $n \geq k$ and $[n-k] = n-k+N$ if $n < k$.

The Discrete Fourier Inverse Formula

We can restate the definition of the discrete Fourier transform viz,

Definition 21. *Let $w = e^{2\pi i/N}$, $\mathbf{a} = (a_0, a_1, ..., a_{N-1}) \in \mathbf{C}^N$, then the discrete Fourier transform of $\mathbf{a}$ is $\hat{\mathbf{a}} =\in \mathbf{C}^N$ with elements*

$$\hat{a}_m = \sum_{n=0}^{N-1} w^{-mn} a_n; \qquad m = 0, \ldots, N-1. \tag{4.12.6}$$

Lemma 11. *For $m = 0, 1, ..., N-1$ let*

$$\mathbf{e}_m = (1, w^m, w^{2m}, ..., w^{(N-1)m}), \quad \text{where} \quad w = e^{2\pi i/N}.$$

Then $\{e_m\}_{m=0}^{N-1}$ is an orthogonal basis for $\mathbf{C}^N$, and $\|\mathbf{e}_m\|^2 = N$ for all m.

Proof. Since $\bar{w}_m = e^{-2\pi imn/N} = \left(e^{2\pi i/N}\right)^{-mn} = w^{-mn}$ we get for $l \neq m$ that the inner product

$$\begin{aligned}\langle \mathbf{e}_l, \mathbf{e}_m \rangle &= \sum_{n=0}^{N-1} (w)^{ln} (\bar{w})^{mn} = \sum_{n=0}^{N-1} (w)^{ln} (w)^{-mn} \\ &= \sum_{n=0}^{N-1} w^{(l-m)n} = \frac{w^{(l-m)N} - 1}{w^{l-m} - 1} = 0,\end{aligned} \tag{4.12.7}$$

since $w^N = \left(e^{2\pi i/N}\right)^N = e^{2\pi i} = 1$ give $w^{(l-m)N} = (w^N)^{l-m} = 1^{l-m} = 1$, and $w^{l-m} \neq 1$ when $l \neq m$.

For $l = m$ we have $w^{(l-m)n} = w^0 = 1$ and

$$\langle \mathbf{e}_m, \mathbf{e}_m \rangle = \|\mathbf{e}_m\|^2 = \sum_{n=0}^{N-1} w^0 = \sum_{n=0}^{N-1} 1 = N \tag{4.12.8}$$

and the proof is complete. □

Now we derive the inversion formula for the discrete Fourier transform.

Theorem 30. *The inversion formula for the discrete Fourier transform is*

$$\begin{aligned}\mathbf{a} &= \frac{1}{N} \sum_{m=0}^{N-1} \hat{a}_m \mathbf{e}_m, \quad \textit{(inverse formula)} \\ a_n &= \frac{1}{N} \sum_{m=0}^{N-1} w^{mn} \hat{a}_m, \quad n = 0, 1, \ldots N.\end{aligned} \tag{4.12.9}$$

$$\sum_{m=0}^{N-1} |\hat{a}_m|^2 = N \sum_{n=0}^{N-1} |a_n|^2, \quad \textit{(Parseval's formula)} \tag{4.12.10}$$

Proof. According to Lemma (11), any $\mathbf{a} \in \mathbf{C}^N$ can be written as

$$\mathbf{a} = \sum_{m=0}^{N-1} \frac{\langle \mathbf{a}, \mathbf{e}_m \rangle \mathbf{e}_m}{\|\mathbf{e}_m\|^2} = \sum_{m=0}^{N-1} \frac{1}{N} \langle \mathbf{a}, \mathbf{e}_m \rangle \mathbf{e}_m, \tag{4.12.11}$$

where by the definition of the discrete Fourier transform

$$\langle \mathbf{a}, \mathbf{e}_m \rangle = \sum_{n=0}^{N-1} a_n \bar{w}^{nm} = \sum_{n=0}^{N-1} a_n w^{-nm} = \hat{a}_m \tag{4.12.12}$$

and thus we have

$$a_n = \frac{1}{N} \sum_{m=0}^{N-1} w^{mn} \hat{a}_m = \frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi i mn/N} \hat{a}_m. \tag{4.12.13}$$

Now Pythagoras theorem for orthogonal vectors give for $\mathbf{a}$ viz (4.12.12) that

$$\|\mathbf{a}\|^2 = \frac{1}{N^2} \sum_{m=0}^{N-1} |\hat{a}_m|^2 \cdot N = \frac{1}{N^2} \sum_{m=0}^{N-1} |\hat{a}_m|^2 \tag{4.12.14}$$

or equivalently

$$N \sum_{n=0}^{N-1} |a_n|^2 = \sum_{m=0}^{N-1} |\hat{a}_m|^2 \tag{4.12.15}$$

and the proof is complete. □

Fast Fourier Transform (FFT)

To proceed we need the following definition:

Definition 22. *An "elementary operation" is a multiplication of two complex numbers followed by an addition of two complex numbers.*

From the definition of $\hat{a}_m$ we have that the calculation of each $\hat{a}_m$ requires N elementary operations. But there are N $\hat{a}_m$'s and hence the calculation of $\hat{a}_m$ requires a total of N^2 elementary operations so the discrete Fourier transform may become computationally unmanageable for large N. When N is prime, not much can be done about this. But when N is composite we can write $N = N_1 N_2$ and the indexes m and n in the definition of $\hat{a}_m$ as multiples of $N_1$and N_2 plus remainders. Let us assume that

$$m = m'N_1 + m'', \quad \text{where} \quad 0 \le m'' \le N_1 - 1 \quad \text{and} \quad 0 \le m' \le N_2 - 1$$

$$n = n'N_2 + n'', \quad \text{where} \quad 0 \le n'' \le N_2 - 1 \quad \text{and} \quad 0 \le n' \le N_1 - 1.$$

Then it follows that

$$e^{-i\frac{2\pi nm}{N}} = e^{-2\pi i\left(\frac{m'n'N_1N_2}{N}+\frac{m'n''N_1}{N}+\frac{m''n'N_2}{N}+\frac{m''n''}{N}\right)} = e^{-2\pi i\left(\frac{m'n''}{N_2}+\frac{m''n'}{N_1}+\frac{m''n''}{N}\right)}.$$

Thus we have

$$\hat{a}_m = \sum_{n''=0}^{N_2-1} C(m'',n'')e^{-2\pi i\left(\frac{m'n''}{N_2}+\frac{m''n''}{N}\right)} \tag{4.12.16}$$

where

$$C(m'',n'') = \sum_{n'=0}^{N_1-1} e^{-2\pi i\frac{m''n'}{N_1}} \cdot a_{n'N_2+n''} = \sum_{n'=0}^{N_1-1} e^{-2\pi i\frac{m''n'}{N_1}} \cdot a_n. \tag{4.12.17}$$

Each $C(m'',n'')$ requires N_1 elementary operations and there are $N_1N_2 = N$ different $C(m'',n'')$'s, so NN_1 elementary operations are needed to calculate them all. Then N_2 elementary operations are required to calculate each $\hat{a}_m$, ($\hat{a}_m = \sum_{n''=0}^{N_2-1}\dots$), and there are N of those, hence NN_2 elementary operations. The total number of elementary operations are thus $NN_1 + NN_2 = N(N_1+N_2)$.

Definition 23. *The resulting algorithm for calculating discrete transforms is called the* **Fast Fourier Transform, FFT**.

Suppose N_1 can be factored further, such that $N_1 = N_{11}N_{12}$. For a fixed n'', $C(m'',n'')$ is a discrete Fourier transform in m''. Then all $C(m'',n'')$ can be calculated with

$$N_2N_1(N_{11}+N_{12}) = N(N_{11}+N_{12}),$$

elementary operations, where N_2 is the number of n'' and N_1 is the number of m'' for a fixed n''.

Totally it requires $N(N_{11}+N_{12})+NN_2 = N(N_{11}+N_{12}+N_2)$ elementary operations, where NN_2 are all $\hat{a}_m : s$.

If $N = N_1N_2\cdot\ldots\cdot N_k$, then it requires $N(N_1+N_2+\ldots+N_k)$ elementary operations. In particular, if N is a power of 2, say $N = 2^k$ it requires $2kN = 2N\log_2 N$ elementary operations.

Chapter 5

Orthogonal sets of functions

We are now in ....

5.1 Function spaces

Consider the space of continuous functions f on the interval $[a,b]$ and think of functions f as infinite-dimensional vectors whose "components" are the values $f(x)$ as x ranges over the interval $[a,b]$.
The operations of *vector* addition and scalar multiplication are just the usual addition of *functions* and multiplication of *functions* by constants.
We first give the definition of the function spaces L_p.

Definition 24. *On an interval $I = (a,b)$ the function spaces $L_p, 1 \le p < \infty$ are defined by*

$$L_p(I) := \Big\{ f : \int_I |f(x)|^p dx < \infty \Big\}. \tag{5.1.1}$$

Thus for $p = 1$ we get

$$L_p(I) := \Big\{ f : \int_I |f(x)| dx < \infty \Big\} \tag{5.1.2}$$

and for $p = 2$ we have

$$L_2(I) := \Big\{ f : \int_I |f(x)|^2 dx < \infty \Big\}. \tag{5.1.3}$$

For $p = \infty$ we define

$$L_\infty(I) := \Big\{ f : \max_I |f(x)| < \infty \Big\}. \tag{5.1.4}$$

Now we give the definition for the *norm* in $L_p p$-spaces.

Definition 25. *The L_p-norm is defined as*

$$\|f\|_{L_p(I)} := \Big(\int_I |f(x)|^p dx \Big)^{1/p}, \quad 1 \le p < \infty \tag{5.1.5}$$

and the L_p-maxnorm ($Ł_\infty$-norm) is defined as

$$\|f\|_{L_\infty(I)} := \max_I |f(x)|. \tag{5.1.6}$$

Theorem 31 (Dominating convergens). *Suppose that $g_n(x) \to g(x)$ when $n \to \infty$, $x \in I$ and $|g_n(x)| \le \varphi(x)$ for all n, $x \in I$, where $\varphi \in L^1(I)$. Then*

$$\int_I g_n(x)\, dx \quad \longrightarrow \int_I g(x)\, dx \quad \text{when} \quad n \to \infty. \tag{5.1.7}$$

In the sequel the functions that we consider are in L_2 (with some exception of L_1 functions). Before we study the L_2-space we recall that for the complex k-dimensional vector space, $\mathbf{C}^k$, the *inner product* is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{k} \mathbf{a}(j)\overline{\mathbf{b}(j)} \tag{5.1.8}$$

and the *norm* of a vector is defined by

$$\|\mathbf{a}\| = \Big(\sum_{j=1}^{k} |\mathbf{a}(j)|^2 \Big)^{1/2}. \tag{5.1.9}$$

To define the *inner product* and the *norm* for the functions in the L_2-space, we simply replace the sum of vectors by the integrals of the functions. Thus:

Definition 26. *The inner product in the L_2-space is defined by*

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)} dx \tag{5.1.10}$$

and the norm in the L_2-space by

$$\|f\| = \langle f, f \rangle^{\frac{1}{2}} = \Big(\int_a^b |f(x)|^2 dx \Big)^{1/2}. \tag{5.1.11}$$

Below we denote by $L_2(a,b)$ the space of *square-integrable* functions on $[a,b]$, that is, the set of all functions on $[a,b]$ whose squares are absolutely (Lebesgue-) integrable over $[a,b]$. Mostly it is enough to think of $f \in PC(a,b)$, which is the space of piecewise continuous functions on the interval $[a,b]$.

We recall the famous *Cauchy-Schwartz inequality*:

$$\Big|\int_a^b f(x)\overline{g(x)}dx\Big| \leq \sqrt{\int_a^b |f(x)|^2dx}\sqrt{\int_a^b |g(x)|^2dx}. \qquad (5.1.12)$$

5.2 Convergence and Completeness

Let us begin with some important definitions.

Definition 27. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $PC(a,b)$, we say that $f_n \longrightarrow f$ in norm, if $\|f_n - f\| \longrightarrow 0$.*

Definition 28. *A sequence of functions, $\{f_n\}_{n=1}^{\infty}$, is called a Cauchy sequence, if $\|a_m - a_n\| \to 0$ as $m,n \to \infty$.*

Definition 29. *A space $\mathbf{S}$ of functions is called complete if every Cauchy sequence in $\mathbf{S}$ has a limit in $\mathbf{S}$.*

Exempel 20. *The function space $L_2(a,b)$ is complete, which means that if $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then there is a function $f \in L_2(a,b)$ such that $f_n \to f$ in norm.*

Weighted $L_2$2 spaces

Definition 30. *Let $w(x)$ be a continuous function on $[a,b]$, such that $w(x) > 0$ for all $x \in [a,b]$. Then we call $w(x)$ a weight function on $[a,b]$.*

Definition 31. *The weighted L_2-space, $L_2^w(a,b)$, is the set of all (Lebesgue measurable) functions on $[a,b]$ such that*

$$\int_a^b |f(x)|^2 w(x)dx < \infty. \qquad (5.2.1)$$

Definition 32. *The $L_2^w(a,b)$ inner product and norm are defined by*

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)dx \tag{5.2.2}$$

and

$$\text{and} \quad \|f\|_w = \langle f, f\rangle_w^{\frac{1}{2}} = \Big(\int_a^b |f(x)|^2 w(x)dx\Big)^{1/2}. \tag{5.2.3}$$

Orthogonal and orthonormal sets

We recall that two k-dimensional complex vectors $\mathbf{a}$ and $\mathbf{b} \in \mathbf{C}^k$, ($\mathbf{a} \neq \mathbf{b}$) are *orthogonal* if $\langle \mathbf{a}, \mathbf{b}\rangle = 0$. There is some work to be done in order to show that we can define orthogonal and orthonormal sets for functions in the same way. For simplicity we trust in the following definitions:

Definition 33. *The functions $\{\varphi_n\}_{n=1}^\infty$ are orthogonal if $\langle \varphi_n, \varphi_m\rangle = 0$, when $n \neq m$. Then we call $\{\varphi_n\}_{n=1}^\infty$ an orthogonal set.*

Definition 34. *If $\{\varphi_n\}_{n=1}^\infty$ is an orthogonal set and $\|\varphi_n\| = 1$ for all n, then $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal set.*

Best approximation in $L_2(a, b)$

If $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal basis for $L_2(a, b)$ then the Fourier series expansion of f with respect to $\{\varphi_n\}_{n=1}^\infty$ is given by $f = \sum_{n=1}^\infty \langle f, \varphi_n\rangle\varphi_n$, where $\langle f, \varphi_n\rangle$ is the projection of f on φ_n. Below we approximate the infinite sum above by a finite sum, viz:

Theorem 32 (The projection theorem). *Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal set of and $\{\varphi_n\}_{n=1}^N \subset \{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for the subspace $U \subset L_2(a, b)$. Then the best approximation to a function $f \in L_2(a, b)$ by a function in U is given by*

$$\tilde{f} = \sum_{n=1}^N \langle f, \varphi_n\rangle\varphi_n, \tag{5.2.4}$$

in the sense that

$$\|f-\tilde{f}\| \le \Big\|f-\sum_{n=1}^{N} C_n\varphi_n\Big\| \quad \textit{for all constants} \quad C_n. \tag{5.2.5}$$

Further the function $(f-\tilde{f})$ *is orthogonal to the subspace* U *and*

$$\|f-\tilde{f}\|^2 = \|f\|^2 - \sum_{n=1}^{N} |\langle f, \varphi_n\rangle|^2. \tag{5.2.6}$$

Proof. From the linear algebra we know that for each k , $1 \le k \le N$,

$$\langle f-\tilde{f}, \varphi_k\rangle = \langle f, \varphi_k\rangle - \langle \tilde{f}, \varphi_k\rangle,$$

and since

$$\langle \tilde{f}, \varphi_k\rangle = \Big\langle \sum_{n=1}^{N} \langle f, \varphi_n\rangle \varphi_n, \varphi_k\Big\rangle. \tag{5.2.7}$$

It follows that

$$\langle \tilde{f}, \varphi_k\rangle = \sum_{n=1}^{N} \langle f, \varphi_n\rangle \langle \varphi_n, \varphi_k\rangle. \tag{5.2.8}$$

Further, since $\{\varphi_n\}_{n=1}^{N}$ is an orthonormal set of functions, we have that

$$\langle \varphi_n, \varphi_k\rangle = \begin{cases} 1, & \text{if} \quad n = k, \\ 0, & \text{if} \quad n \ne k, \end{cases} \tag{5.2.9}$$

i.e., for $\langle \tilde{f}, \varphi_k\rangle = \langle f, \varphi_k\rangle$ and hence $\langle f-\tilde{f}, \varphi_k\rangle = 0$. Thus $(f-\tilde{f})$ is orthogonal to all linear combinations of φ_k:s which means that $(f-\tilde{f})$ is orthogonal to U. Let $g \in L_2$ be an arbitrarily function and write

$$f-g = (f-\tilde{f}) + (\tilde{f}-g).$$

Using Pythagoras theorem (as for the mutually orthogonal vectors) we get

$$\|f-g\|^2 = \|(f-\tilde{f}) + (\tilde{f}-g)\|^2 = \|f-\tilde{f}\|^2 + \underbrace{\|\tilde{f}-g\|^2}_{\ge 0} \ge \|f-\tilde{f}\|^2.$$

Thus we have $\|f-g\| \ge \|f-\tilde{f}\|$. Similarly writing $f = f-\tilde{f}+\tilde{f}$ and using Pythagoras theorem (see the Fig.) it follows that

$$\|f\|^2 = \|f-\tilde{f}+\tilde{f}\|^2 = \|f-\tilde{f}\|^2 + \|\tilde{f}\|^2.$$

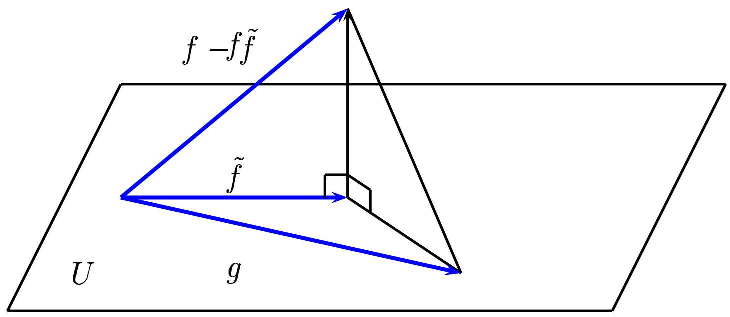


Figure 5.1:

Thus

$$\|f - \tilde{f}\|^2 = \|f\|^2 - \|\tilde{f}\|^2. \tag{5.2.10}$$

Recall that

$$\tilde{f} = \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n,$$

Thus by the orthogonality

$$\|\tilde{f}\|^2 = \Big\| \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n \Big\|^2 = \sum_{n=1}^{N} |\langle f, \varphi_n \rangle|^2 \|\varphi_n\|^2. \tag{5.2.11}$$

Hence the orthonormalithy of $\{\varphi_n\}$, i.e., $\|\varphi_n\| = 1$, and (5.2.10) implies that

$$\|f - \tilde{f}\|^2 = \|f\|^2 - \sum_{n=1}^{N} |\langle f, \varphi_n \rangle|^2. \tag{5.2.12}$$

And the proof is complete. □

Now since $\|f - \tilde{f}\|^2 \geq 0$ we have using (5.2.12) that

$$0 \leq \|f\|^2 - \sum_{n=1}^{N} |\langle f, \varphi_n \rangle|^2,$$

and if we now let $n \to \infty$ we get the following well-known result:

Bessel's Inequality: If $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal set in $L_2(a,b)$ and $f \in L_2(a,b)$, then

$$\sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2 \leq \|f\|^2. \tag{5.2.13}$$

To proceed we shall need the following lemma:

Lemma 12. *If $f \in L_2(a,b)$ and $\{\varphi_n\}_{n=1}^{\infty}$ is any orthonormal set in $L_2(a,b)$, then the series*

$$\sum_{n=1}^{\infty} \langle f, \varphi_n\rangle \varphi_n \quad \textit{converges in norm,} \quad \textit{and} \quad \Big\|\sum_{n=1}^{\infty} \langle f, \varphi_n\rangle \varphi_n\Big\| \leq \|f\|.$$

Proof. Bessel's inequality gives that the series $\sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2$ converges. Using the Pythagorean theorem for orthonormal vectors we obtain

$$\Big\|\sum_{k=m}^{n} \langle f, \varphi_k\rangle \varphi_k\Big\|^2 = \sum_{k=m}^{n} |\langle f, \varphi_k\rangle|^2 \to 0 \quad \text{when} \quad m, n \to \infty. \tag{5.2.14}$$

Thus the partial sums of the series $\sum_{m=1}^{n} \langle f, \varphi_n\rangle \varphi_n$ form a Cauchy sequence, and since $L_2(a,b)$ is complete, the series converges. By another use of the Pythagorean theorem and Bessel's inequality we get

$$\begin{aligned} \Big\|\sum_{n=1}^{\infty} \langle f, \varphi_n\rangle \varphi_n\Big\|^2 &= \lim_{N\to\infty} \Big\|\sum_{n=1}^{N} \langle f, \varphi_n\rangle \varphi_n\Big\|^2 = \lim_{N\to\infty} \sum_{n=1}^{N} |\langle f, \varphi_n\rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2 \leq \|f\|^2. \end{aligned} \tag{5.2.15}$$

And the proof is complete. □

Now we are ready to prove following important theorem.

Theorem 33. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal set in $L_2(a,b)$, then the following conditions are equivalent:*

(a) *If $\langle f, \varphi_n\rangle = 0$ for all n, then $f = 0$.*

(b) *For every $f \in L_2(a,b)$ we have $f = \sum_{n=1}^{\infty} \langle f, \varphi_n\rangle \varphi_n$, where the series converges in norm.*

(c) *For every $f \in L^2(a,b)$ we have* **Parseval's equation***:*

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n\rangle|^2.$$

Proof. **(c)** $\Rightarrow$ **(a):** If $\langle f, \varphi_n \rangle = 0$ for all n, and **(c)** holds, we have

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2 = 0 \quad \text{and} \quad f = 0.$$

(a) $\Rightarrow$ **(b):** Given $f \in L_2(a, b)$, by Lemma (12) the series $f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$ converges in norm and $\left\| \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n \right\| \leq \|f\|$. We can see that its sum is f by showing that the difference $g = f - \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$ is zero. But

$$\langle g, \varphi_m \rangle = \langle f, \varphi_m \rangle - \sum_{n=1}^{\infty} \langle f, \varphi_m \rangle \langle \varphi_n, \varphi_m \rangle = \langle f, \varphi_m \rangle - \langle f, \varphi_m \rangle = 0,$$

for all m, and if **(a)** holds then $g = 0$. Thus we have $f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$. **(b)** $\Rightarrow$ **(c)** If $f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$, then by Pythagorean theorem,

$$\|f\|^2 = \lim_{N \to \infty} \left\| \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n \right\|^2 = \lim_{N \to \infty} \sum_{n=1}^{N} |\langle f, \varphi_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2,$$

since $\|\varphi_m\| = 1$ and thus **(c)** holds. And the proof is complete. □

Definition 35. *An orthonormal set, $\{\varphi_n\}_{n=1}^{\infty}$, that possesses the properties* **(a)** **-(c)** *in Theorem* (33) *is called a complete orthonormal set or orthonormal basis for $L_2(a, b)$.*

Definition 36. *If $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L_2(a, b)$ and $f \in L_2(a, b)$, the numbers $\langle f, \varphi_n \rangle$ are called the (generalized) Fourier coefficients of f with respect to $\{\varphi_n\}_{n=1}^{\infty}$. The series $\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$ is called the (generalized) Fourier series of f.*

Often it is more convenient *not* to require the elements of basis to be *unit* vectors. Suppose $\{\psi_n\}_{n=1}^{\infty}$ is an orthogonal set. Let

$$\varphi_n = \frac{\psi_n}{\|\psi_n\|},$$

then $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal set and the corresponding expansion formula for $f \in L_2(a, b)$ and the Parseval relation take the following form

$$f = \sum_{n=1}^{\infty} \frac{\langle f, \psi_n \rangle}{\|\psi_n\|^2} \psi_n, \qquad \|f\|^2 = \sum_{n=1}^{\infty} \frac{|\langle f, \psi_n \rangle|^2}{\|\psi_n\|^2}. \tag{5.2.16}$$

Definition 37. *$\{\psi_n\}_{n=1}^{\infty}$ is a complete orthogonal set or an orthogonal basis for $L_2(a,b)$ if $\{\varphi_n\}_{n=1}^{\infty} = \{\psi_n/\|\psi_n\|\}_{n=1}^{\infty}$ is an orthonormal basis of $L_2(a,b)$. The Fourier coefficients for f with respect to the orthogonal basis $\{\psi_n\}_{n=1}^{\infty}$ is*

$$C_n = \frac{\langle f, \psi_n \rangle}{\|\psi_n\|^2}.$$

Now we can rewrite the Parseval equation in the following way

$$\|f\|^2 = \sum_{n=1}^{\infty} \frac{|\langle f, \psi_n \rangle|^2}{\|\psi_n\|^2} = \sum_{n=1}^{\infty} \|\psi_n\|^2 |C_n|^2. \tag{5.2.17}$$

Orthogonal bases for $L_2(-\pi,\pi)$.

Theorem 34. *The sets*

$$\{e^{inx}\}_{n=-\infty}^{\infty} \quad \textit{and} \quad \{\cos(nx)\}_{n=0}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$$

are orthogonal bases for $L_2(-\pi,\pi)$. The sets

$$\{\cos(nx)\}_{n=0}^{\infty} \quad \textit{and} \quad \{\sin(nx)\}_{n=1}^{\infty}$$

are orthogonal bases for $L_2(0,\pi)$.

Let us now calculate a Parseval equation for $L_2(-\pi,\pi)$, using (5.2.15):

$$\|\psi_n\|^2 = \int_{-\pi}^{\pi} \psi_n(x)\overline{\psi_n(x)}dx = \int_{-\pi}^{\pi} |\psi_n(x)|^2 dx$$

For $\psi_n(x) = e^{inx}$ we get

$$\|\psi_n\|^2 = \int_{-\pi}^{\pi} e^{inx}e^{-inx}dx = \int_{-\pi}^{\pi} e^0 dx = 2\pi \quad \text{for all} \quad n. \tag{5.2.18}$$

$\psi_n(x) = e^{inx}$, $n = -\infty, \ldots, \infty$ form basis functions and the Fourier series expansion of f:

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx} \quad \text{converges in norm.}$$

$$C_n = \frac{\langle f, \psi_n \rangle|}{\|\psi_n\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{\psi_n(x)}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx. \tag{5.2.19}$$

Thus we get the Parseval equation in $L_2(-\pi,\pi)$:

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} \|\psi_n\|^2|C_n|^2 = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2. \tag{5.2.20}$$

5.3 Regular Sturm-Liouville problems

There is a large class of boundary value problems on an interval $[a, b]$ that lead to orthogonal bases for $L_2(a, b)$. These problems are the subject of this section. To start we provide some basic definitions:

Definition 38. *A linear transformation $T : \mathbf{C}^k \to \mathbf{C}^k$ is called* self-adjoint *or* Hermitian *if*

$$\langle Ta, b\rangle = \langle a, Tb\rangle \quad \text{for all} \quad a, b \in \mathbf{C}^k. \tag{5.3.1}$$

Let S and T be two linear operators that are defined on the subspaces $\mathcal{D}_S$ and $\mathcal{D}_T$ of $L_2(a, b)$ and map them into $L_2(a, b)$. We say that S and T are adjoint *to each other if*

$$\langle S(f), g\rangle = \langle f, T(g)\rangle \quad \text{for all} \quad f \in \mathcal{D}_S \quad \text{and} \quad g \in \mathcal{D}_T. \tag{5.3.2}$$

S is called self-adjoint *or* Hermitian *if*

$$\langle S(f), g\rangle = \langle f, S(g)\rangle \quad \text{for all} \quad f, g \in \mathcal{D}_S. \tag{5.3.3}$$

Theorem 35 (Lagrange's Indentity). *Let $L(f) = (rf')' + pf$ and $L(g) = (rg')' + pg$, where r, q and p are real functions of class $C^{(2)}$ on $[a, b]$, then L is "formally" self-adjoint:*

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \Big[r(f'\bar{g} - f\bar{g}')\Big]_a^b. \tag{5.3.4}$$

Proof. If we write out the integral defining $\langle L(f), g\rangle$, we can move the derivate from f to g by integration by parts, viz:

$$\begin{aligned}
\langle L(f), g\rangle &= \int_a^b \Big((rf')'\bar{g} + pf\bar{g}\Big)dx = \Big[rf'\bar{g}\Big]_a^b - \int_a^b rf'\bar{g}'dx + \int_a^b pf\bar{g}dx \\
&= \Big[rf'\bar{g}\Big]_a^b - \Big[fr\bar{g}'\Big]_a^b + \int_a^b \Big((f(r\bar{g}')' + pf\bar{g}\Big)dx \\
&= \Big[r(f'\bar{g} - f\bar{g}')\Big]_a^b + \int_a^b f\Big((r\bar{g}')' + p\bar{g}\Big)dx.
\end{aligned}$$

For r and p real we have that

$$(r\bar{g}')' + p\bar{g} = \overline{(rg')' + pg} = \overline{L(g)}, \tag{5.3.5}$$

and hence

$$\int_a^b f\Big((r\bar{g}')' + p\bar{g}\Big)dx = \int_a^b f\overline{L(g)}dx = \langle f, L(g)\rangle. \qquad (5.3.6)$$

Thus we have

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + \Big[r(f'\bar{g} - f\bar{g}')\Big]_a^b.$$

and the proof is complete. □

L is obviously *self-adjoint* if

$$\Big[r(f'\bar{g} - f\bar{g}')\Big]_a^b = 0, \qquad (5.3.7)$$

which is determined by the endpoints a and b. Thus we have a boundary value problem.

Let now the boundary conditions be of the form:

$$\alpha f(a) + \alpha' f'(a) = 0 \qquad \beta f(b) + \beta' f'(b) = 0,$$
$$\alpha g(a) + \alpha' g'(a) = 0 \qquad \beta g(b) + \beta' g'(b) = 0.$$

where $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$, $(\alpha, \alpha') \neq (0,0)$ and $(\beta, \beta') \neq (0,0)$.

If $\alpha' = 0$ then $\alpha \neq 0$ and we have $f(a) = g(a) = 0$. Thus $f'\bar{g} - f\bar{g}' = 0$ in $x = a$, since if $g(a) = 0$ we have $\bar{g}(a) = 0$.

If $\alpha' \neq 0$, we have $\alpha f(a) = -\alpha' f'(a)$ and $\alpha g(a) = -\alpha' g'(a)$. Thus

$$f'(a) = -\frac{\alpha}{\alpha'} f(a) \quad \text{and} \quad g'(a) = -\frac{\alpha}{\alpha'} g(a). \qquad (5.3.8)$$

Let now $c = -\frac{\alpha}{\alpha'}$, then

$$f'(a)\overline{g(a)} - f(a)\overline{g(a)}' = cf(a)\overline{g(a)} - f(a)c\overline{g(a)} = 0 \qquad (5.3.9)$$

In the same way we obtain $f'(b)\overline{g(b)} - f(b)\overline{g(b)}' = 0$.

For a second-order operator L we impose two independent boundary conditions of the form

$$B_1(f) = \alpha_1 f(a) + \alpha_1' f'(a) + \beta_1 f(b) + \beta_1' f'(b),$$

$$B_2(f) = \alpha_2 f(a) + \alpha_2' f'(a) + \beta_2 f(b) + \beta_2' f'(b),$$

where α, α', β and β' are constants.

Almost all the boundary conditions that arise in practice are of the form

$$\alpha f(a) + \alpha' f'(a) = 0 \qquad \beta f(b) + \beta' f'(b) = 0$$

where $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$, $(\alpha, \alpha') \neq (0,0)$ and $(\beta, \beta') \neq (0,0)$.
These boundary conditions are called *separated*, since each one involves a condition at only one endpoint.

Now we give the definition for a regular Sturm-Liouville problem.

Definition 39. *A regular Sturm-Liouville problem on the interval* $[a,b]$ *is specified by the following:*
(i) a formally self adjoint differential operator L *defined by* $L(f) = (rf')' + pf$*, where* r, r' *and* p *are real and continuous on* $[a,b]$ *and* $r > 0$ *on* $[a,b]$.
(ii) a set of self-adjoint boundary conditions, $B_1(f) = 0$ *and* $B_2(f) = 0$*, for the operator* L.
(iii) a positive continuous function w *on* $[a,b]$.

Now we want to find all the solutions f of the boundary value problem

$$L(f) + \lambda w f = 0, \quad \text{where} \quad B_1 = B_2 = 0, \tag{5.3.10}$$

i.e.,

$$\begin{cases} [r(x)f'(x)]' + p(x)f(x) + \lambda w(x) f(x) = 0, & a < x < b, \\ \alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0, & \end{cases} \tag{5.3.11}$$

where λ is an arbitrary constant.

For most values of λ, the only solution of this boundary problem is the trivial one, $f(x) \equiv 0$.

Definition 40. *If the Sturm-Liouville problem has nontrivial solutions,* λ *is called an* eigenvalue *and the corresponding nontrivial solutions are called* eigenfunctions.

We summarize the properties of eigenvalues and eigenfunctions in a theorem, which displays the importance of eigenfunctions from the point of view of *orthogonal sets*. Recall that if $w > 0$ is a weight function on $[a,b]$ the weighted inner product, $\langle f, g\rangle_w$ is defined as

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)dx = \langle wf, g\rangle = \langle f, wg\rangle. \tag{5.3.12}$$

Theorem 36. *Given a regular Sturm-Liouville problem defined as in* (5.3.11) *we have that:*
(a) *All eigenvalues are real.*
(b) *Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function w; that is, if f and g are eigenfunctions with eigenvalues λ and μ, $\lambda \neq \mu$ then*

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)dx = 0. \tag{5.3.13}$$

(c) *The eigenspace for any eigenvalue λ is at most 2-dimensional. (If the boundary conditions are separated, it is always 1-dimensional).*

Proof. Suppose that $L(f) + \lambda wf = 0$ and $L(g) + \mu wg = 0$, where $f, g \neq 0$. Using Lagrange's Identity we have $\langle L(f), g\rangle = \langle f, L(g)\rangle$ and since $L(f) = -\lambda wf$ and $L(g) = -\mu wg$ we obtain $-\langle \lambda wf, g\rangle = -\langle f, \mu wg\rangle$ and thus

$$\lambda\langle f, g\rangle_w = \bar{\mu}\langle f, g\rangle_w. \tag{5.3.14}$$

(a) Let $g = f$ and $\mu = \lambda$. Then since $\langle f, f\rangle_w = \|f\|_w^2$, we have that $\lambda\|f\|_w^2 = \bar{\lambda}\|f\|_w^2$ and since $\|f\|_w^2 \neq 0$ it follows that $\lambda = \bar{\lambda}$ and hence the eigenvalue λ is real.
(b) From **(a)** we have that $\mu = \bar{\mu}$ and thus (5.3.14)is written as $\lambda\langle f, g\rangle_w = \mu\langle f, g\rangle_w$. Therefore if $\lambda \neq \mu$ we obtain $\langle f, g\rangle_w = 0$.
(c) The proof is left for the reader, (see, e.g., Folland, Fourier analysis and its applications, page 90). □

It is not evident that a given Sturm-Liouville problem has any eigenfunctions at all. But, in fact, there are as many as anyone could wish for. Let us study the $L_2^w(a,b)$-space using the following theorem:

Theorem 37. *For every regular Sturm-Liouville problem on $[a,b]$:*

$$(rf')' + pf + \lambda wf = 0, \qquad B_1(f) = B_2(f) = 0, \tag{5.3.15}$$

(a) *There is an orthonormal basis $\{\varphi\}_{n=1}^{\infty}$ of $L_2^w w(a,b)$ consisting of eigenfunctions.*
(b) *if λ_n is the eigenvalue for φ_n, then $\lim_{n\to\infty}\lambda_n = +\infty$.*
(c) *If $f \in C^{(2)}$ and satisfies the boundary conditions $B_1(f) = B_2(f) = 0$, then the series*

$$\sum_{n=1}^{\infty} \langle f, \varphi_n\rangle \varphi_n \tag{5.3.16}$$

converges uniformly in norm to f.

Exempel 21. *Determine the orthonormal bases for the following Sturm-Liouville problem*

$$\begin{cases} f'' + \lambda f = 0, & 0 < x < a \\ f(0) = 0, & f'(a) = 0. \end{cases} \tag{5.3.17}$$

If $\underline{\lambda = 0}$ we obtain $f'' = 0$ and the general solution is $f(x) = c_1 + c_2 x$. Then $f(0) = c_1 = 0$ and $f'(a) = c_2 = 0$. Hence there are no eigenfunctions.
If $\underline{\lambda \neq 0}$ the general solution (for both $\lambda > 0$ and $\lambda < 0$) is

$$f(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \tag{5.3.18}$$

This holds also for $\lambda < 0$, since $\sin(i\beta) = -\frac{1}{i}\sinh(\beta)$ and $\cos(i\beta) = cosh(\beta)$. Then

$$f'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

The boundary conditions give us

$$f(0) = c_1\cos(0) = c_1 = 0 \quad \textit{and} \quad f'(a) = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}a) = 0.$$

Since $c_2 \neq 0$ we have $\cos(\sqrt{\lambda}a) = 0$ and thus $\sqrt{\lambda}a = (n+\frac{1}{2})\pi$, where $n = 0, 1, \ldots$. Then

$$\lambda := \lambda_n = \Big[(n+\frac{1}{2})\frac{\pi}{a}\Big]^2, \tag{5.3.19}$$

and we get

$$f_n(x) := \varphi_n(x) = c_2 \sin\Big((n+\frac{1}{2})\frac{\pi}{a}x\Big). \tag{5.3.20}$$

But since we want $\{\varphi\}_{n=1}^{\infty}$ to be an orthonormal bases, we require $\|\varphi_n\|^2 = 1$, i.e.,

$$\|\varphi_n\|^2 = c_2^2 \int_0^a \sin^2\Big((n+\frac{1}{2})\frac{\pi}{a}x\Big)dx = c_2^2\frac{a}{2} = 1,$$

and hence $c_2 = \sqrt{\frac{2}{a}}$. Inserting c_2 in (5.3.20) we finally get

$$\varphi_n(x) = \sqrt{\frac{2}{a}}\sin\Big((n+\frac{1}{2})\frac{\pi}{a}x\Big). \tag{5.3.21}$$

Lemma 13 (The Normalizing Lemma). *If $\sin(2\beta x) = 0$ for $x = a$, $x = b$ and $\beta \neq 0$, then*

$$\int_b^a \sin^2(\beta x)dx = \int_b^a \cos^2(\beta x)dx = \frac{b-a}{2} \tag{5.3.22}$$

Proof.

$$\begin{aligned}\int_b^a \sin^2(\beta x)dx = \int_b^a \cos^2(\beta x)dx &= \int_b^a \frac{1 \pm \cos(2\beta x)}{2}dx \\ &= \frac{b-a}{2} \pm \Big[\frac{1}{4\beta}\sin(2\beta x)\Big]_a^b = \frac{b-a}{2}.\end{aligned}$$

□

Note! If the eigenfunctions φ_n are not normalized the Fourier series extension have the form

$$f = \sum_{n=1}^{\infty} C_n\varphi_n, \qquad C_n = \frac{1}{\|\varphi_n\|_w^2}\langle f, \varphi_n\rangle_w = \frac{1}{\|\varphi_n\|_w^2}\int_a^b f(x)\varphi_n(x)w(x)dx,$$

where φ_n is real and

$$\|\varphi_n\|^2 = \int_a^b |\varphi_n|^2 w(x)dx.$$

Chapter 6

Boundary value problems

In this chapter we will give three techniques to solve Sturm-Liuville problems.

6.1 Fourier techniques for inhomogeneous PDE

Technique 1: To solve an inhomogeneous problem with time-independent data, reduce to the homogeneous case by finding a steady-state solution.

To illustrate this technique we apply it to the following example: a problem describing heat flow on an interval $[0, l]$:

$$\begin{cases} L(u) := u_t - k u_{xx} = x & 0 < x < l \\ u(0,t) = 0, \qquad u_x(l,t) = 2 & \\ u(x,0) = 1. & \end{cases} \tag{6.1.1}$$

Let $u(x,0) = u_0(x)$ satisfy the time independent Steady-state problem

$$\begin{cases} L(u_0) = -k u_0'' = x & t = 0 \\ u_0(0) = 0, \qquad u_0'(l) = 2. & \end{cases} \tag{6.1.2}$$

Then $-k u_0'' = x$ give us $u_0'' = -\frac{x}{k}$ and after twice integration we get

$$u_0' = -\frac{x^2}{2k} + C_1 \quad \text{and} \quad u_0 = -\frac{x^3}{6k} + C_1 x + C_2.$$

Now invoking the boundary values ew have

$$u_0(0) = C_2 = 0 \quad \text{and} \quad u_0'(l) = -\frac{l^2}{2k} + C_1 = 2 \Longrightarrow C_1 = 2 + \frac{l^2}{2k}.$$

Thus the solutions is

$$u_0(x) = -\frac{x^3}{6k} + \Big(2 + \frac{l^2}{2k}\Big)x. \tag{6.1.3}$$

Let now $v(x,t) = u(x,t) - u_0(x)$ and we have

$$v_t = u_t, \qquad v_{xx} = u_{xx} - \frac{6x}{6k} = u_{xx} - \frac{x}{k}.$$

Thus we can rewrite the inhomogeneous partial differential equation in (6.1.1) as

$$u_t - ku_{xx} - x = v_t - kv_{xx} = 0, \tag{6.1.4}$$

which is a homogeneous partial differential equation for v. Further the boundary values for v will take the following form

$$\begin{cases} v(0,t) = u(0,t) - u_0(0) = 0, \\ v_x(l,t) = u_x(l,t) - u_0'(l) = 2 - 2, \\ v(x,0) = u_x(x,0) - u_0(x) = 1 + \frac{3x^2}{6k} - \Big(2 + \frac{l^2}{2k}\Big)x := v_0(x) \end{cases} \tag{6.1.5}$$

Summing up we have the homogeneous problem for v, viz,

$$\begin{cases} v_t - kv_{xx} = 0 & 0 < x < l \\ v(0,t) = 0 \qquad v_x(l,t) = 0 & \\ v_0(x) = 1 + \frac{3x^2}{6k} - \Big(2 + \frac{l^2}{2k}\Big)x & 0 < x < l, \end{cases} \tag{6.1.6}$$

which is solved by the method of separation of variables. Insert $v(x,t) = X(x)T(t) \neq 0$,in the differential equation in (6.1.6) to get

$$XT' = kX''T \quad \text{i.e.} \quad \frac{X''}{X} = \frac{T}{kT} = -\lambda, \qquad (\lambda > 0).$$

Now using the normal procedure of the separation of variables we have the following two problems:

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X(0) = X'(l) = 0 \end{cases} \qquad \text{and} \qquad T' = -\lambda kT, \tag{6.1.7}$$

which we have solved earlier. For the equation for X we get the eigenvalues and eigenfunctions as

$$\lambda = \lambda_n = \Big[\Big(n + \frac{1}{2}\Big)\frac{\pi}{l}\Big]^2, \qquad X_n(x) = \sin\Big(n + \frac{1}{2}\Big)\frac{\pi x}{l}, \qquad n \geq 0, \tag{6.1.8}$$

where $\{X_n\}_{n=0}^{\infty}$ are the orthogonal basis for $L_2(0,l)$, with $\|X_n\|^2 = l/2$.

The equation for T has the solution $T_n(t) = e^{-k\lambda_n t}$.

Finally Super-positioning gives that

$$v(x,t) = \sum_{n=0}^{\infty} C_n e^{-k\left[\left(n+\frac{1}{2}\right)\frac{\pi}{l}\right]^2 t} \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l}, \tag{6.1.9}$$

where the initial condition:

$$v(x,0) = \sum_{n=0}^{\infty} C_n \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l} = v_0(x), \tag{6.1.10}$$

yields that C_n, the Fourier coefficients of $v_0(x) = 1 - u_0(x)$, are given by

$$\begin{aligned} C_n &= \frac{2}{l}\int_0^l v_0(x) \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{l} dx \\ &= \frac{2}{l}\int_0^l \left\{1 + \frac{x^3}{6k} - \left(2 + \frac{l^2}{2k}\right)x\right\} \sin(\beta_n x)\, dx, \end{aligned} \tag{6.1.11}$$

where $\beta_n = \left(n + \frac{1}{2}\right)\frac{\pi}{l}$. Using partial integration we get

$$C_n = \frac{2}{l}\Big[\frac{1}{\beta_n} - (-1)^n \frac{2 + \frac{l^2}{2k}}{\beta_n^2} - (-1)^n \frac{1}{k\beta_n^4}\Big]. \tag{6.1.12}$$

A common heat flow equation

A common heat flow equation is of the form

$$\sigma(x)u_t = \Big(k(x)u_x\Big)_x + R(x,t) - h(x)(u - u_0). \tag{6.1.13}$$

$\sigma(x)$: is the heat capacity,
$k(x)$: is the thermal conductivity,
$R(x,t)$: is the produced heat,
$h(x)(u-u_0)$: is the heat loss along the bar.
u_0 : is a given temperature.

The corresponding homogeneous differential equation is

$$\sigma u_t = (k u_x)_x - hu. \tag{6.1.14}$$

Using separation of variables, where $u = XT \neq 0$, gives

$$\sigma X T' = (kX')T - hXT \quad \Longrightarrow \frac{T'}{T} = \frac{(kX')'T - hX}{\sigma X} = -\lambda. \tag{6.1.15}$$

Thus we get ,e.g., then the eigenvalue problem

$$(kX')' - hX + \lambda\sigma X = 0, \qquad + \text{ boundary conditions.} \tag{6.1.16}$$

We demonstrate **Techniques 2** and **3** in the next section:

6.2 Inhomogeneous Sturm-Liouville problems

We consider the linear inhomogeneous differential equation:

$$L(u) = u_t - L_1(u) = F(x,t) \tag{6.2.1}$$

where L_1 is a linear differential operator in x. To solve this problem we first have to solve the homogeneous problem

$$u_t - L_1(u) = 0. \tag{6.2.2}$$

Separation of variables give us the eigenvalue problem

$$L_1(X) + \lambda X = 0 \qquad + \text{ boundary conditions,} \tag{6.2.3}$$

where we can compute the eigenvalues λ_n and eigenfunctions $X_n(x)$.

Now we expand everything in Fourier series with respect to the basis $\{X_n(x)\}_1^\infty$: We get

$$F(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n(x), \qquad a_n(t) = \frac{1}{\|X_n\|_\sigma^2} \int_a^b F(x,t) X_n(x) \sigma(x) dx. \tag{6.2.4}$$

Let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x) \tag{6.2.5}$$

then we have that (6.2.1) is equivalent to

$$\sum_{n=1}^{\infty} u_n'(t)X_n(x) - \sum_{n=1}^{\infty} u_n(t)L_1(X_n) = \sum_{n=1}^{\infty} a_n(t)X_n(x), \tag{6.2.6}$$

where $L_1(X_n) = -\lambda_n X_n(x)$ and thus

$$\sum_{n=1}^{\infty} [u_n'(t) + \lambda_n u_n(t)]X_n(x) = \sum_{n=1}^{\infty} a_n(t)X_n(x). \tag{6.2.7}$$

Now, identifying the coefficients in (6.2.7) we have the ODE-problem $u_n'(t) + \lambda_n u_n(t) = a_n$.

Technique 2: The Sturm-Liouville expansions used to solve $L(u) = 0$ with homogenous boundary conditions $B(u) = 0$ can also be used to solve the inhomogeneous equation $L(u) = F(x,t)$ with the same boundary conditions.

Exempel 22. *Solve the Sturm-Liouville problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = tx & 0 < x < l, \quad t > 0, \\ u(0,t) = 0, & u(l,t) = 0 \\ u(x,0) = x^2, & u_t(x,0) = 0. \end{cases} \tag{6.2.8}$$

We use separation of variables to solve first the homogeneous problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(0,t) = v(l,t) = 0. \end{cases} \tag{6.2.9}$$

Insert $v(x,t) = X(x)T(t) \neq 0$ *in* (6.2.9) *to get*

$$XT'' = c^2 X'' T \quad \textit{and thus} \quad \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda. \tag{6.2.10}$$

For the eigenvalues problem:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0, \end{cases} \tag{6.2.11}$$

we get, as earlier, the eigenvalues and eigenfunctions, viz,

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \qquad n \geq 1, \tag{6.2.12}$$

where $\{X_n(x)\}_{n=1}^{\infty}$ *is an orthogonal basis for* $L_2(0,l)$*, where* $\|X_n(x)\|^2 = l/2$.

Below we expand tx *and* x^2 *in Fourier series with respect to* $X_n(x)$*:*

$$tx = \sum_{n=1}^{\infty} \alpha_n \sin\Big(\frac{n\pi x}{l}\Big), \qquad \alpha_n = \frac{2}{l}\int_0^l (tx)\sin\Big(\frac{n\pi x}{l}\Big)dx, \tag{6.2.13}$$

partial integration gives

$$\alpha_n = t(-1)^{n-1}\frac{2l}{n\pi} = t\beta_n, \quad \textit{where} \quad (-1)^{n-1}\frac{2l}{n\pi} = \beta_n. \tag{6.2.14}$$

Similarly we have

$$x^2 = \sum_{n=1}^{\infty} \gamma_n \sin\Big(\frac{n\pi x}{l}\Big), \qquad \gamma_n = \frac{2}{l}\int_0^l x^2\sin\Big(\frac{n\pi x}{l}\Big)dx, \tag{6.2.15}$$

where partial integration yields

$$\gamma_n = \frac{4l^2}{n^3\pi^3}\Big[(-1)^n - 1\Big] - \frac{2l^2}{n\pi}(-1)^n. \tag{6.2.16}$$

Let now

$$u(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin\Big(\frac{n\pi x}{l}\Big), \tag{6.2.17}$$

then

$$\begin{cases} u_{tt} - c^2u_{xx} & = \sum_{n=1}^{\infty}\Big[w_n''(t) + c^2\Big(\frac{n\pi x}{l}\Big)^2 w_n(t)\Big] = \sum_{n=1}^{\infty}\beta_n t\sin\Big(\frac{n\pi x}{l}\Big) = tx \\ u(x,0) & = \sum_{n=1}^{\infty} w_n(0)\sin\Big(\frac{n\pi x}{l}\Big) = \sum_{n=1}^{\infty}\gamma_n\sin\Big(\frac{n\pi x}{l}\Big) = x^2 \\ u_t(x,0) & = \sum_{n=1}^{\infty} w_n'\sin\Big(\frac{n\pi x}{l}\Big) = 0, \end{cases}$$

And we get, identifying the coefficients, a sequence of ordinary differential equations, viz

$$\begin{cases} w_n''(t) + \Big(\frac{cn\pi}{l}\Big)^2 w_n(t) = \beta_n t \\ w_n(0) = \gamma_n \qquad w_n'(0) = 0. \end{cases} \tag{6.2.18}$$

Now there are two inhomogeneous terms to deal with: $w_n''(t) + \Big(\frac{cn\pi}{l}\Big)^2 w_n(t) = \beta_n t$ *and* $w_n(0) = \gamma_n$*. Suppose we can solve two new problems obtained by replacing one by* $\beta_n t = 0$ *and one by* $\gamma_n = 0$*:*

$$\begin{cases} g_n''(t) + \Big(\frac{cn\pi}{l}\Big)^2 w_n(t) = 0 \\ g_n(0) = \gamma_n, \qquad g_n'(0) = 0 \end{cases} \tag{6.2.19}$$

and

$$\begin{cases} h_n''(t) + \left(\frac{cn\pi}{l}\right)^2 h_n(t) = \beta_n t \\ h_n(0) = 0, \qquad g_n'(0) = 0. \end{cases} \tag{6.2.20}$$

If $g_n(t)$ *and* $h_n(t)$ *are solutions to the equations* (6.2.19) *and* (6.2.20), *respectively, then by the superposition principle* $w_n = g_n + h_n$ *will be the solution to the equation* (6.2.18). *We know that the solution* $g_n(t)$ *will be*

$$g_n(t) = \gamma_n \cos \frac{cn\pi}{l}. \tag{6.2.21}$$

Using Laplace transform we can solve h_n *and thus also* $w_n(t)$.

Technique 1: *Use the superposition principle to deal with inhomogeneous terms one at time.*

6.3 The Dirichlet problem

The Dirichlet problem is to find a solution of Laplace's equation in a region $\mathbf{D}$ that assumes given values on the boundary $\partial\mathbf{D}$ of $\mathbf{D}$:

$$\nabla^2 u(r,\theta) = 0, \quad \text{in} \quad \mathbf{D}, \qquad u(x) = f(x) \quad \text{for} \quad x \in \partial\,\mathbf{D}. \tag{6.3.1}$$

This can be interpreted physically as finding the steady-state temperature in $\mathbf{D}$, when the temperature on $\partial\mathbf{D}$ is known.

Let us now solve the following Dirichlet problem in polar coordinates.

$$\begin{cases} \Delta u = \nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r_0 < r < r_1, \quad 0 \le \theta \le 2\pi \\ u(r_1,\theta) = f(\theta) \\ u(r_0, g) = g(\theta) \end{cases}$$

Using separation of variables $u(r,\theta) = R(r)\Theta(\theta)$, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \text{and} \quad \frac{r^2R'' + rR'}{R} = \frac{-\Theta''}{\Theta} = \lambda.$$

Then we solve the equation $\Theta'' + \lambda\Theta = 0$, where Θ and Θ' are $2\pi-$periodic. $\lambda = \nu^2$ gives solutions on the form $\Theta(\theta) = e^{\pm i\nu\theta}$ and since the function is $2\pi-$periodic $\pm\nu = n$ must be an integer, i.e, $\Theta_n(\theta) = e^{in\theta}$. Now since $\lambda = \nu^2 = n^2$ we get the equation

$$r^2R'' + rR'' - n^2R = 0, \tag{6.3.2}$$

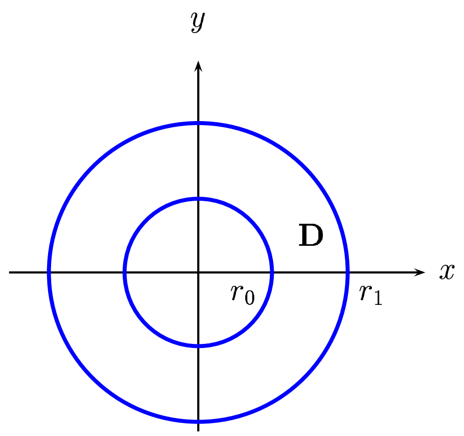


Figure 6.1: **D** is the ring $1 \leq r^2 = x^2 + y^2 \leq 2$.

which is an *Euler equation* with the solutions on the form $R = r^p$. Inserting $R = r^p$ in (6.3.2) it follows that

$$r^2 p(p-1)r^{p-2} + rpr^{p-1} - n^2 r^p = r^p(p^2 - p + p - n^2) = 0,$$

and since $r^p \neq 0$, we have $p^2 = n^2$ and $p = \pm n$. Thus we have

$$R_n(r) = a_n r^n + b_n r^{-n} \quad \text{for} \quad n \neq 0. \tag{6.3.3}$$

If $n = 0$ we get $r^2 R'' + rR' = r(rR'' + R') = 0$ and thus $\frac{d}{dr}(rR') = 0$. Hence $rR' = b_0$, which gives $R' = \frac{b_0}{r}$ and consequently

$$R_0 = a_0 + b_0 \ln r. \tag{6.3.4}$$

Summing up

$$u(r, \theta) = a_0 + b_0 \ln r + \sum_{n \neq 0} \Big(a_n r^n + b_n r^{-n} \Big) e^{in\theta}. \tag{6.3.5}$$

Using boundary data we have

$$\begin{cases} u(r_1, \theta) = a_0 + b_0 \ln r_1 + \sum_{n \neq 0} \Big(a_n r_1^n + b_n r_1^{-n} \Big) e^{in\theta} = f(\theta) = \sum_{-\infty}^{\infty} \alpha_n e^{in\theta}, \\ u(r_0, \theta) = a_0 + b_0 \ln r_0 + \sum_{n \neq 0} \Big(a_n r_0^n + b_n r_0^{-n} \Big) e^{in\theta} = g(\theta) = \sum_{-\infty}^{\infty} \beta_n e^{in\theta}. \end{cases}$$

Identifying the coefficients we get the following system of equations

$$\begin{cases} a_0 + b_0 \ln r_1 = \alpha_0 \\ a_0 + b_0 \ln r_0 = \beta_0, \end{cases} \tag{6.3.6}$$

and

$$\begin{cases} a_n r_1^n + b_n r_1^{-n} = \alpha_n, & n \neq 0 \\ a_n r_0^n + b_n r_0^{-n} = \beta_n, & n \neq 0. \end{cases} \tag{6.3.7}$$

If we let the inner radius r_0 tend to zero, then the functions will blow up at $r = 0$ unless the terms involving $\ln r$ or negative powers of r vanish. Requiring continuity of the product solutions $u(r, \theta) = R\Theta$ at $r = 0$, we thus let $b_0 = 0$, $b_n = 0$ for $n > 0$ and $a_n = 0$ for $n < 0$. It follows that

$$u(r, \theta) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta} \qquad (r_0 = 0). \tag{6.3.8}$$

Thus

$$u(r_1, \theta) = \sum_{-\infty}^{\infty} c_n r_1^{|n|} e^{in\theta} = f(\theta) = \sum_{-\infty}^{\infty} \alpha_n e^{in\theta}, \tag{6.3.9}$$

which means that the numbers $c_n r^{|n|}$ are the Fourier coefficients of $f(\theta)$. To simplify the calculation, we shall take $r_1 = 1$. We recall that the Fourier coefficients of $f(\varphi)$ are given by

$$c_n r_1^{|n|} = c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi. \tag{6.3.10}$$

Substituting (6.3.10) into the formula (6.3.8) for $u(r, \theta)$ we get

$$u(r, \theta) = \sum_{-\infty}^{\infty} \frac{1}{2\pi} r^{|n|} e^{in\theta} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \sum_{-\infty}^{\infty} r^{|n|} e^{in(\theta - \varphi)} d\varphi.$$

Let $\psi = \theta - \varphi$, then

$$\sum_{-\infty}^{\infty} r^{|n|} e^{in\psi} = \sum_{0}^{\infty} r^n e^{in\psi} + \sum_{1}^{\infty} r^n e^{-in\psi} = \sum_{0}^{\infty} \left(re^{i\psi}\right)^n + \sum_{1}^{\infty} \left(re^{-i\psi}\right)^n.$$

For $r < 1$ the geometric series converge and this fact justifies the interchange of integration and summation. Thus we have

$$\begin{aligned} \sum_{-\infty}^{\infty} r^{|n|} e^{in\psi} &= \frac{1}{1 - re^{i\psi}} + \frac{re^{-i\psi}}{1 - re^{-i\psi}} = \frac{1 - re^{-i\psi} + (1 - re^{i\psi}) re^{-i\psi}}{(1 - re^{i\psi})(1 - re^{-i\psi})} \\ &= \frac{1 - r^2}{1 + r^2 - r(e^{i\psi} + e^{-i\psi})} = \frac{1 - r^2}{1 + r^2 - 2r\cos\psi} := P(r, \psi), \end{aligned}$$

where $P(r,\psi)$ is the *Poisson kernel*. Now we have

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\sum_{-\infty}^{\infty} r^{|n|}e^{in\psi}d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)P(r,\psi)d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos\psi}f(\varphi)d\varphi. \end{aligned} \tag{6.3.11}$$

which is the *Poisson integral formula* with $r_1 = 1$.

6.4 Heat flow

A general 3-D heat flow equation can be represented as

$$\frac{\partial}{\partial t}\int\int\int_D q dV = -\int\int_{\partial D}\mathbf{j}\cdot\hat{n}ds + \int\int\int_D F dV. \tag{6.4.1}$$

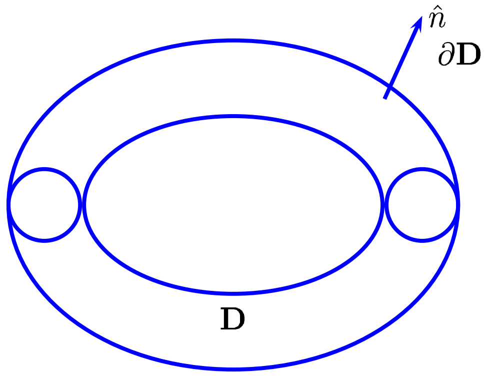


Figure 6.2: a two dimensional cross-section of **D**.

Using the divergence theorem:

$$\int\int_{\partial D}\mathbf{j}\cdot\hat{n}\,ds = \int\int\int_D \nabla\cdot\mathbf{j}\,dV, \tag{6.4.2}$$

the equation (6.4.1) can be rewritten as

$$\int\int\int_D \Big(\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} - F\Big) dV = 0, \tag{6.4.3}$$

where

$q = q(\mathbf{x}, t)$,	is the energy (heat condensation)	$[J/m^3]$
$\rho = \rho(\mathbf{x}, t)$,	is the density	$[kg/m^3]$
$c = c(\mathbf{x}, t)$,	is the heat capacity	$[J/kgK]$
$u = u(\mathbf{x}, t)$,	is the temperature	$[K]$
$\mathbf{j} = \mathbf{j}(\mathbf{x}, t)$,	the density of the flow	$[J/m^2s]$
$F = F(\mathbf{x}, t)$,	is the heat gain	$[J/m^3s]$
$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = F$,	is the continuity equation,	
$\mathbf{j} = -r \cdot \nabla u$,	is Fourier's law,	
$r = r(\mathbf{x}, t)$,	the heat transferability	$[J/msK]$.

Finally using the relations

$$\Delta q = c\rho \cdot \nabla u \quad \text{and} \quad \frac{\partial q}{\partial t} = c\rho \frac{\partial u}{\partial t}, \tag{6.4.4}$$

we get that

$$c\rho \frac{\partial u}{\partial t} - \nabla(r \cdot \nabla u) = F. \tag{6.4.5}$$

One-dimensional heat flow

Let $F(x,t)$ model the effect of some mechanism that adds or subtracts heat from the rod, perhaps a nuclear reaction within the rod itself. Suppose that we lose heat from the system, which is proportional to $u(x,t)$, where $F(x,t) = -p(x)u(x,t)$. Suppose also that the density function is given be $c\rho(x,t) = \alpha(t)w(x)$ and the hear transferability function is $r = r(x)$. Then we have

$$\alpha(t)w(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(r(x)\frac{\partial u}{\partial x}\Big) + p(x)u = 0 \tag{6.4.6}$$

Let now $u = u(x,t) = X(x)T(t)$, then we have

$$\alpha(t)w(x)XT' = \frac{\partial}{\partial x}\Big(r(x)X'\Big)T - p(x)XT, \tag{6.4.7}$$

Thus

$$\frac{\alpha T'}{T} = -\frac{\left(rX'\right)' - pX}{wX} = -\lambda, \tag{6.4.8}$$

and we have the following equations for X and T,

$$\begin{cases} \left(rX'\right)' - pX + \lambda wX = 0, & \text{an Sturm-Liouville problem} \\ \alpha T' + \lambda T = 0, & \text{an ODE problem.} \end{cases} \tag{6.4.9}$$

The boundary condition for the Sturm-Liouville problem is given by

$$\mathbf{j} \cdot \hat{n} = -r(x)\frac{\partial}{\partial n} = ku \quad \Longrightarrow \quad ku + r(x)\frac{\partial u}{\partial n} = 0, \qquad k \geq 0 \tag{6.4.10}$$

Suppose $a \leq x \leq b$, and assume the homogeneous boundary conditions as

$$\begin{cases} \alpha_0 X(a) + \alpha_1 X'(a) & = 0, \\ \beta_0 X(b) + \beta_1 X'(b) & = 0, \end{cases} \qquad \alpha_0,\ \alpha_1\ \beta_0,\ \beta_1 \geq 0. \tag{6.4.11}$$

The eigenvalues are $\lambda = \lambda_n \geq 0$ associated with the eigenfunctions $X_n(x) = \varphi_n(x)$, where $\{\varphi_n\}_{n=1}^{\infty}$ are orthogonal basis for $L_2^w(a,b)$, and the solution of the Sturm-Liouville problem is continued as earlier.
The ODE for T is written as

$$T'(t) = -\frac{\lambda_n}{\alpha(t)}T(t), \tag{6.4.12}$$

which has the general solution of the form

$$T(t) = T_n(t) = C_n e^{-\lambda_n \int_0^t \frac{1}{\alpha(t)}dt}, \qquad C_n = T_n(0). \tag{6.4.13}$$

Now since

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty} T_n(t)\varphi_n(x), \tag{6.4.14}$$

we get for $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\varphi_n(x) = \sum_{n=1}^{\infty} C_n\varphi_n(x) = f(x), \tag{6.4.15}$$

where C_n are the Fourier coefficients of f with respect to $\varphi_n(x) = X_n(x)$:

$$C_n = \frac{1}{\|\varphi_n\|^2}\int_a^b f(x)\varphi_n(x)w(x)dx. \tag{6.4.16}$$

Sturm-Liouoville problem with periodic boundary conditions

Consider the following Sturm-Liouoville problem with periodic boundary conditions:

$$\begin{cases} X''(x) + \lambda X(x) = 0, & -l \leq x \leq l \\ X(-l) = X(l), & X'(-l) = X'(l). \end{cases} \tag{6.4.17}$$

Let as usual $\lambda = \nu^2$, then we know that

$$X(x) = C_1 e^{i\nu x} + C_2 e^{-i\nu x}. \tag{6.4.18}$$

Thus the boundary conditions yield the following system of equations for the coefficients C_1 and C_2:

$$\begin{cases} C_1 e^{-i\nu l} + C_2 e^{i\nu l} = C_1 e^{i\nu l} + C_2 e^{-i\nu l} \\ i\nu\Big(C_1 e^{-i\nu l} - C_2 e^{i\nu l}\Big) = i\nu\Big(C_1 e^{i\nu l} - C_2 e^{-i\nu l}\Big) \end{cases} \tag{6.4.19}$$

From this equation system we get

$$\begin{cases} 2C_1 e^{-i\nu l} = 2C_1 e^{i\nu l} \\ 2C_2 e^{-i\nu l} = 2C_2 e^{i\nu l} \end{cases} \qquad \Longrightarrow \qquad e^{2i\nu l} = e^{-2i\nu l} = 1. \tag{6.4.20}$$

Thus we have $2\nu l = 2n\pi$, where n is an integer, hence $\nu = \frac{n\pi}{l}$ and the eigenfunctions are given by

$$\varphi(x) = e^{\frac{in\pi x}{l}}, \quad \text{for} \quad n = 0. \pm 1, \pm 2, ... \tag{6.4.21}$$

Consequently, in the periodic case the trigonometric Fourier series for f, see (6.4.15), would be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \varphi_n(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}, \tag{6.4.22}$$

with the Fourier coefficients

$$C_n = \frac{1}{\|\varphi_n\|^2}\langle f, \varphi_n\rangle, \qquad \|\varphi_n\|^2 = \int_{-l}^{l} |\varphi_n(x)|^2 dx = \int_{-l}^{l} |e^{in\pi x/l}|^2 dx = 2l.$$

Thus we have

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\overline{\varphi(x)}dx = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l}\, dx. \tag{6.4.23}$$

We can alternatively use the real Fourier coefficients, viz,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\Big), \qquad \text{where} \tag{6.4.24}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \qquad n = 0, 1, 2, ..., \tag{6.4.25}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \qquad n = 1, 2, ... \tag{6.4.26}$$

In general if $f \in L_2(-l, l)$ the the Fourier series for f, given by (6.4.22) or (6.4.24), converges to f in norm. If f is piecewise smooth then these series would converge to $\frac{1}{2}[f(x+) + f(x-)]$.

If $\{\varphi_k\}_{n=0}^{\infty}$ is an orthogonal basis then using Parseval's equation we have

$$\|f\|^2 = \sum_{-\infty}^{\infty}|C_n|^2\|\varphi_n\|^2 \quad\Longleftrightarrow\quad \int_{-l}^{l}|f(x)|^2dx = 2l\sum_{-\infty}^{\infty}|C_n|^2. \tag{6.4.27}$$

Further if $\{\varphi_k\}_{n=0}^{\infty}$ form an orthonormal basis then by the projection theorem the projection error:

$$\|f - \sum_{k=1}^{N} a_k\varphi_k\|, \tag{6.4.28}$$

is minimal when

$$a_k = C_k = \frac{1}{\|\varphi\|^2}\langle f, \varphi_k\rangle, \tag{6.4.29}$$

and the minimum value (of the projection error) is given by

$$\|f\|^2 - \sum_{n=1}^{N}|c_k|^2\|\varphi_k\|^2. \tag{6.4.30}$$

Inhomogeneous heat flow problem

Consider the following inhomogeneous heat flow problem:

$$\begin{cases} \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 y, & 0 < x < 1, \qquad 0 < y < 2 \\ u(0, y) = y, & u(1, y) = 1 - 2y^2 \\ u(x, 0) = 0, & u(x, 2) = 1. \end{cases} \tag{6.4.31}$$

We write $u = v + w$ and consider the same partial differential equation for v with homogeneous data in x and the same data in y as in the original problem for u. As for w we use the homogeneous version of the differential equations with homogeneous data in y and the data of the original problem in x:

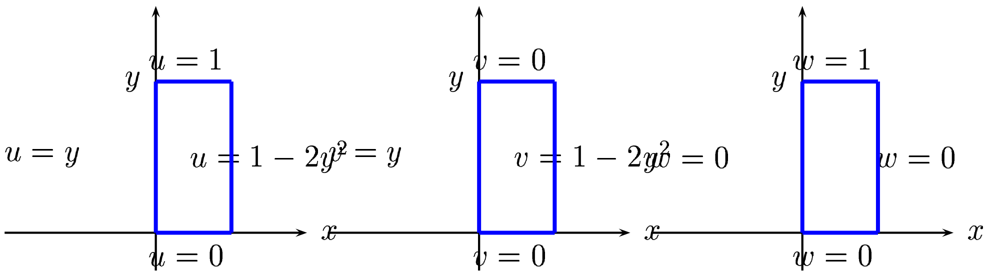


Figure 6.3: The splitting $u(x, u) = v(x, y) + w(x, y)$.

Thus we write $u(x, y) = v(x, y) + w(x, y)$, with

$$\begin{cases} \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = x^2 y, & 0 < x < 1, \qquad 0 < y < 2 \\ v(0, y) = y, & v(1, y) = 1 - 2y^2 \\ v(x, 0) = 0, & v(x, 2) = 0 \end{cases} \tag{6.4.32}$$

and

$$\begin{cases} \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, & 0 < x < 1, \qquad 0 < y < 2 \\ w(0, y) = 0, & w(1, y) = 0 \\ w(x, 0) = 0, & w(x, 2) = 1. \end{cases} \tag{6.4.33}$$

We start solving the homogeneous equation for w. Let $w(x, y) = X(x)Y(y) \neq 0$, which give us

$$\nabla^2 w = X''Y + XY'', \quad \text{and} \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \tag{6.4.34}$$

Then, for X, we have the ODE

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < 1 \\ X(0) = X(1) = 0, \end{cases} \tag{6.4.35}$$

having the eigenvalues $\lambda = \lambda_n = (n\pi)^2$ and the corresponding eigenfunctions $X_n(x) = \sin(n\pi x)$ for $n \geq 1$. Then the ODE for Y will be

$$\begin{cases} Y'' - (n\pi)^2 Y = 0, & 0 < y < 2 \\ Y(0) = 0, \end{cases} \tag{6.4.36}$$

with the eigenfunctions given by $Y_n(y) = C_n \sinh(n\pi y)$. Thus

$$w(x, y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi y) \sin(n\pi x). \tag{6.4.37}$$

To determine the coefficients C_n we use the boundary data viz,

$$w(x, 2) = \sum_{n=1}^{\infty} C_n \sinh(2n\pi) \sin(n\pi x) = 1. \tag{6.4.38}$$

The Fourier coefficients are

$$C_n \sinh(2n\pi) = \frac{1}{1/2} \int_0^1 \sin(n\pi x) dx = 2\Big[- \frac{\cos(n\pi x)}{n\pi} \Big]_0^1 = \frac{2}{n\pi}\Big(1 - (-1)^n\Big).$$

The problem (6.4.33) for v is homogeneous in y, and the corresponding eigenfunctions are $\sin(n\pi y/2)$. Let now

$$v(x, y) = \sum_1^{\infty} v_n(x) \sin\left(\frac{n\pi y}{2}\right). \tag{6.4.39}$$

Then we have

$$\nabla^2 v = \sum_1^{\infty} \left[v_n''(x) - \left(\frac{n\pi}{2}\right)^2 v_n(x) \right] \sin\left(\frac{n\pi y}{2}\right) = x^2 y = x^2 \sum_1^{\infty} \alpha_n \sin\left(\frac{n\pi y}{2}\right).$$

Now we invoke the boundary data in v:

$$\begin{aligned} v(0, y) &= \sum_1^{\infty} v_n(0) \sin\left(\frac{n\pi y}{2}\right) = y = \sum_1^{\infty} \alpha_n \sin\left(\frac{n\pi y}{2}\right). \\ v(1, y) &= \sum_1^{\infty} v_n(1) \sin\left(\frac{n\pi y}{2}\right) = 1 - 2y^2 = \sum_1^{\infty} \gamma_n \sin\left(\frac{n\pi y}{2}\right). \end{aligned} \tag{6.4.40}$$

To find out $v_n(x)$ we identify the coefficients in both sides of the last three formulas. Then it follows that $v_n(x)$ satisfies the ODE:

$$\begin{cases} v_n'' - -\left(\frac{n\pi}{2}\right)^2 v_n = \alpha_n x^2 \\ v_n(0) = \alpha_n \\ v_n(1) = \gamma_n \end{cases} \tag{6.4.41}$$

Solving (6.4.41) we finally can get $u(x, y) = v(x, y) + w(x, y)$.

Chapter 7

Bessel functions

Bessel functions are solutions for the Bessel's differential equation. This equation and its variants arise in many problems in physics and engineering, particularly where some sort of circular symmetry is involved.

We start with the two-dimensional wave equation in polar coordinates:

$$u_{tt} = c^2\nabla^2 u(r,\theta,t) = c^2\Big(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\Big). \tag{7.0.1}$$

By the separation of variables, $u(r,\theta) = R(r)\Theta(\theta)T(t) \neq 0$, this equation is rewritten as

$$T''R\Theta = c^2\Big(TR''\Theta + \frac{1}{r}TR'\Theta + \frac{1}{r^2}TR\Theta''\Big), \tag{7.0.2}$$

which, in the usual way, yields

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\mu^2. \tag{7.0.3}$$

Thus we have *one* ordinary differential equation for T, viz

$$T'' + c^2\mu^2 T = 0, \tag{7.0.4}$$

and the other equation, involving both R and Θ is

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\mu^2, \tag{7.0.5}$$

which multiplying through by r^2, is written in the following, separated form:

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \mu^2r^2 = -\frac{\Theta''}{\Theta} = \nu^2. \tag{7.0.6}$$

Now we have another ordinary differential equation for Θ, viz

$$\Theta'' + \nu^2\Theta = 0, \tag{7.0.7}$$

The left hand side gives the third ordinary differential equation for R:

$$r^2R'' + rR + (\mu^2r^2 - \nu^2)R = 0. \tag{7.0.8}$$

Now we simplify the equation (7.0.8) using the change of variable $r = x/\mu$, and introducing the new function $f(x) := R(r)$, hence

$$R(r) = f(\mu r), \qquad R'(r) = \mu f'(\mu r), \qquad R''(r) = \mu^2 f''(\mu r). \tag{7.0.9}$$

Thus (7.0.8) is rewritten as

$$\left(\frac{x}{\mu}\right)^2 \mu^2 f''(x) + \frac{x}{\mu}\mu f'(x) + (x^2 - \nu^2)f(x) = 0, \tag{7.0.10}$$

or equivalently

$$x^2 f''(x) + xf'(x) + (x^2 - \nu^2)f(x) = 0. \tag{7.0.11}$$

This is the *Bessel's differential equation of order* ν. It and its variants arise in many problems in physics and engineering, particularly where some sort of circular symmetry is involved.

7.1 Solutions of Bessel's differential equation

The Bessel's equation has a *regular singular point* at $x = 0$, so we expect to find solutions of the form

$$f(x) = \sum_{j=0}^{\infty} a_j x^{j+b}, \quad a_0 \neq 0, \tag{7.1.1}$$

which we substitute into Bessel's equation and, when ν is not a negative integer, we get the relation on the coefficients a_k as:

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k}k!(1+\nu)(2+\nu)\cdot\ldots\cdot(k+\nu)}, \qquad a_{2k+1} = 0. \tag{7.1.2}$$

Thus, in short, except when ν is a negative integer we have the solution

$$f(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k} k!(1+\nu)(2+\nu) \cdot \ldots \cdot (k+\nu)}. \tag{7.1.3}$$

The standard choice for a_0 is

$$a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}, \tag{7.1.4}$$

where the *gamma function*, $\Gamma(z)$, is defined in the complex half-plane, $Re z > 0$, by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{7.1.5}$$

From the definition above, we extract some of the most important basic properties of the gamma function:
The second functional equation:

$$\Gamma(z+1) = z\Gamma(z) \tag{7.1.6}$$

The second functional equation:

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z). \tag{7.1.7}$$

Let $z = 1$. Then we get

$$\Gamma(1) = \int_0^{\infty} t^0 e^{-t} dt = \Big[-e^{-t} \Big]_0^{\infty} = 1. \tag{7.1.8}$$

Then the first functional equation is giving us for $z = 1$,

$$\Gamma(n+1) = 1 \cdot 2 \cdot \ldots \cdot n\Gamma(1) = n!. \tag{7.1.9}$$

The first functional equation can be iterated and consequently we get

$$\Gamma(z+n) = z(z+1) \cdot \ldots \cdot (z+n-1)\Gamma(z). \tag{7.1.10}$$

Thus $\Gamma(z)$, or rather $\Gamma(z+1)$, provides a natural extension of the factorial function to numbers other than integers:

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1) \cdot \ldots \cdot (z+n-1)}. \tag{7.1.11}$$

Another important result is that

$$\Gamma\Big(n+\frac{1}{2}\Big)=\frac{1}{2}\Big(\frac{3}{2}\Big)\cdot\ldots\cdot\Big(n-\frac{1}{2}\Big)\sqrt{\pi}. \tag{7.1.12}$$

Now we return to solution (7.1.13), of the Bessel's equation (7.0.11), where a_0 is given by (7.1.4). This choice of a_0 give us the following theorem, where $f(x) = J_\nu(x)$:

Theorem 38. *A solution to Bessel's differential equation of order ν is given by*

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}. \tag{7.1.13}$$

$J_\nu(x)$ is called the Bessel function of order ν.

Proof. To show that $J_\nu(x)$ is a solution to the Bessel's equation of order ν; (7.0.8) we let

$$y(x)=J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}.$$

First we assume that ν is not a negative integer: $\nu \neq -1, -2, \ldots.$ Then the series converges for all $x \neq 0$ and we have that

$$y'(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(2k+\nu)x^{2k+\nu-1}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}, \tag{7.1.14}$$

$$y''(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(2k+\nu)(2k+\nu-1)x^{2k+\nu-2}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}. \tag{7.1.15}$$

Thus letting $A(x) := x^2y''(x) + xy'(x) - \nu^2y(x)$ we get

$$\begin{aligned}A(x)&=\sum_{k=0}^{\infty}\frac{(-1)^k[(2k+\nu)(2k+\nu-1)+2k+\nu-\nu^2]}{k!\,\Gamma(k+\nu+1)2^{2k+\nu}}x^{2k+\nu}\\&=\sum_{k=0}^{\infty}\frac{(-1)^k4k(k+\nu)}{k!\Gamma(k+\nu+1)2^{2k+\nu}}x^{2k+\nu}.\end{aligned}$$

Now since $(k+\nu)\Gamma(k+\nu) = \Gamma(k+\nu+1)$ it follows that

$$A(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 4k(k+\nu)}{k!(k+\nu)\Gamma(k+\nu)2^{2k+\nu}} x^{2k+\nu} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+\nu}}{(k-1)!\Gamma(k+\nu)2^{2(k-1)+\nu}}$$

$$\sum_{k=0}^{\infty} \frac{-(-1)^k x^{2k+2+\nu}}{k!\Gamma(k+1+\nu)2^{2k+\nu}} - x^2 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{k!\Gamma(k+1+\nu)2^{2k+\nu}} = -x^2 y,$$

where on the last line the sum is rewritten using the shift law: $(k-1) \to k$. Thus we have that $y(x) = J_\nu(x)$ satisfies the Bessel's equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2) y(x) = 0.$$

□

Lemma 14. *If $\nu = n$ is an integer we have*

$$J_{-n}(x) = (-1)^n J_n(x). \tag{7.1.16}$$

Proof. Let $\nu = -n$, where $n = 1, 2, \ldots$. For $k = 0, 1, \ldots, (n-1)$ we have by the basic properties of gamma functions that

$$\frac{1}{\Gamma(k-n+1)} = 0. \tag{7.1.17}$$

This simply follows from (7.1.11) with $z = k-n+1$, i.e.,

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdot \ldots \cdot (z+n-1)}$$

implies that

$$\frac{1}{\Gamma(k-n+1)} = \frac{(k-n+1)(k-n+2)\ldots k}{\Gamma(k+1)} = 0, \qquad k = 0, 1, \ldots, n-1.$$

Thus, the first n terms in the series

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-n+1)} \left(\frac{x}{2}\right)^{2k-n},$$

vanish. Using $\Gamma(j+1) = j!$, and then setting $\ell = k-n$ we find that

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!(k-n)!} \left(\frac{x}{2}\right)^{2k-n} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(\ell+n)!} \left(\frac{x}{2}\right)^{2\ell+n} = (-1)^n J_n(x),$$

and the proof is complete. □

If ν is not a integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are linear independent and the general solution is the linear combination

$$C_1 J_\nu(x) + C_2 J_{-\nu}(x). \tag{7.1.18}$$

Thus we want to find a solution, $Y_\nu(x)$, to the Bessel's equation, which is a linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$. The standard solution is given in the following definition.

Definition 41. *The Weber function $Y_\nu(x)$, (which is also known as Neumann function or Bessel function of the second kind), is defined by*

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad \textit{where} \quad \nu \neq \textit{integer}, \tag{7.1.19}$$

$$\lim_{\nu \to n} Y_\nu(x) = Y_n(x). \tag{7.1.20}$$

7.2 The Recurrence Formulas

There is a set of algebraic identities relating the Bessel's function J_ν and its derivate to the functions $J_{\nu-1}$ and $J_{\nu+1}$. In this regard we have the following *recurrence formulas*:

Theorem 39. *For all x and ν, we have*

$$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x), \tag{7.2.1}$$

$$[x^{\nu} J_\nu(x)] = x^{\nu} J_{\nu-1}(x), \tag{7.2.2}$$

$$x[J_{\nu-1}(x) + J_{\nu+1}(x)] = 2\nu J_\nu(x), \tag{7.2.3}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x)] = 2J'_\nu(x), \tag{7.2.4}$$

$$xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x), \tag{7.2.5}$$

$$xJ'_\nu(x) + \nu J_\nu(x) = xJ_{\nu-1}(x). \tag{7.2.6}$$

Proof. Below we give the proofs for (7.2.2), (7.2.5) and (7.2.6), since the proof of formulas (7.2.1) and (7.2.2) are near similar, and (7.2.3) and (7.2.4) follow by subtracting and adding (7.2.5) and (7.2.6). Thus we start to prove formula (7.2.2) using the power series

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} k!\Gamma(k+\nu+1)}.$$

Multiplying by x^ν and differentiating give us

$$\frac{d}{dx}[x^\nu J_\nu(x)] = \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2\nu}}{2^{2k+\nu}k!\Gamma(k+\nu+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k 2(k+\nu)x^{2k+2\nu-1}}{2^{2k+\nu}k!\Gamma(k+\nu+1)}.$$

Now since $\Gamma(k+\nu+1) = (k+\nu)\Gamma(k+\nu)$ it follows that

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+\nu-1}}{2^{2k+\nu-1}k!\Gamma(k+\nu)} = x^\nu J_{\nu-1}(x),$$

and the proof of (7.2.2) is complete.

To prove (7.2.5): performing the indicated differentiation in (7.2.1), we obtain

$$\frac{d}{dx}[x^{-\nu} J_\nu(x)] = -\nu x^{-\nu-1} J_\nu(x) + x^{-\nu} J_\nu'(x).$$

Thus by formula (7.2.2) we have

$$-\nu x^{-\nu-1} J_\nu(x) + x^{-\nu} J_\nu'(x) = -x^{-\nu} J_{\nu+1}(x),$$

and multiplying by $x^{\nu+1}$, we obtain

$$xJ_\nu'(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x),$$

which is the formula (7.2.5).

Formula (7.2.6) is proved similarly: Performing the indicated differentiation in (7.2.2), we obtain

$$\frac{d}{dx}[x^\nu J_\nu(x)] = \nu x^{\nu-1} J_\nu(x) + x^\nu J_\nu'(x).$$

Thus using formula (7.2.2) we get

$$\nu x^{\nu-1} J_\nu(x) + x^\nu J_\nu'(x) = x^\nu J_{\nu-1}(x),$$

which multiplying by $x^{-\nu+1}$, yields

$$xJ_\nu'(x) + \nu J_\nu(x) = xJ_{\nu-1}(x)$$

and hence the proof of (7.2.6) is complete. □

Bessel fictions of half-integer order

Bessel functions of half-integer order can be expressed in Taylor series of $\cos(x)$ and $\sin(x)$,

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos(x) \quad \text{and} \quad J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin(x). \tag{7.2.7}$$

Using the recurrence formula (7.2.3), we obtain

$$J_{\frac{3}{2}}(x) = x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x). \tag{7.2.8}$$

Bessel functions of integer order

We recall that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of numbers, the *generating function* for a_n is the power series $\sum_{n=0}^{\infty} a_n z^n$.

Theorem 40. *The generating function for $J_n(x)$ for all x and all $z \neq 0$ is*

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}. \tag{7.2.9}$$

Proof. We first note that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{x}{2}z} \cdot e^{-\frac{x}{2z}}. \tag{7.2.10}$$

Using the Taylor series for e^x give us absolutely convergent series viz,

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{xz}{2}\right)^j \cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-x}{2z}\right)^k = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{j!k!}\left(\frac{x}{2}\right)^{j+k} z^{j-k}.$$

Let now $j = k + n$. We know also that

$$\frac{1}{(k+n)!} = \frac{1}{\Gamma(k+n+1)} = 0, \quad \text{when} \quad k+n < 0.$$

Then

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] z^n = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

and the proof is complete. □

Bessel's Integral Formulas

In the generating function for $J_n(x)$ formula, z can be any nonzero complex number. Let $z = e^{i\theta} = \cos\theta + i\sin\theta$ and $z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta$, then

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{ix\sin\theta}. \tag{7.2.11}$$

Using the the generating function (7.2.9) for $J_n(x)$, we get

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta}. \tag{7.2.12}$$

Hence $J_n(x)$, are given by the Fourier coefficients for $e^{ix\sin\theta}$,

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}\cdot e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\theta - n\theta)}d\theta. \tag{7.2.13}$$

Because of symmetry we replace θ by $-\theta$, in (7.2.13), then

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta}\cdot e^{in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(x\sin\theta - n\theta)}d\theta. \tag{7.2.14}$$

Thus adding up we get

$$2J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\theta - n\theta)}d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(x\sin\theta - n\theta)}d\theta. \tag{7.2.15}$$

Now using $e^{i(x\sin\theta - n\theta)} = \cos(x\sin\theta - n\theta) + i\sin(x\sin\theta - n\theta)$ we get

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta)d\theta, \tag{7.2.16}$$

and since the integrand in (7.2.16) is even in θ it follows that

$$J_n(x) = \frac{1}{\pi}\int_{0}^{\pi} \cos(x\sin\theta - n\theta)d\theta, \tag{7.2.17}$$

which is one of *Bessel's Integral Formulas.*

An alternative proof of Bessel's theorem. Below, using Bessel's recurrence formula (7.2.5), we give an easier way to prove the theorem of Bessel's differential equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{7.2.18}$$

We write

$$\begin{aligned} x^2J_\nu'' + xJ_\nu' &= x(xJ_\nu'' + J_\nu') = x(xJ_\nu')' \\ &= x(\nu J_\nu - xJ_{\nu+1})' = \nu xJ_\nu' - xJ_{\nu+1} - x^2J_{\nu+1}'. \end{aligned} \tag{7.2.19}$$

Using (7.2.5) once again we get $\nu xJ_\nu' = \nu(\nu J_\nu - xJ_{\nu+1})$ thus

$$x^2J_\nu''+xJ_\nu' = \nu(\nu J_\nu - xJ_{\nu+1}) - xJ_{\nu+1} - x^2J_{\nu+1}' = \nu^2J_\nu - \nu xJ_{\nu+1} - xJ_{\nu+1} - x^2J_{\nu+1}'.$$

Hence

$$x^2J_\nu'' + xJ_\nu' - \nu^2J_\nu = -x(\nu J_{\nu+1} + J_{\nu+1} + xJ_{\nu+1}'). \tag{7.2.20}$$

Further using recurrence formula (7.2.6) with ν replaced by $\nu + 1$ we get $xJ_{\nu+1}' = xJ_\nu - (\nu+1)J_{\nu+1}$ and consequently

$$\begin{aligned} x^2J_\nu'' + xJ_\nu' - \nu^2J_\nu &= -x(\nu J_{\nu+1} + J_{\nu+1} + xJ_\nu - (\nu+1)J_{\nu+1}) \\ &= -x((\nu+1)J_{\nu+1} + xJ_\nu - (\nu+1)J_{\nu+1}) \\ &= -x^2J_\nu, \end{aligned} \tag{7.2.21}$$

so that

$$x^2J_\nu'' + xJ_\nu' + (x^2 - \nu^2)J_\nu = 0 \tag{7.2.22}$$

and the proof is complete. □

Exempel 23. *Show that*

$$\int x^3J_0(x)dx = (x^3 - 4x)J_1(x) + 2x^2J_0(x) + C. \tag{7.2.23}$$

Solution. The recurrence formula (7.2.1) *with* $\nu = 0$ *yields*

$$\frac{d}{dx}[x^0J_0(x)] = -x^0J_1(x) \quad \textit{or} \quad J_0'(x) = -J_1(x). \tag{7.2.24}$$

Hence

$$\int J_1(x)dx = J_0(x) + C. \tag{7.2.25}$$

We write

$$\int x^3 J_0(x)dx = \int x^2 \cdot xJ_0(x)dx, \tag{7.2.26}$$

and use the recurrence formula (7.2.2) *with* $\nu = 1$ *to obtain*

$$\int x^3 J_0(x)dx = \int \left[x^2 \cdot \frac{d}{dx}[xJ_1(x)]\right] dx. \tag{7.2.27}$$

Integrating by parts we have

$$\int \left[x^2 \cdot \frac{d}{dx}[xJ_1(x)]\right] dx = x^3 J_1(x) - 2\int x^2 J_1(x)dx. \tag{7.2.28}$$

The recurrence formula (7.2.2) *(*$\nu = 2$*) give now*

$$\int x^3 J_0(x)dx = x^3 J_1(x) - 2\int x^2 J_1(x)dx = x^3 J_1(x) - 2x^2 J_2(x) + C. \tag{7.2.29}$$

Further by $\nu = 1$ *the recurrence formula* (7.2.2) *(*$\nu = 1$*),*

$$xJ_0(x) + xJ_2(x) = 2J_1(x), \tag{7.2.30}$$

thus

$$2xJ_2(x) = 4J_1(x) - 2xJ_0(x). \tag{7.2.31}$$

Then summing up we have

$$\int x^3 J_0(x)dx = x^3 J_1(x) - x(4J_1(x) - 2xJ_0(x)) + C \tag{7.2.32}$$

and finally the desired result (7.2.23)*:*

$$\int x^3 J_0(x)dx = (x^3 - 4x)J_1(x) - 2x^2 J_0(x) + C.$$

7.3 Orthogonal sets of Bessel functions

We rewrite the Bessel's differential equation (7.0.8) for R:

$$r^2 R'' + rR + (\mu^2 r^2 - \nu^2)R = 0, \tag{7.3.1}$$

as

$$x^2 f''(x) + xf'(x) + (\mu^2 x^2 - \nu^2)f(x) = 0, \tag{7.3.2}$$

where we used the identifications $x \equiv r$ and $f \equiv R$. Let $\lambda = \mu^2$ and assume that $x \neq 0$, then multiplying by $1/x$ yields

$$\frac{d}{dx}(xf'(x)) - \frac{\nu^2}{x}f(x) + \lambda x f(x) = 0, \qquad (7.3.3)$$

which associated with appropriate boundary conditions is is a Sturm-Liouville equation with

$$r(x) = x, \quad p(x) = -\frac{\nu^2}{x} \quad \text{and} \quad w(x) = x.$$

The equation (**??**) is a *regular* Sturm-Liouville problem on an interval $[a, b]$ if $0 < a < b < \infty$.

On an interval of the form $[0, b]$ and under the assumption $\nu \geq 0$ we get a *singular* Sturm-Liouville problem which, by the way, is a more interesting case. Below we examine this case: The general solution to the differential equation(**??**) is then the eigenfunctions of the form

$$f(x) = C_1 J_\nu(\mu x) + C_2 Y_\nu(\mu x). \qquad (7.3.4)$$

The boundary condition at $x = 0$ is interpreted as that $f(0+)$ exists, which means that $C_1 \neq 0$ and $C_2 = 0$, whereas the boundary condition at $x = b$ can be taken as $\beta f(b) + \beta' f'(b) = 0$, which means that

$$\beta J_\nu(\mu b) + \beta' \mu J'_\nu(\mu b) = 0. \qquad (7.3.5)$$

Let now $f(x) = J_\nu(\mu_j x)$ and $g(x) = J_\nu(\mu_k x)$ be the eigenfunctions, then

$$\langle L(f), g\rangle = \langle f, L(g)\rangle, \quad \text{where} \quad L(f) = \frac{d}{dx}(xf') - \frac{\nu^2}{x}f. \qquad (7.3.6)$$

As in the PDE chapter, we still have real eigenvalues and the eigenfunctions are orthogonal with respect to the weight function $w(x) = x$,.

Let $\beta \neq 0$ and $\beta' \neq 0$, which is the most usual case. Then all the eigenvalues are nonnegative and thus for

$$\lambda > 0, \quad \lambda_k = \mu_k^2 = \left(\frac{\alpha_k}{b}\right), \qquad (7.3.7)$$

where $\alpha_k \geq 0$ are the roots of the equation

$$\beta J_\nu(\alpha) + \frac{\beta'}{b} J'_\nu(\alpha) = 0, \qquad (7.3.8)$$

where $\mu b = \alpha$ and the eigenfunctions are

$$\varphi_k(x) = J_\nu\Big(\frac{\alpha_k x}{b}\Big). \tag{7.3.9}$$

We summarize this section formulating some fundamental results (without proofs)

Lemma 15. *If $\beta = 0$ and $\nu = 0$, then $\lambda_0 = 0$ also is an eigenvalue with the eigenfunction $\varphi_0(x) = 1$.*

Lemma 16. *If $\mu > 0$, $b > 0$ and $\nu \geq 0$, then*

$$\int_0^b J_\nu(\mu x)^2\, x\, dx = \frac{b^2}{2} J_\nu'(\mu b)^2 + \frac{\mu^2 b^2 - \nu^2}{2\mu^2} J_\nu(\mu b)^2. \tag{7.3.10}$$

Theorem 41. *Suppose $\nu \geq 0$, $b > 0$ and $w(x) = x$. Let $\{\lambda_k\}_1^\infty$ be the positive zeros of $J_\nu(x)$ and let $\phi_k(x) = J_\nu(\lambda_k x/b)$. Then $\{\phi_k\}_1^\infty$ is an orthogonal basis for $L_w^2(0,b)$ and*

$$\|\phi\|_w^2 = \frac{b^2}{2} J_{\nu+1}(\lambda_k)^2. \tag{7.3.11}$$

Heat flow in a cylinder

Exempel 24. *Let us consider following heat flow problem in a cylinder.*

$$\begin{cases} u_t &= k\nabla^2 u, \quad 0 < r \leq b, \quad 0 < \theta \leq 2\pi, \quad z_1 \leq z \leq z_2 \\ u &= A, \quad r = b, \\ u_z' &= 0, \quad z = z_1 \quad \text{and} \quad z = z_2 \\ u &= B, \quad t = 0, \end{cases} \tag{7.3.12}$$

where $u = u(r,\theta,z,t)$ and $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$. This is a Dirichlet problem in the cylinder

$$D = \{(r,\theta,z) : 0 < r \leq b, \quad 0 < \theta \leq 2\pi \quad , z_1 \leq z \leq z_2\} \tag{7.3.13}$$

Let us assume that $u(r,\theta,z,t)$ is independent of θ and z. Then we get the following problem for $u = u(r,t)$:

$$\begin{cases} u_t &= k(u_{rr} + \frac{1}{r}u_r) \\ u(b,t) &= A, \quad u(0+,t) \quad \text{exists} \\ u(r,0) &= B. \end{cases} \tag{7.3.14}$$

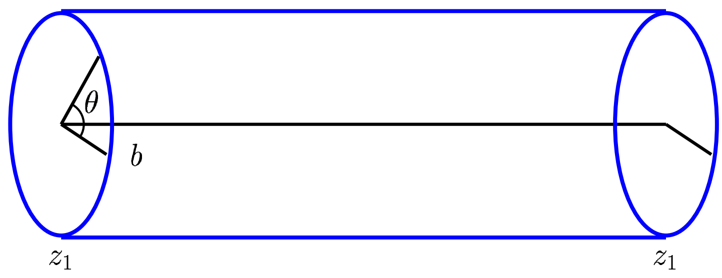


Figure 7.1: Cylindrical space domain.

For further simplification first we study the steady state problem, where $u_t = 0$. *To this approach let* $\widetilde{u}(r)$ *satisfy the differential equation*

$$\widetilde{u}'' + \frac{1}{r}\widetilde{u}' = 0, \quad \text{or equivalently} \quad \frac{d}{dr}(r\widetilde{u}') = 0. \tag{7.3.15}$$

Then we have $r\widetilde{u}' = C_1$, *i.e.,* $\widetilde{u}' = \frac{C_1}{r}$. *After integration we get*

$$\widetilde{u} = C_1 \ln r + C_2. \tag{7.3.16}$$

That $\widetilde{u}(0+)$ *exists yields* $C_1 = 0$, *and the boundary condition* $\widetilde{u}(b) = A$ *for gives that* $\widetilde{u}(r) = A$.

Let now $v(r,t) = u(r,t) - \widetilde{u}(r) = u(r,t) - A$. *Then* v *satisfies the problem:*

$$\begin{cases} v_t & = k(v_{rr} + \frac{1}{r}v_r) \\ v(b,t) & = 0, \qquad v(0+,t) \quad \text{exists} \\ v(r,0) & = B - A \end{cases} \tag{7.3.17}$$

Using separation of variables $v(r,t) = R(r)T(t) \neq 0$, *it follows that* $RT' = k\Big(R'' + \frac{1}{r}R'\Big)T$ *and hence*

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T'}{kT} = -\mu^2. \tag{7.3.18}$$

Then we have to solve the following two problems:

$$(1) \quad \begin{cases} R'' + \frac{1}{r}R' + \mu^2 R = 0 \\ R(b) = 0, \qquad R(0+) \quad \text{exists}, \end{cases} \tag{7.3.19}$$

and

$$T' = -K\mu^2 T. \tag{7.3.20}$$

For the first problem, which is a Sturm-Liouville problem, we know that the eigenfunctions will be of the form

$$R(r) = C_1 J_0(\mu r) + C_2 Y_0(\mu r). \tag{7.3.21}$$

That $R(0+)$ *exists yields* $C_2 = 0$ *(see definition of* $Y_0(\mu r)$ *and* $R(b) = 0$ *gives* $J_0(\mu b) = 0$. *Let now* $\alpha_1, \alpha_2, \ldots$ *be the positive zeros of* $J_0(x)$. *Then we have*

$$\mu b = \alpha_n \quad \text{and} \quad \mu = \mu_n = \frac{\alpha_n}{b}, \quad \text{for} \quad n \geq 1. \tag{7.3.22}$$

Thus the eigenfunctions are

$$R_n(r) = J_0\Big(\frac{\alpha_n r}{b}\Big). \tag{7.3.23}$$

Correspondingly the solution for the differential equation $T' = -K\mu_n^2 T$ *is*

$$T_n(t) = C_n e^{-k\mu_n^2 t}. \tag{7.3.24}$$

By the superposition we have

$$v(r,t) = \sum_{n=1}^{\infty} C_n e^{-k\mu_n^2 t} J_0\Big(\frac{\alpha_n r}{b}\Big). \tag{7.3.25}$$

For $t = 0$ *we get*

$$v(r,0) = \sum_{n=1}^{\infty} C_n J_0\Big(\frac{\alpha_n r}{b}\Big) = B - A. \tag{7.3.26}$$

Recalling that $\{R_n(r)\}_{n=1}^{\infty}$ *is an orthogonal basis for* $L_2^w(0,b)$, *where* $w(r) = r$, *the Fourier coefficients* C_n *are*

$$C_n = \frac{1}{\|R_n\|_w^2} \int_0^b (B - A) J_0\Big(\frac{\alpha_n r}{b}\Big) r dr. \tag{7.3.27}$$

Thus by the previous theorem

$$\|R_n\|_w^2 = \frac{b^2}{2} J_1^2(\alpha_n). \tag{7.3.28}$$

Making the substitution of variable $x = \frac{\alpha_n r}{b}$*, i.e.,* $r = \frac{bx}{\alpha_n}$*, and integrating it follows that*

$$\int_0^b J_0\Big(\frac{\alpha_n r}{b}\Big) r dr = \Big(\frac{b}{\alpha_n}\Big)^2 \int_0^{\alpha_n} J_0(x) x dx. \tag{7.3.29}$$

The second recurrence formula:

$$\int_0^{\alpha_n} J_0(x) x dx = \Big[xJ_1(x)\Big]_0^{\alpha_n} = \alpha_n J_1(\alpha_n), \tag{7.3.30}$$

yields that

$$C_n = \frac{2}{b^2 J_1^2(\alpha_n)} \cdot (B - A) \frac{b^2 J_1^2(\alpha_n)}{\alpha_n} = \frac{2(B-A)}{\alpha_n J_1(\alpha_n)}. \tag{7.3.31}$$

And finally we have, since $u_0 = \widetilde{u} = A$ *and* $J_n(\alpha_n) = 0$*, that*

$$u(r,t) = A + 2(B - A) \sum_{n=1}^{\infty} \frac{1}{\alpha_n J_1(\alpha_n)} e^{-k\frac{\alpha_n^2 t}{b^2}} J_0\Big(\frac{\alpha_n r}{b}\Big). \tag{7.3.32}$$

Exempel 25. *Solve the following heat equation in the plane:*

$$\begin{cases} u_t = k\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, & 0 < r < b, \quad 0 < \theta < 2\pi \\ u(b,\theta,t) = 0, & u(r,\theta,0) = f(r,\theta). \end{cases} \tag{7.3.33}$$

Using separation of variables $u(r,\theta,t) = R(r)\Theta(\theta)T(t) \neq 0$ *we get*

$$R\Theta T' = (R'' + \frac{1}{r}R')\Theta T + \frac{1}{r^2}R\Theta'' T. \tag{7.3.34}$$

Multiplying (7.3.34) *by* $\frac{r^2}{R\Theta T}$ *we have*

$$r^2\frac{T'}{T} - \frac{r^2 R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\nu^2. \tag{7.3.35}$$

This give us two differential equations. The differential equation for Θ *reads*

$$\Theta'' + \nu^2\Theta = 0, \tag{7.3.36}$$

where the 2π*-periodicity of* $\Theta(\theta)$ *and* $\Theta'(\theta)$ *implies that* $\nu = n =$ *integer. Thus*

$$\Theta_n(\theta) = e^{in\theta}. \tag{7.3.37}$$

The other equation is

$$r^2\frac{T'}{T} - \frac{r^2R'' + rR'}{R} = -\nu^2 = -n^2, \tag{7.3.38}$$

which in turn can be separated into two new equations, viz,

$$\frac{T'}{T} = \frac{r^2R'' + rR'}{r^2R} - \frac{n^2}{r^2} = -\mu^2. \tag{7.3.39}$$

Let us first solve R:s equation, which is a Bessel's differential equation of order n, where $\mu > 0$.

$$\begin{cases} r^2R'' + rR' + (\mu^2r^2 - n^2)R = 0 \\ R(r) \quad \text{bounded as} \quad r \to 0+, \qquad\qquad R(b) = 0. \end{cases} \tag{7.3.40}$$

As before the solution to the Bessel's equation of order n is given by

$$R(r) = C_1J_n(\mu r) + C_2Y_n(\mu r), \tag{7.3.41}$$

where since $R(r)$ is bounded as $r \to 0+$, it follows that $C_2 = 0$. Further $R(b) = 0$ yields $J_n(\mu b) = 0$. Let now α_{n_k} be the positive zeros to $J_n(x)$. Then

$$\mu b = \alpha_{n_k} \quad \text{and} \quad \mu = \mu_{n_k} = \frac{\alpha_{n_k}}{b}. \tag{7.3.42}$$

Thus we obtain, for $C_1 = 1$

$$R_{n_k}(r) = J_n\Big(\frac{\alpha_{n_k}r}{b}\Big). \tag{7.3.43}$$

The equation for T: $T' = -\mu_{n_k}^2T$, has the solution

$$T = T_{n_k}(t) = C_{n_k}e^{-\mu_{n_k}^2t}. \tag{7.3.44}$$

The superposition principle give us now the solution to the original equation in the form of

$$u(r,\theta,t) = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}T_{n_k}(t)R_{n_k}(r)\Theta_n(\theta) = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}C_{n_k}e^{-\mu_{n_k}^2t}J_n\Big(\frac{\alpha_{n_k}r}{b}\Big)e^{in\theta}$$

and

$$u(r,\theta,0) = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}C_{n_k}J_n\Big(\frac{\alpha_{n_k}r}{b}\Big)e^{in\theta} = f(r,\theta), \tag{7.3.45}$$

where C_{n_k} *are the Fourier coefficients of* f *in the multiple orthogonal basis* $J_n\left(\frac{\alpha_{n_k} r}{b}\right)e^{in\theta}$. *To calculate these coefficients we first need to calculate the following normalization factor:*

$$\left\| J_n\left(\frac{\alpha_{n_k} r}{b}\right)e^{in\theta}\right\|^2 = \int_0^b J_n^2\left(\frac{\alpha_{n_k} r}{b}\right) r dr \int_0^{2\pi} d\theta, \tag{7.3.46}$$

which using the recurrence formula yields

$$\left\| J_n\left(\frac{\alpha_{n_k} r}{b}\right)e^{in\theta}\right\|^2 = 2\pi\frac{b^2}{2}\Big(J_{n+1}(\alpha_{n_k})\Big)^2. \tag{7.3.47}$$

Thus the Fourier coefficients are

$$C_{n_k} = \frac{1}{\pi b^2\Big[J_{n+1}(\alpha_{n_k})\Big]^2}\int_0^{2\pi}\int_0^b f(r,\theta) J_n\left(\frac{\alpha_{n_k} r}{b}\right)e^{-in\theta} r dr d\theta. \tag{7.3.48}$$

Note! *If* D *is the circular domain* $x^2 + y^2 < b^2$, *then we continue computing of* C_n, *viz,*

$$\int_0^{2\pi}\int_0^b F r dr d\theta = \int\int_D F dx dy.$$

Chapter 8

Orthogonal polynomials

The orthogonal polynomials are ...

8.1 Legendre polynomials

Some of the most useful orthogonal bases for L_2 spaces consist of polynomial functions. One of these polynomial functions is the *Legendre polynomial*,

Definition 42. *The nth Legendre polynomial, denoted by P_n, is defined by*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n. \tag{8.1.1}$$

For the first three values of $n = 0, 1, 2$ we have

$$P_0(x) = (x^2-1)^0 = 1, \quad P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x,$$

and

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{8}\frac{d}{dx}(4x^3-4x) = \frac{1}{8}4(3x^2-1) = \frac{1}{2}(3x^2-1).$$

We have that $(x^2-1)^n$ is of degree 2^n with leading term x^{2n}. After n derivations the leading term is x^n and thus $P_n(x)$ is a polynomial of degree n. The leading coefficient of $P_n(x)$ is

$$\frac{1}{2^n n!}(2n)(2n-1)\cdot \ldots \cdot (n+1) = \frac{(2n)!}{2^n(n!)^2}. \tag{8.1.2}$$

In the sequel we use frequently the following:

Lemma 17. *Suppose $\{p_n\}_0^\infty$ is a sequence of polynomials such that p_n is of degree n for all n. Then every polynomial of degree k $(k = 0, 1, 2, \dots)$ is a linear combination of $p_0, \dots, p_k$.*

Theorem 42. *The Legendre polynomials $\{P_n\}_0^\infty$ are orthogonal in $L_2(-1, 1)$ and*

$$\|P_n\|^2 = \frac{2}{2n+1}. \tag{8.1.3}$$

Proof. If f is any function of class $C^{(n)}$ on the interval $[-1, 1]$, we have

$$2^n n! \langle f, P_n \rangle = \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx. \tag{8.1.4}$$

Using partial integration the right hand side in (8.1.4) is

$$\int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2-1)^n dx = \left[f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx.$$

Note, that $f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n = 0$, for $x = \pm 1$. After n-fold integration by parts we get

$$2^n n! \langle f, P_n \rangle = (-1)^n \int_{-1}^{1} f^{(n)}(x) (x^2 - 1)^n dx. \tag{8.1.5}$$

Now if $f(x)$ is a polynomial of a degree $m < n$, we have that $f^{(n)}(x) \equiv 0$, so $\langle f, P_n \rangle = 0$ and thus $\langle P_m, P_n \rangle = 0$. By the same reasoning with m and n interchanged, we also have $\langle P_n, P_m \rangle = 0$ for $m > n$, so we have proved that the P_n's are mutually orthogonal.

Let now $f = P_n$ and then we have

$$\|P_n\|^2 = \frac{1}{2^n n!} (-1)^n \int_{-1}^{1} P_n^{(n)}(x) (x^2 - 1)^n dx. \tag{8.1.6}$$

Since the leading term for $P_n(x)$ is $\frac{(2n)!}{2^n (n!)^2} x^n$, we have that

$$P_n^{(n)}(x) = \frac{(2n)!}{2^n (n!)^2} \cdot n!. \tag{8.1.7}$$

Then we get

$$\|P_n\|^2 = \frac{(-1)^n}{2^n n!} \cdot \frac{(2n)!}{2^n (n!)^2} \cdot n! \int_{-1}^{1} (x^2 - 1)^n dx. \tag{8.1.8}$$

Note that

$$\begin{aligned}\int_{-1}^{1}(x^2-1)^n dx &= \int_{-1}^{1}(x+1)^n(x-1)^n\,dx\\ &= \Big[\frac{(x+1)^{n+1}}{n+1}(x-1)^n\Big]_{-1}^{1} - \int_{-1}^{1}\frac{(x+1)^{n+1}}{n+1}n(x-1)^{n-1}dx,\end{aligned}$$

where the first term on the right hand side is identically zero. After n-fold integration by parts we have

$$\begin{aligned}\int_{-1}^{1}(x^2-1)^n\,dx &= (-1)^n\int_{-1}^{1}\frac{(x+1)^{2n}}{(n+1)(n+2)\cdot\ldots\cdot(2n)}n(n-1)\cdot\ldots\cdot 1\,dx\\ &= (-1)^n\frac{(n!)^2}{(2n)!}\Big[\frac{(x+1)^{2n+1}}{2n+1}\Big]_{-1}^{1} = (-1)^n\frac{(n!)^2}{(2n)!}\cdot\frac{2^{2n+1}}{2n+1}.\end{aligned}$$

Inserting in (8.1.8) we finally get

$$\|P_n\|^2 = \frac{(-1)^n}{2^n n!}\cdot\frac{(2n)!}{2^n(n!)^2}\cdot n!\cdot(-1)^n\frac{(n!)^2}{(2n)!}\cdot\frac{2^{2n+1}}{2n+1} = \frac{2}{2n+1}. \tag{8.1.9}$$

and the proof is complete. □

Below we state (without proof) the most important property of the Legendre polynomials:

Theorem 43. *$\{P_n\}_0^\infty$ is an orthogonal base for $L_2(-1,1)$.*

Thus every polynomial of degree k, can be written as a linear combination of Legendre polynomials P_n, $n = 0,\ldots,k$. For instance the polynomial $3x^2 - 4x + 1$ can be written as a linear combination of $P_0(x), P_1(x)$ and $P_2(x)$, viz

$$2P_2(x) - 4P_1(x) + 2P_0(x) = 3x^2 - 1 - 4x + 2 = 3x^2 - 4x + 1.$$

We next derive the differential equation satisfied by the Legendre polynomials.

Theorem 44. *For all $n \geq 0$ we have*

$$\Big[(1-x^2)P_n'(x)\Big]' + n(n+1))P_n(x) = 0. \tag{8.1.10}$$

Proof. Let $g(x) = \Big[(1-x^2)P_n'(x)\Big]'$. Since $x^2P_n'(x)$ is of degree $n+1$ we have that $g(x)$ is of degree n. Now lemma 6.1 gives that

$$g(x) = \sum_{j=0}^{n} C_j P_j(x), \quad \text{where} \quad C_j = \frac{1}{\|P_j\|^2}\langle g, P_j\rangle. \tag{8.1.11}$$

For $j < n$ we have

$$\begin{aligned}\langle g, P_j\rangle &= \int_{-1}^{1} \Big[(1-x^2)P_n'(x)\Big]' P_j(x)dx \\ &= \Big[(1-x^2)P_n'(x)P_j(x)\Big]_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'(x)P_j'(x)dx \\ &= \Big[-P_n(x)(1-x^2)P_j'(x)\Big]_{-1}^{1} + \int_{-1}^{1} P_n(x)[(1-x^2)P_j'(x)]'dx = 0.\end{aligned}$$

In the last step we used the fact that $h_j(x) = [(1-x^2)P_j'(x)]'$ is a polynomial of degree j and the orthogonality give us

$$\langle g, P_j\rangle = \int_{-1}^{1} P_n(x)h_j(x)dx = 0. \tag{8.1.12}$$

Consequently

$$C_j = 0 \quad \text{when} \quad j < n. \tag{8.1.13}$$

Thus we have

$$g(x) = C_n P_n(x). \tag{8.1.14}$$

Let now $\alpha_n x^n$ be the leading term for $P_n(x)$. Then the leading term for $g(x) = \Big[(1-x^2)P_n'(x)\Big]'$ is

$$-\frac{d}{dx}[x^2\alpha_n n x^{n-1}] = -\frac{d}{dx}(\alpha_n n x^{n+1}) = -\alpha_n n(n+1)x^n. \tag{8.1.15}$$

Since the leading term for $g(x) = C_nP_n(x)$ also is $C_n\alpha_n x^n$ we have

$$-\alpha_n n(n+1)x^n = C_n\alpha_n x^n \quad \text{and} \quad C_n = -n(n+1). \tag{8.1.16}$$

Then $g(x) = -n(n+1)P_n(x)$ and thus

$$\Big[(1-x^2)P_n'(x)\Big]' + n(n+1))P_n(x) = 0. \tag{8.1.17}$$

and the proof is complete. □

Generating function for Legendre polynomials

Below we formulate the generating function for the Legendre polynomials.

Theorem 45. *For $-1 \le x \le 1$ and $|z| < 1$ (here z may be complex), the Legendre polynomials satisfy*

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{\sqrt{1 - 2xz + z^2}}. \tag{8.1.18}$$

(8.1.18) can be derived, e.g., by means of contour integrals, see Folland. We shall use (8.1.18) to evaluate the integral

$$\int_0^1 P_{2n+1}(x)\, dx. \tag{8.1.19}$$

To this approach, first we compute $P_n(0)$ using the Taylor expansion:

$$\sum_{n=0}^{\infty} P_n(0) z^n = \frac{1}{\sqrt{1 + z^2}} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} z^{2k}. \tag{8.1.20}$$

Consequently

$$P_{2k+1}(0) = 0, \tag{8.1.21}$$

and

$$\begin{aligned} P_{2k}(0) = \binom{-\frac{1}{2}}{k} &= \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdot \ldots \cdot (-\frac{1}{2} - k + 1)}{k!} \\ &= \frac{(-1)^k \cdot 1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k k!} = \frac{(-1)^k (2k)!}{2^k k! \cdot 2 \cdot 4 \cdot \ldots \cdot (2k)} \\ &= \frac{(-1)^k (2k)!}{2^{2k} (k!)^2}. \end{aligned} \tag{8.1.22}$$

Now we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \Big\{ \int_0^1 P_k(x) dx \Big\} z^n &= \int_0^1 (1 - 2xz + z^2)^{-\frac{1}{2}}\, dx = \Big[\frac{2}{-2z} (1 - 2xz + z^2)^{\frac{1}{2}} \Big]_0^1 \\ &= -\frac{1}{z}(1 - 2xz + z^2)^{\frac{1}{2}} + \frac{1}{z}(1 + z^2)^{\frac{1}{2}} = -\frac{1}{z}(1 - z) + \frac{1}{z} \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} z^{2k} \\ &= 1 - \frac{1}{z} + \frac{1}{z} + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} z^{2k-1} = 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} z^{2k-1}. \end{aligned}$$

Thus we have

$$\int_0^1 P_0(x)dx = 1, \qquad \int_0^1 P_{2k}(x)dx = 0, \tag{8.1.23}$$

and

$$\begin{aligned}\int_0^1 P_{2k-1}(x)dx &= \binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdot\ldots\cdot(\frac{1}{2}-k+1)}{k!} \\ &= (-1)^{k-1}\frac{1\cdot 1\cdot 3\cdot\ldots\cdot(2k-3)}{2^k k!}.\end{aligned} \tag{8.1.24}$$

Now transferring k to $k+1$, it follows that

$$\begin{aligned}\int_0^1 P_{2k+1}(x)dx &= \frac{(-1)^k\cdot 1\cdot 1\cdot 3\cdot\ldots\cdot(2k-1)}{2^{k+1}(k+1)!} \\ &= \frac{P_{2k}(0)}{2(k+1)} = \frac{(-1)^k(2k)!}{2^{k+1}(k!)^2(k+1)}.\end{aligned} \tag{8.1.25}$$

We conclude this section with a formula relating the Legendre polynomials and their derivatives.

Theorem 46. *For all $n \geq 1$ we have*

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{8.1.26}$$

Proof. The second derivative of $(x^2-1)^{n+1}$ is

$$\begin{aligned}\frac{d^2}{dx^2}(x^2-1)^{n+1} &= \frac{d}{dx}\left[2(n+1)x(x^2-1)^n\right] \\ &= 2(n+1)\left[(x^2-1)^n + 2nx^2(x^2-1)^{n-1}\right] \\ &= 2(n+1)(2n+1)(x^2-1)^n + 4n(n+1)(x^2-1)^{n-1},\end{aligned}$$

where the last step is derived writing the factor x^2 as $x^2 = (x^2-1)+1$. Now by the definition of Legendre polynomials (8.1.1) and using the relation above we write

$$\begin{aligned}P_{n+1}(x) &= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1} \\ &= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n-1}}{dx^{n-1}}\cdot\left(\frac{d^2}{dx^2}(x^2-1)^{n+1}\right) \\ &= \frac{(2n+1)}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n + \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} \\ &= \frac{(2n+1)}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n + P_{n-1}(x).\end{aligned}$$

Finally, differentiating both sides we obtain the desired result:

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = (2n+1)P_n(x),$$

and complete the proof. □

Spherical coordinates and Legendre functions

Recall that the spherical coordinates of a point $\mathbf{x}= (x, y, z) \in \mathbb{R}^3$ are given by

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad \text{and} \quad z = r\cos\theta \tag{8.1.27}$$

and that the Laplacian operator in these coordinates is given by

$$\nabla^2 u(r,\phi,\theta) = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\, u_\theta) + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}. \tag{8.1.28}$$

As an application of Legendre polynomials we consider the Dirichlet problem for the unit ball in $\mathbb{R}^3$, viz

$$\nabla^2 u(r,\phi,\theta) = 0, \quad u(1,\phi,\theta) = f(\phi,\theta), \tag{8.1.29}$$

where

$$r_0 < r < r_1, \qquad 0 \le \theta \le \pi, \qquad -\pi < \phi \le \pi. \tag{8.1.30}$$

Applying the method of separation of variables we write the solution u as $u(r,\phi,\theta) = R(r)\Phi(\phi)\Theta(\theta) \neq 0$, and insert it on the right hand side of (8.1.28) to obtain

$$\Big(R'' + \frac{2}{r}R'\Big)\Theta\Phi + \frac{1}{r^2\sin\theta}\Big(\sin\theta\cdot\Theta'\Big)'R\Phi + \frac{1}{r^2\sin^2\theta}R\Theta\Phi'' = 0. \tag{8.1.31}$$

Multiplying by $\frac{r^2\sin^2\theta}{R\Theta\Phi}$ we get

$$r^2\sin^2\theta\frac{R'' + (2/r)R'}{R} + \sin\theta\frac{(\sin\theta\,\Theta')'}{\Theta} = -\frac{\Phi''}{\Phi} = m^2. \tag{8.1.32}$$

Thus $\Phi'' + m^2\Phi = 0$ and hence

$$\Phi(\phi) = ae^{im\phi} + be^{-im\phi}. \tag{8.1.33}$$

Since $\Phi(\phi)$ is 2π-periodic hence m must be an integer, which we may take to be nonnegative.

Now we separate Θ and$R$ on the left side of (8.1.32), viz

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\theta} - \frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta} = \lambda, \tag{8.1.34}$$

and write the equations for Θ and R as

$$\frac{1}{\sin\theta}(\sin\theta\,\Theta')' - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0, \tag{8.1.35}$$

and

$$r^2R'' + 2rR' - \lambda R = 0. \tag{8.1.36}$$

Now the equation for Θ can be transformed into a close relative of the Legendre equation by the substitutions $s = \cos\theta$ and $S(s) = \Theta(\theta)$, where $-1 \le s \le 1$. Recalling the chain rule:

$$\frac{d}{d\theta} = \frac{d}{ds}\frac{ds}{d\theta} = -\sin\theta\frac{d}{ds}, \tag{8.1.37}$$

and since $\sin^2\theta = 1 - s^2$, it follows that

$$\frac{d}{ds}\Big((1-s^2)S'\Big) - \frac{m^2}{1-s^2}S + \lambda S = 0, \tag{8.1.38}$$

which in general is called the *associated Legendre equation* of order m.

Definition 43. *The associated Legendre equations are defined by*

$$P_n^m(s) = (1-s^2)^{\frac{m}{2}}\frac{d^mP_n^m(s)}{ds^m}. \tag{8.1.39}$$

Below we state (without proof) some properties of the associated Legendre equations:

Theorem 47. *For $\lambda = n(n+1)$ the associated Legendre equations have the non trivial solutions given by:*

$$S(s) = P_n^m(s), \quad \textit{where} \quad n \ge m. \tag{8.1.40}$$

Theorem 48. *For each positive integer m,$\{P_n^m\}_{n=m}^{\infty}$ is an orthogonal basis for $L_2(-1,1)$ and*

$$\|P_n^m\|^2 = \frac{(n+m)!}{(n-m)!}\cdot\frac{2}{2n+1}. \tag{8.1.41}$$

Now we return to the equation (8.1.36) and let $\lambda = n(n+1)$ to get

$$r^2R'' + 2rR' - n(n+1)R = 0. \tag{8.1.42}$$

This is an Euler equation with the solutions of the form r^p, giving

$$r^2p(p-1)r^{p-2} + 2rpr^{p-1} - n(n-1)r^p = 0, \tag{8.1.43}$$

which is is the same equation as

$$(p^2 - p)r^p + 2pr^p - n(n-1)r^p = 0. \tag{8.1.44}$$

After dividing by r^p we get

$$p^2 - p + 2p - n(n-1) = 0 \quad \Longleftrightarrow \quad p^2 + p - n(n+1) = 0 \tag{8.1.45}$$

and we have

$$p = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + n^2 + n} = -\frac{1}{2} \pm (n + \frac{1}{2}).$$

Thus $P_1 = n$ and $P_2 = -n - 1$ and the general solution for R is given by

$$R_n(r) = A_nr^n + Br^{-n-1}. \tag{8.1.46}$$

Hence the Dirichlet problem (8.1.2): $\nabla^2 u = 0$, on the unit sphere has a solution of the form

$$u(r,\phi,\theta) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} = C_{mn}\Big(A_nr^n + B_nr^{-n-1}\Big)e^{im\phi}P_n^{|m|}(\cos\theta), \tag{8.1.47}$$

where $Y_{mn}(\theta,\phi) := e^{im\phi}P_n^{|m|}(\cos\theta)$ are called the *spherical harmonics*. If $u(r,\phi,\theta) = u(r,\phi)$ is independent of ϕ (rotationally invariant) then the solutions are

$$u(r,\theta) = \sum_{n=0}^{\infty}\Big(A_nr^n + B_nr^{-n-1}\Big)P_n(\cos\theta). \tag{8.1.48}$$

Spherical harmonics

Definition 44. *The spherical harmonics or spherical surface functions are defined by*

$$Y_{mn}(\theta,\phi) = e^{im\phi}P_n^{|m|}(\theta), \quad \textit{where} \quad n = 0,1,2,\dots \quad \textit{and} \quad |m| \le n, \tag{8.1.49}$$

From above it follows that the functions $Y_{mn}(\theta, \phi)$ considered as function on the unit sphere S in $\mathbb{R}^3$, form an *orthogonal basis* of $L_2(S)$ with respect to the surface measure $d\sigma(\theta, \phi) = \sin\theta d\theta d\phi$. Moreover the normalization constants are

$$\|Y_{mn}\|^2 = \frac{4\pi}{2n+1}\frac{(n+|m|)!}{(n-|m|)!} \tag{8.1.50}$$

and the coefficients C_{mn} are given by

$$C_{mn} = \frac{\langle f, Y_{mn}\rangle}{\|Y_{mn}\|^2}. \tag{8.1.51}$$

Exempel 26. *Solve the following, radial dependent Laplace equation in the spherical coordinates:*

$$\begin{cases} \nabla^2 u = \frac{1}{r}, & r > R, \\ u = \cos\theta - \cos^3\theta, & r = R. \end{cases} \tag{8.1.52}$$

This is formally valid since if $u(r) \to 0$ when $R \to \infty$ (normal requirement is: $\nabla^2 u = \frac{1}{rP}$, where $P > 2$ and $P \neq 3$).

Solution. First we find a particular solution $\tilde{u}(r, 0, 0)$, satisfying $\nabla^2\tilde{u} = \frac{1}{r}$. This implies that (see Folland, page 406),

$$\nabla^2 u(r, 0, 0) = \frac{1}{r^2}\frac{d}{dr}(r^2\tilde{u}') = \frac{1}{r} \quad \textit{thus} \quad \frac{d}{dr}(r^2\tilde{u}') = r. \tag{8.1.53}$$

Integrating we get

$$r^2\tilde{u}' = \frac{r^2}{2} + C_1. \iff \tilde{u}' = \frac{1}{2} + \frac{C_1}{r^2}. \tag{8.1.54}$$

Integrating once again we have

$$\tilde{u} = \frac{r}{2} - \frac{C_1}{r} + C_2. \tag{8.1.55}$$

Let now $C_2 = 0$ and choose C_1 such that $\tilde{u}(R) = 0$. We get $\frac{R}{2} - \frac{C_1}{R} = 0$ and thus $C_1 = \frac{R^2}{2}$. Hence

$$\tilde{u}(r) = \frac{r}{2} - \frac{R^2}{2r}. \tag{8.1.56}$$

Note! *that since $\tilde{u}(r) \to \infty$, as $r \to \infty$ therefor $\nabla^2 u = \frac{1}{r}$ has no physical meaning. Loosely speaking, it corresponds to an ongoing charge with no interruption and consequently would mean unbounded potential.*

Let now $v = u - \tilde{u}$ (this removes the unboundedness as $r \to \infty$). Then

$$\begin{cases} \nabla^2 v = 0, & r > R, \\ v = \cos\theta - \cos^3\theta, & r = R, \end{cases} \tag{8.1.57}$$

with a solution given by

$$v(r,\theta) = \sum_{n=0}^{\infty} \Big(A_n r^n + B_n r^{-n-1}\Big) P_n(\cos\theta). \tag{8.1.58}$$

Now since v is bounded as $r \to \infty$ thus $A_n = 0$. Hence for $r = R$ we have

$$v(R,\theta) = \sum_{n=0}^{\infty} B_n R^{-n-1} P_n(\cos\theta) = \cos\theta - \cos^3\theta. \tag{8.1.59}$$

The substitution $s = \cos\theta$, gives that

$$v(R,\theta) = \sum_{n=0}^{\infty} B_n R^{-n-1} P_n(s) = s - s^3. \tag{8.1.60}$$

Recalling the Legendre polynomials

$$P_0(s) = 1, \quad P_1(s) = s, \quad P_2(s) = \frac{1}{2}(3s^2 - 1), \quad P_3(s) = \frac{1}{2}(5s^3 - 3s), \quad \ldots$$

we write, using $P_3(s)$ and $P_1(s) = s$ that $s^3 = \frac{2}{5}P_3 + \frac{3}{5}s = \frac{2}{5}P_3 + \frac{3}{5}P_1$. Thus

$$s - s^3 = P_1 - \frac{2}{5}P_3 - \frac{3}{5}P_1 = \frac{2}{5}P_1 - \frac{2}{5}P_3. \tag{8.1.61}$$

Inserting (8.1.61) in (8.1.60) and identifying the coefficients we get

$$B_1 R^{-2} = \frac{2}{5} \quad \text{and} \quad B_3 R^{-4} = -\frac{2}{5}, \quad \text{otherwise} \quad B_n R^{-n-1} = 0.$$

Thus we have

$$B_1 = \frac{2}{5}R^2 \quad \text{and} \quad B_3 = -\frac{2}{5}R^4, \quad \text{otherwise} \quad B_n = 0. \tag{8.1.62}$$

Hence it follows that

$$\begin{aligned} v(r,\theta) &= \frac{2}{5}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{2}{5}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) \\ &= \frac{2}{5}\left(\frac{R}{r}\right)^2 \cos\theta - \frac{2}{5}\left(\frac{R}{r}\right)^4 \cdot \frac{1}{2}(5\cos^3\theta - 3\cos\theta) \end{aligned} \tag{8.1.63}$$

and consequently

$$u(r,\theta) = \frac{r}{2} - \frac{R^2}{2r} + \left[\frac{2}{5}\left(\frac{R}{r}\right)^2 + \frac{3}{5}\left(\frac{R}{r}\right)^4\right]\cos\theta - \left(\frac{R}{r}\right)^4 \cos^3\theta. \tag{8.1.64}$$

8.2 Hermite polynomials

Another useful orthogonal basis for $L_2(\mathbb{R})$ and $L_2^w(\mathbb{R})$, where $w(x) = e^{-x^2}$, is the *Hermite polynomials*.

Definition 45. *The nth Hermite polynomial, $H_n(x)$, is defined by*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{8.2.1}$$

The recursion formula for Hermite polynomials is

$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x). \tag{8.2.2}$$

Simple calculations show that

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2x, \quad \text{and} \quad H_3(x) = 8x^3 - 12x.$$

For the Hermite polynomials we have the following orthogonality properties which we state without proof:

Theorem 49. *The Hermite polynomials $\{H_n\}_0^\infty$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$ and*

$$\|H_n\|^2 = 2^n n! \sqrt{\pi}. \tag{8.2.3}$$

Theorem 50. *The set of Hermite polynomials, $\{H_n\}_0^\infty$, is an orthogonal basis for $L_2^w(\mathbf{R})$.*

Theorem 51. *For any $x \in \mathbb{R}$ and $z \in \mathbf{C}$ we have the generating function*

$$\sum_{0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}. \tag{8.2.4}$$

For many purposes it is preferable to replace the Hermit polynomials by the *Hermite functions,*$h_n(x)$.

Definition 46. *The Hermite functions are defined by*

$$h_n(x) = e^{-\frac{x^2}{2}} H_n(x). \tag{8.2.5}$$

Differential equations:

The Hermite polynomials, $Y(x) = H_n(x)$, where $x \in \mathbb{R}$, are eigenfunctions to the Sturm-Liouville problem

$$\frac{d}{dx}(e^{-x^2}Y') + \lambda e^{-x^2}Y = 0. \tag{8.2.6}$$

The Hermite functions, $y(x) = h_n(x)$, where $x \in \mathbb{R}$, are eigen functions to the Sturm-Liouville problem

$$y'' - x^2 y + \lambda y = 0, \quad \text{called Hermit's equations.} \tag{8.2.7}$$

8.3 Laguerre polynomials

Definition 47. *Let α be a real number such that $\alpha > -1$, then the nth Laguerre polynomial L_n^α corresponding to the parameter α is defined by*

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x}). \tag{8.3.1}$$

For the Laguerre polynomial we give the following theorems Without proof

Theorem 52. *The Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$ are a complete orthogonal set on $(0,\infty)$ with respect to the weight function*

$$w(x) = x^\alpha e^{-x}, \tag{8.3.2}$$

and their norms are given by

$$\|L_n^\alpha\|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}. \tag{8.3.3}$$

Theorem 53. *The Laguerre polynomial L_n^α satisfies the Laguerre equation*

$$\frac{d}{dx}(x^{\alpha+1}e^{-x}y') + nx^\alpha e^{-x}y = 0, \tag{8.3.4}$$

which can be written in the form

$$xy'' + (\alpha+1-x)y' + ny = 0. \tag{8.3.5}$$

Theorem 54. *The generating function for Laguerre polynomials is for $x > 0$ and $|z| < 1$,*

$$\sum_0^\infty L_n^\alpha(x)z^n = \frac{e^{\frac{-xz}{1-z}}}{(1-z)^{\alpha+1}}. \tag{8.3.6}$$

Chapter 9

Distribution theory

Heaviside and Dirac functions are distributions, ...

9.1 Test functions

Let us first introduce the *impulse train*, which we write in the form

$$u(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT). \tag{9.1.1}$$

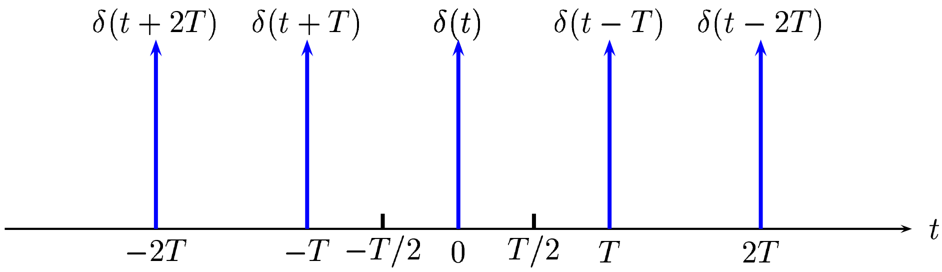


Figure 9.1: A sequence of delta functions $\delta(t - mT),\ m = 0, \pm 1, \pm 2$.

The impulse train (9.1.1) can formally be we written as a Fourier series expansion

$$u(t) \sim \sum_{n=-\infty}^{\infty} u_n(t) e^{jn\Omega t}, \tag{9.1.2}$$

where the Fourier coefficients are given by

$$u_n(t) = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-jn\Omega t} = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jn\Omega t} = \frac{1}{T}. \tag{9.1.3}$$

Thus we have

$$u(t) \sim \sum_{n=-\infty}^{\infty} \frac{1}{T}e^{jn\Omega t}. \tag{9.1.4}$$

This series is divergent for all t. We computed its partial sum in Chapter 5: *the Dirichlet kernel*

$$S_N(t) = \sum_{n=-N}^{N} \frac{1}{T}e^{jn\Omega t} = \frac{\sin(N + \frac{1}{2}\Omega t)}{T\sin(\frac{1}{2}\Omega t)} := D_N(t), \tag{9.1.5}$$

Below we define some basic concepts:

Definition 48. *Smooth T-periodic functions are called test functions.*

Definition 49. *If $f(t)$ is a T-periodic function with the Fourier series expansion*

$$f(t) \sim \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}, \tag{9.1.6}$$

then, we say that, f:s partial sum converges to f weakly and write

$$S_N(t) = \sum_{n=-N}^{N} C_n e^{jn\Omega t} \to f(t) \quad \text{weakly}$$

if for all test functions $\varphi(t)$

$$\lim_{N\to\infty}\int_a^{a+T} S_N(t)\varphi(t)dt = \int_a^{a+T} f(t)\varphi(t)dt. \tag{9.1.7}$$

Then we may write

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}, \quad \text{with the convergence in the weak sense.} \tag{9.1.8}$$

For the *impulse train* we have the approximation (9.1.4). Therefor a variational form of its partial sum can be written as:

$$\int_a^{a+T} S_N(t)\varphi(t)dt = \sum_{-N}^{N} \frac{1}{T} \int_a^{a+T} \varphi(t)e^{jn\Omega t}. \tag{9.1.9}$$

Let C_n be the Fourier coefficients for $\varphi(t)$, then

$$C_n = \frac{1}{T} \int_a^{a+T} \varphi(t)e^{-jn\Omega t}. \tag{9.1.10}$$

Obviously we can write $\sum_{-N}^{N} C_{-n} = \sum_{-N}^{N} C_n$. Then since $\varphi(t)$ is smooth we have

$$\varphi(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}, \qquad \forall t, \tag{9.1.11}$$

and in particular

$$\varphi(0) = \sum_{n=-\infty}^{\infty} C_n. \tag{9.1.12}$$

Taking the limit in (9.1.9) and using (9.1.12) we get

$$\lim_{N\to\infty} \int_a^{a+T} S_N(t)\varphi(t)dt = \varphi(0). \tag{9.1.13}$$

On the other hand we have

$$\int_a^{a+T} u(t)\varphi(t)dt = \int_{-T/2}^{T/2} \delta(t)\varphi(t)dt = \varphi(0). \tag{9.1.14}$$

Then we have combining (9.1.13) and (9.1.14), that

$$\lim_{N\to\infty} \int_a^{a+T} S_N(t)\varphi(t)dt = \int_a^{a+T} u(t)\varphi(t)dt, \qquad \forall\varphi(t). \tag{9.1.15}$$

Thus we have finally, recalling the right hand side of (9.1.9)

$$u(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\Omega t} \qquad \text{for weak convergence.} \tag{9.1.16}$$

Exempel 27. *Expand the function $f(t) = 1 - t$ for $0 < t < 2$ in complex Fourier series. ($T = 2$)*

Solution: With the weak convergence we have

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}, \quad \textit{where} \quad \Omega = \frac{2\pi}{T} = \pi. \tag{9.1.17}$$

Then the Fourier coefficients C_n are

$$C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\Omega t}dt = \frac{1}{2}\int_0^2 (1-t)e^{-jn\pi t}dt. \tag{9.1.18}$$

For $n \neq 0$ and using partial integration we get

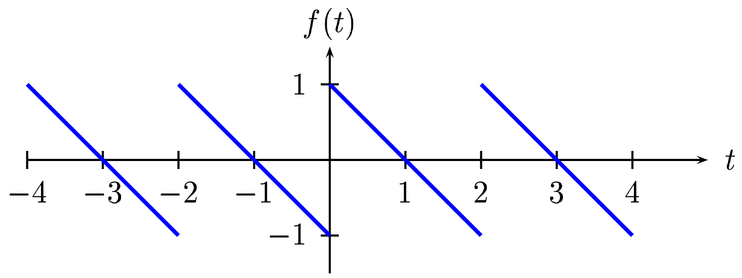
$$\begin{aligned} C_n &= \frac{1}{2}\Big[(1-t)\frac{e^{-jn\pi t}}{-jn\pi}\Big]_0^2 - \frac{1}{2}\int_0^2 (-1)\frac{e^{-jn\pi t}}{-jn\pi}dt \\ &= \frac{1}{2}\Big(-\frac{e^{-2jn\pi}}{-jn\pi} - \frac{1}{-jn\pi}\Big) + \frac{1}{2}\Big[\frac{e^{-jn\pi t}}{(-jn\pi)^2}\Big]_0^2 = \frac{1}{jn\pi}, \end{aligned} \tag{9.1.19}$$

where we have used $e^{-2jn\pi} = 1$ and $\Big[\frac{e^{-jn\pi t}}{(-jn\pi)^2}\Big]_0^2 = 0$. For $n = 0$ we have

$$C_0 = \frac{1}{2}\int_0^2 (1-t)\,dt = \frac{1}{2}\Big[-\frac{1}{2}(1-t)^2\Big]_0^2 = 0. \tag{9.1.20}$$

Consequently

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{jn\pi}e^{jn\pi t}, \qquad n \neq 0. \tag{9.1.21}$$



Exempel 28. *Let $u(t) = 1 - t$. Then using the previous example we get*

$$u(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega t}, \quad \Omega = \pi, \quad \text{and} \quad C_n = \begin{cases} 0, & n = 0, \\ \frac{1}{jn\pi}, & n \neq 0. \end{cases} \tag{9.1.22}$$

We may assume an electrical circuit associated to $u(t)$ as in the figure below:

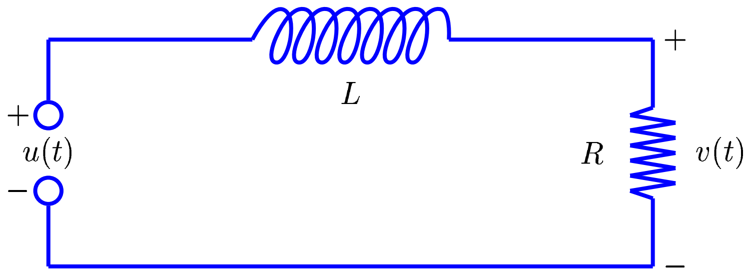


Figure 9.2: An electrical circuit.

or in the dynamic form: Then using the superposition principle each fre-

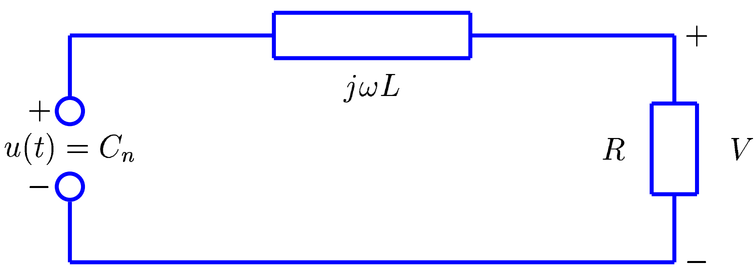


Figure 9.3: The dynamic form circuit

quency can be treated separately. The component $C_n e^{jn\Omega t}$ gives an output *signal which can be computed by the $j\omega$-method, where $\omega = n\Omega$. Now by the well-known formulas:*

$$u = (j\omega L + R)I \quad \Longleftrightarrow \quad I = \frac{u}{j\omega L + R}, \tag{9.1.23}$$

and

$$V = RI = \frac{uR}{j\omega L + R} = \frac{C_n R}{jn\omega L + R}, \tag{9.1.24}$$

the output signal is

$$u_n(t) := \frac{C_n R}{jn\omega L + R} e^{jn\Omega t}, \tag{9.1.25}$$

and finally we have

$$u(t) = \sum_{n=-\infty}^{\infty} \frac{C_n R}{jn\Omega L + R} e^{jn\Omega t} = \sum_{n=-\infty}^{\infty} \frac{1}{jn\pi} \cdot \frac{R}{jn\pi L + R} e^{jn\pi t}. \tag{9.1.26}$$

9.2 Delta functions

To get a test function we can use the Dirac's delta function. Below we list some of the properties of the delta functions (see also Chapters 1 and 5):

Definition 50. *Dirac's delta function is defined as*

$$\begin{cases} \delta(x) = 0 & \text{for } x \neq 0, \\ \int_{-a}^{a} \delta(x)dx = 1 & \text{for all} \quad a > 0. \end{cases} \tag{9.2.1}$$

Within the traditional realm of functions, the Dirac function does not make sense. But we want to look at $\delta(x)$ as a idealized limit element to sequences of the form $\delta_n(x)$: or

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x). \tag{9.2.2}$$

To do this we expand the concept in the definition of a function:

Instead of describe a function by the functions values $f(x)$ we give the values of the integrals

$$\int_{-\infty}^{\infty} f(x)\varphi(x)dx, \tag{9.2.3}$$

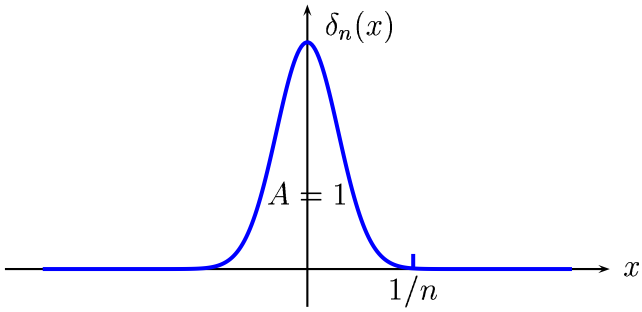


Figure 9.4: The Dirac function $\delta_n(x)$

for a sufficiently large class of *generalized functions* $\varphi(x)$. If $\varphi = \varphi_n$ were as the above the δ_n functions, but centered about x_0, then $\int f\varphi_n dx$ will be a weighted mean value of the function values about x_0 and

$$\int_{-\infty}^{\infty} f(x)\varphi_n(x)dx \to f(x_0), \qquad n \to \infty, \tag{9.2.4}$$

if $f(x)$ is uniformly continuous. Note that $F[\varphi] := \int f\varphi dx$ is linearly depending on $\varphi(x)$ and thus defines a *linear functional*. There are also other linear functionals.

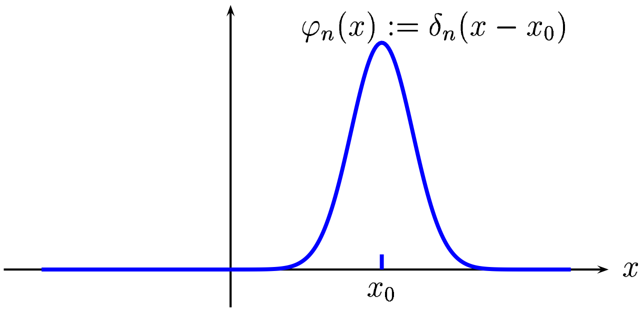


Figure 9.5: The Dirac function centered on x_0 : $\delta_n(x - x_0)$

Below we give some more examples of delta functions:

The square pulse function. As we have seen in the introduction Chapter the square pulse function is defined as

$$\delta_\varepsilon(x) = \begin{cases} 1/\varepsilon & \text{for} \quad 0 \le x \le \epsilon \\ 0 & \qquad\quad \text{otherwise,} \end{cases} \tag{9.2.5}$$

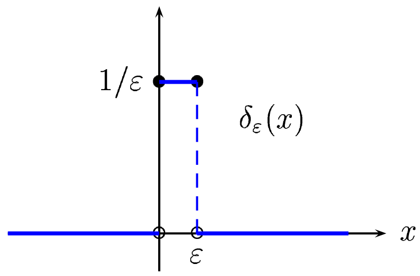


Figure 9.6: The square pulse function $\delta_\varepsilon(t)$.

The Gauss pulse function The Gauss pulse function is defined as

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\varepsilon^2}}, \tag{9.2.6}$$

which we illustrate in the following figure:

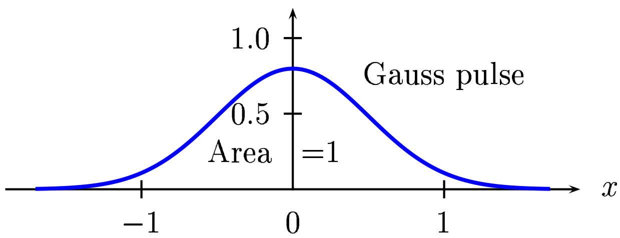


Figure 9.7: The Gauss pulse $\delta_\varepsilon(x) = \sqrt{2/\pi}\, e^{-x^2/(2\varepsilon^2)}$.

The exponential pulse function The Exponential pulse function is defined as

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon} e^{-\frac{x}{2}}. \tag{9.2.7}$$

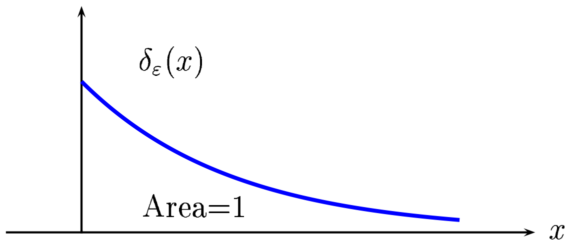


Figure 9.8: The exponential pulse $\delta_\varepsilon(x) = \frac{1}{\varepsilon} e^{-\frac{x}{2}}$.

The triangular pulse function The triangular pulse function is defined as

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}x & \text{for} \quad -\varepsilon < x \le 0, \\ \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2}x & \text{for} \quad 0 \le x < \varepsilon. \end{cases} \tag{9.2.8}$$

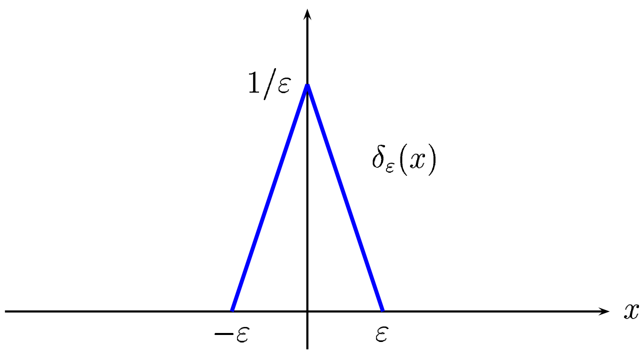


Figure 9.9: The triangle pulse function $\delta_\varepsilon(x)$.

Now we need to introduce some terminology.

Definition 51. *If f is a function on $\mathbb{R}^n$, then its support is the closure of the set of all points $\mathbf{x}$, such that $f(\mathbf{x}) \neq 0$, in other words, the smallest closed set outside of which f vanishes identically.*

We let $C_0^{(\infty)}(\mathbb{R}^n)$ *denote the space of all functions on* $\mathbb{R}^n$ *whose (partial) derivatives of all orders exist and are continuous on* $\mathbb{R}^n$ *and whose support is a bounded subset of* $\mathbb{R}^n$

Exempel 29. *Let*

$$\psi(x) = \begin{cases} e^{-1/x}, & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x < 0. \end{cases} \tag{9.2.9}$$

Show that

$$\varphi(x) = \psi(2(1+x)) \cdot \psi(2(1-x)) \in C_0^{(\infty)}. \tag{9.2.10}$$

Solution: Since

$$\frac{d^n}{dx^n}\left(e^{-1/x}\right) = \{\text{polynomial in } \frac{1}{x}\} \cdot e^{-1/x} \to 0, \quad \text{as} \quad x \to 0^+, \tag{9.2.11}$$

therefore $\psi(x)$ *have derivatives of all orders, and we have*

$$\varphi(x) = e^{-\frac{1}{2(1+x)}} \cdot e^{-\frac{1}{2(1-x)}} = e^{-\frac{1}{1-x^2}}. \tag{9.2.12}$$

For $|x| \geq 1$ *we have* $(1-x^2) < 0$ *and hence* $\varphi(x) = \psi(1-x^2) = 0$*. It follows that*

$$\varphi(x) = \begin{cases} e^{-1/(1-x^2)}, & \text{for} \quad |x| < 1, \\ 0, & \text{for} \quad |x| \geq 1, \end{cases} \tag{9.2.13}$$

and hence $\varphi(x) \in C_0^{(\infty)}$.

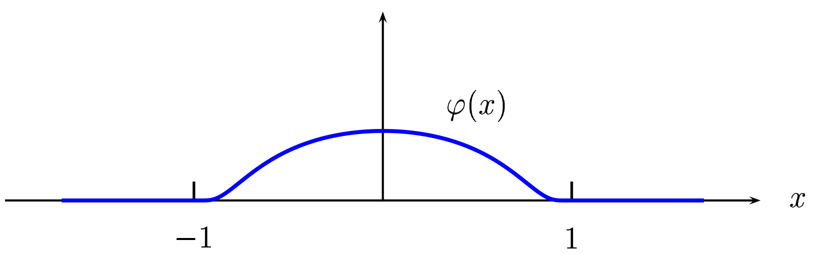


Figure 9.10: The function $\varphi(x) = e^{-1/(1-x^2)}, \ |x| < 1, \quad \varphi(x) = 0, \ |x| > 1.$

9.3 Distributions

A continuous function f on $\mathbb{R}^n$ is specified by giving its values $f(x)$ at all points $\mathbf{x} \in \mathbb{R}^n$, but f can equally well be specified by giving the values of the integrals

$$\int_{-\infty}^{\infty} f(\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}, \quad \text{where} \quad \varphi \in C_0^{(\infty)}. \tag{9.3.1}$$

For a particular $\varphi \in C_0^{(\infty)}$ such that

$$\int_{-\infty}^{\infty} \varphi(\mathbf{y})d\mathbf{y} = 1, \tag{9.3.2}$$

we get for all $\mathbf{x}$ and ε

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(\mathbf{y})\varphi_{\mathbf{x},\varepsilon}(\mathbf{y})d\mathbf{y} = f(\mathbf{x}), \tag{9.3.3}$$

and therefor we know $f(\mathbf{x})$ for all $\mathbf{x}$. (See Folland, Theorem 7.7). In other words we can think of $f(\mathbf{x})$ as a limit of $\int_{-\infty}^{\infty} f(\mathbf{y})\varphi_{\mathbf{x},\varepsilon}(\mathbf{y})d\mathbf{y}$ and we introduce the following notation:

$$f[\varphi] = \int_{-\infty}^{\infty} f(\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}, \quad \text{where} \quad \varphi \in C_0^{(\infty)}. \tag{9.3.4}$$

There are *other* linear functionals on $C_0^{(\infty)}$ that are not given by integration against a function f and these functionals will be our *generalized functions* or *distributions* they are commonly called.

Now we can give the definition of a *distribution*.

Definition 52. *A distribution is a mapping $F : C_0^{(\infty)} \to \mathbf{C}$ that satisfies the following conditions:*

(i) Linearity: $\forall \varphi_1, \varphi_2 \in C_0^{(\infty)}$, and $\forall c_1, c_2 \in \mathbf{C}$ we have that

$$F[c_1\varphi_1 + c_2\varphi_2] = c_1F[\varphi_1] + c_2F[\varphi_2]. \tag{9.3.5}$$

(ii) Continuity: Suppose $\{\varphi_k\}$ is a sequence in $C_0^{(\infty)}$ such that supp(φ_k) is contained in a fixed bounded set D for all k, and suppose that the functions φ_k and all derivatives $\partial^\alpha \varphi_k$ converge uniformly to zero as $k \to \infty$. Then $F[\varphi_k] \to 0$. Shortly

$$\varphi_k \to 0 \quad \Longrightarrow \quad F[\varphi_k] \to 0. \tag{9.3.6}$$

(iii) Equality:

$$F_1 = F_2 \iff F_1[\varphi] = F_2[\varphi], \quad \forall\varphi. \tag{9.3.7}$$

Notation. The space of all distributions on $\mathbb{R}^n$ is denoted by $\mathcal{D}'(\mathbb{R}^n)$ or for short $\mathcal{D}'$. The prime is an indication that $\mathcal{D}'$ is a space of linear functionals.

A distribution F is like a function, but it may be too singular for the point-wise values $F(\mathbf{x})$ to make sense, whereas only the smeared-out values $F[\varphi]$ are well-defined. For convenience we write

$$F[\varphi] = \int_{-\infty}^{\infty} F(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}. \tag{9.3.8}$$

Example on distributions

Exempel 30. *Let $f(x)$ be a piecewise continuous function and $\varphi(x)$ a test function. Then*

$$F[\varphi] = \int_{-\infty}^{\infty} f(x)\varphi(x)dx \tag{9.3.9}$$

is a distribution and $F \in \mathcal{D}'$.

Exempel 31. *The simplest example of a distribution that is not a function is the* **Dirac delta-function** *δ, which is defined by*

$$\delta[\varphi] = \varphi(0). \tag{9.3.10}$$

Operations on distributions

It is possible to extend the operation of differentiation from functions to distributions in such way that every distribution possesses derivatives of all orders that are also distributions. To this approach let $f(x)$ be a continuously differentiable function on $\mathbb{R}$ and $\varphi \in C_0^{(\infty)}(\mathbb{R})$. Then using (9.3.9) we have

$$F'[\varphi] = \int_{-\infty}^{\infty} f'(x)\varphi(x)dx. \tag{9.3.11}$$

Integrating by parts we get

$$F'[\varphi] = [f(x)\varphi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\varphi'(x)dx, \tag{9.3.12}$$

and since $\varphi(x) = 0$ when $|x|$ is large, it follows that

$$F'[\varphi] = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx = -F[\varphi']. \tag{9.3.13}$$

Hence we have the following definition:

Definition 53. *For any distribution F on $\mathbb{R}$ we define the distribution derivative F' by*

$$F'[\varphi] = -F[\varphi'] \quad \text{where} \quad F \in \mathcal{D}'(\mathbb{R}) \quad \text{and} \quad \varphi \in C_0^{(\infty)}(\mathbb{R}). \tag{9.3.14}$$

Exempel 32. *Show that $\theta'(x) = \delta(x)$, where*

$$\theta(x) = \begin{cases} 1, & \text{for} \quad x > 0 \\ 0, & \text{for} \quad x < 0 \end{cases} \tag{9.3.15}$$

and $\delta(x)$ is Dirac's delta function.

Solution: We know that

$$\theta[\varphi] = \int_{-\infty}^{\infty} \theta(x)\varphi(x)dx = \int_{0}^{\infty} \varphi(x)dx. \tag{9.3.16}$$

Using the definition and the previous example it follows that

$$\theta'[\varphi] = -\theta[\varphi'] = -\int_{0}^{\infty} \varphi'(x)dx = -\Big[\varphi(x)\Big]_0^{\infty} = \varphi(0) = \delta[\varphi]. \tag{9.3.17}$$

Hence we get $\theta'[\varphi] = \delta[\varphi]$ and thus

$$\theta'(x) = \delta(x). \tag{9.3.18}$$

Jump discontinuities

Theorem 55. *Suppose f is piecewise smooth on $\mathbf{R}$ with discontinuities at $x_1, x_2, ..., x_n$. Let $f^{(1)}$ denote the pointwise smooth derivate of f, which exists and is continuous except at the x_j's and perhaps some points where it has jump discontinuities, and let f' denote the* distribution *derivate of f. Then for any test function φ we have*

$$f'[\varphi] = \int_{-\infty}^{\infty} f^{(1)}\varphi(x) + \sum_{j=1}^{n} \Big[f(x_j+) - f(x_j-)\Big]\varphi(x_j), \tag{9.3.19}$$

in other words

$$f'(x) = f^{(1)}(x) + \sum_{j=1}^{n} \Big[f(x_j+) - f(x_j-)\Big]\delta(x - x_j). \tag{9.3.20}$$

Proof. Exercise! □

Exempel 33. *Let f be piecewise smooth on $\mathbb{R}$ with a jump discontinuity x_0, then we have the distribution derivative*

$$f'(x) = f^{(1)}(x) + \Big[f(x_0+) - f(x_0-)\Big]\delta(x - x_0) = f^{(1)}(x) + \sigma\delta(x - x_0),$$

where $\delta(x - x_0)$ is Dirac's delta function, where $x = x_0$.

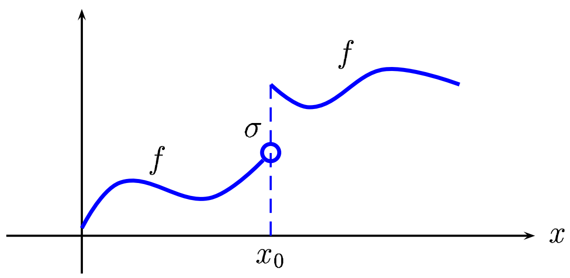


Figure 9.11: The function $f(x)$ with a jump discontinuity at x_0.

Multiplication by smooth functions

For multiplying distributions by a smooth function we give the definition:

Definition 54. *Suppose g is an infinitely differentiable function on $\mathbb{R}^n$, then $g\varphi$ is a test function whenever φ is, and is defined as*

$$(gF)[\varphi] = F[g\varphi], \tag{9.3.21}$$

and since $g\varphi \in C_0^{(\infty)}$ we have that $gF \in \mathcal{D}'$.

To calculate the derivative of the distribution gF we have the theorem:

Theorem 56. *If $g \in C_0^{(\infty)}$ and $F \in \mathcal{D}'$, then*

$$(gF)' = g'F + gF'. \tag{9.3.22}$$

Proof. Using the definition of the distribution derivative and the definition above we have that

$$(gF)'[\varphi] = -(gF)[\varphi'] = -F[g\varphi']. \tag{9.3.23}$$

But since $(g\varphi)' = g'\varphi + g\varphi'$ we obtain

$$\begin{aligned}-F[g\varphi'] &= -F[(g\varphi)' - g'\varphi] = -F[(g\varphi)'] + F[g'\varphi] \\ &= F'[g\varphi] + (g'F)[\varphi] = (gF')[\varphi] + (g'F)[\varphi] = (gF' + g'F))[\varphi]\end{aligned}$$

thus using (9.3.23) the proof is complete. □

Convergence of distributions

We only give a definition of *weakly convergence* for distributions and a theorem for differentiation of distributions with weak convergence.

Definition 55. *A sequence $\{F_n\}_0^\infty$ of distributions converges weakly to a distribution F (we write $F_n \to F$ weakly), if*

$$F_n[\varphi] \to F[\varphi], \qquad \forall \varphi \in C_0^{(\infty)}. \tag{9.3.24}$$

Theorem 57. *Differentiation is continuous with respect to weak convergence:*

$$F_n \longrightarrow F \quad \textit{weakly} \quad \Longrightarrow \quad F_n' \longrightarrow F' \quad \textit{weakly}. \tag{9.3.25}$$

Proof. For any test function φ we have that $F_n'[\varphi] = -F_n[\varphi']$. Further $-F_n[\varphi'] \to -F[\varphi']$ and $-F[\varphi'] = F'[\varphi]$. Thus we have that $F_n'[\varphi] \to F'[\varphi]$ and the proof is complete. □

Periodic distributions

Definition 56. *A distribution F on $\mathbb{R}$ is called periodic with period P if $F(x+P) = F(x)$, where*

$$F\Big[\varphi(x-P)\Big] = F\Big[\varphi(x)\Big], \qquad \forall \varphi \in C_0^{(\infty)}. \tag{9.3.26}$$

Theorem 58. *If F is any periodic distribution, then F can be expanded in a weakly convergent Fourier series, viz*

$$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}, \tag{9.3.27}$$

that is,

$$F[\varphi] = \sum_{k=-\infty}^{\infty} c_k \int_{-P}^{P} \varphi(x) e^{ikx} dx, \qquad \forall\, \varphi \in C_0^{(\infty)}, \tag{9.3.28}$$

where the coefficients c_k satisfy

$$c_k \leq C\Big(1 + |k|\Big)^N, \tag{9.3.29}$$

for some C, $N \geq 0$. Conversely, if $\{c_k\}_{-\infty}^{\infty}$ is any sequence satisfying the estimate (9.3.29), then the series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges weakly to a periodic distribution.

The most fundamental example of a *periodic distribution* that is not a function is the *periodic delta function*:

$$\delta_{per}(x) = \sum_{-\infty}^{\infty} \delta(x - 2k\pi). \tag{9.3.30}$$

Exempel 34. *Calculate the first and second derivatives for the 2π-periodic distribution given by*

$$f(x) = \frac{1}{2\pi}\Big(\pi x - \frac{x^2}{2}\Big), \quad 0 < x < 2\pi. \tag{9.3.31}$$

Solution: We have earlier seen that for an even Fourier series expansion and weak convergence we have

$$f(x) = \frac{a_0}{2} - \sum_{k=1}^{\infty} \frac{\cos kx}{\pi k^2}. \tag{9.3.32}$$

Then with weak convergence we have the first derivative

$$f'(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{\pi k}. \tag{9.3.33}$$

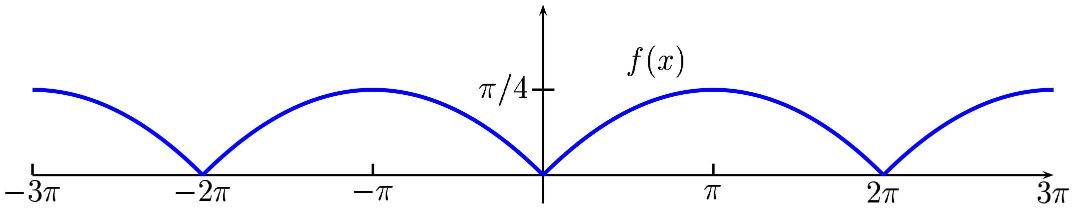


Figure 9.12: The function $f(x) = \frac{1}{2\pi}\left(\pi x - \frac{x^2}{2}\right), \ |x| < 3\pi.$

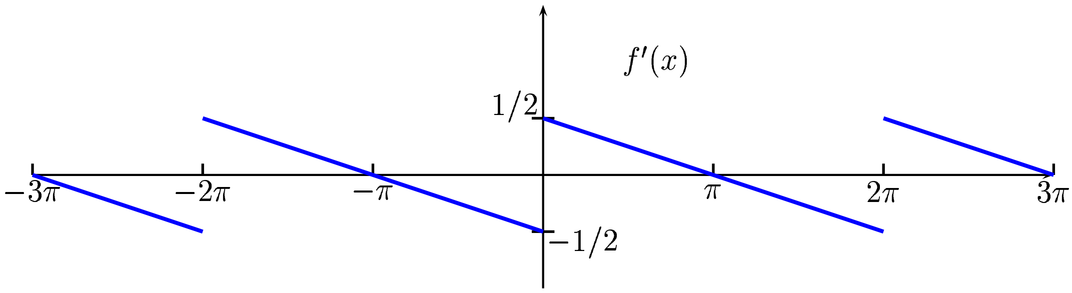


Figure 9.13: The function $f'(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{\pi k}, \ |x| < 3\pi.$

The second derivative is

$$f''(x) = -\frac{1}{2\pi} + \sum_{-\infty}^{\infty} \delta(x - 2k\pi). \tag{9.3.34}$$

But with the weak convergence we get

$$\delta_{per}(x) = \sum_{-\infty}^{\infty} \delta(x - 2k\pi) = \frac{1}{2\pi} + \sum_{k=1}^{\infty} \frac{\cos kx}{\pi} = \sum_{-\infty}^{\infty} \frac{1}{2\pi} e^{inx}, \tag{9.3.35}$$

having the Fourier coefficients

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta_{per.}(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x)dx = \frac{1}{2\pi}.$$

Thus we have

$$f''(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{\pi}. \tag{9.3.36}$$

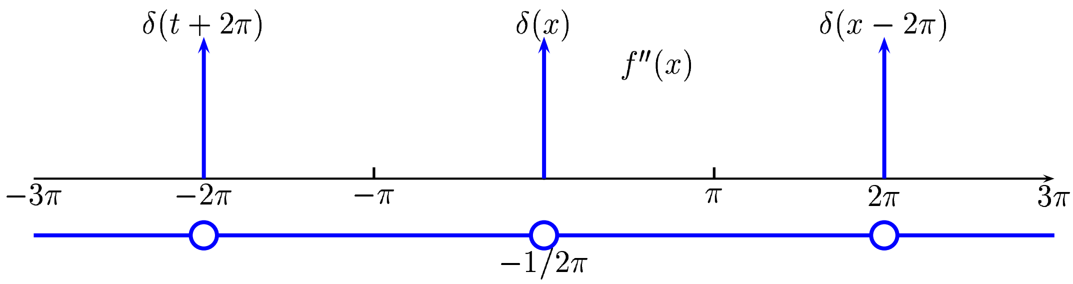


Figure 9.14: The functions $f''(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{\pi}, \ |x| < 3\pi$.

Exempel 35. *Solve the differential (wave) equation*

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0, & x \in \mathbb{R}, \\ u(x,0) = f(x), & u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{9.3.37}$$

Solution: Let

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big]. \tag{9.3.38}$$

The weak solution is in C^2 and we have that $\forall\, \varphi \in C_0^{(\infty)}\Big(\mathbb{R} \times (0,\infty)\Big)$,

$$(u_{tt} - c^2 u_{xx})[\varphi] = \int_0^{\infty} \int_{-\infty}^{\infty} u(x,t)(\varphi_{tt} - c^2 \varphi_{xx}) dx dt = 0. \tag{9.3.39}$$

We introduce the change of variables viz $\eta = x - ct$ and $\zeta = x + ct$. Then

$$\begin{cases} x = \frac{\eta+\zeta}{2}, \\ t = \frac{\zeta-\eta}{2c}, \end{cases} \qquad \text{and} \qquad \frac{d(x,t)}{d(\eta,\zeta)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2c} & \frac{1}{2c} \end{vmatrix} = \frac{1}{2c}, \tag{9.3.40}$$

is the Jacobian of this transformation which yields $dxdt = \frac{1}{2c} d\eta\zeta$. Now $t > 0 \iff \eta < \zeta$, and letting $\varphi(x,t) = \psi(\eta,\zeta)$, it follows that

$$\int_0^{\infty} \int_{-\infty}^{\infty} f(x-ct)(\varphi_{tt} - c^2\varphi_{xx}) dx dt = \int_{\eta} \int_{\zeta} f(\eta)[-4c^2 \psi_{\eta\zeta}] \frac{1}{2c} d\eta\zeta$$
$$= -2c \int_{-\infty}^{\infty} f(\eta) \Big\{ \int_{(\eta or 0)}^{\infty} \psi_{\eta\zeta} d\zeta \Big\} dy = -2c \int_{-\infty}^{\infty} f(\eta) \Big[\psi_{\eta}(\eta,\zeta)\Big]_{(\eta or 0)}^{\infty} d\eta = 0,$$

where the last step is due to the boundary condition and the fact that ψ has compact support (see Figures below).

The term corresponding to $f(x+ct)$ is treated analogously.

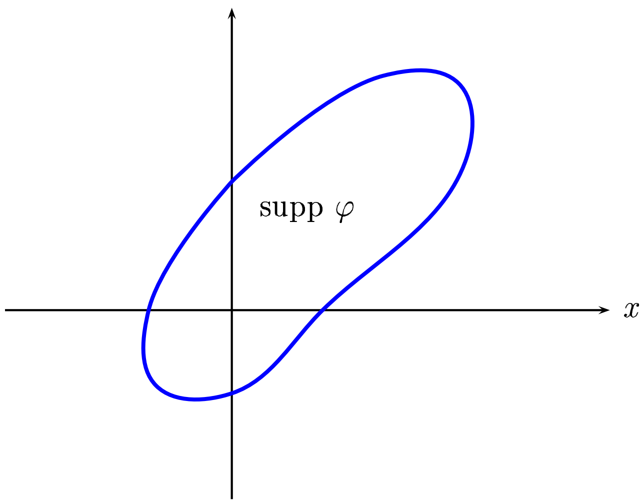


Figure 9.15: The support of $\varphi(x)$ containing the origin.

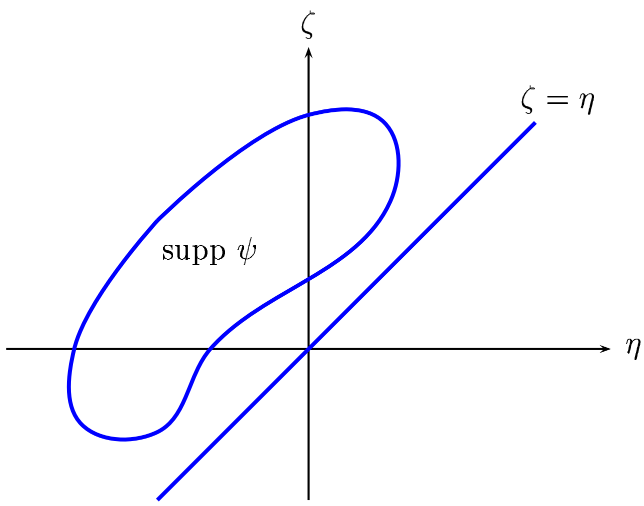


Figure 9.16: The support of $\psi(x)$ on the left side of the line $\zeta = \eta$.

Chapter 10

Derivations of Some PDEs

In this appendix we discuss some of the important partial differential equations and derive a few of them based on the fundamental laws of physics. Here, to give an concise outline for a derivation procedure, we focus on derivation of the principle equations of fluid dynamics: the Navier-Stokes equations. For more discussions and derivation of other equations we refer the reader to the seemingly rich literature in PDE.

Remark We point out that the Laplacian ∇^2 commutes with all rigid motion of Euclidean space; that is, if $\mathcal{T}$ denotes any translation or rotation of n-space, then $\nabla^2(f \circ \mathcal{T}) = \nabla^2(f) \circ \mathcal{T}$ for all function f. Moreover, the only linear differential operators of order ≤ 2 with this property are $\alpha\nabla^2 + \beta$ where α and β are constants. Hence, the differential equation describing any process that is spatially symmetic (i.e., unaffected by translation and rotation) is likely to involve the Laplacian ∇^2.

10.1 Some important equations

Below we shall introduce some of the important equations modelling physics that govern, e.g., the motion of fluids and gases. These equations result from the conservation laws and constitutive relations based on some macroscopic entities described, mainly, by the following space-time fuctions:

- the density function $\rho(\mathbf{x}, t)$
- the velocity vector $u(\mathbf{x}, t)$
- the pressure $p(\mathbf{x}, t)$.

10.2 The Incompressible Navier-Stokes

The most important equations of the fluid dynamics are the *Incompressible Navier-Stokes equations* (NS). NS equations consists of a system of partial differential equations described by

(1) the moment equation, Newton's second law

(2) the constinuity equation, conservation of mass.

The first derivation of NS was given 1822 by Claude Navier (and later, 1845, by George Stokes).

• Continuity equation

Consider a fluid element occupying an arbitrary, fixed domain (volume) Ω. We can describe the variation of the mass in this volym by

$$\frac{\partial}{\partial t}\int_\Omega \rho\, d\mathbf{x} = \int_\Omega \frac{\partial \rho}{\partial t} d\mathbf{x}. \tag{10.2.1}$$

The flow of mass out Ω per area unit is then $\rho \cdot u\mathbf{n}$, where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Using conservation of mass (the increase of mass inside Ω is identical to outflow of the mass through the boundary $\partial\Omega$):

$$\frac{\partial}{\partial t}\int_\Omega \rho\, d\mathbf{x} = -\int_{\partial\Omega} \rho u \cdot \mathbf{n}\, dS. \tag{10.2.2}$$

Using Gauss divergent theorem, we rewrite the equation (10.2.2) as

$$\int_\Omega \Big(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u)\Big)\, d\mathbf{x} = 0. \tag{10.2.3}$$

Since Ω is arbitrary, using lemma 1, this yields *continuity equation* known as the *transport equation*:

$$\rho_t + \nabla \cdot (\rho\, u) = 0. \tag{10.2.4}$$

If ρ is constant (homogeneous media), then we get the simpler equation

$$\nabla \cdot u = 0. \tag{10.2.5}$$

Exempel 36. *We justify* (10.2.4) *for a one-dimensional model problem for traffic flow where we assume that* $\rho = \rho(x,t)$, *the density of cars, satisfies* $0 \le \rho \le 1$. *For a highway path* (a, b) *the difference between the traffic inflow*

$u(a)\rho(a)$ at the (boundary) point $x = a$ and outflow $u(b)\rho(b)$ at the (boundary) point $x = b$ gives the density variation on the interval (a, b)

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \int_a^b \dot{\rho}(x,t)\,dx = u(a)\rho(a) - u(b)\rho(b) = -\int_a^b (u\rho)'\,dx,$$

or equivalently

$$\int_a^b \{\dot{\rho} + (u\rho)'\}dx = 0. \tag{10.2.6}$$

Now since a and b are choosen arbitray, hence by the above lemma,

$$\dot{\rho} + (u\rho)' = 0, \tag{10.2.7}$$

which is just the $1 - D$ version of the transport equation (10.2.4).

Exempel 37. *We may relate the velocity vector u and the density function ρ in a variety of ways. For instance if we choose $u = 1 - \rho$ in* (10.2.7), *then we get the* nonlinear convection equation

$$\dot{\rho} + (1 - 2\rho)\rho' = 0. \tag{10.2.8}$$

On the other hand, choosing $u = c - \varepsilon(\rho'/\rho)$, $c > 0, \varepsilon > 0$, then (10.2.7) *yields*

$$\dot{\rho} + \Big((c - \varepsilon\frac{\rho'}{\rho})\rho\Big)' = 0, \tag{10.2.9}$$

which, in its simplified form

$$\dot{\rho} + c\rho' - \varepsilon\rho'' = 0, \tag{10.2.10}$$

is a one dimensional convection -diffusion *equation. If $c > \varepsilon$, then* (10.2.10) *is a* convection-dominated *convection-diffusion equation. Finally a change of notation $\rho \curvearrowright u$ and $c \curvearrowright \beta$ gives*

$$\dot{u} + \beta u' - \varepsilon u'' = 0. \tag{10.2.11}$$

Equation (10.2.11) can be compared with the homogenoeous *Navier-Stokes* equations for an incompressible flow

$$\begin{aligned} \dot{u} + (\beta \cdot \nabla)\,u - \varepsilon\Delta u + \nabla p &= 0, \\ \nabla u &= 0. \end{aligned} \tag{10.2.12}$$

where $\beta = u$, $u = (u_1, u_2, u_3)$:=(mass, momentum, energy) is the velocity vector, p is the pressure and $\varepsilon = 1/Re$ with Re denoting the Reynold's number. These equations are not easily solvable, for $\varepsilon > 0$ *small*, because of the appearence of *boundary layer* and *turbulence*. A typical range for Re is between 10^5 and 10^7. To derive the Navier-Stokes equations we shall also need the *equation of moment:*

• **Equation of moment**

We denote by $X(t)$ the trajectory of a fluid particle (or a very small portion of the fluid which we call the fluid particle), in Lagrange coordinates, in Ω. The velocity, $u(X(t), t)$, of this fluid particle is given by

$$u(X(t), t) = \frac{dX(t)}{dt},$$

and its acceleration $a(X(t), t)$, through the chain rule, by

$$\frac{d^2}{dt^2}X(t) = \frac{d}{dt}u(X(t), t) = \frac{\partial u}{\partial t} + \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}u_i = \frac{\partial u}{\partial t} + (u \cdot \nabla)u. \tag{10.2.13}$$

It is customary to denote the right hand side by the so called mass derivative of u: Du/Dt, viz

$$\frac{d^2}{dt^2}X(t) = \frac{d}{dt}u(X(t), t) = \frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + (u \cdot \nabla)u. \tag{10.2.14}$$

There are a number of different forces acting on a fluid particle such as the *pressure*, *viscosity forces* (or *internal forces*) and *volum forces* such *gravity*. The pressure p is acting on the surface of the fluid particle in the direction of the normal to surface. The viscosity forses, which are denoted by a $d \times d$ matrix σ, are also acting on the surface of the fluid particles, but both in the normal and the tangent direction to the surface. The matrix σ, which is called the *stress tensor*, must be a matrix that represents the forces acting on all and d directions. These forces appear because of the motion (*diffusion*) of atoms with different kinetic enegy in and out of the fluid particle. One may think this phenomenon as frictions between different fluid particles moving with different velocities. The volum force f is a force per unit of mass, i.e. a force action on the whole volume of the particle. We sum all forces acting on a particle, and use Gauss theorem, to obtain

$$\int_{\partial\Omega} (-p\,\mathbf{n} + \sigma \cdot \mathbf{n})\, dS + \int_{\Omega} \rho f\, d\mathbf{x} = \int_{\Omega} (-\nabla p + \nabla \cdot \sigma) + \rho f\, d\mathbf{x}. \tag{10.2.15}$$

This means that the sum of forces acting on the fluid per unit volume is given by

$$-\nabla p + \nabla \cdot \sigma + \rho f. \tag{10.2.16}$$

Thus with equations (10.2.14) and (10.2.16) the Newton second law, for volume unit in a fluid, becomes

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \sigma + \rho f. \tag{10.2.17}$$

It remains to find an expression for the stress tensor σ. The viscos forces appear as the fluid particles move with different velocities. Hence, these forces must depend on the spatial derivatives of the velocity field. In general (i) the velocity gradient is assumed to be small. (ii) it is assumed that there is an approximative linear relation relation between the stress tensor and the first derivatives of the velocity vector, viz $\sigma_{ij} \propto \partial u_i/\partial x_j$. Note that, with this assumption, if u is constant, then $\sigma = 0$. Further that σ should depend *only* on first derivatives means that there will not be any contributions from the constants. Finally, σ should vanish for the fluids rotating with a constant velocity (angular frequency) ω. It appears that the linear combination of $\partial u_i/\partial x_j + \partial u_j/\partial x_i$ vanishes for the velocities of the form and therefore σ should contain such a linear combinations. Thus, we have a suggestion for the form of the stress tensor as

$$\sigma_{ij} = \eta \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big), \tag{10.2.18}$$

where η is called the *viscosity*. Hence, assuming that η is constant, the viscous term in the equation (10.2.17) can be written as

$$\nabla \cdot \sigma = \eta \frac{\partial}{\partial x_j} \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big) = \eta \Delta u. \tag{10.2.19}$$

Inserting (10.2.19) in the equation (10.2.17) the moment equation becomes

$$\rho \frac{Du}{Dt} = -\nabla p + \eta \Delta u + \rho f. \tag{10.2.20}$$

• **The continum hyphotesis**

The above reasoning is based on the hypothesis of considering the fluid as a *continum*. This is under the hyphothesis that the size (volume) of the fluid under consideration is much larger that the size of the fluid atoms/moleculs. This is the so called *continum hyphotesis*.

• Boundary value problem and Navier-Stokes equations
We consider a bounded spatial domain $\Omega = \Omega(t) \subset \mathbb{R}^d$, a time interval $I = [0, T]$, and assume that η and ρ are constants. Then the equations (10.2.5) and (10.2.20) give the following Navier-Stokes equations:

$$\begin{aligned} u_t + (u \cdot \nabla)u - \nu\Delta u + \frac{1}{\rho}\nabla p = f, \qquad & x \in \Omega(t), \quad t \in I, \\ \nabla u = 0, \qquad & x \in \Omega(t), \quad t \in I, \\ u = u_0, \qquad & x \in \Omega(t), \quad t = 0, \\ u = g, \qquad & x \in \Gamma, \qquad t \in I, \end{aligned} \tag{10.2.21}$$

where $\Gamma = \partial\Omega$ is the boundary of Ω, $\nu = \eta/\rho$ is the kinetic viscosity, u_0 is the initial data and g is the velocity at the boundary. Note that here are no *slip boundary conditions*, i.e., the fluid at the boundary and the boundary itself have the same velovity.

• Approximatios of Navier-Stokes equations

Stokes equations:

$$\begin{aligned} -\nu\Delta u + \frac{1}{\rho}\nabla p = f, \qquad & x \in \Omega(t), \\ \nabla u = 0 \qquad & x \in \Omega(t), \end{aligned} \tag{10.2.22}$$

The Potential flow:

$$-\nu\Delta u = f, \qquad x \in \Omega(t). \tag{10.2.23}$$

•Other models
Navier-Stokes and Stokes equations are the so-called continum models. In contrast to these there are particle models that describe how the particles (atoms, moleculs) interact in a microscopic level. *Boltzmann equation* is an example of a *particle-based* model equation.

• Dimensionless form of the Navier-Stokes equations
Below we shall derive the Navier-Stokes equations in the dimensionless form. To this approach we introduce a characteristic lenght L, and a characteristic speed U, (then we get a characteristic time $T = L/U$). The choice of L and

U can be based of a form of averaging. We scale the Navier-Stokes equations using the following scaling of the involved parameters:

$$x_i' = x_i/L, \quad u_i' = u_i/U, \quad t' = t/T, \quad p' = p/(\rho U^2).$$

Now using termwise chain rule in the Navier-Stokes equations (10.2.21), and lengthy calculations yield

$$\begin{aligned}
\frac{\partial u}{\partial t} &= U\frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial t} = \frac{U}{T}\frac{\partial u'}{\partial t'},\\
(u\cdot\nabla)u = u_i\frac{\partial u}{\partial x_i} &= U^2 u_i\frac{\partial u'}{\partial x_i'}\frac{\partial x_i'}{\partial x_i} = \frac{U^2}{L}u_i'\frac{\partial u'}{\partial x_i'} = \frac{U^2}{L}(u'\cdot\nabla')u',\\
\nabla p &= \rho U^2\frac{\partial p'}{\partial x_i'}\frac{\partial x_i'}{\partial x_i} = \frac{\rho U^2}{L}\frac{\partial p'}{\partial x_i'} = \frac{\rho U^2}{L}\nabla' p',\\
\Delta u &= \frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_i} = U\frac{\partial}{\partial x_i'}\frac{\partial x_i'}{\partial x_i}\frac{\partial u'}{\partial x_i'}\frac{\partial x_i'}{\partial x_i} = \frac{U}{L^2}\frac{\partial}{\partial x_i'}\frac{\partial u'}{\partial x_i'} = \frac{U}{L^2}\Delta' u'.
\end{aligned} \tag{10.2.24}$$

Here ∇' and Δ' denote the gradient and Laplacian in the dimensionless variables. Now for simplicity we supress the $'$ s and get the following dimensionless Navier-stokes equations:

$$\begin{aligned}
u_t + (u\cdot\nabla)u - \frac{1}{Re}\Delta u + \nabla p = f, &\qquad x\in\Omega(t), \quad t\in I,\\
\nabla u = 0, &\qquad x\in\Omega(t), \quad t\in I,
\end{aligned} \tag{10.2.25}$$

where $Re = LU/\nu$ is the so called *Reynold number*.

- **Laminary and turbulent fluids**

Solutions to the Navier-Stokes equations depend on the size of the Reynold number and may look completely different for different ranges of the size of the Renold number Re. For instance *large Reynold numbers,* as $Re \gtrsim 100$, correspond to turbulent fluids having irregular, chaotic, "stochastic", solutions, whereas the *small Reynold numbers,* represent *laminary fluids*, with regular solutions.

- **Existence and uniqueness questions**

We point out that in the case of the **Stokes** equations we have existence of unique solutions. As for the **Navier-Stokes** equations it is *hard* to give a simple answer. See
www.claymath.org/Millennium_Price_Problems (a one-million $ problem!).

• Boundary value problem for the pressure

The pressure is determined by the velocity field. This is justified through taking the divergence of the moment equation in Navier-Stokes. A detailed term- and stepwise calculation yields

$$\begin{cases} \nabla \cdot u_t = \dfrac{\partial}{\partial t} \nabla \cdot u = \{\nabla \cdot u = 0\} = 0, \\ \nabla \cdot \Delta u = \dfrac{\partial^2}{\partial x_i \partial x_j} \nabla \cdot u = 0, \\ \nabla \cdot \nabla p = \Delta p, \\ \nabla \cdot \{(u \cdot \nabla) u\} = \dfrac{\partial}{\partial x_i}\Big(u_j \dfrac{\partial u_i}{\partial x_j}\Big) = \dfrac{\partial u_j}{\partial x_i} \dfrac{\partial u_i}{\partial x_j}. \end{cases} \tag{10.2.26}$$

The moment equation now becomes

$$\Delta p = \nabla \cdot f - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}. \tag{10.2.27}$$

In the of *no-slip* case (assume $g = 0$) we can derive the boundary conditions through the scalar product of the moment equation in the Navier-Stokes with the normal vector at the boundary:

$$\begin{cases} u_t \cdot \mathbf{n} = \dfrac{\partial}{\partial t}(u \cdot \mathbf{n}) = 0, \\ \nabla p \cdot \mathbf{n} = \dfrac{\partial p}{\partial \mathbf{n}}, \\ \{(u \cdot \nabla) u\} \cdot \mathbf{n} = u_j \dfrac{\partial u_i}{\partial x_j} \mathbf{n}_j = 0. \end{cases} \tag{10.2.28}$$

We obtain that

$$\frac{\partial p}{\partial \mathbf{n}} = \{f + \nu \Delta u\} \cdot \mathbf{n}. \tag{10.2.29}$$

Equations (10.2.27) and (10.2.29) give a boundary value problem for the pressure with the Neumann boundary condition (10.2.29).

10.2.1 The weak formulation of the Navier-Stokes

First we shall assume that in the equation (10.2.21), $g \equiv 0$ and introduce some function spaces. Let $\Omega \subset \mathbb{R}^d$, for the functions q, $v : \Omega \to \mathbb{R}$ we define

$$\begin{aligned} L^2(\Omega) &= \{q : \int_\Omega q^2\, d\mathbf{x} < \infty\}, \\ L_0^2(\Omega) &= \{q \in L^2(\Omega) : \int_\Omega q\, d\mathbf{x} = 0\}, \\ H^1(\Omega) &= \{v : \int_\Omega (|\nabla v|^2 + v^2)\, d\mathbf{x} < \infty\}, \\ H_0^1(\Omega) &= \{v \in H^1(\Omega) : v|_\Gamma = 0\}. \end{aligned} \tag{10.2.30}$$

We shall use $L_0^2(\Omega)$ for the pressure. The pressure in the NS is in determined modulo constants (if p is a solution for the pressure then also $p + c$, where c is a constant, is a solution). The conditoion : $\int_\Omega q\, d\mathbf{x} = 0$ gives a uniquelly determined pressure. The space $H_0^1(\Omega)$ is employes for the velocity field. For $u \in H_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ we multiply the Navier-Stokes equation (10.2.21) by $v \in H_0^1(\Omega)$ and the continuity equation by $q \in L_0^2(\Omega)$ and integrate over Ω. The termwise, step-by-step, calculation yields

$$\begin{aligned} \int_\Omega \frac{\partial u}{\partial t} \cdot v\, d\mathbf{x} &= \frac{d}{dt}\int_\Omega u \cdot v\, d\mathbf{x}, \\ -\int_\Omega \Delta u \cdot v\, d\mathbf{x} &= \{PI\} = -\int_\Gamma \frac{\partial u_i}{\partial \mathbf{n}} v_i\, d\Gamma + \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, d\mathbf{x}, \\ &= \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, d\mathbf{x}, \qquad (v|_\Gamma = 0), \\ \int_\Omega \nabla u \cdot v\, d\mathbf{x} &= \{PI\} = \int_\Gamma pv \cdot \mathbf{n}\, d\Gamma - \int_\Omega p\nabla \cdot v\, d\mathbf{x} \\ &= -\int_\Omega p\nabla \cdot v. \end{aligned} \tag{10.2.31}$$

Now the final weak formulation is: find $(u, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u \cdot v\, d\mathbf{x} + \frac{1}{Re}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} + \int_\Omega u_i \frac{\partial u_j}{\partial x_i} v_j d\mathbf{x} + \int_\Omega p\nabla \cdot v d\mathbf{x} &= \int_\Omega f \cdot v\, d\mathbf{x}, \\ \int_\Omega q\nabla \cdot u\, d\mathbf{x} &= 0, \end{aligned} \tag{10.2.32}$$

for all $(v, q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$.

Further, introducing the following notation

$$\begin{aligned} (u, v) &= \int_\Omega u \cdot v \, d\mathbf{x}, \\ a(u, v) &= \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x}, \\ b(p, v) &= -\int_\Omega p \nabla \cdot v \, d\mathbf{x}, \\ c(u, u, v) &= \int_\Omega u_i \frac{\partial u_j}{\partial x_i} v_j \, d\mathbf{x} \end{aligned} \tag{10.2.33}$$

we may write the equation (10.2.32) in the following concise form: find $(u, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$\begin{aligned} \frac{d}{dt}(u, v) + \frac{1}{Re} a(u, v) + c(u, u, v) + b(p, v) &= (f, v) \\ b(q, u) &= 0, \end{aligned} \tag{10.2.34}$$

for all $(v, q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$.

Exempel 38. *Below we give an alternative derivation, (a heuristic one), of Navier-Stokes equations. For a more detailed derivation and motivation of the assumpltions see PDE and Fluid dynamics litterature.*

Now we return to the transport equation (10.2.4). *Requiring that the fluid satisfies the law of Conservation of momentum. Then, the forces acting on the fluid in* Ω *are the pointwise gravity* g *and the pressure* p *applied at the boundary* $\partial\Omega$. *Now, neglecting the friction forces beween fluid molecules, Newton's law of motion implies an equality between the change in the fluid momentum and the total forces acting on the fluid.*

$$\frac{\partial}{\partial t} \int_\Omega \rho u \, d\mathbf{x} = -\int_{\partial\Omega} p\mathbf{n} \, dS + \int_\Omega \rho g \, d\mathbf{x}. \tag{10.2.35}$$

Now, using (10.2.4) *we get*

$$\int_\Omega [\rho \dot{u} + \rho(u \cdot \nabla u) u] \, d\mathbf{x} = \int_\Omega (-\nabla p + \rho g) \, d\mathbf{x}, \tag{10.2.36}$$

which yields Euler equation:

$$u_t + (u \cdot \nabla) \, u = -\frac{1}{\rho} \nabla p + g, \tag{10.2.37}$$

where g is the density vector of the gravitational force. Let now ν be the fluid's viscosity coefficient, then using the relation between g and the fluid viscosity

$$\rho\, g = \nu \Delta u, \tag{10.2.38}$$

and assuming a gradient-free flow ($\nabla u = 0$) we get the Navier-Stokes equations:

$$\begin{aligned} \rho(u_t + (u \cdot \nabla)\, u) &= \nu \Delta u - p, \\ \nabla u &= 0. \end{aligned} \tag{10.2.39}$$

10.3 Further real world equations

• The minimal surface equation: Lagrange equation

Lagrange showed in 1760 that the surface area of the membrane is smaller than the surface area of any other surface that is a small perturbation of it. Such special surfaces are called the *minimal surfaces*. Lagrange further demonstrated that the graph of a minimal surface satisfies the following second-order nonlinear partial differential equation:

$$(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0. \tag{10.3.1}$$

For minimal surfaces with small slope (i.e. u_x, $u_y \ll 1$), then the equation (10.3.1) can be approximated by the two-dimensional Laplace equation.

• The biharmonic equation

The equilibrium state of a thin elastic plate is provided by its amplitude $u(x, y)$ and satisfies the *biharmonic equation*:

$$\Delta^2 u = \Delta(\Delta u) = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0. \tag{10.3.2}$$

The unknown function $u(x, y)$ describes the deviation of the plate from its horizontal position. Note that in contrast to all important equations, which are of first or second order, the biharmonic equation is a PDE of order four. There are other biharmonic equations, e.g. the *Cohan-Hilliard equations*, to mention one.

• The Schrödinger equation

One of the fundamental equations of quantum mechanics, derived in 1926 by Erwin Schrödinger, governs by the evolution of the wave function u of a particle in a potential field V:

$$i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\Delta u + Vu, \tag{10.3.3}$$

where m is the particle's mass, V is a given function, and $\hbar = \vartheta/2\pi$, with ϑ being the Planck's constant.

Chapter 11

Solved Problems

Example 1. The function $f(x)$ given by $f(x) = (x+1)^2$ for $-1 < x < 1$, is 2-periodic. Expand $f(x)$ in complex trigonometric Fourier series. Determine a 2-periodic solution to the equation

$$2y'' - y' - y = f(x). \tag{11.0.1}$$

Solution: Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\Omega x}$. Since $T = 2$ we get $\Omega = 2\pi/T = \pi$. Thus we have $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$, where the Fourier coefficients C_n are computed as follows: For $n \neq 0$, using repeated partial integration

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-1}^{1} f(x)e^{-in\pi x}dx = \frac{1}{2}\int_{-1}^{1}(x+1)^2 e^{-in\pi x}dx \\ &= \frac{1}{2}\Big[(x+1)^2\frac{e^{-in\pi x}}{-in\pi}\Big]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} 2(x+1)\frac{e^{-in\pi x}}{-inx}dx \\ &= 2\frac{(-1)^n}{-in\pi} - \Big[(x+1)\frac{e^{-in\pi x}}{(-in\pi)^2}\Big]_{-1}^{1} + \int_{-1}^{1}\frac{e^{-in\pi x}}{(-in\pi)^3}dx \\ &= \frac{2i(-1)^n}{n\pi} - \frac{2(-1)^n}{-n^2\pi^2} + 0 = \frac{2(-1)^n(1+in\pi)}{n^2\pi^2}. \end{aligned} \tag{11.0.2}$$

For $n = 0$ we get

$$c_0 = \frac{1}{2}\int_{-1}^{1}(x+1)^2 e^0 dx = \Big[\frac{1}{2}\frac{(x+1)^3}{3}\Big]_{-1}^{1} = \frac{8}{6} = \frac{4}{3}.$$

Then the complex Fourier series expansion for the function $f(x)$ is given by

$$f(x) = \frac{4}{3} + \frac{2}{\pi^2}\sum_{n=-\infty, n\neq 0}^{\infty}\frac{(-1)^n(1+in\pi)}{n^2}e^{in\pi x}.$$

Now to solve the equation (11.0.1) we may rewrite it as

$$2y'' - y' - y = f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}, \tag{11.0.3}$$

and denote the complex Fourier series expansion for the solution y by $y = \sum_{n=-\infty}^{\infty} y_n e^{in\pi x}$. Now our task is to determine the unknown Fourier Coefficients y_n. To this end we note that, performing derivations on y, (11.0.3) can be written as

$$2y'' - y' - y = \sum_{n=-\infty}^{\infty} [2(in\pi)^2 - (in\pi) - 1] y_n e^{in\pi x} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}, \tag{11.0.4}$$

where identification of coefficients yields

$$[2(in\pi)^2 - (in\pi) - 1] y_n = c_n \qquad \forall n. \tag{11.0.5}$$

Thus for $n = 0$ we get $-y_0 = c_0$ and $y_0 = -c_0 = -\frac{4}{3}$.
As for $n \neq 0$ we have from (11.0.5) that

$$-[2n^2\pi^2 + in\pi + 1] y_n = \frac{2}{\pi^2} \frac{(-1)^n (1 + in\pi)}{n^2} \iff y_n = \frac{2}{\pi^2} \frac{(-1)^{n-1}(1 + in\pi)}{n^2(2n^2\pi^2 + in\pi + 1)}.$$

Thus the solution to the original equation (11.0.1) is

$$y(x) = -\frac{4}{3} + \frac{2}{\pi^2} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^{n-1}(1 + in\pi)}{n^2(2n^2\pi^2 + in\pi + 1)} e^{in\pi x}. \tag{11.0.6}$$

Example 2. The function $f(t)$ is 3-periodic, where

$$f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } 1 < t < 2 \\ 3 - t & \text{for } 2 \leq t \leq 3 \end{cases}$$

Determine a periodic solution to the differential equation $y'' + 3y' = f(t)$ in the form of a trigonometric Fourier series

Solution: We have that the function $f(t)$ is even, (see the Figure below). For $T = 3$ we get $\Omega = 2\pi/T = 2\pi/3$. Thus in the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t, \quad \text{we have} \quad b_n = 0, \quad \forall n. \tag{11.0.7}$$

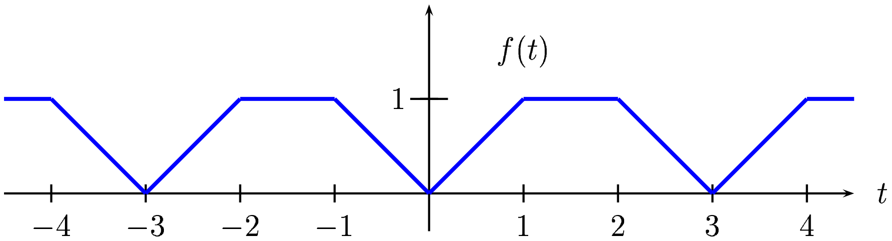


Figure 11.1: The 3-periodic function $f(t)$.

As an appropriate interval we choose $t \in [-T/2, T/2]$ then, since $f(t) = t,\ t \in [0,1]$ and $f(t) = 1,\ t \in [1, 3/2]$, it follows that, for $n \geq 1$,

$$
\begin{aligned}
a_n &= \frac{T}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\Omega t\,dt = \frac{2}{3}\cdot 2\int_0^{\frac{T}{2}} f(t)\cos n\Omega t\,dt \\
&= \frac{4}{3}\Big\{\int_0^1 t\cos n\Omega t\,dt + \int_1^{3/2}\cos n\Omega t\,dt\Big\} \\
&= \frac{4}{3}\Big\{\Big[t\frac{\sin n\Omega t}{n\Omega}\Big]_0^1 - \int_0^1 \frac{\sin n\Omega t}{n\Omega}\,dt + \Big[\frac{\sin n\Omega t}{n\Omega}\Big]_1^{3/2}\Big\} \\
&= \frac{4}{3}\Big\{\frac{\sin n\Omega}{n\Omega} + \Big[\frac{\cos n\Omega}{(n\Omega)^2}\Big]_0^1 + \frac{\sin n\Omega\frac{3}{2}}{n\Omega} - \frac{\sin n\Omega}{n\Omega}\Big\} \\
&= \frac{4}{3}\Big\{\frac{\cos n\Omega - 1}{(n\Omega)^2} + \frac{\sin n\Omega\frac{3}{2}}{n\Omega}\Big\} = \frac{3}{\pi^2}\frac{\cos\frac{2n\pi}{3} - 1}{n^2} \quad \text{for} \quad n \geq 1.
\end{aligned}
\tag{11.0.8}
$$

where in the last step we substituted $\Omega = \frac{2\pi}{3}$!

$$
a_0 = \frac{4}{3}\{\int_0^1 t\,dt + \int_1^{3/2} dt\} = \frac{4}{3}\Big[\frac{t^2}{2}\Big]_0^1 + \frac{3}{2} - 1 = \frac{4}{3}. \tag{11.0.9}
$$

The expand of the given function $f(t)$ in a Fourier series is thus

$$
f(t) = \frac{2}{3} - \frac{3}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - \cos\frac{2n\pi}{3}}{n^2}\cos\frac{2n\pi t}{3}. \tag{11.0.10}
$$

To solve the differential equation we let $y(t) = y_0 + \sum_{n=1}^{\infty} y_n \cos\frac{2n\pi t}{3}$. Then

$$
y'' + 3y = 3y_0 + \sum_{n=1}^{\infty}\Big(-\frac{4n^2\pi^2}{9} + 3\Big)y_n\cos\frac{2n\pi t}{3} = f(t). \tag{11.0.11}
$$

Identification of the coefficients of (11.0.10) and (11.0.11) we get

$$3y_0 = 2/3, \quad \text{and} \quad \left(3 - \frac{4n^2\pi^2}{9}\right) y_n = -\frac{3}{\pi^2} \frac{1 - \cos\frac{2n\pi}{3}}{n^2}.$$

Thus

$$y(t) = \frac{2}{9} - \frac{3}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos\frac{2n\pi}{3}}{n^2\left(3 - \frac{4n^2\pi^2}{9}\right)} \cos\frac{2n\pi t}{3}. \tag{11.0.12}$$

Example 3. Expand the function $g(x) = \cos x$ in Fourier sine series at the interval $(0, \pi/2)$. Use the result to calculate the sum

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2 - 1)^2}. \tag{11.0.13}$$

Solution: We expand the function $g(x) = \cos x$ to a $(2L = \pi)$-periodic *odd* function f at the interval $[-\pi/2, \pi/2]$, see the figure below:

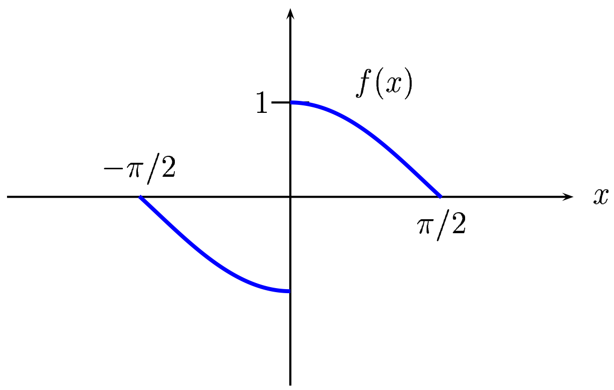


Figure 11.2: The π-periodic function $f(x)$.

We know that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\Omega x + b_n \sin n\Omega x\right).$$

Since $f(t)$ is an odd function we have $a_n = 0$ for all $n \geq 0$. Further $T = \pi = 2L \Longrightarrow \Omega = 2\pi/T = 2 = \pi/L$. Hence,

$$\begin{aligned} b_n &= \frac{2}{L}\int_L^1 f(x)\sin\frac{n\pi x}{L}\,dx = \frac{4}{\pi}\int_0^{\pi/2}\cos x\,\sin\frac{n\pi x}{\pi/2}\,dx \\ &= \frac{4}{\pi}\int_0^{\pi/2}\cos x\,\sin 2nx\,dx = \frac{4}{\pi}\cdot\frac{1}{2}\int_0^{\pi/2}[\sin(2n+1)x + \sin(2n-1)x]\,dx \\ ,&= \frac{2}{\pi}\Big(\Big[\frac{-1}{2n+1}\cos(2n+1)x\Big]_0^{\pi/2} - \Big[\frac{1}{2n-1}\cos(2n-1)x\Big]_0^{\pi/2}\Big) \\ &= \frac{2}{\pi}\Big(\frac{1}{2n+1} + \frac{1}{2n-1}\Big) = \frac{8n}{\pi(4n^2-1)}. \end{aligned}$$

Thus we have an expansion of the odd function $f(x)$ in Fourier sinus series as

$$f(x) \approx \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx, \tag{11.0.14}$$

and hence the function $g(x) = \cos x$ has an odd expansion on $(0, \pi/2)$ viz,

$$\cos x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx. \qquad x \in (0, \pi/2). \tag{11.0.15}$$

To compute the sum (11.0.13) we multiply the equation(11.0.15) by $\cos x$ and integrate the result over the interval $(0, \pi/2)$. Then changing the order of sum and integral on the right side we get

$$\int_0^{\pi/2}\cos x\cos x\,dx = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\int_0^{\pi/2}\sin 2nx\cos x\,dx \tag{11.0.16}$$

Note that the integral on the right hand side is $\frac{\pi}{4}b_n$ which we have already calculated above:

$$\frac{4}{\pi}\int_0^{\pi/2}\cos x\,\sin 2nx\,dx = \frac{8n}{\pi(4n^2-1)} \quad\Longrightarrow\quad \int_0^{\pi/2}\cos x\,\sin 2nx\,dx = \frac{2n}{4n^2-1}.$$

Consequently (11.0.16) can be written as

$$\int_0^{\pi/2}\Big(\frac{1}{2} + \frac{1}{2}\cos 2x\Big)dx = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\cdot\frac{2n}{4n^2-1}, \tag{11.0.17}$$

which yields

$$\frac{\pi}{4} = \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2}, \quad i.e. \quad \sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}. \tag{11.0.18}$$

Example 4. Let $f(t) = 1-t^2$ be a 2-periodic function for $|t| \le 1$. Determine a bounded solution for the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & x > 0, \quad -\infty < t < \infty \\ u(0,t) = f(t), & \quad -\infty < t < \infty \end{cases} \tag{11.0.19}$$

Solution: Expand the function $f(t) = 1 - t^2$ in a complex trigonometric Fourier series, $f(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega t}$, where for $T = 2$ have $\Omega = 2\pi/T = \pi$. For $n \neq 0$, using repeated partial integration, we get the Fourier coefficients

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1}(1-t^2)e^{-in\pi t}dt = \frac{1}{2}\underbrace{\left[(1-t^2)\frac{e^{-in\pi t}}{-in\pi}\right]_{-1}^{1}}_{=0} - \frac{1}{2}\int_{-1}^{1}(-2t)\frac{e^{-in\pi t}}{-in\pi}dt \\ &= \left[t\frac{e^{-in\pi t}}{(-in\pi)^2}\right]_{-1}^{1} - \int_{-1}^{1}\frac{e^{-in\pi t}}{(-in\pi)^2}dt = \frac{-1}{n^2\pi^2}(e^{-in\pi} + e^{in\pi}) - \underbrace{\left[\frac{e^{-in\pi t}}{(-in\pi)^3}\right]_{-1}^{1}}_{=0} \\ &= \frac{-2(-1)^n}{n^2\pi^2}. \end{aligned}$$

For $n = 0$ we get

$$c_0 = \frac{1}{2}\int_{-1}^{1}(1-t^2)\,dt = \int_{0}^{1}(1-t^2)\,dt = \frac{2}{3}.$$

Thus the complex trigonometric expansion of the function $f(t)$ is

$$f(t) = \frac{2}{3} - \frac{2}{\pi^2}\sum_{-\infty, n\neq 0}^{\infty}\frac{(-1)^n}{n^2}e^{in\pi t}. \tag{11.0.20}$$

Let now

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x)e^{in\pi t}, \tag{11.0.21}$$

be a periodic solution for (11.0.19) with bounded Fourier coefficients $u_n(x) = \frac{1}{2}\int_{-1}^{1} u(x,t)e^{-in\pi t}$. Then the condition $u(0,t) = f(t)$ yields

$$u(0,t) = \sum_{-\infty}^{\infty} u_n(0)e^{in\pi t} = f(t) = \sum_{-\infty}^{\infty} c_n e^{in\pi t}, \tag{11.0.22}$$

and thus $u_n(0) = c_n$ for all n. Now we may rewrite the pde: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ as

$$\sum_{n=-\infty}^{\infty} u_n(x)(in\pi)e^{in\pi t} = \sum_{n=-\infty}^{\infty} u_n''(x)e^{in\pi t}, \tag{11.0.23}$$

where, identifying the coefficients gives the ordinary differential equations,

$$\begin{cases} (in\pi)u_n(x) = u_n'', \\ u_n(0) = c_n, \end{cases} \qquad \forall n, \quad (n \neq 0). \tag{11.0.24}$$

The characteristic equation for (11.0.24) $r^2 = in\pi$ has the roots

$$\begin{cases} r_{2,1} = \pm\frac{1}{\sqrt{2}}(1+i)\sqrt{n\pi} & n > 0 \\ r_{2,1} = \pm\frac{1}{\sqrt{2}}(1-i)\sqrt{|n|\pi} & n < 0, \end{cases} \tag{11.0.25}$$

which we identify by r_1 and r_2, with $Re\, r_1 < 0$ and $Re\ r_2 > 0$. More specifically,

$$r_1 = -\frac{1}{\sqrt{2}}(1+i\, sign\, n)\sqrt{|n|\pi} \quad \text{and} \quad r_2 = \frac{1}{\sqrt{2}}(1+i\, sign\, n)\sqrt{|n|\pi}. \tag{11.0.26}$$

Note that $\sqrt{\pm i} = \frac{1}{\sqrt{2}}(1 \pm i)$ since $\pm i = \frac{1}{2}(1 \pm i)^2 = \frac{1}{2}(1 + i^2 \pm 2i) = \pm i$.
The general solution of the ode (11.0.24) is then given by

$$u_n(x) = A_n e^{r_1 x} + B_n e^{r_2 x}. \tag{11.0.27}$$

Since $Re\ r_2 > 0$, $|e^{r_2 x}| \to \infty$, as $x \to \infty$. Thus, since the solution is bounded, we have $B_n = 0$.
As for $n = 0$, the ode: $u_n'' = 0$ has the solution $u_0 = A_0 + B_0 x$. As above, bounded solution yields, $B_0 = 0$. Hence we have

$$\begin{cases} u_n(x) = A_n e^{-\frac{1}{\sqrt{2}}(1+i\, sign\, n)\sqrt{|n|\pi}\, x}, \\ u_n(0) = A_n = c_n, \end{cases} \qquad n \in \mathbf{Z}, \tag{11.0.28}$$

which yields

$$u_n(x) = c_n e^{-\sqrt{\frac{|n|\pi}{2}}(1+i\, sign\, n)\, x}. \tag{11.0.29}$$

The periodic solution to the pde is now

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} c_n e^{-\sqrt{\frac{|n|\pi}{2}}(1+i\, sign\, n)\, x} e^{in\pi t} \\ &= c_0 + \sum_{n=1}^{\infty} \left\{ c_n e^{-\sqrt{\frac{n\pi}{2}}} e^{i(n\pi t - \sqrt{\frac{n\pi}{2}}\, x)} + c_{-n} e^{-\sqrt{\frac{n\pi}{2}}} e^{i(-n\pi t + \sqrt{\frac{n\pi}{2}}\, x)} \right\} \\ &= c_0 + \sum_{n=1}^{\infty} \left\{ c_n e^{-\sqrt{\frac{n\pi}{2}}} 2\cos(n\pi t - \sqrt{n\pi/2}\, x) \right. \end{aligned}$$

Recalling that $c_n = \frac{-2(-1)^n}{n^2\pi^2} = \frac{2(-1)^{n+1}}{n^2\pi^2}$ and $c_0 = \frac{2}{3}$ we finally have the bounded solution for the pde (11.0.19) given by

$$u(x,t) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2} e^{-\sqrt{\frac{n\pi}{2}}} \cos\left(n\pi t - \sqrt{n\pi/2}\; x\right). \tag{11.0.30}$$

Example 5. Solve the following Laplace equation on the circular ring described by $1 < r < 2$ in polar coordinates:

$$\begin{cases} \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 1 < r < 2 \\ u(1,\theta) = 0 & \\ u(2,\theta) = f(\theta) = 1 - \frac{\theta^2}{\pi^2}, & |\theta| \le \pi. \end{cases} \tag{11.0.31}$$

where the function $f(\theta)$ is 2π-periodic.

Solution: For a fixed r the solution, $u(r,\theta)$, is 2π-periodic in θ, hence we may write

$$u(r,\theta) = \sum_{-\infty}^{\infty} u_n(r) e^{in\theta}. \tag{11.0.32}$$

Inserting in the differential equation yields

$$\sum_{-\infty}^{\infty} \left[(u_n'' + \frac{1}{r}u_n') e^{in\theta} - \frac{1}{r^2} n^2 u_n e^{in\theta} \right] = 0. \tag{11.0.33}$$

Thus we have

$$u_n'' + \frac{1}{r}u_n' - \frac{1}{r^2}n^2u_n = 0 \Longrightarrow r^2u_n'' + ru_n' - n^2u_n = 0. \qquad (11.0.34)$$

This is the Euler equation of order n, which has a solution of the form $u_n(r) = r^p$, i.e.,

$$r^2p(p-1)r^{p-2} + rpr^{p-1} - n^2r^p = 0. \qquad (11.0.35)$$

Hence we have $p^2 - p + p - n^2 = 0$, thus $p = \pm n$ and, for $n \neq 0$, we have the solution $u_n(r) = a_nr^n + b_nr^{-n}$. For $n = 0$ the equation would become $u_n'' + \frac{1}{r}u_n' = 0$, which has a solution of the form $u_0(r) = a_0 + b_0 \ln r$. Thus

$$u(r,\theta) = \sum_{-\infty}^{\infty} u_n(r)e^{in\theta} = a_0 + b_0 \ln r + \sum_{-\infty, n\neq 0}^{\infty} (a_nr^n + b_nr^{-n})e^{in\theta}, \quad (11.0.36)$$

and hence, invoking the boundary conditions we get

$$\begin{cases} u(1,\theta) = a_0 + \displaystyle\sum_{-\infty,n\neq 0}^{\infty} (a_n + b_n)e^{in\theta} = 0, \\ u(2,\theta) = a_0 + b_0 \ln 2 + \displaystyle\sum_{-\infty,n\neq 0}^{\infty} (a_n2^n + b_n2^{-n})e^{in\theta} = f(\theta) = \displaystyle\sum_{-\infty}^{\infty} c_ne^{in\theta}. \end{cases}$$

Now we identify the coefficients to get

$$\begin{cases} a_0 = 0 & a_n + b_n = 0, & \text{for} \quad n \neq 0. \\ a_0 + b_0 \ln 2 = c_0, & a_n2^n + b_n2^{-n} = c_n, & \text{for} \quad n \neq 0. \end{cases} \qquad (11.0.37)$$

This implies that

$$a_0 = 0, \quad b_0 = \frac{c_0}{ln2}, \qquad a_n = \frac{c_n}{2^n - 2^{-n}}, \quad b_n = -a_n, \text{for} \quad n \neq 0. \quad (11.0.38)$$

Now we determine complex Fourier coefficients c_n for the function $f(\theta)$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(1 - \frac{\theta^2}{\pi^2}\Big)e^{-in\theta}\, d\theta. \qquad (11.0.39)$$

This yields, for $n = 0$,

$$c_0 = \frac{1}{\pi}\int_0^{\pi} \Big(1 - \frac{\theta^2}{\pi^2}\Big)e^{-in\theta}\, d\theta = \frac{2}{3}, \qquad (11.0.40)$$

and for $n \neq 0$ we get

$$\begin{aligned} c_n =& \frac{1}{2\pi}\left[\left(1-\frac{\theta^2}{\pi^2}\right)\frac{e^{-in\theta}}{-in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\theta}{\pi^2}\cdot\frac{e^{-in\theta}}{-in}\,d\theta \\ =& \frac{1}{2\pi}\left[\frac{2\theta}{\pi^2}\cdot\frac{e^{-in\theta}}{(-in)^2}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2}{\pi^2}\cdot\frac{e^{-in\theta}}{-n^2}\,d\theta \\ =& \frac{1}{2\pi}\cdot\frac{2\pi}{\pi^2}\cdot\frac{e^{-in\pi}}{-n^2}\cdot 2 = \frac{2(-1)^{n+1}}{\pi^2 n^2}. \end{aligned} \tag{11.0.41}$$

Finally we have the solution to the given Laplace equation

$$u(r,\theta) = \frac{2}{3\ln 2}\ln r + \frac{2}{\pi^2}\sum_{n=-\infty, n\neq 0}^{\infty}\frac{(-1)^{n+1}}{n^2(2^n - 2^{-n})}(r^n - r^{-n})e^{in\theta}. \tag{11.0.42}$$

Example 6. Compute the Fourier transform of the following rational functions:

$$\begin{array}{llll} \textbf{i)} & \frac{t}{(t^2+a^2)^2}, & \textbf{ii)} & \frac{1}{(t^2+a^2)^2}, \\ \textbf{iii)} & \frac{t}{(t^2+1)(t^2+2t+5)}, & \textbf{iv)} & e^{-a|t|}\sin bt, \quad a>0, b>0. \end{array} \tag{11.0.43}$$

Solution: **i)** We start from the Fourier transform $\mathcal{F}\left[\frac{1}{t^2+a^2}\right] = \frac{\pi}{2a}e^{-a|\omega|}$, which we may also denote using the notation

$$\frac{1}{t^2+a^2} \supset^{\mathcal{F}} \frac{\pi}{2a}e^{-a|\omega|}. \tag{11.0.44}$$

Differentiating the left side of (11.0.44) with respect to t and using the Fourier transform formula for the derivatives we get that

$$\frac{-2t}{(t^2+a^2)^2} \supset^{\mathcal{F}} (i\omega)\frac{\pi}{a}e^{-a|\omega|} \quad \Longrightarrow \quad \frac{t}{(t^2+a^2)^2} \supset^{\mathcal{F}} -i\frac{\pi}{2a}\omega e^{-a|\omega|}. \tag{11.0.45}$$

ii) By the definition of the Fourier transform we have

$$\int_{-\infty}^{\infty}\frac{1}{t^2+a^2}e^{-i\omega t}\,dt = \frac{\pi}{a}e^{-a|\omega|}. \tag{11.0.46}$$

Differentiating both sides in (11.0.46) with respect to a, gives

$$\int_{-\infty}^{\infty}\frac{-2a}{(t^2+a^2)^2}e^{-i\omega t}\,dt = -2a\int_{-\infty}^{\infty}\frac{1}{(t^2+a^2)^2}e^{-i\omega t}\,dt = -\frac{\pi}{a^2}e^{-a|\omega|} - \frac{\pi}{a}|\omega|e^{-a|\omega|}.$$

Hence, once again, by the definition of the Fourier transform we get

$$\mathcal{F}\Big[\frac{1}{(t^2+a^2)^2}\Big] = \frac{\pi}{2a^3}e^{-a|\omega|} + \frac{\pi}{2a^2}|\omega|e^{-a|\omega|} = \frac{\pi}{2a^3}(1+a|\omega|)e^{-a|\omega|}. \quad (11.0.47)$$

iii) For this expression we use partial fractions to write

$$\frac{t}{(t^2+1)(t^2+2t+5)} = \frac{A+Bt}{t^2+1} + \frac{C+Dt}{t^2+2t+5} := f(t). \quad (11.0.48)$$

To determine the constants A, B, C and D, we let first $t=0$ and get $0 = A + C/5$, thus

$$C = -5A. \quad (11.0.49)$$

Further multiplying both sides of (11.0.48) by t and letting $t \to \infty$, it follows that

$$B + D = 0. \quad (11.0.50)$$

Next we multiply both sides of (11.0.48) by (t^2+1) and let $t = i$, to get

$$\frac{i}{-1+2i+5} = A+Bi \Longrightarrow A+Bi = \frac{i}{2(2+i)} = \frac{i(2-i)}{2(2+i)(2-i)} = \frac{1}{10}(2i+1),$$

where, identifying the coefficients gives $A = \frac{1}{10}$ and $B = \frac{1}{5}$. Thus we have $C = -5A = -\frac{1}{2}$ and $D = -B = -\frac{1}{5}$, and we can rewrite the function $f(t)$ as

$$f(t) = \frac{\frac{1}{10}+\frac{1}{5}t}{t^2+1} - \frac{\frac{1}{2}+\frac{1}{5}t}{t^2+2t+5} = \frac{1}{10}\cdot\frac{1+2t}{t^2+1} - \frac{\frac{3}{10}+\frac{1}{5}(t+1)}{(t+1)^2+4}. \quad (11.0.51)$$

Here we use the following chain of known Fourier transforms:

$$\frac{1}{t^2+1} \supset^{\mathcal{F}} \frac{\pi}{2a}e^{-|\omega|}. \quad (11.0.52)$$

By the symmetry rule we have that

$$e^{-|t|}sign\, t \supset^{\mathcal{F}} \frac{-2i\omega}{1+\omega^2}e^{-|\omega|} \Rightarrow \frac{t}{t^2+1} \supset^{\mathcal{F}} \frac{i}{2}\cdot 2\pi e^{-|\omega|}sign\,(-\omega) = -\pi e^{-|\omega|}sign\,(\omega).$$

Further the scaling would give

$$\frac{t/2}{1+(t/2)^2} = \frac{2t}{t^2+4} \supset^{\mathcal{F}} -2i\pi e^{-2|\omega|}sign\,(2\omega) \Longrightarrow \frac{t}{t^2+4} \supset^{\mathcal{F}} -i\pi e^{-2|\omega|}sign\,(\omega),$$

which by the substitution; $t \to t+1$, yields

$$\frac{t+1}{(t+1)^2+4} \supset^{\mathcal{F}} -i\pi e^{i\omega} e^{-2|\omega|} sign\,(\omega). \tag{11.0.53}$$

Similarly, by the same change of variable:$t \to t+1$, in (11.0.46) for $a = 2$, we have

$$\frac{1}{t^2+4} \supset^{\mathcal{F}} \frac{\pi}{2} e^{-2|\omega|} \Longrightarrow \frac{1}{(t+1)^2+4} \supset^{\mathcal{F}} \frac{\pi}{2} e^{i\omega} e^{-2|\omega|} \tag{11.0.54}$$

Summing up the Fourietransform $\hat{f}(\omega)$ of the function $f(t)$ is

$$\hat{f}(\omega) = \frac{\pi}{10} e^{-|\omega|}\Big(1 - 2i\,sign\,(\omega)\Big) - \frac{\pi}{10} e^{-2|\omega|} e^{i\omega}\Big(\frac{3}{2} - 2i\,sign\,(\omega)\Big). \tag{11.0.55}$$

iv) The repeated use of the Fourier transform

$$e^{-a|t|} \supset^{\mathcal{F}} \frac{2a}{\omega^2+a^2}, \tag{11.0.56}$$

and the relation $\sin bt = \Big(e^{ibt} - e^{-ibt}\Big)/2i$, gives that

$$\begin{aligned} e^{-a|t|} \sin bt = e^{-a|t|} \cdot \frac{1}{2i}\Big(e^{ibt} - e^{-ibt}\Big) &\supset^{\mathcal{F}} \frac{1}{2i}\Big(\frac{2a}{(\omega-b)^2+a^2} - \frac{2a}{(\omega+b)^2+a^2}\Big) \\ &= ia\Big(\frac{1}{\omega^2+2b\omega+b^2+a^2} - \frac{1}{\omega^2-2b\omega+b^2+a^2}\Big). \end{aligned}$$

Thus we have

$$e^{-a|t|} \sin bt \supset^{\mathcal{F}} \frac{-4iab\omega}{(\omega^2+2b\omega+b^2+a^2)(\omega^2-2b\omega+b^2+a^2)}. \tag{11.0.57}$$

Example 7. The Fourier transform of the function $f(t)$ is given as

$$\hat{f}(\omega) = \frac{\omega}{1+\omega^4}. \tag{11.0.58}$$

Compute the following quantities:

$$\text{a)} \quad \int_{-\infty}^{\infty} t f(t)\,dt, \qquad\qquad \text{b)} \quad f'(0). \tag{11.0.59}$$

Solution: a) Using the formula and the definition of the Fourier transform for the function $tf(t)$ we have that

$$tf(t) \supset^{\mathcal{F}} i\hat{f}'(\omega) \quad \Longleftrightarrow \quad i\hat{f}'(\omega) = \int_{-\infty}^{\infty} tf(t)e^{-i\omega t}\,dt. \tag{11.0.60}$$

We let $\omega = 0$ in (11.0.60), and evaluate the derivative of $\hat{f}(\omega)$, given by (11.0.58), at $\omega = 0$ to get

$$\int_{-\infty}^{\infty} tf(t)\,dt = i\hat{f}'(0) = i\Big[\frac{d}{d\omega}\Big(\frac{\omega}{1+\omega^4}\Big)\Big]_{\omega=0} = i\Big[\frac{1+\omega^4-4\omega^4}{(1+\omega^4)^2}\Big]_{\omega=0} = i.$$

b) Similar to part a) we consider now the formula and the definition of the Fourier transform for the function $f'(t)$ viz,

$$f'(t) \supset^{\mathcal{F}} i\omega\hat{f}(\omega) \quad \Longleftrightarrow \quad f'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\hat{f}(\omega)e^{i\omega t}\,d\omega. \tag{11.0.61}$$

Now evaluating (11.0.61) at $t = 0$ yields

$$f'(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\hat{f}(\omega)\,d\omega = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2}{1+\omega^4}\,d\omega. \tag{11.0.62}$$

We compute the integral on the right hand side of (11.0.62) using the residual calculus viz: The function $g(z) = \frac{z^2}{1+z^4}$ has pols in $\{z_k; 1+z_k^4 = 0\}$, i.e.,

$$z_k^4 = -1 = e^{i\pi(2k+1)} \quad \Longleftrightarrow \quad z_k = e^{i(2k+1)\pi/4}, \quad k = 0,1,2,3. \tag{11.0.63}$$

From these poles $z_0 = e^{i\pi/4}\frac{1}{\sqrt{2}}(1+i)$ and $z_1 = e^{i3\pi/4} = \frac{1}{\sqrt{2}}(-1+i)$ are in the upper half plane and their residuals are

$$Res_{z=z_0}\, g(z) = \Big[\frac{z^2}{4z^3}\Big]_{z=z_0} = \frac{1}{4z_0} = \frac{\bar{z}_0}{4} = \frac{1}{4\sqrt{2}}(1-i). \tag{11.0.64}$$

Similarly

$$Res_{z=z_1}\, g(z) = \frac{1}{4z_1} = \frac{1}{4\sqrt{2}}(-1-i). \tag{11.0.65}$$

Now we can calculate the integral

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx = 2\pi i\Big(Res_{z=z_0}\, g(z) + Res_{z=z_1}\, g(z)\Big) = \frac{2\pi i}{4\sqrt{2}}(1-i-1-i) = \frac{\pi}{\sqrt{2}}.$$

Inserting in (11.0.62) we finally get

$$f'(0) = \frac{i}{2\pi}\frac{\pi}{\sqrt{2}} = \frac{i}{2\sqrt{2}}. \tag{11.0.66}$$

Example 8. The function $f(t)$ has the Fourier transform

$$\hat{f}(\omega) = \frac{1-i\omega}{1+i\omega}\cdot\frac{\sin\omega}{\omega}. \tag{11.0.67}$$

Compute the following integral

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt. \tag{11.0.68}$$

Solution: By the Parseval's relation we have that

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1-i\omega}{1+i\omega}\right|^2\frac{\sin^2\omega}{\omega^2}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)^2 d\omega,\end{aligned} \tag{11.0.69}$$

where we used $\left|\frac{1-i\omega}{1+i\omega}\right| = 1$. Further since $\chi_1(t) = \theta(t+1) - \theta(t-1) \supset^{\mathcal{F}} 2\frac{\sin\omega}{\omega}$. Thus

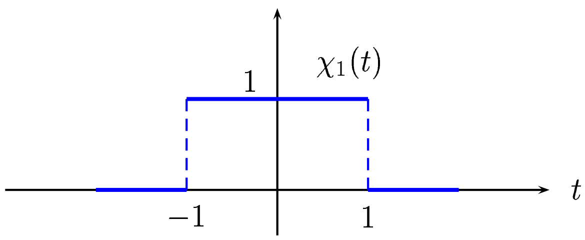


Figure 11.3: The function $\chi_1(t) = \theta(t+1) - \theta(t-1)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)^2 d\omega = \int_{-\infty}^{\infty}\left\{\frac{1}{2}\Big[\theta(t+1) - \theta(t-1)\Big]\right\}^2 dt = \frac{1}{4}\int_{-1}^{1} dt = \frac{1}{2}.$$

Consequently

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2}. \tag{11.0.70}$$

Example 9. Use the Fourier transform to compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x\,(x2+1)} \, dx \tag{11.0.71}$$

Solution: We have the Fourier transform of the rectangular pulse function

$$\chi_a(x) = \theta(x+a) - \theta(x-a) \supset^{\mathcal{F}} hat\chi_a(\xi) = \frac{2\sin a\xi}{\xi}. \tag{11.0.72}$$

Thus by the symmetry rule

$$\frac{2\sin ax}{x} \supset^{\mathcal{F}} 2\pi\chi_a(-\xi) = 2\pi\chi_a(\xi) = 2\pi\Big[\theta(\xi+a) - \theta(\xi-a)\Big], \tag{11.0.73}$$

where for $a = 1$ we get

$$\frac{\sin x}{x} \supset^{\mathcal{F}} \pi\Big[\theta(\xi+1) - \theta(\xi-1)\Big]. \tag{11.0.74}$$

Further, we have the following Fourier transform:

$$\frac{1}{x^2+1} \supset^{\mathcal{F}} \pi e^{-|\xi|}. \tag{11.0.75}$$

Thus by the Plancherel Theorem it follows that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin x}{x} \cdot \frac{1}{x^2+1} \, dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\Big[\theta(\xi+1) - \theta(\xi-1)\Big]\pi e^{-|\xi|} \, d\xi \\ &= \frac{\pi}{2}\int_{-1}^{1} e^{-|\xi|} \, d\xi = \pi\int_{0}^{1} e^{-|\xi|} \, d\xi = \Big[-e^{-\xi}\Big]_0^1 = \pi(1-e^{-1}).\end{aligned}$$

Example 10. The function $f(t)$ has the Fourier transform $\hat{f}(\omega) = \frac{1}{|\omega|^3+1}$. Compute

$$\int_{-\infty}^{\infty} |f * f'|^2 \, dt, \tag{11.0.76}$$

where $f * f'$ is the convolution between the functions f and f'.

Solution: We have that $f(t) \supset^{\mathcal{F}} \hat{f}(\omega) \Longrightarrow f'(t) \supset^{\mathcal{F}} (i\omega)\hat{f}(\omega) = \hat{f}'(\omega)$. Thus using Parsevals formula and $(f * f')(x) \supset^{\mathcal{F}} \hat{f}(\xi)\hat{g}(\xi)$ it follows that

$$\begin{aligned}
\int_{-\infty}^{\infty} |f * f'|^2\, d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |(f * f')\hat{}|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}\cdot\hat{f}'|^2\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{1}{|\omega|^3+1}\cdot\frac{i\omega}{|\omega|^3+1}\Big)\Big(\frac{1}{|\omega|^3+1}\cdot\frac{\overline{i\omega}}{|\omega|^3+1}\Big)\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2}{(|\omega|^3+1)^4}\, d\omega = \frac{1}{\pi}\int_0^{\infty}\frac{\omega^2}{(\omega^3+1)^4}\, d\omega \\
&= \{\omega^3 = z \Rightarrow 3\omega^2\, d\omega = dz\} = \frac{1}{3\pi}\int_0^{\infty}\frac{1}{(z+1)^4}\, dz \\
&= \Big[-\frac{1}{9\pi}(z+1)^{-3}\Big]_0^{\infty} = \frac{1}{9\pi}.
\end{aligned}$$

Example 11. Determine the Fourier transform of the function

$$f(t) = \int_0^2 \frac{\sqrt{\omega}}{1+\omega} e^{i\omega t}\, d\omega, \tag{11.0.77}$$

and then compute the integrals

$$\text{a)}\quad \int_{-\infty}^{\infty} f(t)\cos t\, dt, \qquad\qquad \text{b)}\quad \int_{-\infty}^{\infty} |f(t)|^2\, dt. \tag{11.0.78}$$

Solution: We identify (11.0.77) and the Fourier Inversion formula:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\, d\omega = \int_0^2\frac{\sqrt{\omega}}{1+\omega}e^{i\omega t}\, d\omega. \tag{11.0.79}$$

to get

$$\hat{f}(\omega) = \begin{cases} \frac{2\pi\sqrt{\omega}}{1+\omega}, & \text{for} \quad 0 < \omega < 2 \\ 0, & \qquad\quad \text{otherwise.} \end{cases} \tag{11.0.80}$$

a) Using the definition of the Fourier Transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt, \tag{11.0.81}$$

for $\omega = 1$ we rewrite the integral in a) as

$$\int_{-\infty}^{\infty} f(t)\cos t\,dt = \int_{-\infty}^{\infty} f(t)\frac{1}{2}(e^{it} + e^{-it})\,dt = \frac{1}{2}[\hat{f}(-1) + \hat{f}(1)]. \quad (11.0.82)$$

Now by (11.0.80) $\hat{f}(-1) = 0$ and $\hat{f}(1) = 2\pi/2 = \pi$, thus

$$\int_{-\infty}^{\infty} f(t)\cos t\,dt = \frac{\pi}{2}. \quad (11.0.83)$$

b) Here, by the Parsevals formula

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_0^2 \frac{(2\pi)^2\omega}{(1+\omega)^2}\,d\omega \\ &= \{1+\omega = y\} = 2\pi\int_1^3 \frac{y-1}{y^2}\,dy = 2\pi\int_1^3 \Big(\frac{1}{y} - \frac{1}{y^2}\Big)\,dy \\ &= 2\pi\Big[\ln y + \frac{1}{y}\Big]_1^3 = 2\pi(\ln 3 + \frac{1}{3} - 0 - 1) = 2\pi(\ln 3 - \frac{2}{3}).\end{aligned}$$

Example 12. Let $f(t) = \int_0^1 \sqrt{\omega}\, e^{\omega^2}\cos\omega t\,d\omega$. Compute $\int_{-\infty}^{\infty} |f'(t)|^2\,dt$.

Solution: Note that, on the interval $[0,1]$, we can write $\omega = |\omega|$, so that the integrand in $f(t)$ can be written as an even function $\sqrt{|\omega|}\, e^{\omega^2}\cos\omega t\,d\omega$. Thus

$$\begin{aligned} f(t) &= \int_0^1 \sqrt{\omega}\, e^{\omega^2}\cos\omega t\,d\omega = \int_0^1 \sqrt{|\omega|}\, e^{\omega^2}\cos\omega t\,d\omega \\ &= \frac{1}{2}\int_{-1}^1 \sqrt{|\omega|}\, e^{\omega^2}(\cos\omega t + i\sin\omega t)\,d\omega = \frac{1}{2}\int_{-1}^1 \sqrt{|\omega|}\, e^{\omega^2} e^{i\omega t}\,d\omega.\end{aligned}$$

where we used the fact that $\sin\omega t$ is an odd function. On the other hand by the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega \equiv \frac{1}{2}\int_{-1}^1 \sqrt{|\omega|}\, e^{\omega^2} e^{i\omega t}\,d\omega. \quad (11.0.84)$$

Hence, identifying the integrands it follows that

$$\hat{f}(\omega) = \begin{cases} \pi\sqrt{|\omega|}\, e^{\omega^2} & \text{for} \quad |\omega| \le 1 \\ 0, & \text{for} \quad |\omega| > 1 \end{cases} \quad (11.0.85)$$

Note! that $\hat{f}$ is not continuous.
Now by the formula $f'(t) \supset^{\mathcal{F}} i\omega\hat{f}(\omega)$ and the Plancherels theorem,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f'(t)|^2\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |i\omega\hat{f}(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} \omega^2\pi^2|\omega|e^{2\omega^2}\,d\omega \\
&= \frac{\pi}{2}\cdot 2\int_0^1 \omega^3 e^{2\omega^2}\,d\omega = \Big[2\omega^2 = v,\ 4\omega\,d\omega = dv\Big] \\
&= \pi\int_0^2 \frac{v}{2}e^v\cdot\frac{1}{4}\,dv = \frac{\pi}{8}\int_0^2 ve^v\,dv \\
&= \frac{\pi}{8}\Big\{\Big[ve^v\Big]_0^2 - \int_0^2 e^v\,dv\Big\} = \frac{\pi}{8}\Big\{2e^2 - (e^2-1)\Big\}.
\end{aligned} \tag{11.0.86}$$

Example 13. Compute a solution to the integro-differential equation below

$$u'(t) + 2u(t) + e^{-2t}\int_{-\infty}^{t} e^{2\tau}u(\tau)\,d\tau = \delta(t). \tag{11.0.87}$$

Solution: To solve the problem we shall apply the Fourier transform operator to both sides in the equation (11.0.87), where for the integral term we use the convolution representation:

$$\int_{-\infty}^{t} e^{2(\tau-t)}u(\tau)\,d\tau = \int_{-\infty}^{\infty}\theta(t-\tau)e^{-2(t-\tau)}u(\tau)\,d\tau = \{\theta(t)e^{-2t}\}*\{u(t)\}.$$

The Fourier transform of the first term on the right hand side: $\theta(t)e^{-2t}$ is

$$\int_{-\infty}^{\infty}\theta(t)e^{-2t}e^{-i\xi t}\,dt = \int_0^{\infty} e^{-i(2+i\xi)t}\,dt = \Big[-\frac{e^{-i(2+i\xi)t}}{2+i\xi}\Big]_0^{\infty} = \frac{1}{2+i\xi}. \tag{11.0.88}$$

Thus the Fourier transform of the integro-differential equation (11.0.87), and consequently the solution function u, is written as

$$\begin{aligned}
&(i\xi)\hat{u}(\xi) + 2\hat{u}(\xi) + \frac{1}{2+i\xi}\hat{u}(\xi) = 1 \iff (i\xi + \frac{1}{2+i\xi} + 2)\hat{u}(\xi) = 1 \\
\iff\quad &\frac{-\xi^2+4i\xi+5}{2+i\xi}\hat{u}(\xi) = 1 \iff \hat{u}(\xi) = \frac{2+i\xi}{-\xi^2+4i\xi+5}.
\end{aligned} \tag{11.0.89}$$

We substitute $i\xi = s$ and write $u(\xi)$ in (11.0.89) in the rational form:

$$\frac{2+s}{s^2+4s+5} = \frac{2+s}{(s+2-i)(s+2+i)} = \frac{A}{s+2-i} + \frac{B}{s+2+i}, \qquad (11.0.90)$$

where, using simple partial fractional techniques, we get

$$A = \frac{i}{2i} = \frac{1}{2} \quad \text{and} \quad B = \frac{-i}{-2i} = \frac{1}{2}. \qquad (11.0.91)$$

Thus, substituting back ($s = i\xi$), we have

$$\hat{u}(\xi) = \frac{1}{2} \cdot \frac{1}{2+i(\xi-1)} + \frac{1}{2} \cdot \frac{1}{2+i(\xi+1)}. \qquad (11.0.92)$$

Further, (11.0.88) implies that

$$\frac{1}{2+i\xi} \subset^{\mathcal{F}} \theta(t)e^{-2t} \quad \Longrightarrow \quad \frac{1}{2+i(\xi \pm 1)} \subset^{\mathcal{F}} e^{\pm it}\theta(t)e^{-2t}. \qquad (11.0.93)$$

Hence, finally we have the solution of the equation (11.0.87) as

$$u(t) = \frac{1}{2} \cdot e^{it}\theta(t)e^{-2t} + \frac{1}{2} \cdot e^{-it}\theta(t)e^{-2t} = \theta(t)e^{-2t}(e^{it} + e^{-it}) = \theta(t)e^{-2t}\cos t.$$

Example 14. Solve the integro-differential equation

$$\int_0^\infty e^{-\tau}u(t-\tau)\,d\tau - \int_{-\infty}^0 e^{\tau}u(t-\tau)\,d\tau = \sqrt{3}u(t) - e^{-|t|}. \qquad (11.0.94)$$

Solution: Using the $sign\,\tau$ the equation can also be written as:

$$\int_{-\infty}^\infty e^{-|\tau|} sign\,\tau\, u(t-\tau)\,d\tau = \sqrt{3}u(t) - e^{-|t|}, \qquad (11.0.95)$$

or in the convolution form as

$$\{e^{-|t|} sign\ t\} * \{u(t)\} = \sqrt{3}u(t) - e^{-|t|}. \qquad (11.0.96)$$

Fourier transforming (11.0.96) yields

$$\frac{-2i\omega}{1+\omega^2}\hat{u}(\omega) = \sqrt{3}\hat{u}(\omega) - \frac{2}{1+\omega^2}, \qquad (11.0.97)$$

and gives

$$\hat{u}(\omega)\Big(\sqrt{3}+\frac{2i\omega}{1+\omega^2}\Big)=\frac{2}{1+\omega^2}\Longrightarrow\quad \hat{u}(\omega)=\frac{2}{\sqrt{3}(1+\omega^2)+2i\omega}. \tag{11.0.98}$$

Substituting $i\omega = s$, we get by the usual partial fractions

$$\hat{u}(s)=-\frac{2}{\sqrt{3}}\frac{1}{s^2-\frac{2}{\sqrt{3}}s-1}=-\frac{2}{\sqrt{3}}\frac{1}{(s+\frac{1}{\sqrt{3}})(s-\sqrt{3})}=\frac{1}{2}\frac{1}{s+\frac{1}{\sqrt{3}}}-\frac{1}{2}\frac{1}{s-\sqrt{3}},$$

which, substituting back, yields

$$\hat{u}(\omega)=\frac{1}{2}\cdot\frac{1}{\frac{1}{\sqrt{3}}+i\omega}+\frac{1}{2}\cdot\frac{1}{\sqrt{3}-i\omega}. \tag{11.0.99}$$

Now we need to compute the function, which has the Fourier transform of the form $1/(a+i\omega)$. To this approach we note that

$$\int_{-\infty}^{\infty} e^{-at}\theta(t)e^{-i\omega t}\,dt=\int_0^{\infty} e^{-(a+i\omega)t}\,dt=\Big[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\Big]_{t=0}^{\infty}=\frac{1}{a+i\omega}.$$

Thus we have

$$e^{-at}\theta(t)\supset^{\mathcal{F}}\frac{1}{a+i\omega}. \tag{11.0.100}$$

Further, by changing $t\to -t$, we get also

$$e^{at}(1-\theta(t))=e^{at}\theta(-t)\supset^{\mathcal{F}}\frac{1}{a-i\omega}. \tag{11.0.101}$$

Hence, combining (11.0.100) and (11.0.101), we get the desired result:

$$u(t)=\frac{1}{2}e^{-\frac{1}{\sqrt{3}}t}\theta(t)+\frac{1}{2}e^{\sqrt{3}t}(1-\theta(t))\begin{cases}\frac{1}{2}e^{\frac{-t}{\sqrt{3}}}, & t\geq 0\\ \frac{1}{2}e^{\sqrt{3}t}, & t<0.\end{cases} \tag{11.0.102}$$

Example 15. Compute a solution to the equation

$$u(t)+\int_{-\infty}^{t} e^{\tau-t}u(\tau)\,d\tau=e^{-2|t|}. \tag{11.0.103}$$

Solution: Using the definition of Heaviside function $\theta(t-\tau)$ and the convolution we have

$$\int_{-\infty}^{t} e^{\tau - t} u(\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau)}\theta(t-\tau)u(\tau)\, d\tau = \{e^{-t}\theta(t)\} * \{u(t)\}. \tag{11.0.104}$$

Now Fourier transform of the equation(11.0.103) yields

$$\hat{u}(\xi) + \frac{1}{1+i\xi}\hat{u}(\xi) = \frac{4}{\xi^2+4} \quad \Longrightarrow \quad \hat{u}(\xi)\Big(1 + \frac{1}{1+i\xi}\Big) = \frac{4}{\xi^2+4} \quad \Longrightarrow$$
$$\hat{u}(\xi) = \frac{4(i\xi+1)}{(i\xi+2)(\xi^2+4)} = 4\cdot\Big(\frac{i\xi+2}{(i\xi+2)(\xi^2+4)} - \frac{1}{(i\xi+2)(\xi^2+4)}\Big).$$

Now we apply partial fractioning to write

$$\frac{1}{(i\xi+2)(\xi^2+4)} = \frac{1}{(2-i\xi)(2+i\xi)^2} = \frac{A}{2-i\xi} + \frac{B}{2+i\xi} + \frac{C}{(2+i\xi)^2}. \tag{11.0.105}$$

The coefficients A and C are determined easily as : $A = 1/16$ and $C = 1/4$. Thus we have

$$\frac{1}{(i\xi+2)(\xi^2+4)} = \frac{\frac{1}{16}(2+i\xi)^2 + B(4+\xi^2) + \frac{1}{4}(2-i\xi)}{(i\xi+2)(\xi^2+4)}. \tag{11.0.106}$$

Identifying the coefficient for the ξ^2-term in both numerators we end up with $-1/16 + B = 0$, i.e, $B = 1/16$. Thus

$$\begin{aligned}
\hat{u}(\xi) &= \frac{4}{\xi^2+4} - \frac{1}{4}\cdot\frac{1}{2-i\xi} - \frac{1}{4}\cdot\frac{1}{2+i\xi} - \frac{1}{(2+i\xi)^2} \\
&= \frac{4}{\xi^2+4} - \frac{1}{4}\cdot\frac{4}{\xi^2+4} + \frac{d}{d\xi}\Big(\frac{i}{2+i\xi}\Big) \\
&= \frac{3}{4}\cdot\frac{4}{\xi^2+4} + \frac{d}{d\xi}\Big(\frac{i}{2+i\xi}\Big)
\end{aligned} \tag{11.0.107}$$

Using Fourier transform formulas we get the solution of (11.0.103) as

$$u(t) = \frac{3}{4}e^{-2|t|} - te^{-2t}\theta(t). \tag{11.0.108}$$

Example 16 The input and out signals for a linear, time invariant system are given by $\frac{1}{1+t^2}$ and $\frac{t}{(4+t^2)^2}$, respectively. Compute the impulse response, $h(t)$ and the response for the input $\cos\omega t$. Is this causally stable.

Solution: We have that

$$\frac{1}{t^2+1} \curvearrowright \frac{t}{(4+t^2)^2} = -\frac{1}{2}\frac{d}{dt}\Big(\frac{1}{4+t^2}\Big). \tag{11.0.109}$$

Recall that the Fourier transform of output signal is the product of the Fourier transforms of the corresponding input signal and the impulse respond. Thus after Fourier transforming in (11.0.109) we get

$$-\frac{1}{2}(i\omega)\frac{\pi}{2}e^{-2|\omega|} = \hat{h}(\omega)\pi e^{-|\omega|}. \tag{11.0.110}$$

Hence we have

$$\hat{h}(\omega) = -\frac{i\omega}{4}e^{-|\omega|} = -\frac{1}{4\pi}(i\omega)\pi e^{-|\omega|} \subset^{\mathcal{F}} -\frac{1}{4\pi}\frac{d}{dt}\Big(\frac{1}{t^2+1}\Big) = h(t), \tag{11.0.111}$$

which give the impulse response

$$h(t) = \frac{1}{2\pi}\cdot\frac{t}{(t^2+1)^2}. \tag{11.0.112}$$

Note that if $h(t)$ is real valued then

$$\mathcal{S}[\cos\omega t] = Re[\hat{h}(\omega)e^{i\omega t}]. \tag{11.0.113}$$

Hence the output signal corresponding to the input signal $\cos\omega t$ is given by

$$\cos\omega t \curvearrowright Re\Big[\frac{-i\omega}{4}e^{-|\omega|}(\cos\omega t + i\sin\omega t)\Big] = \frac{\omega}{4}e^{-|\omega|}\sin\omega t. \tag{11.0.114}$$

Finally, we observe that $h(t) = \frac{1}{2\pi}\cdot\frac{t}{(t^2+1)^2} \neq 0$ for $t<0$, hence the system is not causal. However, since

$$\int_{-\infty}^{\infty} h(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{t}{(t^2+1)^2} = -\frac{1}{4\pi}\Big[\frac{1}{t^2+1}\Big]_{-\infty}^{\infty} = 0 < \infty \tag{11.0.115}$$

thus the system is a stable.

Example 17. A linear time invariant system has the impulse response, $h(t) = e^{-4t^2}$. Let $y(t)$ be the response to the input signal $x(t) = e^{-t^2}$. Compute the integral

$$\int_{-\infty}^{\infty} e^{it}h(t)y(t)\,dt. \tag{11.0.116}$$

Solution: We use the Fourier transform for $e^{-at^2/2}$: i.e.

$$e^{-at^2/2} \supset^{\mathcal{F}} \sqrt{\frac{2\pi}{a}}\, e^{-\omega^2/2a}. \tag{11.0.117}$$

Then for $a/2 = 4 \Rightarrow a = 8$ we get

$$e^{-4t^2} \supset^{\mathcal{F}} \sqrt{\frac{2\pi}{8}}\, e^{-\omega^2/16} = \frac{\sqrt{\pi}}{2}\, e^{-\omega^2/16} = \hat{h}(\omega). \tag{11.0.118}$$

Using (11.0.117) with $a/2 = 1 \Rightarrow a = 2$ we have

$$x(t) = e^{-t^2} \supset^{\mathcal{F}} \sqrt{\pi} e^{-\omega^2/4} = \hat{x}(\omega), \tag{11.0.119}$$

and the corresponding output response $y(t)$ viz,

$$y(t) = h(t) * x(t) \supset^{\mathcal{F}} \hat{h}(\omega)\hat{x}(\omega) = \frac{\sqrt{\pi}}{2}\, e^{-\omega^2/16} \cdot \sqrt{\pi} e^{-\omega^2/4} = \frac{\sqrt{\pi}}{2} \cdot \sqrt{\pi} e^{-5\omega^2/16}.$$

Now using (11.0.117) once again, this time with $\frac{1}{2a} = \frac{5}{16}$, i.e., $a = 8/5$ and $\sqrt{a} = 2\sqrt{2/5}$, it follows that

$$\hat{y}(\omega) = \frac{\sqrt{\pi}}{2}\sqrt{\frac{a}{2}} \cdot \sqrt{\frac{2\pi}{a}} e^{-5\omega^2/16} \subset^{\mathcal{F}} \sqrt{\frac{\pi}{5}} e^{-\frac{4}{5}t^2} = y(t). \tag{11.0.120}$$

Now to compute we write

$$\begin{aligned} \int_{-\infty}^{\infty} e^{it} h(t) y(t)\, dt &= \sqrt{\frac{\pi}{5}} \int_{-\infty}^{\infty} e^{-4t^2} \cdot e^{-\frac{4}{5}t^2} \cdot e^{it}\, dt \\ &= \sqrt{\frac{\pi}{5}} \int_{-\infty}^{\infty} e^{-\frac{24}{5}t^2} \cdot e^{it}\, dt = \sqrt{\frac{\pi}{5}} \mathcal{F}[e^{-\frac{24}{5}t^2}](-1). \end{aligned} \tag{11.0.121}$$

Here we have in (11.0.117) $a/2 = 24/5$, and thus $a = 48/5$ and $1/2a = 5/96$. Consequently

$$\int_{-\infty}^{\infty} e^{it} h(t) y(t)\, dt = \left[\sqrt{\frac{\pi}{5}} \sqrt{\frac{10\pi}{48}} e^{-\frac{5}{96}\omega^2}\right]_{\omega=-1} = \frac{\pi}{2\sqrt{6}} e^{-\frac{5}{96}}. \tag{11.0.122}$$

Example 18. For a linear, time invariant system the input signal $x(t) = \frac{1}{4+t^2}$ yields the output signal $y(t) = e^{-2t^2}$. Compute the impulse response and also the output signal, in the form of a complex Fourier series, corresponding to the input signal given by the following impulse series:

$$\sum_{n=-\infty}^{\infty} [2\delta(t-2n) - \delta(t-2n-1)]. \tag{11.0.123}$$

Solution: Recall that

$$x(t) = \frac{1}{4+t^2} \supset^{\mathcal{F}} \frac{\pi}{2} e^{-2|\omega|} = \hat{x}(\omega). \tag{11.0.124}$$

Further, recalling (11.0.117) with $a = 4$, it follows that

$$e^{-at^2/2} \supset^{\mathcal{F}} \sqrt{\frac{2\pi}{a}} e^{-\omega^2/2a} \Longleftrightarrow y(t) = e^{-2t^2} \supset^{\mathcal{F}} \sqrt{\frac{\pi}{2}} e^{-\omega^2/8} = \hat{y}(\omega). \tag{11.0.125}$$

Thus, using $\hat{y}(\omega) = \hat{h}(\omega)\hat{x}(\omega)$ we get

$$\hat{h}(\omega) = \frac{\hat{y}(\omega)}{\hat{x}(\omega)} = \sqrt{\frac{2}{\pi}} e^{-\omega^2/8+2|\omega|}. \tag{11.0.126}$$

Now we consider the inpulse series signal

$$x(t) = \sum_{n=-\infty}^{\infty} [2\delta(t-2n) - \delta(t-2n-1)]. \tag{11.0.127}$$

Now we expand $x(t)$ in the complex trigonometric Fourier series with the period $T = 2$ and $\Omega = 2\pi/T = \pi$ as $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t}$, where the complex Fourier coefficients are given by

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-1/2}^{3/2} x(t) e^{-in\Omega t}\, dt = \frac{1}{2}\int_{-1/2}^{3/2} [2\delta(t) - \delta(t-1)]) e^{-in\pi t}\, dt \\ &= \frac{1}{2}(2e^0 - e^{-in\pi}) = 1 - \frac{1}{2}(-1)^n. \end{aligned} \tag{11.0.128}$$

Now the formula

$$e^{in\omega t} \curvearrowright \hat{h}(\omega) e^{in\omega t}, \tag{11.0.129}$$

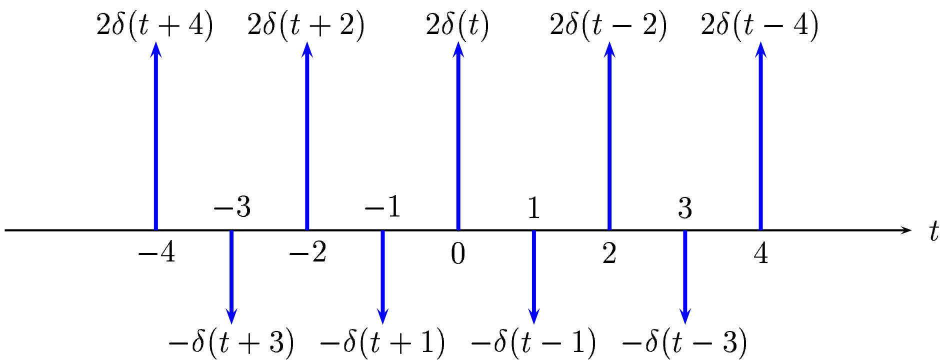


Figure 11.4: The functions $x(t) = \sum_{n=-\infty}^{\infty}[2\delta(t-2n) - \delta(t-2n-1)]$.

with $\omega = n\pi$ implies that

$$e^{in\pi t} \curvearrowright \hat{h}(n\pi)e^{in\pi t}. \tag{11.0.130}$$

In other words, the input signal $e^{i\omega t}$ gives the output signal $\hat{h}(\omega)e^{i\omega t}$. Hence

$$x(t) \curvearrowright \sum_{n=-\infty}^{\infty} c_n \hat{h}(n\pi)e^{in\pi t}, \tag{11.0.131}$$

i.e.,

$$y(t) = \sqrt{\frac{2}{\pi}} \sum_{n=-\infty}^{\infty} \Big(1 - \frac{1}{2}(-1)^n\Big)e^{-\frac{n^2\pi^2}{8}+2|n|\pi}e^{in\pi t}. \tag{11.0.132}$$

Example 19. Let $X(n)$ be a N-periodic function defined by

$$X(n) = \begin{cases} 1, & \text{for} \quad 0 \le n \le k-1 \\ 0, & \text{for} \quad k \le n \le N-1. \end{cases} \tag{11.0.133}$$

Compute the discrete Fourier transform, $\hat{X}(\mu)$, for $X(n)$ and use Parseval's formula to evaluate the sum

$$\sum_{\mu=1}^{N-1} \frac{1 - \cos\frac{2\pi\mu k}{N}}{1 - \cos\frac{2\pi\mu}{N}}. \tag{11.0.134}$$

Solution: By the definition of the discrete Fourier transform, substituting $e^{-i\frac{2\pi}{N}} = w$, we get

$$\hat{X}(\mu) = \sum_{n=0}^{N-1} X(n) w^{\mu n}. \tag{11.0.135}$$

Inserting $X(n)$ from (11.0.133) yields

$$\hat{X}(\mu) = \sum_{n=0}^{k-1} (w^{\mu})^n = \begin{cases} k, & \text{for} \quad \mu = 0 \\ \frac{w^{\mu k}-1}{w^{\mu}-1}, & \text{for} \quad \mu = 1, 2, \ldots, N-1. \end{cases} \tag{11.0.136}$$

Further by the Parseval's theorem for discrete Fourier transform

$$\sum_{n=0}^{N-1} \Big(X(n)\Big)^2 = \frac{1}{N} \sum_{\mu=0}^{N-1} \Big|\hat{X}(\mu)\Big|^2. \tag{11.0.137}$$

Thus, using (11.0.136), with $w = e^{-i\frac{2\pi}{N}}$, it follows that

$$\begin{aligned}
\sum_{\mu=0}^{N-1} \Big|\hat{X}(\mu)\Big|^2 &= k^2 + \sum_{\mu=1}^{N-1} \frac{\Big|e^{-i\frac{2\pi\mu k}{N}} - 1\Big|^2}{\Big|e^{-i\frac{2\pi\mu}{N}} - 1\Big|^2} \\
&= k^2 + \sum_{\mu=1}^{N-1} \frac{\Big(\cos\frac{2\pi\mu k}{N} - 1\Big)^2 + \sin^2\frac{2\pi\mu k}{N}}{\Big(\cos\frac{2\pi\mu}{N} - 1\Big)^2 + \sin^2\frac{2\pi\mu}{N}} \\
&= k^2 + \sum_{\mu=1}^{N-1} \frac{2\Big(1 - \cos\frac{2\pi\mu k}{N}\Big)}{2\Big(1 - \cos\frac{2\pi\mu}{N}\Big)} = N \sum_{n=0}^{N-1} |X(n)|^2 = Nk.
\end{aligned} \tag{11.0.138}$$

where in the last two steps we use the Parseval's relation (11.0.137), followed by the definition of $X(n)$ viz (11.0.133). Consequently,

$$\sum_{\mu=1}^{N-1} \frac{2\Big(1 - \cos\frac{2\pi\mu k}{N}\Big)}{2\Big(1 - \cos\frac{2\pi\mu}{N}\Big)} = Nk - k^2 = k(N-k). \tag{11.0.139}$$

Example 20. Compute the discrete Fourier transform for the, N-periodic, signal (sequence) given by

$$X(n) = \sin\frac{n\pi}{N}, \quad n = 0, \ldots, N-1. \tag{11.0.140}$$

Solution: The period $T = N$ yields $\Omega = \frac{2\pi}{T} = \frac{2\pi}{N}$. Letting $w = e^{\frac{2\pi i}{N}}$, the discrete Fourier transform of $X(n)$ is written as

$$\hat{X}(\mu) = \sum_{n=0}^{N-1} X(n)e^{-\mu n} = \frac{1}{2i}\sum_{n=0}^{N-1}\left(e^{\frac{in\pi}{N}} - e^{-\frac{-in\pi}{N}}\right)e^{-i\frac{2\pi\mu n}{N}}. \tag{11.0.141}$$

Let now $v = e^{-i\frac{\pi}{N}}$. Then

$$\hat{X}(\mu) = \frac{1}{2i}\sum_{n=0}^{N-1}\left(v^{-n} - v^{n}\right)v^{2\mu n} = \frac{1}{2i}\sum_{n=0}^{N-1}\left[v^{(2\mu-1)n} + v^{(2\mu+1)n}\right]. \tag{11.0.142}$$

Note that $v^N = (e^{-i\frac{\pi}{N}})^N = e^{-i\pi} = -1$ and since $2\mu - 1$ is an odd integer, hence $v^{2\mu-1} = e^{-i(2\mu-1)\pi/N} \neq 1$. Thus, using the formula for the sum of geometric series, it follows that

$$\sum_{n=0}^{N-1} v^{(2\mu-1)n} = \sum_{n=0}^{N-1}(v^{2\mu-1})^n = \frac{(v^{2\mu-1})^N - 1}{v^{2\mu-1} - 1} = \frac{-2}{v^{2\mu-1} - 1}. \tag{11.0.143}$$

The substitution $\mu \to \mu + 1$ in (11.0.143) gives

$$\sum_{n=0}^{N-1} v^{(2\mu+1)n} = \frac{-2}{v^{2\mu+1} - 1}. \tag{11.0.144}$$

Inserting (11.0.143) and (11.0.144) in (11.0.142) we have that

$$\begin{aligned}\hat{X}(\mu) &= \frac{1}{2i}(-2)\Big(\frac{1}{v^{2\mu-1} - 1} - \frac{-2}{v^{2\mu+1} - 1}\Big) = -\frac{1}{i}\frac{v^{2\mu+1} - v^{2\mu-1}}{v^{4\mu} + 1 - (v^{2\mu+1} + v^{2\mu-1})} \\ &= -\frac{1}{i}\frac{v - v^{-1}}{v^{2\mu} + v^{-2\mu} - (v + v^{-1})} = -\frac{1}{i}\frac{e^{-i\pi/N} - e^{i\pi/N}}{e^{-i2\mu\pi/N} + e^{i2\mu\pi/N} - (e^{-i\pi/N} + e^{i\pi/N})},\end{aligned}$$

and finally

$$\hat{X}(\mu) = \frac{\sin \pi/N}{\cos 2\mu\pi/N - \cos \pi/N}. \tag{11.0.145}$$

Example 21. Show that the functions $\varphi_n(x) = \frac{\sin x/2}{\pi x}e^{inx}$ are mutually orthogonal in $L_2(\mathbb{R})$. Compute the coefficients c_n minimizing the integral:

$$\int_{-\infty}^{\infty}\Big|\frac{1}{1 + x^2} - \sum_{n=-N}^{N} c_n\varphi(x)\Big|^2 dx. \tag{11.0.146}$$

Solution: Recall the Fourier transform of $\frac{\sin ax}{x}$:

$$\frac{\sin ax}{x} \supset^{\mathcal{F}} \pi \chi_a(\xi) \tag{11.0.147}$$

and let $a = 1/2$ to get

$$\frac{\sin x/2}{\pi x} \supset^{\mathcal{F}} \chi_{1/2}(\xi) = \begin{cases} 1, & \text{for} \quad -1/2 < \xi < 1/2 \\ 0, & \text{for} \quad |\xi| > 1/2. \end{cases} \tag{11.0.148}$$

Then using a shift Fourier transform formula we get

$$\varphi_n(x) = \frac{\sin x/2}{\pi x} e^{inx} \supset^{\mathcal{F}} \chi_{1/2}(\xi - n), \tag{11.0.149}$$

where

$$\chi_{1/2}(\xi - n) = \begin{cases} 1, & -1/2 \le \xi - n \le 1/2 \iff n - 1/2 < \xi < n + 1/2 \\ 0, & \text{else.} \end{cases}$$

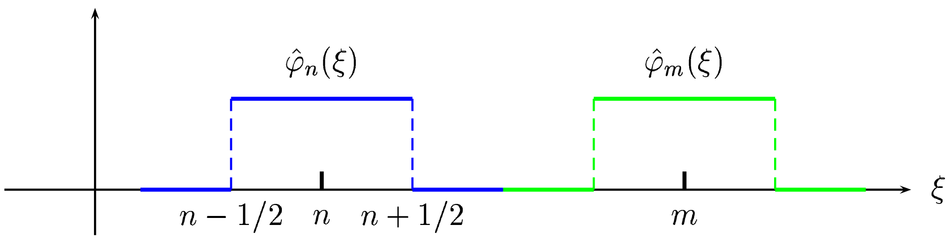


Figure 11.5: The functions $\hat{\varphi}_n(\xi)$ and $\hat{\varphi}_m(\xi)$.

Hence, (see figure)

$$\int_{-\infty}^{\infty} \hat{\varphi}_n(\xi) \overline{\hat{\varphi}_m(\xi)} \, d\xi = \begin{cases} 1, & \text{for} \quad m = n \\ 0, & \text{for} \quad m \neq n. \end{cases} \tag{11.0.150}$$

Thus by the Plancherel's formula we have

$$\int_{-\infty}^{\infty} \varphi_n(x) \overline{\varphi_m(x)} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}_n(\xi) \overline{\hat{\varphi}_m(\xi)} \, d\xi = \begin{cases} 1/2\pi, & m = n \\ 0, & m \neq n. \end{cases}$$

Hence $\{\varphi_n\}_{-\infty}^{\infty}$ is an orthogonal set.

Let now $f(x) = 1/(x^2+1)$, i.e. $\hat{f}(\xi) = \pi e^{-|\xi|}$. Then, the integral

$$\int_{-\infty}^{\infty} \Big| f(x) - \sum_{n=-N}^{N} c_n \varphi(x) \Big|^2 \, dx \tag{11.0.151}$$

will be minimal if and only if c_n are the Fourier coefficients of f on the base $\{\varphi_n\}_{-\infty}^{\infty}$:

$$\begin{aligned} c_n &= \frac{1}{\|\varphi_n\|^2}\langle f, \varphi_n\rangle = 2\pi\langle f, \varphi_n\rangle = \langle \hat{f}, \hat{\varphi}_n\rangle = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{\varphi}_n(\xi)}\, d\xi \\ &= \int_{n-1/2}^{n+1/2} \hat{f}(\xi)\, d\xi = \int_{n-1/2}^{n+1/2} \pi e^{-|\xi|}, d\xi. \end{aligned} \tag{11.0.152}$$

Thus, for $n \geq 1$ we have $n - 1/2 > 0$ and

$$\begin{aligned} c_n &= \int_{n-1/2}^{n+1/2} \pi e^{-\xi}\, d\xi = \pi\Big[-e^{-\xi}\Big]_{n-1/2}^{n+1/2} = \pi\Big(e^{-n+1/2} - e^{-n-1/2}\Big) \\ &= \pi e^{-n}\Big(e^{1/2} - e^{-1/2}\Big). \end{aligned} \tag{11.0.153}$$

As for $n \leq -1$, i.e. $n + 1/2 < 0$:

$$c_n = \int_{n-1/2}^{n+1/2} \pi e^{\xi}\, d\xi = \pi\Big(e^{n+1/2} - e^{n-1/2}\Big) = \pi e^{n}\Big(e^{1/2} - e^{-1/2}\Big). \tag{11.0.154}$$

Finally, for $n = 0$ we have

$$c_0 = \int_{-1/2}^{1/2} \pi e^{-|\xi|}\, d\xi = 2\int_{0}^{1/2} \pi e^{-\xi}\, d\xi = 2\pi\Big(1 - e^{-1/2}\Big). \tag{11.0.155}$$

Summing up we have

$$c_n = \begin{cases} \pi e^{n}\Big(e^{1/2} - e^{-1/2}\Big)e^{-|n|}, & \text{for} \quad n \neq 0 \\ 2\pi\Big(1 - e^{-1/2}\Big), & \text{for} \quad n = 0. \end{cases} \tag{11.0.156}$$

Example 22. Compute the solution $y(x)$ for the differential equation $y'' - y = 0$, that minimize the following integral.

$$\int_{-1}^{1} 1\Big[1 + x - y(x)\Big|^2 \, dx. \tag{11.0.157}$$

Solution: The differential equation $y'' - y = 0$ has the general solution

$$y(x) = c_1 \cosh x + c_2 \sinh x. \tag{11.0.158}$$

Since $\cosh x \sinh x$ is an odd function we have, over the symmetric interval $[-1, 1]$, that

$$\int_{-1}^{1} \cosh x \, \sinh x \, dx = 0. \tag{11.0.159}$$

Thus $\varphi_1(x) = \cosh x$ and $\varphi_2(x) = \sinh x$ form an orthogonal base for the solution space viewed as a subspace of $L_2(-1, 1)$. Therefore

$$\int_{-1}^{1} \Big[(1+x) - \Big(c_1\varphi_1(x) + c_2\varphi_2(x)\Big)\Big]^2 dx \tag{11.0.160}$$

is minimal when

$$c_k = \frac{1}{\|\varphi_k\|^2}\langle 1 + x, \varphi_k\rangle, \qquad k = 1, 2. \tag{11.0.161}$$

By straightforward computation we get

$$\|\varphi_1\|^2 = \int_{-1}^{1} \cosh^2 x \, dx = 2\int_0^1 \frac{1 + \cosh 2x}{2} dx = 1 + \frac{1}{2}\sinh 2,$$
$$\|\varphi_2\|^2 = \int_{-1}^{1} \sinh^2 x \, dx = 2\int_0^1 \frac{\cosh 2x - 1}{2} dx = \frac{1}{2}\sinh 2 - 1,$$

and

$$\int_{-1}^{1} (1+x)\cosh x \, dx = 2\int_0^1 \cosh x \, dx = 2\Big[\sinh x\Big]_0^1 = 2\sinh 1,$$
$$\int_{-1}^{1} (1+x)\sinh x \, dx = 2\int_0^1 x\sinh x \, dx = 2\Big[x\cosh x\Big]_0^1 - 2\int_0^1 1\cdot\cosh x \, dx$$
$$= 2(\cosh 1 - \sinh 1) = 2e^{-1}.$$

Inserting in (11.0.161) we get the coefficients c_k, $k = 1, 2$. Finally, using (11.0.158), it follows that

$$y(x) = \frac{2\sinh 1}{\frac{1}{2}\sinh 2 + 1}\cosh x + \frac{2e^{-1}}{\frac{1}{2}\sinh 2 - 1}\sinh x. \tag{11.0.162}$$

Example 23. Compute all eigenvalues and, their corresponding, eigenfunctions to the Sturm-Liuoville problem

$$\begin{cases} f'' + \lambda f = 0, & 0 < x < a \\ f(0) - f'(0) = 0, & f(a) + 2f'(a) = 0. \end{cases} \tag{11.0.163}$$

Solution: **Case I:** $\lambda = 0$, then $f'' = 0$ and $f(x) = c_1 x + c_2$. Inserting the boundary points yields

$$\begin{cases} f(0) - f'(0) = c_2 - c_1 = 0, \\ f(a) + 2f'(a) = c_1 a + 3c_2 = 0. \end{cases} \tag{11.0.164}$$

Thus, for $a \neq -3$, we have $c_1 = c_2 = 0$ and hence $f \equiv 0$. In other words we get a trivial solution and therefore $\lambda = 0$ is not an eigenvalue.
Case II: $\lambda \neq 0$. Let $\lambda = \nu^2$ and set, $\nu = \sqrt{\lambda} > 0$ if $\lambda > 0$ and $\nu = \sqrt{-\lambda} = i\mu$ if $\lambda < 0, (\mu > 0)$.
If $\lambda < 0$, then using the general solution form

$$f(x) = c_1 \cos \nu x + c_2 \sin \nu x \qquad \Longrightarrow \qquad f'(x) = \nu(c_2 \cos \nu x - c_1 \sin \nu x),$$

it follows that

$$\begin{aligned} f(0) - f'(0) &= c_1 - \nu c_2 = 0, \quad \Longrightarrow \quad c_1 = \nu c_2, \qquad \text{and} \\ f(a) + 2f'(a) &= c_1 \cos \nu a + c_2 \sin \nu a + 2\nu(c_2 \cos \nu a - c_1 \sin \nu a) \\ &= c_2 \Big(3\nu \cos \nu a + (1 - 2\nu^2) \sin \nu a \Big) = 0. \end{aligned} \tag{11.0.165}$$

Now if $c_2 \neq 0$, then we have

$$3\nu \cos \nu a = (2\nu^2 - 1) \sin a\nu \qquad \Longrightarrow \qquad \tan \nu a = \frac{3\nu}{2\nu^2 - 1}. \tag{11.0.166}$$

For $\nu > 0$, we let ν_1, ν_2, ν_3, $\dots$, be the positive rots to equation (11.0.166), see figure below: If $\nu = i\mu$ we get the equation

$$i \tanh \mu a = \frac{3i\mu}{-2\mu^2 - 1} \qquad \Longrightarrow \qquad \tanh \mu a = -\frac{3\mu}{2\mu^2 + 1}, \tag{11.0.167}$$

which has no positive roots $\mu > 0$ due to the fact that then the left hand side becomes positive and the right hand side is negative.

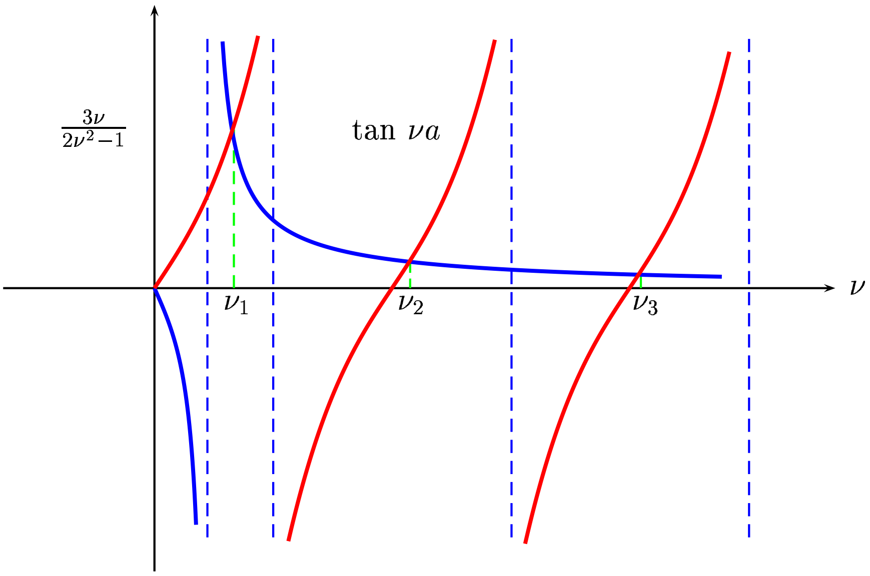


Figure 11.6: The functions $\frac{3\nu}{2\nu^2-1}$ (blue) and $\tan \nu a$ (red).

Now since $c_1 = \nu c_2$ we can rewrite the solution $f(x)$ as

$$f(x) = c_1 \cos \nu x + c_2 \sin \nu x = c_2(\nu \cos \nu x + \sin \nu x). \qquad (11.0.168)$$

Summing up the eigenvalues: ν_k, $k = 1, 2, \ldots$ are the positive roots of the equation $\tan \nu a = \frac{3\nu}{2\nu^2-1}$ with the corresponding eigenfunctions being $f_k(x) = \nu_k \cos \nu_k x + \sin \nu_k x$.

Example 24. a) Compute all eigenvalues and eigenfunctions to the Sturm-Liuoville problem

$$\begin{cases} -e^{-4x}\frac{d}{dx}\left(e^{4x}\frac{du}{dx}\right) = \lambda u, & 0 < x < a \\ u(0) = 0, & u'(1) = 0. \end{cases} \qquad (11.0.169)$$

b) Expand the function e^{-2x} in a Fourier series with respect to the eigenfunctions.

Solution:(a). We rewrite the differential equation as

$$e^{-4x}\left(e^{4x}u'\right)'+\lambda u = e^{-4x}\left(4e^{4x}u'+e^{4x}u''\right)+\lambda u = u''+4u'+\lambda u = 0. \quad (11.0.170)$$

Equation (11.0.170) has the characteristic equation $r^2+4r+\lambda = 0$, with the roots $r_{1,2} = -2 \pm \sqrt{4-\lambda}$.

Case I: Let $\lambda < 4$ and set $\beta = \sqrt{4-\lambda}$. Then the general solution to the equation has the form

$$u = e^{-2x}(c_1 \cosh \beta x + c_2 \sinh \beta x). \quad (11.0.171)$$

Then we have $u(0) = c_1 = 0$ and $u'(1) = c_2 e^{-2}\beta \cosh \beta - 2e^{-2}c_2 \sinh \beta = 0$. If now $c_2 \neq 0$, then $\tanh \beta = \frac{\beta}{2}$. This equation has only one positive root: $\beta_1 > 0$, (see the figure below):

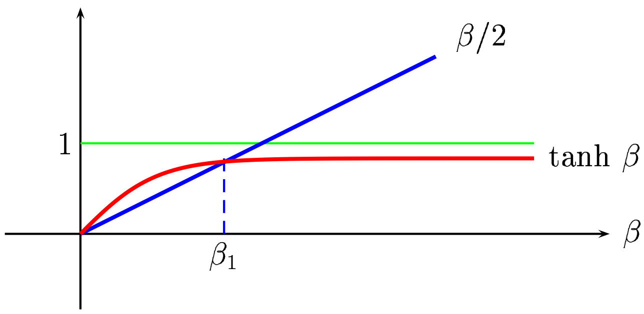


Figure 11.7: The functions $\beta/2$ (blue) and $\tanh \beta$ (red).

The eigenvalue is $\lambda_1 = 4 - \beta_1^2$ with the eigenfunction $u_1(x) = e^{-2} \sinh \beta_1 x$.

Case II: $\lambda = 4$. For $\lambda = 4$ we get $r_1 = r_2 = -2$ and therefore the solution $u(x) = (c_1 + c_2 x)e^{-2x}$. Here the data $u(0) = 0$ gives $c_1 = 0$ and $u'(1) = 0$ yields $-c_2 e^{-2} = 0$, thus we have $c_2 = 0$. Since $c_1 = c_2 = 0$ there are no eigenfunctions corresponding to $\lambda = 4$.

Case III: $\lambda > 4$. Then $\beta = \sqrt{\lambda - 4} > 0$. Then the general soution is

$$u(x) = e^{-2x}(c_1 \cos \beta x + c_2 \sin \beta x). \quad (11.0.172)$$

Hence $u(0) = c_1 = 0$, which give us $u(x) = c_2 e^{-2x} \sin \beta x$. We assume that $c_2 \neq 0$ then the eigenfunctions are $u_n(x) = e^{-2x} \sin \beta x, \quad n \geq 2$. Now, invoking the data $u'(1) = 0$ we get the equation

$$u'(1) = c_2(e^{-2}\beta\cos\beta - 2e^{-2}\sin\beta) = 0 \qquad (11.0.173)$$

with solutions satisfying $\tan\beta = \beta/2, \quad \beta > 0$ (note that $c_2 \neq 0$). Let now $\beta_2, \beta_2, \ldots$ be the positive rots of the equation (11.0.173) ($\beta_1 \equiv 0$), see fig. below. The eigenvalues are $\lambda_n = 4 + \beta_n^2, \quad n \geq 2$, with the corresponding eigenfunctions: $u_n(x) = e^{-2x}\sin\beta_n x$.
Note that, in case I, we have treated $\beta_1, +, \lambda_1$ and u_1, corresponding to $\lambda < 4$.

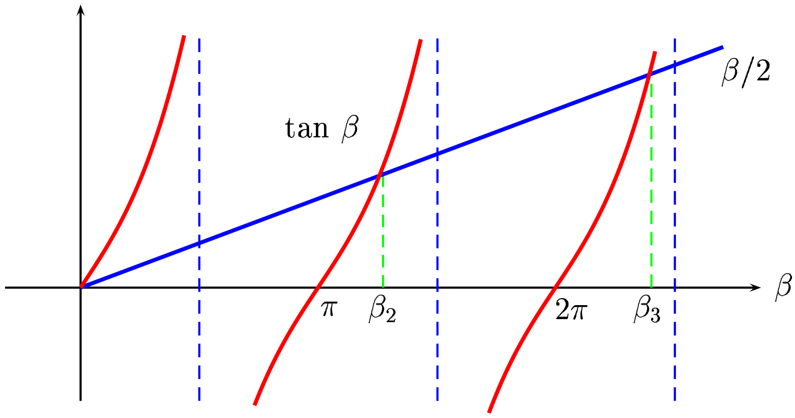


Figure 11.8: The functions $\beta/2$ (blue) and $\tan\beta$ (red).

Solution:(24b). The eigenfunctions $\{u_n\}_{n=1}^{\infty}$ form a complete orthogonal system of functions at the interval $(0,1)$, with the weight function $w(x) = e^{4x}$. We expand $f(x)$ in terms of u_n, viz,

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x), \quad \text{with} \quad c_n = \frac{1}{\rho_n}\int_0^1 f(x)u_n(x)e^{4x}\,dx, \qquad (11.0.174)$$

where

$$\rho_n := \|u_n\|_w^2 = \int_0^1 u_n^2(x)e^{4x}\,dx. \qquad (11.0.175)$$

We thus compute

$$\begin{aligned}
\rho_1 :&= \int_0^1 \sinh^2 \beta_1 x \, dx = \int_0^1 \frac{1}{2}\Big(\cosh 2\beta_1 x - 1\Big)\, dx = \frac{1}{4\beta_1}\sinh 2\beta_1 - \frac{1}{2} \\
&= \frac{1}{4\beta_1} \cdot \frac{2\sinh\beta_1 \, \cosh\beta_1}{\cosh^2\beta_1 - \sinh^2\beta_1} - \frac{1}{2} = \frac{1}{2\beta_1} \cdot \frac{\tanh\beta_1}{1-\tanh^2\beta_1} - \frac{1}{2} \\
&= \frac{1}{2\beta_1} \cdot \frac{\beta_1/2}{1-\beta_1^2/4} - \frac{1}{2} = \frac{1}{4-\beta_1^2} - \frac{1}{2} = \frac{1}{\lambda_1} - \frac{1}{2} = \frac{2-\lambda_1}{2\lambda_1}.
\end{aligned}$$

As for $n \geq 2$ we get

$$\begin{aligned}
\rho_n := \|u_n\|_w^2 &= \int_0^1 \sin^2 \beta_n x \, dx = \int_0^1 \frac{1}{2}\Big(1 - \cos\, 2\beta_n x\Big)\, dx = \frac{1}{2} - \frac{1}{4\beta_n}\sin\, 2\beta_n \\
&= \frac{1}{2} - \frac{1}{4\beta_n} \cdot \frac{2\sin\beta_n \, \cos\beta_n}{\cos^2\beta_n + \sin^2\beta_n} = \frac{1}{2} - \frac{1}{2\beta_n} \cdot \frac{\tan\beta_n}{1+\tan^2\beta_n} \\
&= \frac{1}{2} - \frac{1}{2\beta_n} \cdot \frac{\beta_n/2}{1+\beta_n^2/4} = \frac{1}{2} - \frac{1}{4+\beta_n^2} = \frac{1}{2} - \frac{1}{\lambda_n} = \frac{\lambda_n - 2}{2\lambda_n}.
\end{aligned}$$

For $n = 1$ and $f(x) = e^{-2x}$ we have

$$\rho_1 c_1 = \int_0^1 f(x) u_1(x) w(x)\, dx = \int_0^1 \sinh\beta_1 x \, dx = \frac{1}{\beta_1}(\cosh\beta_1 - 1). \tag{11.0.176}$$

Now we calculate $\cosh\beta_1$, viz,

$$\begin{aligned}
\cosh^2\beta_1 &= \frac{\cosh^2\beta_1}{\cosh^2\beta_1 - \sinh^2\beta_1} = \frac{1}{1-\tanh^2\beta_1} = \frac{1}{1-\beta_1^2/4} \\
&= \frac{4}{4-\beta_1^2} = \frac{4}{\lambda_1} \quad \Longrightarrow \quad \cosh\beta_1 = \frac{2}{\pm\sqrt{\lambda_1}},
\end{aligned} \tag{11.0.177}$$

where only $\cosh\beta_1 = 2/\sqrt{\lambda_1}$ is acceptable. Inserting (11.0.177) in (11.0.176) we get

$$\rho_1 c_1 = \frac{1}{\beta_1}\Big(\frac{2}{\sqrt{\lambda_1}} - 1\Big) = \frac{2-\sqrt{\lambda_1}}{\beta_1\sqrt{\lambda_1}}, \tag{11.0.178}$$

which, recalling ρ_1, yields

$$c_1 = \frac{2-\sqrt{\lambda_1}}{\beta_1\sqrt{\lambda_1}} \cdot \frac{2\lambda_1}{2-\lambda_1} = \frac{2\lambda_1(2-\sqrt{\lambda_1})}{\beta_1(2-\lambda_1)}. \tag{11.0.179}$$

For $n \geq 2$, the same procedure yields

$$\rho_n c_n = \int_0^1 f(x)u_n(x)w(x)\,dx = \int_0^1 \sin\beta_n x\,dx = \frac{1}{\beta_n}(1-\cos\beta_n). \quad (11.0.180)$$

To calculate $\cos\beta_n$ we use

$$\cos^2\beta_n = \frac{1}{1+\tan^2\beta_n} = \frac{1}{1+\beta_n^2/4} = \frac{4}{4+\beta_n^2} = \frac{4}{\lambda_n},$$

which give

$$\cos\beta_n = \pm\frac{2}{\sqrt{\lambda_n}} = (-1)^{n-1}\frac{2}{\sqrt{\lambda_n}}, \quad (11.0.181)$$

where, in the last step, we use the fact that $\cos\beta_2 < 0$ and the alternating sign. Inserting (11.0.181) in (11.0.178) it follows that

$$\rho_n c_n = \frac{1}{\beta_n}\Big(1-(-1)^{n-1}\frac{2}{\sqrt{\lambda_n}}\Big) = \frac{\sqrt{\lambda_n}+2(-1)^n}{\beta_n\sqrt{\lambda_n}},$$

which gives that

$$c_n = \frac{[\sqrt{\lambda_n}+2(-1)^n](2\sqrt{\lambda_n})}{\beta_n\sqrt{\lambda_n}(\lambda_n-2)} = \frac{2\lambda_n[\sqrt{\lambda_n}+2(-1)^n]}{\beta_n(\sqrt{\lambda_n}-2)} \quad (11.0.182)$$

Note that formula (11.0.182) yields for $n=1$ as well, giving the value of c_1. Summing up we have the expand of the function e^{-2x} in a Fourier series as

$$e^{-2x} = \sum_{n=1}^{\infty}\frac{2\lambda_n[\sqrt{\lambda_n}+2(-1)^n]}{\beta_n(\sqrt{\lambda_n}-2)}u_n(x). \quad (11.0.183)$$

Example 25. Solve the following boundary value problem:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = y, & 0<x<2, \qquad 0<y<1 \\ u(x,0)=0, & u(x,1)=0 \\ u(0,y)=y-y^3, & u(2,y)=0 \end{cases} \quad (11.0.184)$$

Solution: Recall the homogenization procedure: We want to determine a function $w(y)$, such that $w''(y)=y$ having the same boundary conditions, as $u(x,y)$, i.e., $w(0)=w(1)=0$.

$$w''(y)=y \quad \Longrightarrow w'(y)=\frac{1}{2}y^2+A \quad \Longrightarrow \quad w(y)=\frac{1}{6}y^3+Ay+B. \quad (11.0.185)$$

Then using the boundary conditions we get

$$w(0) = 0 \Longrightarrow B = 0 \quad \text{and} \quad w(1) = 0 \Longrightarrow \frac{1}{6} + A = 0 \Longrightarrow \quad A = -\frac{1}{6}.$$

Thus we have

$$w(y) = \frac{1}{6}(y^3 - y). \tag{11.0.186}$$

Let now $v(x, y) = u(x, y) - w(y)$, then

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} - y. \tag{11.0.187}$$

and

$$\begin{cases} v(0, y) = u(0, y) - w(y) = y - y^3 - \frac{1}{6}(y - y^3) = \frac{7}{6}(y - y^3), \\ v(2, y) = u(2, y) - w(y) = \frac{1}{6}(y - y^3) \end{cases} \tag{11.0.188}$$

Now inserting (11.0.187) and (11.0.188) in (11.0.184) we obtain the homogeneous boundary value problem for v viz,

$$\begin{cases} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, & 0 < x < 2, & 0 < y < 1 \\ v(x, 0) = 0, & v(x, 1) = 0 & \\ v(0, y) = \frac{7}{6}(y - y^3), & v(2, y) = \frac{1}{6}(y - y^3). & \end{cases} \tag{11.0.189}$$

Using separation of variables: let $v(x, .y) = X(x, y)Y(x, y) \neq 0$, then the differential equation (11.0.189) yields

$$X''Y + XY'' = 0 \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda. \tag{11.0.190}$$

Thus we end up with the two eigenvalue problems:

$$\begin{cases} -Y'' = \lambda Y \\ Y(0) = Y(1) = 0, \end{cases} \quad \text{and} \quad \begin{cases} X'' = \lambda X \\ X(0) = X(2) = 0. \end{cases} \tag{11.0.191}$$

The Sturm-Liouville problem for Y has the eigenvalues and eigenfunctions

$$\lambda = \lambda_n = (n\pi)^2, \qquad Y = Y_n(y) = \sin n\pi y, \quad n \geq 1. \tag{11.0.192}$$

The general solution for the equation for X, with $\lambda = \lambda_n$, can be written as

$$X_n(x) = A_n \sinh n\pi x + = B_n \sinh n\pi(2-x)x. \tag{11.0.193}$$

Now we may write $v(x, y)$ as a superposition of $X_n(x)Y_n(y)$:

$$v(x,y) = \sum_{n=1}^{\infty} \Big[A_n \sinh n\pi x + B_n \sinh n\pi(2-x)\Big] \sinh n\pi y, \tag{11.0.194}$$

where, the boundary conditions are given by

$$\begin{aligned} v(0,y) &= \sum_{n=1}^{\infty} B_n \sinh 2n\pi \cdot \sin n\pi y = \frac{7}{6}(y-y^3), \\ v(2,y) &= \sum_{n=1}^{\infty} A_n \sinh 2n\pi \cdot \sin n\pi y = \frac{1}{6}(y-y^3). \end{aligned} \tag{11.0.195}$$

We expand the function $\frac{1}{6}(y-y^3)$ in Fourier series with respect to the complete orthonormal system $\{\sin n\pi y\}_{n=1}^{\infty}$:

$$\frac{1}{6}(y-y^3) = \sum_{n=1}^{\infty} C_n \sin n\pi y, \quad \text{with} \quad C_n = \frac{1}{1/2}\int_0^1 \frac{1}{6}(y-y^3)\sin n\pi y \; dy.$$

Identifying the coefficients C_n with those of equation (11.0.195), it follows that

$$A_n \sinh 2n\pi = C_n \quad \text{and} \quad B_n = 7A_n = \frac{7c_n}{\sinh 2n\pi}. \tag{11.0.196}$$

We compute C_n by repeated use of partial integration, viz

$$\begin{aligned} C_n &= \frac{1}{3}\Big[(y-y^3)\frac{-\cos n\pi y}{n\pi}\Big]_0^1 + \frac{1}{3}\int_0^1 (1-3y^2)\frac{\cos n\pi y}{n\pi} \; dy \\ &= \frac{1}{3}\Big[(1-3y^2)\frac{\sin n\pi y}{(n\pi)^2}\Big]_0^1 - \frac{1}{3}\int_0^1 (-6y)\frac{\sin n\pi y}{(n\pi)^2} \; dy \\ &= -2\Big[y\frac{\cos n\pi y}{(n\pi)^3}\Big]_0^1 + 2\int_0^1 \frac{\cos n\pi y}{(n\pi)^3} \; dy = -2\frac{\cos n\pi}{(n\pi)^3} \\ &= \frac{2(-1)^{n-1}}{(n\pi)^3}. \end{aligned} \tag{11.0.197}$$

Thus by (11.0.196) we have

$$A_n = \frac{2(-1)^{n-1}}{(n\pi)^3 \sinh 2n\pi} \quad \text{and} \quad B_n = \frac{14(-1)^{n-1}}{(n\pi)^3 \sinh 2n\pi}. \tag{11.0.198}$$

Hence, the solution to the original problem (11.0.184) is given by

$$u(x,y) = \frac{1}{6}(y^3 - y) + \frac{2}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \sinh 2n\pi}\Big[\sinh n\pi x + 7\sinh n\pi(2-x)\Big]\sin n\pi y.$$

An alternative Solution for Example 25. We solve the problem by using Fourier series in product form: Let

$$v(x,y) = u(x,y) - \varphi(x,y), \tag{11.0.199}$$

so that $\varphi(x,y)$ is a polynomial in x and y with $\varphi_{xx} = 0$, $\varphi_{xx} = Ay + B$ and φ satisfies the boundary conditions. This yields

$$\varphi(x,y) = (ax+b)(py^3 + qy^2 + ry + s). \tag{11.0.200}$$

Then

$$\begin{cases} \varphi(x,0) & = (ax+b)s = 0 \quad \forall x, & \Rightarrow \quad s = 0, \\ \varphi(x,1) & = (ax+b)(p+q+r) = 0, \quad \forall x, & \Rightarrow \quad p+q+r = 0, \\ \varphi(0,y) & = b(py^3 + qy^2 + ry) = y - y^3, & \Rightarrow \quad p = -\frac{1}{b},\ q = 0,\ r = \frac{1}{b}, \\ \varphi(2,y) & = (\frac{2a}{b} + 1)(-y^3 + y) = 0 \quad \forall y, & \Rightarrow \quad b = -2a. \end{cases}$$

Thus we have

$$\varphi(x,y) = (ax+b)(py^3+qy^2+ry+s) = (ax-2a)\cdot\frac{1}{b}(-y^3+y) = \frac{1}{2}(2-x)(y-y^3).$$

Therefore $\varphi_{xx} = 0$, $\varphi_{yy} = \frac{1}{2}(2-x)(-6y) = -6y + 3xy$, and hence

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} - (\varphi_{xx} + \varphi_{yy}) = y - (-6y + 3xy) = y(7-3x). \tag{11.0.201}$$

Consequently we have the following boundary value problem for v:

$$\begin{cases} v_{xx} + v_{yy} = y(7-3x), & 0 < x < 1, \quad 0 < y < 2 \\ v(x,0) = v(x,1) = 0, & 0 < x < 1, \\ v(0,y) = v(2,y) = 0, & \quad 0 < y < 2. \end{cases} \tag{11.0.202}$$

Using separation of variables, viz $v(x,y) = X(x,y)Y(x,y) \neq 0$, we get, for the homogeneous differential equation, that

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda, \quad \Rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0. \quad (11.0.203)$$

So that we have two Sturm-Liouville problems:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(2) = 0 \end{cases} \Longrightarrow \lambda_m = \Big(\frac{m\pi}{2}\Big)^2, \quad X_m(x) = \sin\frac{m\pi}{2}x, \; m \geq 1. \tag{11.0.204}$$

and

$$\begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = Y(1) = 0 \end{cases} \Longrightarrow \quad Y_n(y) = a_n \sin n\pi y, \; n \geq 1. \tag{11.0.205}$$

Now we use multiple Fourier sinus series (technique II for the homogeneous boundary conditions) to write

$$v(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{2} \sin n\pi y. \tag{11.0.206}$$

Thus

$$v_{xx} + v_{yy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \underbrace{a_{mn}\Big(-\frac{m^2\pi^2}{4} - n^2\pi^2\Big)}_{:=d_{mn}} \sin\frac{m\pi x}{2} \sin n\pi y = y(7-3x).$$

Here d_{mn} are the Fourier coefficient for $y(7-3x)$, with respect to the multiple basis $\sin\frac{m\pi x}{2}\sin n\pi y$, which we compute below:

$$\begin{aligned} d_{mn} =& \frac{4}{1/2}\int_0^1 y \sin n\pi y \, dy \int_0^2 (7-3x)\sin\frac{m\pi x}{2}\, dx \\ =& 2\Big[y\frac{-\cos\pi y}{n\pi}\Big|_0^1 - \underbrace{\int_0^1 \frac{-\cos n\pi y}{n\pi}\, dy}_{=0}\Big] \cdot \\ &\cdot \Big[(7-3x)\frac{-2}{m\pi}\cos\frac{m\pi x}{2}\Big|_0^2 - \underbrace{\int_0^2 (-3)\frac{-2}{m\pi}\cos\frac{m\pi x}{2}\, dx}_{=0}\Big] \\ =& \frac{-2}{n\pi}\cos n\pi \cdot \frac{-2}{m\pi}\Big((7-3\cdot 2)\cos m\pi - 7\Big) = \frac{4(-1)^n}{nm\pi^2}\Big((-1)^m - 7\Big). \end{aligned}$$

This yields

$$d_{mn} = a_{mn}\Big(-\frac{m^2\pi^2}{4} - n^2\pi^2\Big) \equiv \frac{4(-1)^n}{nm\pi^2}\Big((-1)^m - 7\Big), \qquad (11.0.207)$$

giving the Fourier coefficients a_{mn}, as

$$a_{mn} = \frac{4(-1)^n\Big(7 - (-1)^m\Big)}{nm\Big(\frac{m^2}{4} + n^2\Big)\pi^4}. \qquad (11.0.208)$$

Finally, inserting a_{mn} from (11.0.208) in (11.0.206) and recalling the computed expression for φ we get the solution of the original problem as

$$u(x,y) = \frac{1}{2}(2-x)(y-y^3) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4(-1)^n\Big(7 - (-1)^m\Big)}{nm\Big(\frac{m^2}{4} + n^2\Big)\pi^4} \sin\frac{m\pi x}{2}\sin n\pi y.$$

Example 26. Solve following initial-boundary value problem:

$$\begin{cases} \sqrt{1+t}\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, & 0 < x < 1, \qquad t > 0 \quad (\Leftrightarrow u''_{xx} = \frac{1}{\sqrt{1+t}}u'_t) \\ u(0,t) = 1, & u(1,t) = 0 \\ u(x,0) = 1 - x^2 \end{cases} \qquad (11.0.209)$$

Solution: Observe that there is an inhomogeneous boundary value. We choose a polynomial $S(x)$ in x such that $S''(x) = 0$ and S satisfies the boundary conditions in x: $S(0) = 1$ and $S(1) = 0$. Then $S(x) = Ax + B$, where $S(x)$ and $S(0) = 1 \Rightarrow B = 1$, whereas $S(1) = 0 \Rightarrow A + B = 0 \Rightarrow A = -1$. Thus $S(x) = 1 - x$.

Now let $v(x,t) = u(x,t) - S(x)$. Then the initial-boundary value problem for v is given by:

$$\begin{cases} v_{xx} = \frac{1}{\sqrt{1+t}}v_t, & 0 < x < 1, \quad t > 0 \\ v(0,t) = 0, & v(1,t) = 0 \quad t > 0 \\ v(x,0) = x - x^2, & 0 < x < 1. \end{cases} \qquad (11.0.210)$$

Using separation of variables $v(x,t) = X(x)T(t) \neq 0$ we get

$$X''T = \frac{1}{\sqrt{1+t}}XT' \quad \Rightarrow \quad \frac{X''}{X} = \frac{1}{\sqrt{1+t}}\frac{T'}{T} = \lambda(<0).$$

The Sturm-Liouville problem for X with its eigenvalues and eigenvectors are:

$$\begin{cases} X'' = \lambda X \\ X(0) = X(1) = 0, \end{cases} \qquad \begin{cases} \lambda_n = -(n\pi)^2 \\ X_n(x) = \sin n\pi x, \quad n \geq 1. \end{cases}$$

To determine the corresponding T_n:s we integrate the t-dependent eigenvalue problem as

$$\ln T = \lambda \int \sqrt{1+t}\, dt + \ln C \Rightarrow \ln \frac{T}{C} = \frac{2}{3}(1+t)^{\frac{3}{2}}\lambda \Rightarrow T(t) = Ce^{\frac{2}{3}\lambda(1+t)^{\frac{3}{2}}}.$$

Inserting $\lambda_n = -(n\pi)^2$ for the eigenvalues, we get

$$T_n(t) = C_n e^{-\frac{2}{3}\pi^2 n^2 (1+t)^{\frac{3}{2}}}. \tag{11.0.211}$$

Thus, by superposition theorem, we have that

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{2}{3}\lambda(1+t)^{\frac{3}{2}}} \sin n\pi x. \tag{11.0.212}$$

Now we need to compute C_n. To this approach we use the initial data, viz

$$v(x,0) = \sum_{n=1}^{\infty} C_n e^{-\frac{2}{3}\lambda} \sin n\pi x = x(x+1), \tag{11.0.213}$$

and compute the Fourier coefficients, $d_n := C_n e^{-\frac{2}{3}\lambda}$, for the function $x(x+1)$ expanded in the orthogonal basis $\{\sin n\pi x\}_{n=1}^{\infty}$:

$$\begin{aligned} d_n =& \frac{1}{1/2}\int_0^1 x(1-x)\sin n\pi x\, dx = 2\underbrace{\Big[x(1-x)\frac{-\cos n\pi x}{n\pi}\Big]_0^1}_{=0} \\ &+2\int_0^1 (1-2x)\frac{\cos n\pi x}{n\pi}\, dx = 2\underbrace{\Big[(1-2x)\frac{\sin n\pi x}{(n\pi)^2}\Big]_0^1}_{=0} -2\int_0^1 (-2)\frac{\sin n\pi x}{(n\pi)^2}\, dx \\ =&4\Big[\frac{-\cos n\pi x}{(n\pi)^3}\Big]_0^1 = \frac{4}{(n\pi)^3}(1-\cos n\pi) = \begin{cases} \frac{8}{\pi^3(2k+1)^3}, & n = 2k+1 \\ 0, & n = 2k. \end{cases} \end{aligned}$$

Consequently inserting $C_n = e^{\frac{2}{3}\lambda_n} d_n$ in (11.0.212) we have

$$v(x,t) = \frac{8}{\pi^3}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} e^{-\frac{2}{3}\lambda_n[(1+t)^{\frac{3}{2}}-1]} \sin(2k+1)\pi x \tag{11.0.214}$$

and finally the solution for the original equation, $u(x,t) = S(x) + v(x,t)$, is given by

$$u(x,t) = 1 - x + \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-\frac{2}{3}(2k+1)^2\pi^2[(1+t)^{\frac{3}{2}}-1]}\sin(2k+1)\pi x.$$

Example 27. Solve the homogeneous boundary value problem:

$$\begin{cases} u''_{xx} + u''_{yy} + 20u = 0 & 0 < x < 1, & 0 < y < 1 \\ u(0,y) = 0, & u(1,y) = 0, & 0 < y < 1 \\ u(x,0) = 0, & u(x,1) = x^2 - x & \end{cases} \tag{11.0.215}$$

Solution: Using Spartan of variables: $u(x,y) = X(x)Y(y) \neq 0$, we get from the differential equation that

$$X''Y + XY'' + 20XY = 0 \qquad \Longrightarrow \qquad \frac{X''}{X} = -\frac{Y''}{Y} - 20 = \lambda. \tag{11.0.216}$$

Invoking also the boundary conditions, the eigenvalue problem for X and its solution are given below

$$\begin{cases} X'' = \lambda X \\ X(0) = X(1) = 0 \end{cases} \quad \text{with} \quad \begin{cases} \lambda_n = -(n\pi)^2 \\ X_n(x) = \sin n\pi x, \quad n \geq 1. \end{cases} \tag{11.0.217}$$

As for the equation for Y:

$$\begin{cases} Y'' = -(20+\lambda)Y = \Big((n\pi)^2 - 20\Big)Y \\ Y(0) = 0, \end{cases} \tag{11.0.218}$$

we note that, for $n = 1$ we have $(n\pi)^2 - 20 = \pi^2 - 20 < 0$, and $(n\pi)^2 - 20 > 0$, when $n \geq 2$. Therefore we treat the case $n = 1$ separately:

For $n = 1$ we rewrite the equation for Y as $Y_1'' = -\beta_1^2 Y_1$ with $\beta_1 = \sqrt{20 - \pi^2}$. The general solution to this equation is given by

$$Y_1(y) = A_1 \cos\beta_1 y + B_1 \sin\beta_1 y, \tag{11.0.219}$$

where the boundary condition $Y_1(0) = 0$ yields $Y_1(0) = A_1 = 0$. Thus

$$Y_1(y) = B_1 \sin\beta_1 y. \tag{11.0.220}$$

For $n \geq 2$ we have the equation $Y_n'' = -\beta_n^2 Y_n$ with $\beta_n = \sqrt{(n\pi)^2 - 20}$, and the general solution

$$Y_n(y) = A_n \cosh \beta_n y + B_n \sinh \beta_n y, \tag{11.0.221}$$

where $Y_n(0) = 0$ gives $Y_n(0) = A_n = 0$ and hence

$$Y_n(y) = B_n \sinh \beta_1 y. \tag{11.0.222}$$

Now by the superposition we have

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y) = B_1 \sin \pi x \cdot \sin \beta_1 y + \sum_{n=2}^{\infty} B_n \sin n\pi x \sinh \beta_n y.$$

It remains to compute B_n, such that

$$u(x,1) = \sum_{n=1}^{\infty} Y_n(1) X_n(x) = x^2 - x. \tag{11.0.223}$$

Recall that $\{X_n(x)\}_{n=1}^{\infty}$ is a complete orthogonal system on the interval $(0,1)$, with

$$M_n := \int_0^1 X_n^2(x)\,dx = \int_0^1 \sin^2 n\pi x\,dx = \frac{1}{2}. \tag{11.0.224}$$

Then the Fourier coefficients, $Y_n(1)$, for the function $x^2 - x$ in (11.0.223) in a Fourier sine series expansion are

$$\begin{aligned} Y_n(1) &= \frac{1}{M_n}\int_0^1 (x^2 - x) X_n(x)\,dx = 2\int_0^1 (x^2 - x)\sin n\pi x\,dx \\ &= 2\underbrace{\Big[(x^2 - x)\frac{-\cos n\pi x}{n\pi}\Big]_0^1}_{=0} + 2\int_0^1 (2x-1)\frac{\cos n\pi x}{n\pi}\,dx \\ &= 2\underbrace{\Big[(2x-1)\frac{\sin n\pi x}{(n\pi)^2}\Big]_0^1}_{=0} - 2\int_0^1 2\frac{\sin n\pi x}{(n\pi)^2}\,dx \\ &= 4\Big[\frac{\cos n\pi x}{(n\pi)^3}\Big]_0^1 = 4\frac{(-1)^n - 1}{(n\pi)^3}. \end{aligned} \tag{11.0.225}$$

Then for $n = 1$ we have $Y_1(1) = -8/\pi^3$ and thus $Y_1(y) = C_1 \sin \beta_1$, give that

$$B_1 = -\frac{8}{\pi^3 \sin \beta_1}. \tag{11.0.226}$$

As for $n \geq 2$ we have $Y_n(1) = 4\dfrac{(-1)^n - 1}{(n\pi)^3} = B_n \sinh \beta_n$ and thus

$$\begin{cases} B_{2k} = 0 & \text{for} \quad k = 0, 1, \dots \\ B_{2k+1} = -\frac{8}{(2k+1)^3\pi^3 \sinh \beta_{2k+1}} & \text{for} \quad k = 1, 2, \dots \end{cases} \tag{11.0.227}$$

Summing up the solution to the equation is given by

$$\begin{aligned} u(x, y) = &- \frac{8}{\pi^3} \sin \pi x \frac{\sin(\sqrt{20 - \pi^2}\; y)}{\sin \sqrt{20 - \pi^2}} \\ &- \frac{8}{\pi^3} \sum_{k=1}^{\infty} \frac{\sin(2k+1)\,\pi x}{(2k+1)^3} \cdot \frac{\sinh \sqrt{(2k+1)^2\pi^2 - 20}\; y}{\sinh \sqrt{(2k+1)^2\pi^2 - 20}}. \end{aligned} \tag{11.0.228}$$

Example 28. Solve the following inhomogeneous heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial^2 x} = t \sin x, & 0 < x < 1, \qquad t > 0 \\ u(0, t) = 0, & u(1, t) = 0 \\ u(x, 0) = \sin 2\pi x. \end{cases} \tag{11.0.229}$$

Solution: Applying separation of variables technique for the corresponding homogeneous heat equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial^2 x} = 0, \tag{11.0.230}$$

with $u(x, y) = X(x)T(t) \neq 0$, yields

$$X(x)T'(t) - X''(x)T(t) = 0 \quad \Longrightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{11.0.231}$$

The eigenvalues and eigenfunctions to the eigenvalue problem:

$$\begin{cases} X'' = \lambda X \\ X(0) = X(1) = 0 \end{cases} \quad \text{are} \quad \begin{cases} \lambda_n = -(n\pi)^2 \\ X_n(x) = \sin n\pi x, \quad n \geq 1, \end{cases} \tag{11.0.232}$$

where $\{\sin n\pi x\}_{n=1}^{\infty}$ is a complete orthogonal system on the interval $(0, 1)$, with

$$M_n = \int_0^1 \sin^2 n\pi x \, dx = \frac{1}{2}. \tag{11.0.233}$$

We expand both $u(x,t)$ and $t\sin x$ as Fourier sine series in the base $\{\sin n\pi x\}_{n=1}^{\infty}$, viz

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin n\pi x, \\ t\,\sin x = \sum_{n=1}^{\infty} t\,b_n \sin n\pi x \quad \Big(\Leftarrow\ \sin x = \sum_{n=1}^{\infty} b_n \sin n\pi x\Big), \end{cases} \tag{11.0.234}$$

The expansion of $u(x,t)$ gives the Fourier expansion for the left hand side of the differential equation (11.0.229):

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial^2 x} = \sum_{n=1}^{\infty}\Big(B_n'(t) + (n\pi)^2 B_n(t)\Big)\sin n\pi x, \tag{11.0.235}$$

Thus the differential equation (11.0.229) can be written as

$$\sum_{n=1}^{\infty}\Big(B_n'(t) + (n\pi)^2 B_n(t)\Big)\sin n\pi x = \sum_{n=1}^{\infty} t\,b_n \sin n\pi x. \tag{11.0.236}$$

From the second expansion in (11.0.234) we compute the Fourier coefficients b_n as

$$\begin{aligned} b_n &= \frac{1}{1/2}\int_0^1 \sin x \sin n\pi x = 2\int_0^1 \Big(\cos(1-n\pi)x - \cos(1+n\pi)x\Big)\,dx \\ &= \frac{\sin(1-n\pi)}{1-n\pi} - \frac{\sin(1+n\pi)}{1+n\pi} = \frac{(-1)^n\sin 1}{1-n\pi} - \frac{(-1)^n \sin 1}{1+n\pi} \\ &= (-1)^n \frac{2n\pi}{1-n^2\pi^2}\sin 1. \end{aligned}$$

Now, identifying the coefficients in (11.0.236) we get the following ode for $B_n(t)$:

$$B_n'(t) + (n\pi)^2 B_n(t) = t\,b_n. \tag{11.0.237}$$

Multiplying (11.0.237) by the integrating factor $e^{(n\pi)^2 t}$ yields

$$e^{(n\pi)^2 t}B_n'(t) + (n\pi)^2 e^{(n\pi)^2 t}B_n(t) = e^{(n\pi)^2 t}t\,b_n \Rightarrow \frac{d}{dt}\Big(e^{(n\pi)^2 t}B_n(t)\Big) = e^{(n\pi)^2 t}t\,b_n.$$

Then, by integrating, it follows that:

$$\begin{aligned} e^{(n\pi)^2 t} B_n(t) &= b_n \int t\, e^{(n\pi)^2 t} dt = b_n \Big(t \frac{e^{(n\pi)^2 t}}{(n\pi)^2} - \int \frac{e^{(n\pi)^2 t}}{(n\pi)^2} dt \Big) + C_n \\ &= b_n t \frac{e^{(n\pi)^2 t}}{(n\pi)^2} - b_n \frac{e^{(n\pi)^2 t}}{(n\pi)^4} + C_n. \end{aligned} \tag{11.0.238}$$

Thus we have

$$B_n(t) = C_n\, e^{-(n\pi)^2 t} + \frac{1}{(n\pi)^2}\, b_n t - \frac{1}{(n\pi)^4}\, b_n. \tag{11.0.239}$$

Now we invoke the initial data

$$u(x,0) = \sum_{n=1}^{\infty} B_n(0) \sin n\pi x = \sin 2\pi x$$

to get

$$\sum_{n=1}^{\infty} \Big(C_n - \frac{1}{(n\pi)^4}\, b_n \Big) \sin n\pi x = \sin 2\pi x, \tag{11.0.240}$$

where for $n = 2$ we have

$$\Big(C_2 - \frac{1}{(2\pi)^4}\, b_2 \Big) \sin 2\pi x = \sin 2\pi x \quad \Rightarrow \quad \Big(C_2 - \frac{1}{(2\pi)^4}\, b_2 \Big) = 1 \tag{11.0.241}$$

whereas,for $n \neq 2$ we have $C_n = \frac{1}{(n\pi)^4}\, b_n$. Thus by (11.0.239) we have

$$\begin{cases} B_2(t) = e^{-4\pi)^2 t} + \Big[\frac{1}{(2\pi)^4} e^{-4\pi)^2 t} + \frac{1}{(2\pi)^2}\, t - \frac{1}{(2\pi)^4} \Big] b_2 \\ B_n(t) = \Big[\frac{1}{(n\pi)^4}\, e^{-(n\pi)^2 t} + \frac{1}{(n\pi)^2}\, t - \frac{1}{(n\pi)^4} \Big] b_n, \quad n \neq 2 \end{cases} \tag{11.0.242}$$

Note that the expression for $B_n(t)$ in (11.0.242) gives the expression for B_2 as well (is valid also for $n = 2$).

Finally we have the solution for the differential equation (11.0.242):

$$\begin{aligned} u(x,t) =& e^{-4\pi^2 t} \sin 2\pi x \\ &+ (2\pi \sin 1) \times \sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{n^2\pi^2 - 1} \Big[\frac{t}{n^2\pi^2} - \frac{1 - e^{-n^2\pi^2 t}}{n^4\pi^4} \Big] \sin n\pi x. \end{aligned}$$

Example 29. Solve the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, & & 0 < x < 1, & t > 0 \\ u(0,t) = t+1, & u(1,t) = 0, & & t > 0 \\ u(x,0) = 1-x, & & 0 < x < 1. & \end{cases} \tag{11.0.243}$$

Solution: Choose a function, which satisfies the given boundary conditions: For example, $\tilde{u}(x,t) = (t+1)(1-x)$. Let $v = u - \tilde{u}$. Then v satisfies

$$\frac{\partial v}{\partial t} - 2\frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial \tilde{u}}{\partial t} - 2\frac{\partial^2 \tilde{u}}{\partial x^2}\right) = 0 - (1-x) - 2\cdot 0 = x - 1.$$

Thus, using the data, we formulate the problem for v, viz

$$\begin{cases} \frac{\partial v}{\partial t} - 2\frac{\partial^2 v}{\partial x^2} = x - 1. & 0 < x < 1, & t > 0 \\ v(0,t) = 0, & v(1,t) = 0, & t > 0 \\ v(x,0) = 1 - x - (1-x) = 0, & 0 < x < 1. & \end{cases} \tag{11.0.244}$$

Using the separation of variables, $v(x,y) = X(x)T(t) \neq 0$, for the corresponding homogeneous equation for v we have that

$$X(x)T'(t) - 2X''(x)T(t) = 0 \quad \Longrightarrow \quad \frac{T'(t)}{T(t)} = 2\frac{X''(x)}{X(x)} = -2\lambda.$$

The eigenvalues and the eigenfunctions of the eigenvalue problem for X:

$$\begin{cases} X'' = -\lambda X \\ X(0) = X(1) = 0 \end{cases} \quad \text{are} \quad \begin{cases} \lambda_n = (n\pi)^2 \\ X_n(x) = \sin n\pi x, \quad n \geq 1, \end{cases}$$

where $\{\sin n\pi x\}_{n=1}^{\infty}$ is a complete orthogonal system on the interval $(0,1)$, with

$$M_n = \int_0^1 \sin^2 n\pi x \, dx = \frac{1}{2}. \tag{11.0.245}$$

Now we expand $x - 1$ in a Fourier sine series with respect to $\{\sin n\pi x\}_{n=1}^{\infty}$:

$$x - 1 = \sum_{n=1}^{\infty} C_n \sin n\pi x \tag{11.0.246}$$

and let

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x. \tag{11.0.247}$$

Inserting the expansions (11.0.246) and (11.0.247) in the differential equation for v we get Then we get the Fourier coefficients, C_n, for $x-1$ as

$$\begin{aligned} C_n &= \frac{1}{\|X_n\|^2}\int_0^1 (x-1)X_n(x)\,dx = \frac{1}{1/2}\int_0^1 (x-1)\sin n\pi x\,dx \\ &= 2\Big[(x-1)\frac{-\cos n\pi x}{n\pi}\Big]_0^1 + 2\int_0^1 \frac{\cos n\pi x}{n\pi}\,dx \\ &= -\frac{2}{n\pi} + 2\underbrace{\left[\frac{\sin n\pi x}{(n\pi)^2}\right]_0^1}_{=0} = -\frac{2}{n\pi}. \end{aligned} \qquad (11.0.248)$$

Inserting the expansions (11.0.246) and (11.0.247) in the differential equation for v it follows that

$$\frac{\partial v}{\partial t} - 2\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty}\Big[v_n'(t) + 2(n\pi)^2 v_n(t)\Big]\sin n\pi x = x-1 = \sum_{n=1}^{\infty}\frac{-2}{n\pi}\sin n\pi x.$$

Further

$$v(x,0) = \sum_{n=1}^{\infty} v_n(0)\sin n\pi x = 0 \quad\Longrightarrow\quad v_n(0) = 0. \qquad (11.0.249)$$

Identifying the coefficients for $\sin n\pi x$ and using (11.0.249) we get the ode for v_n as

$$\begin{cases} v_n'(t) + 2(n\pi)^2 v_n(t) = \frac{-2}{n\pi}, \\ v_n(0) = 0. \end{cases} \qquad (11.0.250)$$

Multiplying the equation for v_n by the integrating factor $e^{2n^2\pi^2 t}$ we get

$$\frac{d}{dt}\Big(e^{2n^2\pi^2 t} v_n(t)\Big) = -\frac{2}{n\pi}e^{2n^2\pi^2 t}. \qquad (11.0.251)$$

Now let $t = s$ in (11.0.251) and integrate over the interval $[0,t]$ to get

$$\Big[e^{2n^2\pi^2 s} v_n(s)\Big]_0^t = -\frac{2}{n\pi}\int_0^t e^{2n^2\pi^2 s}\,ds \qquad \Longrightarrow \qquad (11.0.252)$$

$$e^{2n^2\pi^2 t} v_n(t) - v_n(0) = -\frac{1}{n^3\pi^3}e^{2n^2\pi^2 t} + \frac{1}{n^3\pi^3}. \qquad (11.0.253)$$

Thus we have

$$v_n(t) = \frac{1}{n^3\pi^3}\Big(e^{-2n^2\pi^2 t} - 1\Big), \qquad (11.0.254)$$

and therefore

$$v(x,t)=\sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}\left(e^{-2n^2\pi^2 t}-1\right)\sin n\pi x, \tag{11.0.255}$$

which gives the solution to the original equation (11.0.243), viz:

$$u(x,t)=(t+1)(1-x)+\sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}\left(e^{-2n^2\pi^2 t}-1\right)\sin n\pi x. \tag{11.0.256}$$

Example 30. Solve the Laplace equation $\Delta u=0$, in the polar coordinates region $0<\theta<\frac{\pi}{4}$, $\quad 1<r<2$, with the boundary conditions given viz,

$$\begin{cases} \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^u}{\partial\theta^2}=0, & 1<r<2, \qquad 0<\theta<\frac{\pi}{4}\\ u(1,\theta)=0, & \frac{\partial u}{\partial r}(2,\theta)=0,\\ u(r,0)=0, & u(r,\frac{\pi}{4})=r-1. \end{cases}$$

Solution: Using separation of variables $u(r,\theta)=R(r)\Theta(\theta)\neq 0$, we rewrite the Laplace equation as two odes:

$$\frac{1}{r}\left(rR'\right)'\Theta+\frac{1}{r^2}R\Theta''=0 \quad\Longrightarrow\quad -\frac{r\left(rR'\right)'}{R}=\frac{\Theta''}{\Theta}=\lambda. \tag{11.0.257}$$

Associated with the boundary conditions we get the Sturm-Liouville problems for R and Θ:

$$(I)\quad \begin{cases} -r\frac{d}{dr}\left(r\frac{dR}{dr}\right)=\lambda R\\ R(1)=0, \quad R'(2)=0. \end{cases} \qquad (II)\quad \begin{cases} \Theta''=\lambda\Theta\\ \Theta(0)=0. \end{cases} \tag{11.0.258}$$

The differential equation (I):

$$r^2R''+rR'+\lambda R=0, \tag{11.0.259}$$

is of Euler type. Let $t=\ln r$, then by the chain rule

$$\frac{d}{dt}=\frac{d}{dr}\cdot\frac{dr}{dt}=r\frac{d}{dr}. \tag{11.0.260}$$

Now since $t = \ln r$ thus $r = e^t$ and hence if we define $T(t) := R(e^t)$, then we have $T' = rR'$, which give us the more easier eigenvalue problem:

$$\begin{cases} -T'' = \lambda T \\ T(0), \quad T'(ln2) = 0, \end{cases} \tag{11.0.261}$$

with the eigenvalues

$$\lambda_n = \left[\left(n + \frac{1}{2}\right)\frac{\pi}{\ln 2}\right]^2 = \beta_n^2, \qquad n = 0, 1, 2, \dots, \tag{11.0.262}$$

and the eigenfunctions $T_n(t) = \sin \beta_n t$. Consequently we have that

$$R_n(r) = \sin(\beta_n \ln r). \tag{11.0.263}$$

Now the differential equation (II) would become $\Theta'' = \beta_n^2 \Theta$, which has the general solution

$$\Theta_n(\theta) = a_n \cosh \beta_n \theta + b_n \sinh \beta_n \theta. \tag{11.0.264}$$

The boundary condition $\Theta_n(0) = 0$ yields $a_n = 0$. Thus the total s solution is written as

$$u(r, \theta) = \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) = \sum_{n=0}^{\infty} b_n \sin(\beta_n \ln r) \cdot \sinh \beta_n \theta. \tag{11.0.265}$$

Using the boundary condition $u(r, \frac{\pi}{4}) = r - 1$ we get

$$u(r, \frac{\pi}{4}) = \sum_{n=0}^{\infty} b_n \sin(\beta_n \ln r) \cdot \sinh \beta_n \frac{\pi}{4} = r - 1. \tag{11.0.266}$$

Since $t = \ln r$ and $r = e^t$, we can rewrite (11.0.266) as

$$u(r, \frac{\pi}{4}) = \sum_{n=0}^{\infty} b_n \sinh \beta_n \frac{\pi}{4} \cdot \sin \beta_n t = e^t - 1, \quad 0 < t < \ln 2. \tag{11.0.267}$$

It remains to compute b_n. To this approach we recall that $\{\sin \beta_n t\}_{n=0}^{\infty}$ is a complete orthogonal system on the interval $(0, \ln 2)$. Thus the Fourier coefficient, C_n, for the function $e^t - 1$ is given by

$$\begin{aligned} C_n &= b_n \sinh \beta_n \frac{\pi}{4} = \frac{1}{\frac{\ln 2}{2}} \int_0^{\ln 2} (e^t - 1) \sin \beta_n t \, dt \\ &= \frac{2}{\ln 2} \underbrace{\left[(e^t - 1) \frac{-\cosh \beta_n t}{\beta_n t} \right]_0^{\ln 2}}_{=0 \text{ since } \beta_n = (n+1/2)\frac{\pi}{ln 2}} + \frac{2}{\ln 2} \int_0^{\ln 2} e^t \frac{\cos \beta_n t}{\beta_n t} \, dt = [\text{by tabell...}] \\ &= \frac{2}{\beta_n \ln 2} \left[\frac{e^t (\cos \beta_n t + \beta_n \sin \beta_n t)}{\beta_n^2 + 1} \right]_0^{\ln 2} = \frac{2}{\ln 2 \cdot \beta_n (\beta_n^2 + 1)} \Big[2\beta_n (-1)^n - 1 \Big]. \end{aligned}$$

Now since $b_n = \frac{C_n}{\sinh(\beta_n \pi / 4)}$ we have the solution to the original Laplace equation given by:

$$\begin{aligned} u(r, \theta) = \sum_{n=0}^{\infty} \frac{2\Big[2(n+\frac{1}{2})\frac{\pi}{\ln 2}(-1)^n - 1\Big]}{(n+\frac{1}{2})\pi\Big[(n+\frac{1}{2})^2(\frac{\pi}{\ln 2})^2 + 1\Big]} \cdot \frac{\sinh(n+\frac{1}{2})\frac{\pi\theta}{\ln 2}}{\sinh(n+\frac{1}{2})\frac{\pi^2}{4\ln 2}} \times \\ \times \sinh\Big((n+\frac{1}{2})\frac{\pi}{\ln 2} \ln r\Big). \end{aligned}$$

Example 31. Expand the function $\sin(2 \sin x)$ in the trigonometric Fourier series, in real form.

Solution: We recall the generalizing function for the Bessel function $\mathcal{J}_n(x)$:

$$e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(x) e^{in\theta}. \tag{11.0.268}$$

We take the imaginary part in (11.0.268), (note that $\mathcal{J}_n(x)$ are real functions), to obtain

$$\sin(x \sin \theta) = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(x) \sin n\theta = \sum_{n=-\infty}^{-1} \mathcal{J}_n(x) \sin n\theta + \sum_{n=1}^{\infty} \mathcal{J}_n(x) \sin n\theta.$$

Changing $n \to -n$ in the first sum and using $\mathcal{J}_{-n}(x) = (-1)^n \mathcal{J}_n(x)$ yields

$$\begin{aligned} \sin(x \sin \theta) &= \sum_{n=1}^{\infty} \mathcal{J}_{-n}(x) \sin(-n\theta) + \sum_{n=1}^{\infty} \mathcal{J}_n(x) \sin n\theta \\ &= \sum_{n=1}^{\infty} \Big[1 - (-1)^n\Big] \mathcal{J}_n(x) \sin n\theta = 2 \sum_{k=0}^{\infty} \mathcal{J}_{2k+1}(x) \sin(2k+1)\theta. \end{aligned}$$

Let now $x = 2$ and then substitute θ by x to get the desired expansion for $\sin(2\sin x)$:

$$\sin(2\sin x) = 2\sum_{k=0}^{\infty} \mathcal{J}_{2k+1}(2)\sin(2k+1)x. \tag{11.0.269}$$

Example 32. A circular membrane with the radius a is effected by a, uniformly distributed, periodic external force $q\sin\omega t$. Hence the transversal vibrations has the equation:

$$\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{q}{S}\sin\omega t, \qquad u|_{r=a} = 0. \tag{11.0.270}$$

Compute the stationary vibration movement, i.e., a solution of the form $u(r,t) = v(r)\sin\omega t$. Which are the resonance angle frequencies?

Solution: In polar coordinates and with $u = u(r,t)$ we get the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{q}{S}\sin\omega t, \quad 0 < r < a, \quad u(a,t) = 0. \tag{11.0.271}$$

Let $u = Im\,\tilde{u}$, where $\tilde{u}$ satisfy the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{u}}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 \tilde{u}}{\partial t^2} = -\frac{q}{S}e^{i\omega t}, \quad \tilde{u}(a,t) = 0. \tag{11.0.272}$$

For this equation: (11.0.272), we seek a solution of the form $\tilde{u}(r,t) = v(r)e^{i\omega t}$, that is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)e^{i\omega t} + \frac{\omega^2}{c^2}ve^{i\omega t} = -\frac{q}{S}e^{i\omega t}. \tag{11.0.273}$$

Thus we have the differential equation for v, viz

$$\begin{cases} \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \frac{\omega^2}{c^2}v = -\frac{q}{S}, & 0 < r < a \\ v(a) = 0, & v(r) \text{ is bounded as } r \to 0^+. \end{cases} \tag{11.0.274}$$

This Imogen equation has a particular solution of the form $v_p = A =$ constant, which insuring in the equation (11.0.274) yields

$$v_p = A = -\frac{q}{S}\left(\frac{c}{\omega}\right)^2. \tag{11.0.275}$$

The homogeneous equation

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{dv}{dr}\Big)+\Big(\frac{\omega}{c}\Big)^2 v=0, \qquad (11.0.276)$$

is a Bessel differential equation of order 0 with the general solution

$$v_h(r)=C_1\mathcal{J}_0\Big(\frac{\omega r}{c}\Big)+C_2Y_0\Big(\frac{\omega r}{c}\Big). \qquad (11.0.277)$$

Since the solution is bounded as $r\to 0^+$ thus we have $C_2=0$, and hence

$$v(r)=v_p+v_h(r)=-\frac{q}{S}\Big(\frac{c}{\omega}\Big)^2+C_1\mathcal{J}_0\Big(\frac{\omega r}{c}\Big). \qquad (11.0.278)$$

The boundary condition $v(a)=0$ gives that

$$C_1=\frac{qc^2}{S\omega^2\mathcal{J}_0(\frac{\omega a}{c})}. \qquad (11.0.279)$$

Thus

$$v(r)=-\frac{q}{S}\Big(\frac{c}{\omega}\Big)^2+\frac{qc^2}{S\omega^2\mathcal{J}_0(\frac{\omega a}{c})}\mathcal{J}_0\Big(\frac{\omega r}{c}\Big)=\frac{q}{S}\Big(\frac{c}{\omega}\Big)^2\Big[\frac{\mathcal{J}_0(\frac{\omega r}{c})}{\mathcal{J}_0(\frac{\omega a}{c})}-1\Big]. \qquad (11.0.280)$$

Inserting $v(r)$ in $\tilde{u}$ we have that

$$\tilde{u}(r,t)=v(r)e^{i\omega t}=v(r)(\cos\omega t+i\sin\omega t). \qquad (11.0.281)$$

Now since $u(r,t)=Im\,\tilde{u}$, we finally get the solution:

$$u(r,t)=v(r)\sin\omega t=\frac{q}{S}\Big(\frac{c}{\omega}\Big)^2\Big[\frac{\mathcal{J}_0(\frac{\omega r}{c})}{\mathcal{J}_0(\frac{\omega a}{c})}-1\Big]\sin\omega t. \qquad (11.0.282)$$

Resonance frequencies are obtained setting $\mathcal{J}_0(\frac{\omega a}{c})=0$. Thus the resonance frequencies are: $\omega=\omega_n=\frac{c}{a}\alpha_{0,n}$, where $\alpha_{0,n}$, $n=1,2,\dots$ are the positive zeros for $\mathcal{J}_0$.

Example 33. Solve the heat equation $u_t=\nabla^2 u$ in a cylinder with the radius b so that: The cross-section surfaces are isolated and the surface $r=b$, (cylinder coordinates), obeys the cooling law $u+2u'_r=0$. Further, the initial temperature is assumed to be $u|_{t=0}=r^2=x^2+y^2$.

Solution: The problem for $u = u(r,t)$ is given by

$$\begin{cases} u_t = \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), & 0 < r < b, \qquad t > 0, \\ u \text{ bounded as} & r \to 0^+, \\ u(b,t) + 2u_r(b,t) = 0, & \\ u(r,0) = r^2 = x^2 + y^2. & \end{cases} \tag{11.0.283}$$

Using the separation of variables, $u(r,t) = R(r)T(t) \neq 0$, we get

$$RT' = \frac{1}{r}\left(rR'\right)'T \quad \Longrightarrow \quad \frac{\frac{1}{r}\left(rR'\right)'}{R} = \frac{T'}{T} = -\lambda, \tag{11.0.284}$$

This yields a singular Sturm-Liuville problem for R with $\lambda > 0$:

$$\begin{cases} -\frac{1}{r}\left(rR'\right)' = \lambda R, & 0 < r < b, \qquad t > 0, \\ R(r) \text{ bounded as} & r \to 0, \\ R(b) + 2R'(b) = 0. & \end{cases} \tag{11.0.285}$$

Let $\lambda = \beta^2$ with $\beta > 0$. Then the general solution for R is

$$R(r) = C_1\mathcal{J}_0(\beta r) + C_2Y_0(\beta r). \tag{11.0.286}$$

Since $R(r)$ is bounded, as $r \to 0$ we have $C_2 = 0$ thus $R(r) = C_1\mathcal{J}_0(\beta r)$. Consequently

$$R(b) + 2R'(b) = C_1[\mathcal{J}_0(\beta b) + 2\beta\mathcal{J}_0'(\beta b)] = 0. \tag{11.0.287}$$

For $C_1 \neq 0$, (and with $\beta b \equiv x$) βb is a zero of the function

$$\mathcal{J}_0(x) + \frac{2}{b}\mathcal{J}_0'(x). \tag{11.0.288}$$

Let now $\alpha_k,\ k = 0, 1, \ldots,$ be the positive zeros for the function (11.0.288). Then $\beta_k b = \alpha_k,\ k = 0, 1, 2, \ldots.$ The eigenfunctions are thus $R_k(r) = \mathcal{J}_0(\beta_k r)$, (we take $C_1 = 1$). $R_k(r) = \mathcal{J}_0(\beta_k r)$.

The second equation in (11.0.283): $T' = -\lambda T$ is thus written as $T_k'(t) = -\beta_k^2 T_k(t)$ with the solution $T_k(t) = a_k e^{-\beta_k^2 t}$. Hence the solution for the original problem is now

$$u(r,t) = \sum_{k=0}^{\infty} T_k(t)R_k(r) = \sum_{k=0}^{\infty} a_k e^{-\beta_k^2 t}\mathcal{J}_0(\beta_k r), \tag{11.0.289}$$

where $\{R_k(r)\}_{k=1}^{\infty} = \{\mathcal{J}_0(\beta_k r)\}_{k=1}^{\infty}$ is a complete orthogonal system, i.e., an orthogonal base on $(0, b)$ with the weight function r.

Now we expand the initial condition $u(r, 0) = r^2$ in a Fourier series with respect to this system. Then the Fourier coefficients are

$$a_k = \frac{1}{\rho_k} \int_0^b r^2 \mathcal{J}_0(\beta_k r) r\, dr, \quad \text{where} \quad \rho_k = \int_0^b \mathcal{J}_0^2(\beta_k r) r\, dr. \qquad (11.0.290)$$

To compute ρ_k, we use *Lemma 14, with* $nu = 0$ *and* $x = \beta_k r$, to get

$$\rho_k = \frac{1}{\beta_k^2} \int_0^{\beta_k b = \alpha_k} \mathcal{J}_0^2(x) x\, dx = \frac{1}{\beta_k^2} \cdot \frac{\alpha_k^2}{2} \Big(\mathcal{J}_0'^2(\alpha_k) + \mathcal{J}_0^2(\alpha_k) \Big). \qquad (11.0.291)$$

Now since $\mathcal{J}_0(\alpha_k) + 2\beta_k \mathcal{J}_0'(\alpha_k) = 0$ we have $\mathcal{J}_0'(\alpha_k) = -\dfrac{\mathcal{J}_0(\alpha_k)}{2\beta_k}$ and thus

$$\rho_k = \frac{b^2}{2} \Big[\mathcal{J}_0^2(\alpha_k) + \frac{\mathcal{J}_0^2(\alpha_k)}{4\beta_k^2} \Big] = \frac{b^2}{2} \cdot \frac{4\beta_k^2 + 1}{4\beta_k^2} \mathcal{J}_0^2(\alpha_k). \qquad (11.0.292)$$

To compute a_k we use the first relation in (11.0.290) and rewrite

$$\rho_k a_k = \int_0^b r^3 \mathcal{J}_0(\beta_k r)\, dr. \qquad (11.0.293)$$

To compute the integral in (11.0.293) we use recurrence formulas for Bessel functions (see the first 2 formulas in theorem 39) and partial integration, to get

$$\begin{aligned}
\int x^3 \mathcal{J}_0(x)\, dx &= \int x^2 \cdot x \mathcal{J}_0(x)\, dx = \int x^2 \cdot \frac{d}{dx}\Big(x \mathcal{J}_1(x) \Big)\, dx \\
&= x^2 \cdot x \mathcal{J}_1(x) - \int 2x \cdot x \mathcal{J}_1(x)\, dx = x^3 \mathcal{J}_1(x) + \int 2x^2 \mathcal{J}_0'(x)\, dx \\
&= x^3 \mathcal{J}_1(x) + 2x^2 \mathcal{J}_0(x) - \int 4x \mathcal{J}_0(x)\, dx = (x^3 - 4x)\mathcal{J}_1(x) + 2x^2 \mathcal{J}_0(x).
\end{aligned}$$

We let $x = \beta_k r$ in (11.0.293) and use the above relation to write

$$\begin{aligned}
\rho_k a_k &= \frac{1}{\beta_k^4}\int_0^{\alpha_k} x^3 \mathcal{J}_0(x)\,dx = \frac{1}{\beta_k^4}\Big[(x^3-4x)\mathcal{J}_1(x) + 2x^2\mathcal{J}_0(x)\Big]_0^{\alpha_k} \\
&= \frac{1}{\beta_k^4}\Big[(\alpha_k^3 - 4\alpha_k)\mathcal{J}_1(\alpha_k) + 2\alpha_k^2\mathcal{J}_0(\alpha_k)\Big] \\
&= \frac{1}{\beta_k^4}\Big[\frac{\alpha_k^3 - 4\alpha_k}{2\beta_k}\mathcal{J}_0(\alpha_k) + 2\alpha_k^2\mathcal{J}_0(\alpha_k)\Big] \\
&= \frac{\mathcal{J}_0(\alpha_k)}{\beta_k^4}\Big[\frac{b}{2}(\alpha_k^2 - 4) + 2\alpha_k^2\Big] = \frac{\mathcal{J}_0(\alpha_k)}{\beta_k^4}\Big[\Big(2+\frac{b}{2}\Big)\alpha_k^2 - 2b\Big],
\end{aligned} \tag{11.0.294}$$

where we also used the relation $\mathcal{J}_1(\alpha_k) = -\mathcal{J}'_0(\alpha_k) = -\mathcal{J}_0(\alpha_k)/2\beta_k$. Inserting ρ_k, from (11.0.293), in equation (11.0.294), it follows that

$$\begin{aligned}
a_k &= \frac{8\beta_k^2}{b^2(4\beta_k^2+1)\mathcal{J}_0^2(\alpha_k)} \cdot \frac{\mathcal{J}_0(\alpha_k)}{\beta_k^4}\Big[\Big(2+\frac{b}{2}\Big)\alpha_k^2 - 2b\Big] \\
&= \frac{8[(2+\frac{b}{2})\alpha_k^2 - 2b]}{b^2\beta_k^2(4\beta_k^2+1)\mathcal{J}_0(\alpha_k)}.
\end{aligned} \tag{11.0.295}$$

Now $\beta_k = \alpha_k/b$ implies that $4\beta_k^2 + 1 = (4\alpha_k^2 + b^2)/b^2$. Thus, finally, we can write the solution as

$$u(r,t) = \sum_{k=1}^{\infty} \frac{8b^2[(2+\frac{b}{2})\alpha_k^2 - 2b]}{\alpha_k^2(4\alpha_k^2+b^2)\mathcal{J}_0(\alpha_k)} e^{-(a_k/b)t}\mathcal{J}_0(\alpha_k r/b), \tag{11.0.296}$$

where $\mathcal{J}_0(\alpha_k) + \frac{2}{b}\mathcal{J}_0'(\alpha_k) = 0$.

Example 34 (a) Give a bounded solution, of the form $u(r,t) = v(r)e^{i\omega t}$, to the equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) - \frac{n^2}{r^2}u, \quad 0 < r < a, \\ u(a,t) = e^{i\omega t}, \end{cases} \tag{11.0.297}$$

where $n \geq 0$ is an integer. For which values of $\omega > 0$ there is a solution of this form?

34 (b) Let ω be chosen so that there exists a solution to problem (11.0.297). Show how to use this solution in oder to solve the following problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) - \frac{n^2}{r^2}u = 0, & 0 < r < a, \qquad t > 0, \\ u(a,t) = \sin\omega t & u \quad \text{is bounded} \quad , \\ u(r,0) = 0, & u_t(r,0) = 0. \end{cases} \tag{11.0.298}$$

Solution (a): Let $u(r,t) = v(r)e^{i\omega t}$ in the equation (11.0.298), then

$$v(r)(i\omega)^2 e^{i\omega t} = -v(r)\omega^2 e^{i\omega t} = \Big[\frac{1}{r}\cdot\frac{d}{dr}\Big(r\frac{dv}{dr}\Big) - \frac{n^2}{r^2}v(r)\Big]e^{i\omega t}, \qquad (11.0.299)$$

and hence

$$\frac{1}{r}\cdot\frac{d}{dr}\Big(r\frac{dv}{dr}\Big) + \Big(\omega^2 - \frac{n^2}{r^2}\Big)v = v'' + \frac{1}{r}v' + \Big(\omega^2 - \frac{n^2}{r^2}\Big)v = 0 \qquad (11.0.300)$$

Equation (11.0.300) is a Bessel differential equation of order n, with general solution

$$v(r) = A\mathcal{J}_n(\omega r) + BY_n(\omega r). \qquad (11.0.301)$$

Since v is bounded (u is bounded) thus we have $B \equiv 0$ and hence $v(r) = A\mathcal{J}_n(\omega r)$. Further using boundary data $u(a,t) = v(a)e^{i\omega t} = e^{i\omega t}$ gives that $v(a) = 1$, whereas by (11.0.301), (note! $B = 0$), $v(a) = A\mathcal{J}_n(\omega a)$. Thus if $\mathcal{J}_n(\omega a) \neq 0$, then $A = 1/\mathcal{J}_n(\omega a)$ and hence we have the following solution for the equation system (11.0.297)

$$u(r,t) = v(r)e^{i\omega t} = \frac{\mathcal{J}_n(\omega r)}{\mathcal{J}_n(\omega a)}e^{i\omega t}. \qquad (11.0.302)$$

Solution (b): We use the function v defined in (a) and note that

$$\tilde{u}(r,t) = Im\Big[v(r)e^{i\omega t}\Big] = v(r)\sin\omega t = \frac{\mathcal{J}_n(\omega r)}{\mathcal{J}_n(\omega a)}\sin\omega t \qquad (11.0.303)$$

is a solution to the first 2 equations in (11.0.298). Let now $w = u - \tilde{u}$. Then w satisfies the initial-boundary value problem:

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = \frac{1}{r}\cdot\frac{\partial}{\partial r}\Big(r\frac{\partial w}{\partial r}\Big) - \frac{n^2}{r^2}w \\ w(a,t) = u(a,t) - \tilde{u}(a,t) = \sin\omega t - \frac{\mathcal{J}_n(\omega a)}{\mathcal{J}_n(\omega a)}\sin\omega t = 0, \\ w(r,0) = 0 \\ w_t(r,0) = u_t(r,0) - \tilde{u}_t(r,0) = -\frac{\omega\mathcal{J}_0(\omega r)}{\mathcal{J}_n(\omega a)}. \end{cases} \qquad (11.0.304)$$

Using separation of variables as, $w(r,t) = R(r)T(t) \neq 0$, we get

$$RT'' = \frac{1}{r}\cdot\frac{d}{dr}(rR')T - \frac{n^2}{r^2}RT \quad\Rightarrow\quad \frac{T''}{T} = \frac{\frac{1}{r}(rR')'}{R} - \frac{n^2}{r^2} = -\lambda, \quad (11.0.305)$$

which yields the following two eigenvalue problems for R and T:

$$\begin{cases} R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2}\right)R = 0, & \\ R(a) = 0, & R \quad \text{is bounded,} \end{cases} \tag{11.0.306}$$

and

$$\begin{cases} T'' = -\lambda T, \\ T(0) = 0 \end{cases} \tag{11.0.307}$$

The equation for R; (11.0.306) is a Bessel eigenvalue problem of order n with $\lambda > 0$, $\mu > 0$, $\lambda = \mu^2$, $(\mu > 0)$,: The general solution to this equation is given by

$$R(r) = A\mathcal{J}_n(\mu r) + BY_n(\mu r), \tag{11.0.308}$$

where, as immediate consequence of boundedness, we have $B = 0$ and therefore $R(r) = A\mathcal{J}_n(\mu r)$. For a non-trivial solution we have $A \neq 0$. Then the boundary data

$$R(a) = A\mathcal{J}_n(\mu a) = 0 \quad \Longrightarrow \quad \mathcal{J}_n(\mu a) = 0. \tag{11.0.309}$$

Let now α_k, $k = 1, 2, \ldots$ be the positive zeros for $\mathcal{J}_n(x)$. Then $\mu a = \alpha_k$, and thus we may write $\mu = \mu_k = \alpha_k/a$.
As for the solution for (11.0.307) we get

$$T = T_k(t) = a_k \sin \mu_k t + b_k \cos \mu_k t, \tag{11.0.310}$$

where $T_k(0) = 0 \Rightarrow b_k = 0$, $k = 1, 2, \ldots$. Thus we have $T_k(t) = a_k \sin \mu_k t$ and the superposition yields

$$w(r,t) = \sum_{k=1}^{\infty} a_k \sin \mu_k t \cdot \mathcal{J}_n(\mu_k r). \tag{11.0.311}$$

From (11.0.311) we get

$$w_t(r,t) = \sum_{k=1}^{\infty} a_k \mu_k (\cos \mu_k t) \cdot \mathcal{J}_n(\mu_k r), \tag{11.0.312}$$

Hence, by the initial data in (11.0.304) it follows that

$$w_t(r,0) = \sum_{k=1}^{\infty} a_k \mu_k \mathcal{J}_n(\mu_k r) = -\frac{\omega \mathcal{J}_n(\omega r)}{\mathcal{J}_n(\omega a)}. \tag{11.0.313}$$

$\{\mathcal{J}_n(\mu_k r)\}_{k=1}^{\infty}$ is an orthogonal basis for the function space $L_2^w(0,a)$, where $w(r)=r$. Thus $a_k\mu_k$ are the Fourier coefficient of $\dfrac{-\omega\mathcal{J}_n(\omega r)}{\mathcal{J}_n(\omega a)}$ with respect to the basis functions $\{\mathcal{J}_n(\mu_k r)\}_{k=1}^{\infty}$, and are computed viz,

$$a_k\mu_k = \frac{1}{\|\mathcal{J}_n(\mu_k r)\|_w^2}\int_0^a \frac{-\omega\mathcal{J}_n(\omega r)}{\mathcal{J}_n(\omega a)}\mathcal{J}_n(\mu_k r) r\,dr. \tag{11.0.314}$$

From theorem 41 we have

$$\|\mathcal{J}_n(\mu_k r)\|_w^2 = \frac{a^2}{2}\Big[\mathcal{J}_{n+1}(\alpha_k)\Big]^2, \tag{11.0.315}$$

which inserting in (11.0.314) yields

$$a_k = -\frac{2\omega}{a^2\mathcal{J}_{n+1}^2(\alpha_k)\mathcal{J}_n(\omega a)}\int_0^a \mathcal{J}_n(\omega r)\mathcal{J}_n(\frac{\alpha_k}{a}r) r\,dr. \tag{11.0.316}$$

The final solution is now given by:

$$u(r,t) = \tilde{u}(r,t) + w(r,t) = \frac{\mathcal{J}_n(\omega r)}{\mathcal{J}_n(\omega a)}\sin\omega t + \sum_{k=1}^{\infty} a_k \sin\frac{\alpha_k}{a}t\mathcal{J}_n(\frac{\alpha_k}{a}r). \tag{11.0.317}$$

Example 35. Solve the Laplace equation $\nabla^2 u = 0$ in the cylinder: $f = \sqrt{x^2+y^2} < R$, $0 < z < L$, where $u = 0$ for $z = 0$ and $z = L$, and $u = \sin\frac{\pi z}{L}(1-\cos\frac{z\pi}{L})$ for $r = R$.

Solution: We reformulate the problem for $u(r,z)$ in the cylindrical coordinates as

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} = 0, & 0 < r < R, \quad 0 < z < L, \\ u(r,0) = 0, & u(r,L) = 0, \\ u(R,z) = \sin\frac{\pi z}{L}(1-\cos\frac{z\pi}{L}). \end{cases} \tag{11.0.318}$$

Separation of variables: $u(r,z) = R(r)Z(z) \neq 0$ gives that

$$\Big(R'' + \frac{1}{r}R'\Big)Z + RZ'' = 0 \quad \Rightarrow \quad \frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = \lambda. \tag{11.0.319}$$

The eigenvalues and eigenfunctions for the Sturm-Liouville problem for Z:

$$\begin{cases} Z'' = -\lambda Z \\ Z(0) = Z(L) = 0 \end{cases} \quad \text{are} \quad \begin{cases} \lambda = \lambda_n = (\frac{n\pi}{L})^2, & n = 1,2,\dots \\ Z(z) = Z_n(z) = \sin\frac{n\pi}{L}z. \end{cases}$$

Therefore the differential equation for R is written as follows

$$R'' + \frac{1}{r}R' - \left(\frac{n\pi}{L}\right)^2 R = 0 \quad \Rightarrow \quad r^2R'' + rR' - \left(\frac{n\pi r}{L}\right)^2 R = 0. \quad (11.0.320)$$

(11.0.320) is a modified Bessel's differential equation, (the usual Bessel's differential equation $x \to ix$), with the general solution

$$R(r) = aI_0\left(\frac{n\pi r}{L}\right) + bK_0\left(\frac{n\pi r}{L}\right), \quad (11.0.321)$$

where

$$\mathcal{I}_\nu(x) = i^{-\nu}\mathcal{J}_\nu(ix) \quad \text{and} \quad \mathcal{K}_\nu(x) = \frac{\pi}{2}\frac{\mathcal{I}_{-\nu}(x) - \mathcal{I}_\nu(x)}{\sin \nu\pi} = \mathcal{Y}_\nu(ix). \quad (11.0.322)$$

That $R(r)$ is bounded as $r \to 0$ implies $b = 0$. Thus we may use the ansatz

$$u(r, z) = \sum_{n=1}^{\infty} a_n I_0\left(\frac{n\pi r}{L}\right) \sin \frac{n\pi z}{L}. \quad (11.0.323)$$

Using the boundary data we get

$$u(R, z) = \sum_{n=1}^{\infty} a_n I_0\left(\frac{n\pi R}{L}\right) \sin \frac{n\pi z}{L} = \sin \frac{\pi z}{L}\left(1 - \cos \frac{\pi z}{L}\right) = \sin \frac{\pi z}{L} - \frac{1}{2}\sin \frac{2\pi z}{L}.$$

Identifying the coefficients of $\sin \frac{n\pi z}{L}$ for $n = 1, 2, \dots$, it follows that

$$\begin{cases} a_1 I_0\left(\frac{\pi R}{L}\right) \sin \frac{\pi z}{L} = \sin \frac{\pi z}{L} & \Rightarrow a_1 I_0\left(\frac{\pi R}{L}\right) = 1, \\ a_2 I_0\left(\frac{2\pi R}{L}\right) \sin \frac{2\pi z}{L} = -\frac{1}{2}\sin \frac{2\pi z}{L} & \Rightarrow a_2 I_0\left(\frac{2\pi R}{L}\right) = -\frac{1}{2}, \\ a_n I_0\left(\frac{n\pi R}{L}\right) \sin \frac{n\pi z}{L} = 0 & \Rightarrow a_n I_0\left(\frac{n\pi R}{L}\right) = 0, \quad n \geq 3. \end{cases} \quad (11.0.324)$$

Thus

$$a_1 = \frac{1}{I_0(\frac{\pi R}{L})}, \qquad a_2 = -\frac{1}{2I_0(\frac{2\pi R}{L})}, \qquad a_n = 0, \quad n \geq 3. \quad (11.0.325)$$

Inserting the values of a_i, $i = 1, 2, \dots$ in (11.0.323) we get the desired solution

$$u(r, z) = \frac{I_0(\frac{\pi r}{L})}{I_0(\frac{\pi R}{L})} \sin\left(\frac{\pi z}{L}\right) - \frac{I_0(\frac{2\pi r}{L})}{2I_0(\frac{2\pi R}{L})} \sin\left(\frac{2\pi z}{L}\right). \quad (11.0.326)$$

Example 36. Find the polynomial $P(x)$ of at most degree 2 that minimizes the integral

$$\int_0^\infty \left[\sqrt{x} - P(x)\right]^2 e^{-x}\,dx. \tag{11.0.327}$$

Solution: We use the Laguerre polynomials, $\{L_n^\alpha\}_0^\infty$, which are orthogonal on $(0,\infty)$ with the weight function $w(x) = x^\alpha e^{-x}$ (in our problem $\alpha = 0$). By the completeness $\{L_n^\alpha\}_0^\infty$ every polynomial $P(x)$ of degree k, $k = 0, 1, 2, \ldots$ is a linear combination of L_0, L_1, $\ldots$, L_k. Thus let

$$P(x) = c_0 L_0(x) + c_1 L_1(x) + c_2 L_2(x) = \sum_{k=0}^{2} c_k L_k(x). \tag{11.0.328}$$

Then the integral is

$$\int_0^\infty \left[\sqrt{x} - \sum_{k=0}^{2} c_k L_k(x)\right]^2 e^{-x}\,dx = \|\sqrt{x} - \sum_{k=0}^{2} C_k L_k(x)\|^2_{w(x)}, \tag{11.0.329}$$

is minimal only if C_k are the Laguerre coefficients for $\sqrt{x}$ with respect to the weight function $w(x) = e^{-x}$:

$$C_k = \frac{1}{\rho_k}\int_0^\infty \sqrt{x} L_k(x) e^{-x}\,dx = \int_0^\infty \sqrt{x} L_k(x) e^{-x}\,dx, \tag{11.0.330}$$

where, for the Gamma functions, we used the property, $\Gamma(z+1) = x\Gamma(z)$, to compute

$$\rho_k = \frac{1}{\|L_k\|_w^2} = \frac{n!}{\Gamma(n+\alpha+1)}\bigg|_{\alpha=0} \frac{n!}{\Gamma(n+1)} = \frac{n!}{n!} = 1. \tag{11.0.331}$$

We recall a few first Laquerre polynomials for $\alpha = 0$,

$$L_0(x) = 1, \quad L_1(x) = 1 - x \quad \text{and} \quad L_2(x) = 1 - 2x + \frac{x^2}{2}. \tag{11.0.332}$$

Thus, to compute C_k, $k = 0$, 1, 2, we need to compute integrals of the form

$$I_n = \int_0^\infty x^{\frac{1}{2}} x^n e^{-x}\,dx = \int_0^\infty x^{n+\frac{3}{2}-1} e^{-x}\,dx, = \Gamma(n+\frac{3}{2}) = (n+\frac{1}{2})\Gamma(n+\frac{1}{2}),$$

which we compute for $n = 0, 1, 2$, viz,

$$I_0 = \int_0^\infty \sqrt{x} e^{-x}\, dx = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi},$$
$$I_1 = \int_0^\infty \sqrt{x}\, x e^{-x}\, dx = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi},$$
$$I_2 = \int_0^\infty \sqrt{x}\, x^2 e^{-x}\, dx = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}\cdot\frac{3}{2}\Gamma(\frac{3}{2}) = \frac{5}{2}\cdot\frac{3}{4}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}.$$

These integrals give us

$$\begin{aligned} C_0 &= \int_0^\infty \sqrt{x}, e^{-x}\, dx = I_0 = \frac{1}{2}\sqrt{\pi}, \\ C_1 &= \int_0^\infty \sqrt{x}(1-x)e^{-x}\, dx = I_0 - I_1 = \frac{1}{2}\sqrt{\pi} - \frac{3}{2}\sqrt{\pi} = -\frac{1}{4}\sqrt{\pi}, \\ C_2 &= \int_0^\infty \sqrt{x}(1-2x+\frac{x^2}{2})e^{-x}\, dx = I_0 - 2I_1 + \frac{1}{2}I_2 \\ &= (\frac{1}{2} - \frac{6}{4} + \frac{15}{16})\sqrt{\pi} = -\frac{1}{16}\sqrt{\pi}. \end{aligned} \tag{11.0.333}$$

Finally, inserting these values in (11.0.328) we get

$$P(x) = \sqrt{\pi}\Big[\frac{1}{2}\cdot 1 - \frac{1}{4}(1-x) - \frac{1}{16}(1-2x+\frac{x^2}{2})\Big] = \frac{\sqrt{\pi}}{16}(3+6x-\frac{1}{2}x^2).$$

Example 37. Find the polynomial $P(x)$ of at most degree 2 that minimizes the integral

$$I = \int_{-\infty}^\infty \Big[x^4 - P(x)\Big]^2 e^{-x^2/2}\, dx. \tag{11.0.334}$$

Solution: To solve the problem we use Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{11.0.335}$$

which are orthogonal on the real line $(-\infty, \infty)$. First we substitute $\frac{x}{\sqrt{2}} = s$, to get

$$I = \int_{-\infty}^\infty \Big[x^4 - P(x)\Big]^2 e^{-x^2/2}\, dx = \sqrt{2}\int_{-\infty}^\infty \Big[4s^4 - Q(s)\Big]^2 e^{-s^2}\, ds, \tag{11.0.336}$$

where $Q(s) = P(\sqrt{2}s)$. Now since the function s^4 is even and the Hermits polynomials $\{H_n\}_0^\infty$ are orthogonal on $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$ we may expand $Q(s)$ in even Hermits polynomials viz,

$$Q(s) = P(\sqrt{2}s) = \sum_{k=0}^{1} C_k H_{2k}(s). \tag{11.0.337}$$

Thus the integral (11.0.334) would become minimum only if C_k, $k = 0, 1$ are the Hermite coefficients of $4s^4$.

$$C_k = \frac{1}{\rho_k}\int_{-\infty}^{\infty} H_{2k}(s)\cdot 4s^4 e^{-s^2}\,ds = \frac{4}{\rho_k}\int_{-\infty}^{\infty} H_{2k}(s)s^4 e^{-s^2}\,ds, \tag{11.0.338}$$

where $\rho_k = (2k)!\,2^{2k}\sqrt{\pi}$. Hence, omitting the details, the coefficients are

$$C_0 = \frac{4}{\sqrt{\pi}}\int_{-\infty}^{\infty} 1\cdot s^4 e^{-s^2}\,ds = 3 \quad\text{and}\quad C_1 = \frac{4}{8\sqrt{\pi}}\int_{-\infty}^{\infty} 2s\cdot s^4 e^{-s^2}\,ds = 3.$$

Inserting in C_0 and C_1 in (11.0.337) we have that

$$P(\sqrt{2}s) = 3H_0(s)+3H_1(s) = 3\cdot 1+3(4s^2-2) = 3(4s^2-1) \Rightarrow P(x) = 3(2x^2-1).$$

Note that computing the above integrals for $C_i, , i = 1, 2$ are rather tedious. An alternative way is to proceed through a Hermit expansion for $f(x) = x^{2m}$:

$$x^{2m} = \sum_{k=0}^{m} \frac{(2m)!H_{2k}(x)}{2^{2m}(2k)!(m-k)!}. \tag{11.0.339}$$

For our case $f(x) = x^4 = x^{2m}$ yields $m = 2$. Thus we get

$$C_0 = 4\cdot\frac{41}{2^4(0!)2!} = 3 \quad\text{and}\quad C_1 = 4\cdot\frac{41}{2^4(2!)1!} = 3, \tag{11.0.340}$$

which are the same as above.

Example 38. Determine the polynomial $P(x)$ of at most degree 2 that minimizes the integral

$$\int_{-\infty}^{\infty}\left[e^{x/4} - P(x)\right]^2 xe^{-x}\,dx. \tag{11.0.341}$$

Solution I: To solve the problem we use the Laguerre polynomials L_n^α, which are orthogonal on the interval $(0,\infty)$ with the weight function $x^\alpha e^{-x}$. (Here, with the weight function xe^{-x} we have $\alpha = 1$).

$$L_n^{(1)}(x) = \frac{x^{-1}e^x}{n!}\cdot\frac{d^n}{dx^n}\Big(x^{n+1}e^{-x}\Big). \tag{11.0.342}$$

Below we compute the Laguerre polynomials up to degree 2:

$$\begin{aligned}
L_0^{(1)}(x) &= x^{-1}e^x \cdot xe^x = 1\\
L_1^{(1)}(x) &= x^{-1}e^x \cdot \frac{d}{dx}(x^2e^{-x}) = x^{-1}e^x(2xe^{-x} - x^2e^{-x}) = 2 - x\\
L_2^{(1)}(x) &= \frac{x^{-1}e^x}{2!}\cdot\frac{d^2}{dx^2}(x^3e^{-x}) = \frac{1}{2}x^{-1}e^x(3x^2e^{-x} - x^3e^{-x})'\\
&= \frac{1}{2}x^{-1}e^x(6x - 3x^2 - 3x^2 + x^3) = 3 - 3x + \frac{1}{2}x^2.
\end{aligned} \tag{11.0.343}$$

We also compute the norm of the Laguerre polynomials $L_n^{(1)}(x)$,

$$\|L_n^{(1)}(x)\|_w^2 = \int_0^\infty \Big[L_n^{(1)}(x)\Big]^2 xe^{-x}\,dx = \frac{\Gamma(n+2)}{n!} = \frac{(n+1)!}{n!} = n+1,$$

where we used Theorem 52 with $\alpha = 1$. Now let

$$P(x) = \sum_{n=0}^{2} C_n L_n^{(1)}(x). \tag{11.0.344}$$

Then the integral (11.0.341) would be minimal if and only if C_n are the Fourier coefficients of $e^{x/4}$, expanded with respect to the Laguerre bases functions $L_n^{(1)}(x)$, $n = 0,1,2$.

$$C_n = \frac{1}{\|L_n^{(1)}(x)\|_w^2}\int_0^\infty e^{x/4}L_n^{(1)}(x)xe^{-x}\,dx = \frac{1}{n+1}\int_0^\infty L_n^{(1)}(x)xe^{-\frac{3}{4}x}\,dx.$$

Note that with the substitution $y = \frac{3}{4}x$ we have

$$\begin{aligned}
I_m &= \int_0^\infty x^m e^{-\frac{3}{4}x}\,dx = \int_0^\infty \left(\frac{4}{3}\right)^m y^m e^{-y}\frac{4}{3}\,dy\\
&= \left(\frac{4}{3}\right)^{m+1}\Gamma(m+1) = \left(\frac{4}{3}\right)^{m+1} m!.
\end{aligned} \tag{11.0.345}$$

Now for $n = 0, 1, 2$ it follows, from (11.0.345), that

$$\begin{aligned}
C_0 &= \frac{1}{0+1}\int_0^\infty xe^{-\frac{3}{4}x}\,dx = I_1 = \Big(\frac{4}{3}\Big)^2 1! = \frac{16}{9},\\
C_1 &= \frac{1}{1+1}\int_0^\infty (2-x)xe^{-\frac{3}{4}x}\,dx = \frac{1}{2}[2I_1 - I_2] = \frac{16}{9} - \frac{1}{2}\Big(\frac{4}{3}\Big)^3(2!)\\
&= \frac{16}{9} - \frac{64}{27} = -\frac{16}{27},\\
C_2 &= \frac{1}{2+1}\int_0^\infty (3 - 3x + \frac{1}{2}x^2)xe^{-\frac{3}{4}x}\,dx = \frac{1}{3}[3I_1 - 3I_2 + \frac{1}{2}I_3]\\
&= I_1 - I_2 + \frac{1}{6}I_3 \quad = \frac{16}{9} - \Big(\frac{4}{3}\Big)^3(2!) + \frac{1}{6}\Big(\frac{4}{3}\Big)^4(3!) = \frac{16}{81}.
\end{aligned}$$

Inserting these values for C_i, $i = 0, 1, 2$ in (11.0.345) it follows that

$$\begin{aligned}
P(x) &= C_0L_0^{(1)}(x) + C_1L_1^{(1)}(x) + C_2L_2^{(1)}(x)\\
&= \frac{16}{9}\cdot 1 - \frac{16}{27}(2-x) + \frac{16}{81}(3 - 3x + \frac{1}{2}x^2) = \frac{8}{81}(x^2+12).
\end{aligned} \tag{11.0.346}$$

Solution II: We could use Theorem 54 and derive the expansion of the function $f(x) = e^{-bx}$, $b > 0$ in a series of Laguerre polynomials with $\alpha = 1$, viz

$$e^{-bx} = \Big(\frac{1}{b+1}\Big)^{\alpha+1}\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\Big(\frac{b}{b+1}\Big)^n. \tag{11.0.347}$$

Actually (11.0.347) can be proved for $b \geq -1/2$. (We have omitted the details in deriving (11.0.347) and refer the reader to the exercises of this section). In our case we have $b = -1/4$ and $\alpha = 1$. Thus using (11.0.347) we get

$$e^{x/4} = \Big(\frac{1}{-\frac{1}{4}+1}\Big)^2\sum_{n=0}^{\infty} L_n^{(1)}(x)\Big(\frac{-\frac{1}{4}}{-\frac{1}{4}+1}\Big)^n = \Big(\frac{4}{3}\Big)^2\sum_{n=0}^{\infty} L_n^{(1)}(x)\Big(-\frac{1}{3}\Big)^n.$$

This would yield an optimal approximation of $e^{x/4}$ in $L_2^w(0,\infty)$, with $w(x) = xe^{-x}$, with a polynomial of degree two consisting of the first three Laguerre polynomials: i.e.,

$$\begin{aligned}
P(x) &= \Big(\frac{4}{3}\Big)^2\Big[L_0^{(1)}(x)\Big(-\frac{1}{3}\Big)^0 + L_1^{(1)}(x)\Big(-\frac{1}{3}\Big)^1 + L_2^{(1)}(x)\Big(-\frac{1}{3}\Big)^2\Big] \\
&= \frac{16}{9}L_0^{(1)}(x) - \frac{16}{27}L_1^{(1)}(x) + \frac{16}{81}L_2^{(1)}(x) \\
&= \frac{16}{9}\cdot 1 - \frac{16}{27}(2-x) + \frac{16}{81}(3 - 3x + \frac{1}{2}x^2) \\
&= \frac{8}{81}(x^2+12).
\end{aligned}$$

Example 39. Determine a polynomial of the form $P(x) = x^3 + ax^2 + bx + c$, which minimizes the integral

$$\int_0^1 [P(x)]^2\,dx. \tag{11.0.348}$$

Solution I: The integral (11.0.348) is over a bounded interval. To solve this problem we use Legendre polynomials, which are orthogonal in $L_2(-1,1)$. Therefore we employ the substitution $t = 2x - 1$ which transfers $x \in [0,1]$ into $t \in [-1,1]$. Thus, since $[P(x)]^2$ is an even function it follows that

$$\begin{aligned}
\int_0^1 [P(x)]^2\,dx &= \frac{1}{2}\int_{-1}^1 P\Big[\frac{1}{2}(t+1)\Big]^2\,dt \\
&= \frac{1}{2}\Big(\frac{1}{8}\Big)^2\int_{-1}^1 \Big[t^3 - (At^2 + Bt + C)\Big]^2\,dt := \frac{1}{128}J.
\end{aligned} \tag{11.0.349}$$

Then J is minimal when

$$J = \int_{-1}^1 \Big[t^3 - \{C_0P_0(t) + C_1P_1(t) + C_2P_2(t)\}\Big]^2\,dt, \tag{11.0.350}$$

where $P_0(t) = 1$, $P_1(t) = t$ and $P_2(t) = \frac{1}{2}(3t^2-1)$ are Legendre polynomials, and C_k, $k = 0,1,2$, are the Fourier coefficient of t^3, expanded on the orthogonal bases $\{P_0(t), P_1(t), P_2(t)\}$ for polynomials of degree ≤ 2:

$$C_k = \frac{1}{\|P_k(t)\|^2}\int_{-1}^1 t^3 P_k(t)\,dt, \quad k = 0,1,2, \quad \text{where} \quad \|P_k(t)\|^2 = \frac{2}{2k+1}.$$

Thus we have the Fourier coefficients

$$\begin{cases} C_0 = \dfrac{1}{\|P_0(t)\|^2}\displaystyle\int_{-1}^{1} t^3 P_0(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^3\cdot 1\,dt = 0, \\ C_1 = \dfrac{3}{2}\displaystyle\int_{-1}^{1} t^3\cdot t\,dt = 3\int_0^1 t^4\,dt = \frac{3}{5}, \\ C_2 = \dfrac{5}{2}\displaystyle\int_{-1}^{1} t^3\frac{1}{2}(3t^2-1)\,dt = 0, \quad \text{odd function !} \end{cases} \tag{11.0.351}$$

Thus J is minimal when

$$P\Big[\frac{1}{2}(t+1)\Big] = \frac{1}{8}\Big[t^3 - \{C_0P_0(t) + C_1P_1(t) + C_2P_2(t)\}\Big] = \frac{1}{8}\Big(t^3 - \frac{3}{5}t\Big). \tag{11.0.352}$$

Consequently, with $t = 2x-1$, we finally get

$$\begin{aligned} P(x) &= P\Big[\frac{1}{2}(t+1)\Big] = \Big(\frac{1}{2}t\Big)^3 - \frac{3}{8\cdot 5}t = \Big(\frac{1}{2}(2x-1)\Big)^3 - \frac{3}{40}(2x-1) \\ &= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}. \end{aligned} \tag{11.0.353}$$

Solution II: An alternative way to solve this problem is to let

$$P(x) = x^3 + ax^2 + bx + c = x^3 - Q(x). \tag{11.0.354}$$

Then the substitution $t = 2x-1$ yields

$$I = \int_0^1 [P(x)]^2\,dx = \int_0^1 [x^3 - Q(x)]^2\,dx = \frac{1}{2}\int_{-1}^{1}\Big[\Big(\frac{t+1}{2}\Big)^3 - Q\Big(\frac{t+1}{2}\Big)\Big]^2\,dt.$$

Now we rewrite the second degree polynomial $Q\Big(\frac{t+1}{2}\Big)$ as

$$Q\Big(\frac{t+1}{2}\Big) = A_0P_0(t) + A_1P_1(t) + A_2P_2(t), \tag{11.0.355}$$

where $P_k(t)$ are Legendre polynomials. Then the integral I would be minimal if and only if A_k are the Fourier coefficients of $[(t+1)/2]^3$ in the Legendre expansion for polynomials of degree ≤ 2:

$$A_k = \frac{2k+1}{2}\int_{-1}^{1}\Big(\frac{t+1}{2}\Big)^3 P_k(t)\,dt. \tag{11.0.356}$$

Substituting $x = \frac{t+1}{2}$ we have $dt = 2\,dx$ and hence

$$A_k = \frac{2k+1}{2} \cdot 2 \int_0^1 x^3 P_k(2x-1)\,dx = (2k+1) \int_0^1 x^3 P_k(2x-1)\,dx.$$

Thus the Fourier coefficients for $k = 0, 1, 2$ are

$$\begin{cases} A_0 = 1 \cdot \displaystyle\int_0^1 x^3 \cdot 1\,dx = \frac{1}{4} \\ A_1 = 3 \displaystyle\int_0^1 x^3(2x-1)\,dx = 3 \int_0^1 (2x^4 - x^3)\,dx = \frac{9}{20} \\ A_2 = 5 \displaystyle\int_0^1 x^3 \frac{1}{2}[(2x-1)^2 - 1]\,dx = \frac{5}{2} \int_0^1 x^3(12x^2 - 12x + 2)\,dx = \frac{1}{4}. \end{cases}$$

It follows that

$$\begin{aligned} Q(x) &= A_0 P_0(2x-1) + A_1 P_1(2x-1) + A_2 P_2(2x-1) \\ &= \frac{1}{4} \cdot 1 + \frac{9}{20}(2x-1) + \frac{1}{4} \cdot \frac{1}{2}[3(2x-1)^2 - 1] \\ &= \frac{3}{2}x^2 - \frac{3}{5}x + \frac{1}{20}, \end{aligned} \tag{11.0.357}$$

and finally the polynomial $P(x)$, which minimize the integral in (11.0.348) is:

$$P(x) = x^3 - Q(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}. \tag{11.0.358}$$

Example 40. Show that

$$\int_0^1 x P_{2m}(x)\,dx = \frac{1}{3} \begin{pmatrix} 3/2 \\ m+1 \end{pmatrix}. \tag{11.0.359}$$

Solution: Recall the generating formula for Legendre polynomials:

$$\sum_{n=0}^{\infty} P_n(x) z^n = (1 - 2xz + z^2)^{-1/2}, \quad -1 \le x \le 1, \quad |z| < 1. \tag{11.0.360}$$

By (11.0.360) and partial integration we have that

$$\begin{aligned}
\sum_{n=0}^{\infty}\Big\{\int_0^1 xP_n(x)\,dx\Big\}z^n &= \int_0^1 \frac{x}{\sqrt{1-2xz+z^2}}\,dx \\
&= \Big[x\Big(-\frac{1}{z}\Big)(1-2xz+z^2)^{1/2}\Big]_{x=0}^{1} + \frac{1}{z}\int_0^1 \sqrt{1-2xz+z^2}\,dx \\
&= -\frac{1}{z}\sqrt{1-2xz+z^2} + \frac{1}{z}\Big[-\frac{1}{2z}\cdot\frac{2}{3}(1-2xz+z^2)^{3/2}\Big]_{x=0}^{1} \\
&= -\frac{1}{z}(1-z) - \frac{1}{3z^2}\underbrace{(1-2z+z^2)^{3/2}}_{=(1-z)^3} + \frac{1}{3z^2}(1+z^2)^{3/2} \\
&= -\frac{1}{3z^2}(1-z)\underbrace{[3z+(1-z)^2]}_{=1+z+z^2} + \frac{1}{3z^2}(1+z^2)^{3/2}, \{\text{mini-telescoping}\}, \\
&= -\frac{1}{3z^2}(1-z^3) + \frac{1}{3z^2}(1+z^2)^{3/2} = \frac{1}{3}z + \frac{1}{3z^2}\Big[(1+z^2)^{3/2}-1\Big].
\end{aligned}$$

Now since $(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ for $-1<x<1$, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\Big\{\int_0^1 xP_n(x)\,dx\Big\}z^n &= \frac{1}{3}z + \frac{1}{3z^2}\Big[\sum_{k=0}^{\infty}\binom{3/2}{k}z^{2k} - 1\Big] \\
&= \frac{1}{3}z + \frac{1}{3z^2}\Big[\underbrace{\binom{3/2}{0}}_{=1} + \sum_{n=1}^{\infty}\binom{3/2}{n}z^{2n} - 1\Big] \\
&= \frac{1}{3}z + \frac{1}{3z^2}\sum_{n=0}^{\infty}\binom{3/2}{n+1}z^{2(n+1)} \\
&= \frac{1}{3}z + \frac{1}{3}\sum_{n=0}^{\infty}\binom{3/2}{n+1}z^{2n}.
\end{aligned} \tag{11.0.361}$$

Identifying the coefficient for z^{2m}, in (11.0.361), completes the proof:

$$\int_0^1 xP_{2m}(x)\,dx = \frac{1}{3}\binom{3/2}{m+1}. \tag{11.0.362}$$

Example 41. The generating function of the Hermite,s polynomial H_n is given by

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}, \quad x \in \mathbb{R}, \quad z \in \mathbf{C}, \tag{11.0.363}$$

Use (11.0.363) to compute $H_n'(0)$.

Solution: Differentiating (11.0.363) with respect to x yields

$$\sum_{n=0}^{\infty} H_n'(x)\frac{z^n}{n!} = 2ze^{2xz-z^2}. \tag{11.0.364}$$

For $x = 0$ we get

$$\sum_{n=0}^{\infty} H_n'(0)\frac{z^n}{n!} = 2ze^{-z^2} = 2z\sum_{k=0}^{\infty}\frac{(-z^2)^k}{k!} = 2\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{k!}. \tag{11.0.365}$$

Identifying the coefficients imply that

$$H_n'(0)\frac{1}{n!} = \begin{cases} \dfrac{2(-1)^k}{k!}, & n = 2k+1, \quad k = 0, 1, \dots \\ 0, & n = 2k. \end{cases} \tag{11.0.366}$$

Thus we have

$$H_n'(0) = \begin{cases} \dfrac{2(2k+1)!(-1)^k}{k!}, & n = 2k+1, \quad k = 0, 1, \dots \\ 0, & n = 2k. \end{cases} \tag{11.0.367}$$

Example 42. Recall the generating function for the Laguerre polynomials:

$$\sum_{n=0}^{\infty} L_{n+1}^{\alpha}(x)z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}}, \qquad x > 0, \quad |z| < 1. \tag{11.0.368}$$

Show that

$$\frac{d}{dx}L_{n+1}^{\alpha}(x) = -L_n^{\alpha+1}(x). \tag{11.0.369}$$

Solution: Differentiating (11.0.368) with respect to x we get

$$\sum_{n=0}^{\infty}\frac{d}{dx}L_n^{\alpha}(x)z^n = -\frac{z}{1-z}\cdot\frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}}, \tag{11.0.370}$$

which can also be written (for $z \neq 0$) as

$$\sum_{n=0}^{\infty} \frac{d}{dx} L_n^{\alpha}(x) z^{n-1} = -\frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+2}}. \tag{11.0.371}$$

Now we may rewrite the equation (11.0.371) as

$$-\sum_{n=0}^{\infty} L_n^{\alpha+1}(x) z^n = \sum_{n=1}^{\infty} \frac{d}{dx} L_n^{\alpha}(x) z^{n-1} = \sum_{n=0}^{\infty} \frac{d}{dx} L_{n+1}^{\alpha}(x) z^n. \tag{11.0.372}$$

Identifying the coefficients for z^n we get the desired relation: (11.0.369).

Example 43. Solve the Laplace equation $\Delta u = 0$ in a spherical domain $x^2+y^2+z^2 < R^2$, and the boundary data $u = z(x^2+y^2)$, for $x^2+y^2+z^2 = R^2$.

Solution: In spherical coordinates we rewrite the Laplace equation as

$$\begin{cases} \Delta u = 0, & r < R, \\ u = z(x^2+y^2) = R^3 \cos\theta \cdot \sin^2\theta, & r = R. \end{cases} \tag{11.0.373}$$

Assuming rotationally invariant solution we may make a φ independent ansatz viz,

$$u(r,\theta) = \sum_{n=0}^{\infty} \Big(A_n r^n + B_n r^{-n-1}\Big) P_n(\cos\theta), \tag{11.0.374}$$

where P_n are Legendre polynomials.

Since u is bounded for $r = 0$ we have $B_n = 0$, for $n = 0, 1, \dots$ and thus

$$u(R,\theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = R^3 \cos\theta \sin^2\theta = R^3(\cos\theta - \cos^3\theta).$$

Let $\xi = \cos\theta$, then

$$u(R,\theta) = R^3(\xi - \xi^3). \tag{11.0.375}$$

Recall now the Legendre polynomials

$$P_1(\xi) = \xi \quad \text{and} \quad P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi), \tag{11.0.376}$$

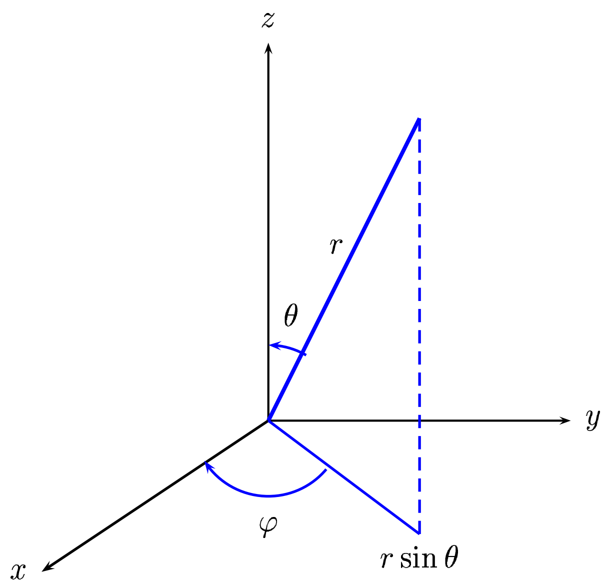


Figure 11.9: The spherical coordinates.

and compute ξ^3 as:

$$\frac{5}{2}\xi^3 = \frac{3}{2}P_1(\xi) + P_3(\xi) \quad \Rightarrow \quad \xi^3 = \frac{3}{5}P_1(\xi) + \frac{2}{5}P_3(\xi). \tag{11.0.377}$$

Thus we have

$$\xi - \xi^3 = P_1(\xi) - \frac{3}{5}P_1(\xi) - \frac{2}{5}P_3(\xi) = \frac{2}{5}P_1(\xi) - \frac{2}{5}P_3(\xi), \tag{11.0.378}$$

and consequently

$$u(R,\theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = \frac{2}{5}R^3 P_1(\xi) - \frac{2}{5}R^3 P_3(\xi). \tag{11.0.379}$$

Identification the coefficients it follows that

$$\begin{aligned} A_1 R = \frac{2}{5}R^3 \quad &\Rightarrow \quad A_1 = \frac{2}{5}R^2 \\ A_3 R^3 = -\frac{2}{5}R^3 \quad &\Rightarrow \quad A_3 = -\frac{2}{5} \\ &\quad\;\; A_n = 0, \quad n \neq 1, 3. \end{aligned} \tag{11.0.380}$$

Thus, using also $r\cos\theta = z$, we finally obtain the solution as

$$\begin{aligned} u(r,\theta) &= A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta) \frac{2R^2}{5} r P_1(\cos\theta) - \frac{2}{5} r^3 P_3(\cos\theta) \\ &= \frac{2R^2}{5} r\cos\theta - \frac{2}{5} r^3 \cdot \frac{5\cos^3\theta - 3\cos\theta}{2} = \frac{2R^2}{5} z - \frac{1}{5}(5z^3 - 3r^2 z) \\ &= \frac{2R^2}{5} z + \frac{3}{5}(x^2+y^2+z^2)z - z^3. \end{aligned}$$

Example 44. Solve the Laplace equation $\Delta u(r,\theta) = 0$, $0 < a < r < b$, (using spherical coordinates), with the boundary conditions given by

$$\begin{cases} u(r,\theta) = 1 + \cos\theta, & \text{for} \quad r = a \\ u(r,\theta) = \cos(2\theta), & \text{for} \quad r = b \end{cases} \tag{11.0.381}$$

Solution: A φ independent solution can be written as

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1} \right) P_n(\cos\theta), \tag{11.0.382}$$

where P_n are Legendre polynomials. Inserting (11.0.382) in the expressions for the boundary data yields

$$\begin{cases} u(a,\theta) = \sum_{n=0}^{\infty} \left(A_n a^n + B_n a^{-n-1} \right) P_n(\cos\theta) = 1 + \cos\theta \\ u(b,\theta) = \sum_{n=0}^{\infty} \left(A_n b^n + B_n b^{-n-1} \right) P_n(\cos\theta) = \cos(2\theta) = 2\cos^2\theta - 1. \end{cases}$$

We recall the few first Legendre polynomials,

$$P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta \quad \text{and} \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1),$$

and rewrite the boundary terms as

$$\begin{cases} u(a,\theta) = 1 + \cos\theta = P_0(\cos\theta) + P_1(\cos\theta), \\ u(b,\theta) = 2\cos^2\theta - 1 = C_0 P_0(\cos\theta) + C_2 P_2(\cos\theta) = C_0 + C_2 \frac{1}{2}(3\cos^2\theta - 1). \end{cases}$$

Identifying the coefficients of the last relation above we get

$$\begin{cases} C_0 \quad -\frac{C_2}{2} = -1 & \Rightarrow \quad C_0 = -1 + \frac{C_2}{2} = -1 + \frac{2}{3} = -\frac{1}{3}, \\ \frac{3}{2} C_2 = 2 & \Rightarrow \quad C_2 = \frac{4}{3}. \end{cases} \tag{11.0.383}$$

Identifying the coefficients of the Fourier-Legendre-series it follows that

$$n=0: \quad \begin{cases} A_0 + \dfrac{B_0}{a} = 1, \\ A_0 + \dfrac{B_0}{b} = -\dfrac{1}{3} \end{cases} \quad \Rightarrow \quad \begin{cases} A_0 = -\dfrac{3a+b}{3(b-a)}, \\ B_0 = \dfrac{4ab}{3(b-a)}. \end{cases} \tag{11.0.384}$$

$$n=1: \quad \begin{cases} A_1 a + \dfrac{B_1}{a^2} = 1, \\ A_1 b + \dfrac{B_1}{b^2} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} A_1 = -\dfrac{a^2}{b^3-a^3}, \\ B_1 = \dfrac{a^2b^3}{b^3-a^3}. \end{cases} \tag{11.0.385}$$

$$n=2: \quad \begin{cases} A_2 a^2 + \dfrac{B_2}{a^3} = 0, \\ A_2 b^2 + \dfrac{B_2}{b^3} = \dfrac{4}{3} \end{cases} \quad \Rightarrow \quad \begin{cases} A_2 = \dfrac{4b^3}{3(b^5-a^5)}, \\ B_2 = -\dfrac{4a^5b^3}{3(b^5-a^5)}. \end{cases} \tag{11.0.386}$$

$$n\geq 3: \quad \begin{cases} A_n a^n + \dfrac{B_n}{a^{n+1}} = 0, \\ A_n b^n + \dfrac{B_n}{b^{n+1}} = 0 \end{cases} \quad \Rightarrow \quad A_n = B_n = 0. \tag{11.0.387}$$

Inserting these coefficients in (11.0.382) we get the solution as

$$\begin{aligned} u(r,\theta) &= \Big(A_0 + \frac{B_0}{r}\Big) P_0(\cos\theta) + \Big(A_1 r + \frac{B_1}{r^2}\Big) P_1(\cos\theta) \\ &\quad + \Big(A_2 r^2 + \frac{B_2}{r^r}\Big) P_2(\cos\theta) \\ &= \frac{1}{3}\frac{1}{b-a}\Big(\frac{4ab}{r} - 3a - b\Big) + \frac{a^2}{b^3-a^3}\Big(\frac{b^3}{r^2} - r\Big)\cos\theta \\ &\quad + \frac{2}{3}\frac{b^3}{b^5-a^5}\Big(r^2 - \frac{a^5}{r^3}\Big)(\cos^2\theta - 1). \end{aligned} \tag{11.0.388}$$

Example 45. Find a solution to the problem

$$\begin{cases} u_t = k u_{xx}, & -\infty < x < \infty \quad t > 0, \\ u(x,0) = (1-2x^2)e^{-x^2}, & -\infty < x < \infty. \end{cases} \tag{11.0.389}$$

Solution: To solve this problem we use Fourier transform, with respect to x to obtain.

$$\hat{u}_t = k(i\xi)^2 \hat{u} = -k\xi^2 \hat{u}. \tag{11.0.390}$$

Hence,

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-k\xi^2 t}, \tag{11.0.391}$$

where

$$\hat{u}(\xi,0) \subset^{\mathcal{F}} (1-2x^2)e^{-x^2} := u_0(x). \tag{11.0.392}$$

Thus, we need to compute the Fourier transform of $u_0(x)$. Recall that

$$\mathcal{F}[e^{-ax^2/2}](\xi) = \sqrt{\frac{2\pi}{a}}e^{-\frac{\xi^2}{2a}}, \quad (a=2) \implies \mathcal{F}[e^{-x^2}](\xi) = \sqrt{\pi}e^{-\xi^2/4}.$$

Further, we compute

$$\begin{aligned}\mathcal{F}[-x^2e^{-x^2}](\xi) &= \mathcal{F}[(-ix)^2e^{-x^2}](\xi) = \Big(\frac{d}{d\xi}\Big)^2\mathcal{F}[e^{-x^2}](\xi) = \sqrt{\pi}(e^{-\xi^2/4})'' \\ &= \sqrt{\pi}\Big(-\frac{\xi}{2}e^{-\xi^2/4}\Big)' = \sqrt{\pi}\Big(\frac{\xi^2}{4}-\frac{1}{2}\Big)e^{-\xi^2/4}.\end{aligned}$$

Thus

$$u_0(x) = (1-2x^2)e^{-x^2} \supset^{\mathcal{F}} \sqrt{\pi}e^{-\xi^2/4} - 2\Big[\sqrt{\pi}\Big(\frac{\xi^2}{4}-\frac{1}{2}\Big)e^{-\xi^2/4}\Big] = \frac{\sqrt{\pi}}{2}\xi^2e^{-\xi^2/4}.$$

and hence (11.0.391) can be rewritten as

$$\hat{u}(\xi,t) = \frac{\sqrt{\pi}}{2}\xi^2e^{-(kt+\frac{1}{4})\xi^2}. \tag{11.0.393}$$

Let now $kt+\frac{1}{4} = \frac{1}{2a}$ and use the Fourier transform for $e^{-ax^2/2}$ to get

$$\begin{aligned}\hat{u}(\xi,t) &= \frac{\sqrt{\pi}}{2}\xi^2e^{-\xi^2/2a} = \frac{1}{2}\sqrt{\frac{a}{2}}\xi^2\sqrt{\frac{2\pi}{a}}e^{-\xi^2/2a} =\subset^{\mathcal{F}} \frac{1}{2}\sqrt{\frac{a}{2}}\Big(i\frac{d}{dx}\Big)^2\Big(e^{-ax^2/2}\Big) \\ &- \frac{1}{2}\sqrt{\frac{a}{2}}\Big(-axe^{-ax^2/2}\Big)' = -\frac{1}{2}\sqrt{\frac{a}{2}}(-a+a^2x^2)e^{-ax^2/2} \\ &= \Big(\frac{a}{2}\Big)^{3/2}(1-ax^2)e^{-ax^2/2}.\end{aligned}$$

Thus substituting back: $\frac{1}{2a} = kt+\frac{1}{4}$, i.e., $a = 2/(4kt+1)$, it follows that

$$\begin{aligned}u(x,t) &= \Big(\frac{1}{4kt+1}\Big)^{3/2}\Big(1-\frac{2x^2}{4kt+1}\Big)e^{-\frac{x^2}{4kt+1}} \\ &= \frac{1}{(4kt+1)^{5/2}}(4kt+1-2x^2)e^{-\frac{x^2}{4kt+1}}.\end{aligned} \tag{11.0.394}$$

Example 46. Solve the problem

$$\begin{cases} u_{xx}+u_{yy}=x, & 0<x<1 \qquad -\infty<y<\infty, \\ u_x(0,y)=0, & u(1,y)=ye^{-|y|} \end{cases} \tag{11.0.395}$$

Solution: To solve this problem we need to homogenize the equation: Let

$$u(x,y)=v(x,y)+S(x), \tag{11.0.396}$$

and determine $S(x)$, so that $S''(x)=x$, $S'(0)=S(1)=0$. It follows that $S'(x)=x^2/2+A$, where $S'(0)=0 \Rightarrow A=0$, i.e., $S(x)=x^3/6+B$. Now $S(1)=1/6+B=0 \Rightarrow B=-1/6$. Thus, we end up with

$$S(x)=\frac{x^3}{6}-\frac{1}{6}. \tag{11.0.397}$$

From this, the problem for v is:

$$\begin{cases} v_{xx}+v_{yy}=0, & 0<x<1, \qquad -\infty<y<\infty, \\ v_x(0,y)=0, & u(1,y)=ye^{-|y|}. \end{cases} \tag{11.0.398}$$

Now we Fourier transform the system (11.0.398) with respect to y, to obtain

$$\begin{cases} \hat{v}_{xx}-\omega^2\hat{v}=0, \\ \hat{v}_x(0,\omega)=0, \\ \hat{v}(1,\omega)=i\frac{d}{dw}\left(\frac{2}{1+\omega^2}\right)=-\frac{4i\omega}{(1+\omega^2)^2}. \end{cases} \tag{11.0.399}$$

where we have used the Fourier transforms $e^{-|y|} \supset^{\mathcal{F}} \frac{2}{1+\omega^2}$ and $xf(x) \supset^{\mathcal{F}} i(\hat{f})'(\xi)$. The general solution to the differential equation for $\hat{v}$ is written as

$$\hat{v}(x,\omega)=A(\omega)\cosh\omega x+B(\omega)\sinh\omega x. \tag{11.0.400}$$

Thus $\hat{v}_x(x,\omega)=\omega A(\omega)\sinh\omega x+\omega B(\omega)\cosh\omega x=0$ and $\hat{v}_x(0,\omega)=0+\omega B(\omega)=0 \Rightarrow B(\omega)=0$. Consequently,

$$\hat{v}(1,\omega)=A(\omega)\cosh\omega=-\frac{4i\omega}{(1+\omega^2)^2} \Rightarrow A(\omega)=-\frac{4i\omega}{(1+\omega^2)^2\cosh\omega}.$$

Inserting the values of $A(\omega)$ and $B(\omega)$ in (11.0.400) and using the Fourier inversion formula we get

$$v(x,y)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4i\omega\cosh\omega x}{(1+\omega^2)^2\cosh\omega}e^{i\omega y}\,d\omega=\frac{4}{\pi}\int_0^{\infty}\frac{\omega\cosh\omega x\sin\omega y}{(1+\omega^2)^2\cosh\omega}\,d\omega.$$

due to the fact that the expression $\frac{4i\omega\cosh\omega x}{(1+\omega^2)^2\cosh\omega}$ in the integrand is odd in ω. Hence we have the solution for the original problem (11.0.395) given as:

$$u(x,y)=\frac{x^3}{6}-\frac{1}{6}+\frac{4}{\pi}\int_0^\infty \frac{\omega\cosh\omega x\sin\omega y}{(1+\omega^2)^2\cosh\omega}\,d\omega. \qquad (11.0.401)$$

Example 47. Let $f\in L^2(\mathbb{R})$ and find a solution to the problem

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, & \infty<x<\infty, \quad 0<y<a,\\ u(x,0)=0, & u(x,a)=f(x).\end{cases} \qquad (11.0.402)$$

Show that

$$\int_{-\infty}^{\infty}|u(x,y)|^2\,dx\le\int_{-\infty}^{\infty}|f(x)|^2\,dx. \qquad (11.0.403)$$

Solution I: We Fourier transforming with respect to x: $\hat{u}(\xi,y)=\mathcal{F}_x\Big[u(x,y)\Big]$ to get the equation

$$(i\xi)^2\hat{u}+\frac{\partial^2\hat{u}}{\partial y^2}=0, \qquad \text{or} \quad \hat{u}_{yy}-\xi^2\hat{u}=0. \qquad (11.0.404)$$

(11.0.404) has the general solution

$$\hat{u}(\xi,y)=C_1(\xi)\sinh\xi y+C_2(\xi)\cosh\xi y. \qquad (11.0.405)$$

The boundary data is transformed as: $\hat{u}(\xi,0)=0$ and $\hat{u}(\xi,a)=\hat{f}(\xi)$. Hence $=C_2(\xi)=\hat{u}(\xi,0)=0$ and

$$\hat{f}(\xi)=\hat{u}(\xi,a)=C_1(\xi)\sinh\xi a \quad\Rightarrow\quad C_1(\xi)=\frac{\hat{f}(\xi)}{\sinh\xi a}. \qquad (11.0.406)$$

Thus we have

$$\hat{u}(\xi,y)=\frac{\sinh\xi y}{\sinh\xi a}\hat{f}(\xi). \qquad (11.0.407)$$

Using the Fourier inversion formula we get the solution viz,

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh\xi y}{\sinh\xi a}\hat{f}(\xi)e^{i\xi x}\,d\xi. \qquad (11.0.408)$$

Solution II: We can also use the Fourier transform

$$\mathcal{F}_t\Big[\frac{\sinh at}{\sinh bt}\Big](\omega) = \frac{\pi \sin \frac{\pi a}{b}}{b \cosh \frac{\pi\omega}{b} + b \cos \frac{\pi a}{b}}, \qquad 0 < a < b. \tag{11.0.409}$$

Let $a = y$ and $b = a$ and use the symmetry rule to get

$$\mathcal{F}\Big[\frac{1}{2a}\cdot\frac{\pi \sin \frac{\pi y}{a}}{\cosh \frac{\pi x}{a} + \cos \frac{\pi y}{a}}\Big] = \frac{\sinh \xi y}{\sinh \xi a}, \tag{11.0.410}$$

such that we can write the solution as

$$u(x,y) = \frac{1}{2a}\int_{-\infty}^{\infty} \frac{\pi \sin \frac{\pi y}{a}}{\cosh \frac{\pi(x-t)}{a} + \cos \frac{\pi y}{a}} f(t)\, dt. \tag{11.0.411}$$

To show (11.0.403) we use (11.0.407) and (twice) Plancherel's theorem, to get

$$\int_{-\infty}^{\infty} |u(x,y)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\xi,y)|^2\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\Big(\frac{\sinh \xi y}{\sinh \xi a}\Big)}_{\leq 1}{}^2 |\hat{f}(\xi)|^2\, d\xi$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

Example 48. Find a periodic solution for the equation

$$y'' - y' + y = f'(t), \tag{11.0.412}$$

where $f'(t)$ is the distribution derivative of the 2-periodic function

$$f(t) = \begin{cases} 0 & \text{for} \quad 0 < t \leq 1, \\ t-1 & \text{for} \quad 1 < t < 2. \end{cases} \tag{11.0.413}$$

Solution: Since the function f is 2-periodic we have $2L = 2$ and thus $L = 1$.
Then, for $0 < t < 2$, we compute the distribution derivative of f, viz

$$f'(t) = -\delta(t) + [\theta(t-1) - \theta(t-2)]. \tag{11.0.414}$$

Using the 2-periodic Fourier series expansion of the function f:

$$f(t) = \sum_{\infty}^{-\infty} C_n(f) e^{in\frac{\pi}{L}t} = \sum_{\infty}^{-\infty} C_n(f) e^{in\pi t}, \tag{11.0.415}$$

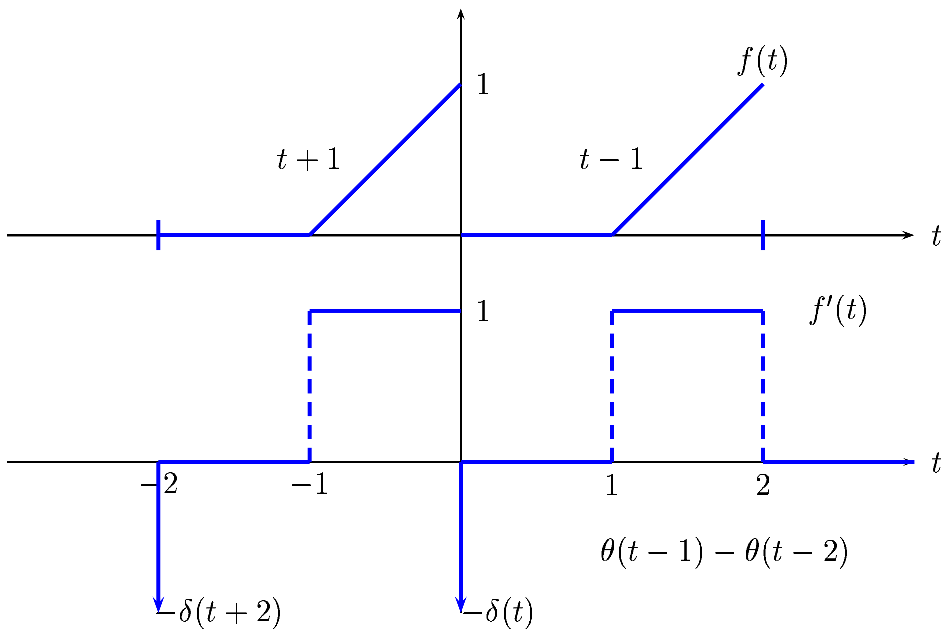


Figure 11.10: The function $f(t)$ and its distribution derivative.

The Fourier series expansion for the distribution derivative f' is given by

$$f'(t) = \sum_{\infty}^{-\infty} in\pi C_n(f) e^{in\pi t} = \sum_{\infty}^{-\infty} C'_n(f) e^{in\pi t}. \tag{11.0.416}$$

Thus

$$C'_n = in\pi C_n. \tag{11.0.417}$$

Hence, to determine C'_n we shall need the Fourier coefficients C_n below:

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-in\pi t}\, dt = \frac{1}{2}\int_{-1}^{0} (t+1) e^{-in\pi t}\, dt. \tag{11.0.418}$$

$$n = 0 \Longrightarrow C_0 = \frac{1}{2}\int_{-1}^{0} (t+1) dt = \frac{1}{2}\Big[\frac{(t+1)^2}{2}\Big]_{-1}^{0} = \frac{1}{4} \Longrightarrow C'_0 = 0. \tag{11.0.419}$$

For $n \neq 0$ we use (11.0.418) and partial integration to compute

$$\begin{aligned} C_n &= \frac{1}{2}\Big[(t+1)\frac{e^{-in\pi t}}{-in\pi}\Big]_{-1}^{0} - \frac{1}{2}\int_{-1}^{0}\frac{e^{-in\pi t}}{-in\pi}\,dt \\ &= \frac{1}{2}\cdot\frac{1}{-in\pi} - \frac{1}{2}\Big[\frac{e^{-in\pi t}}{(-in\pi)^2}\Big]_{-1}^{0} = -\frac{1}{2}\Big(\frac{1}{in\pi} + \frac{1}{(-in\pi)^2} - \frac{(-1)^n}{(-in\pi)^2}\Big) \\ &= \frac{1}{in\pi}\cdot\Big(-\frac{1}{2}\Big)\Big(1 + \frac{1}{in\pi} - \frac{(-1)^n}{in\pi}\Big). \end{aligned}$$

Thus, for $n \neq 0$, we have

$$C_n' = in\pi C_n = -\frac{1}{2}\Big(1 + \frac{1}{in\pi} - \frac{(-1)^n}{in\pi}\Big) = \frac{1}{2in\pi}\Big(-in\pi - 1 + (-1)^n\Big). \tag{11.0.420}$$

Summing up we have

$$n \neq 0, \quad C_n' = \frac{1}{2in\pi}\Big[-n\pi - i\Big((-1)^n - 1\Big)\Big] \quad \text{and} \quad C_0' = 0. \tag{11.0.421}$$

Now we let $y(t)$ be a 2-periodic function with a general Fourier series viz

$$y(t) = \sum_{-\infty}^{\infty} d_n e^{in\pi t}. \tag{11.0.422}$$

Thus

$$y'(t) = \sum_{-\infty}^{\infty} in\pi\, d_n e^{in\pi t} \qquad \text{and} \quad y''(t) = \sum_{-\infty}^{\infty} -n^2\pi^2\, d_n e^{in\pi t}. \tag{11.0.423}$$

Now inserting the expansions for y, y', y'', and f' in (11.0.412), it follows that

$$\sum_{-\infty}^{\infty} d_n\Big[1 - in\pi - n^2\pi^2\Big]ne^{in\pi t} = \sum_{-\infty}^{\infty} C_n' e^{in\pi t}. \tag{11.0.424}$$

Identifying the coefficients we get

$$d_n = \frac{C_n'}{1 - in\pi - n^2\pi^2} = \frac{1}{2n\pi}\cdot\frac{-n\pi - i\Big((-1)^n - 1\Big)}{1 - in\pi - n^2\pi^2} \quad \text{and} \quad d_0 = 0.$$

Consequently, the solution is given by

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n\pi}\cdot\frac{n\pi - i\Big(1 - (-1)^n\Big)}{n^2\pi^2 - 1 + in\pi} e^{in\pi t}. \tag{11.0.425}$$

Example 49. Determine the distribution derivative $f'(t)$ of the 3-periodic function

$$f(t) = \begin{cases} -1 & \text{for} \quad 0 < t < 1, \\ 1 & \text{for} \quad 1 < t < 3. \end{cases} \tag{11.0.426}$$

Expand $f'(t)$ in complex trigonometric Fourier series and use the result to compute the Fourier series expansion of $f(t)$ itself.

Solution: As we can see in the figure $f'(t)$ can be written as

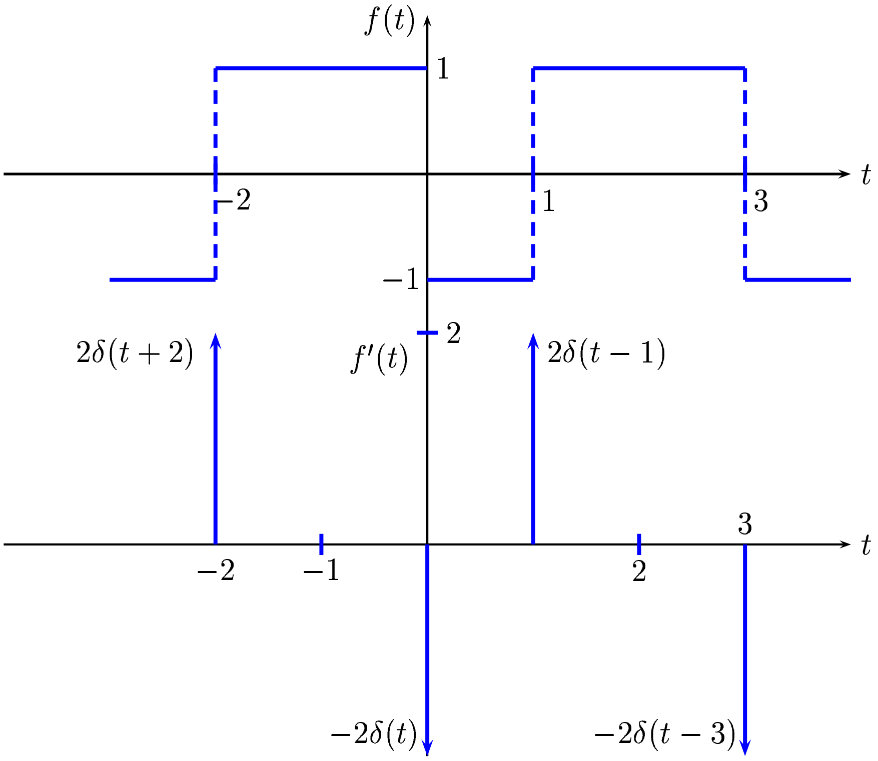


Figure 11.11: The function $f(t)$ and its distribution derivative.

$$f'(t) = \sum_{n=-\infty}^{\infty} 2\delta(t-3n-1) - \sum_{n=-\infty}^{\infty} 2\delta(t-3n) = 2\sum_{n=-\infty}^{\infty} [\delta(t-3n-1) - \delta(t-3n)].$$

Since $T = 3$, we have that $\Omega = 2\pi/T = 2\pi/3$. Then the Fourier series expansion of the function $f'(t)$ is:

$$f'(t) = \sum_{-\infty}^{\infty} C_n \, e^{in\frac{2\pi}{3}t}, \tag{11.0.427}$$

where the Fourier coefficients C_n are given by

$$\begin{aligned} C_n &= \frac{1}{3}\int_{-1}^{2} f'(t)\, e^{-in\frac{2\pi}{3}t}\, dt = \frac{1}{3}\int_{-1}^{2} [2\delta(t-1) - 2\delta(t)]e^{-in\frac{2\pi}{3}t} \\ &= \frac{2}{3}\Big(e^{-in\frac{2\pi}{3}t} - e^{0}\Big) = \frac{2}{3}\Big(e^{-in\frac{2\pi}{3}t} - 1\Big) \quad \Rightarrow \quad C_0 = 0. \end{aligned} \tag{11.0.428}$$

Thus the complex Fourier series expansion of f' in (11.0.427) is given by

$$f'(t) = \frac{2}{3}\sum_{-\infty, n\neq 0}^{\infty} \Big(e^{-in\frac{2\pi}{3}t} - 1\Big)\, e^{in\frac{2\pi}{3}t}. \tag{11.0.429}$$

Note that we separate $n = 0$, to determine the constant term in the Fourier series expansion for f, viz

$$f(t) = \sum_{-\infty, n\neq 0}^{\infty} \frac{C_n}{in\frac{2\pi}{3}}\, e^{in\frac{2\pi}{3}t} + A, \tag{11.0.430}$$

The coefficient A, corresponding to $n = 0$, is computed as

$$A = \frac{1}{3}\int_{0}^{3} f(t)e^{0}\, dt = \frac{1}{3}(-1+2) = \frac{1}{3}. \tag{11.0.431}$$

Hence, it follows that the complex Fourier series expansion of the function $f(t)$ is:

$$f(t) = \frac{1}{3} + \sum_{-\infty, n\neq 0}^{\infty} \frac{e^{-in\frac{2\pi}{3}t} - 1}{in\pi}\, e^{in\frac{2\pi}{3}t}. \tag{11.0.432}$$

Example 50. Compute the complex Fourier series of the 2π-periodic function $f(x) = x(x^2 - \pi^2)$ on the interval $(-\pi, \pi)$. What is the sum of this series at the points 2π and $3\pi/2$?

Solution I: Since the function $f(x)$ is 2π-periodic we have $\Omega = 2\pi/2\pi = 1$, and thus the complex Fourier series expansion for $f(x)$ is given by

$$f(x) = x(x^2 - \pi^2) = \sum_{-\infty}^{\infty} C_n \, e^{inx}, \tag{11.0.433}$$

with the Fourier coefficients

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)\, e^{-inx}\, dx. \qquad (11.0.434)$$

Thus for $n = 0$ we get

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(x^2-\pi^2)\, dx = 0, \qquad (11.0.435)$$

and for $n \neq 0$ using partial integration it follows that

$$\begin{aligned}
C_n &= \frac{1}{2\pi}\Big[x(x^2-\pi^2)\frac{1}{-in}e^{-inx}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(3x^2-\pi^2)\frac{1}{-in}e^{-inx}\, dx\Big] \\
&= -\frac{1}{2\pi}\Big[(3x^2-\pi^2)\frac{e^{-inx}}{(-in)^2}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6x\cdot\frac{e^{-inx}}{(-in)^2}\, dx\Big] \\
&= -\frac{1}{2\pi}(-6)\Big[x\frac{e^{-inx}}{(-in)^3}\Big|_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi}\frac{e^{-inx}}{(-in)^3}\, dx}_{=0}\Big] \\
&= \frac{3}{\pi}\cdot\frac{1}{(-in)^3}\Big[\pi e^{-in\pi} - (-\pi)e^{in\pi}\Big] = \frac{3}{(-in)^3}2\cos n\pi \\
&= \frac{6}{(-in)^3}(-1)^n = \frac{6(-1)^{n+1}}{n^3}\, i.
\end{aligned} \qquad (11.0.436)$$

Hence we have that

$$f(x) = x(x^2-\pi^2) \approx 6i \sum_{-\infty,\, n\neq 0}^{\infty} \frac{(-1)^{n+1}}{n^3} e^{inx}. \qquad (11.0.437)$$

Solution II: Since $f(x)$ is an odd function we have the Fourier coefficients

$a_n = 0$ and

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi x(x^2-\pi^2)\sin nx\,dx \\
&= \frac{2}{\pi}\underbrace{\left[x(x^2-\pi^2)\frac{-\cos nx}{n}\right]_0^\pi}_{=0} + \frac{2}{\pi n}\int_0^\pi (3x^2-\pi^2)\cos nx\,dx \\
&= \frac{2}{n\pi}\underbrace{\left[(3x^2-\pi^2)\frac{\sin nx}{n}\right]_0^\pi}_{=0} - \frac{12}{\pi n^2}\int_0^\pi x\sin nx\,dx \qquad (11.0.438)\\
&= -\frac{12}{\pi n^2}\left[x\frac{-\cos nx}{n}\right]_0^\pi - \frac{12}{\pi n^3}\underbrace{\int_0^\pi \cos nx\,dx}_{=0} \\
&= \frac{12\pi}{\pi n^3}\cos n\pi = \frac{12(-1)^n}{n^3}.
\end{aligned}
$$

Thus the complex Fourier series coefficients are given by

$$C_n = \frac{1}{2}(a_n - ib_n) = \frac{12(-1)^n}{n^3}\,i \qquad (11.0.439)$$

and we get the same complex Fourier series expansion for the function $f(x) = x(x^2-\pi^2)$ as in the *solution I* above.

Further, since the function $f(x)$ is 2π-periodic the sum of the Fourier series in $x = 2\pi$ is

$$f(2\pi) = f(0) = 0. \qquad (11.0.440)$$

For $x = 3\pi/2$ we compute the sum viz

$$f\left(\frac{3\pi}{2}\right) = f\left(\frac{-\pi}{2}\right) = \frac{-\pi}{2}\left(\frac{\pi^2}{4}-\pi^2\right) = \frac{3\pi^3}{8}. \qquad (11.0.441)$$

Example 51. Solve the following problem

$$
\begin{cases}
u_{xx} + 1 = \frac{1}{4}u_{tt}, & 0 < x < 2, \quad t > 0, \quad (DE)\\
u(0,t) = 0, & u(2,t) = -2, \quad (BC)\\
u_t(x,0) = 0, & u(x,0) = x - x^2. \quad (IC)
\end{cases} \qquad (11.0.442)
$$

Solution: To make the problem homogeneous we let

$$u(x,t) = w(x,t) + S(x), \qquad (11.0.443)$$

such that

$$\begin{cases} S''(x) = -1, \\ S(0) = 0, \qquad S(2) = -2. \end{cases} \tag{11.0.444}$$

Solving (11.0.444) we get

$$\begin{aligned} S''(x) = -1 \quad &\Rightarrow \quad S(x) = -\frac{x^2}{2} + Ax + B, \\ S(0) = 0 \quad &\Rightarrow \quad B = 0 \\ S(2) = -2 \quad &\Rightarrow \quad -2 + 2A = -2 \Rightarrow A = 0, \\ &\Rightarrow \quad S(x) = -\frac{x^2}{2}. \end{aligned} \tag{11.0.445}$$

Now we get a homogeneous equation for $w(x,t)$, viz

$$\begin{cases} w_{xx} = \frac{1}{4}w_{tt} = 0, & 0 < x < 2, \qquad t > 0, & (DE) \\ w(0,t) = 0, & w(2,t) = 0 & (BC) \\ w_t(x,0) = 0, & w(x,0) = u(x,0) - S(x) = x - \frac{x^2}{2}. & (IC) \end{cases} \tag{11.0.446}$$

Using separation of variables, $w(x,t) = X(x)T(t) \neq 0$, we get

$$\frac{X''}{X} = \frac{T''}{4T} = \lambda. \tag{11.0.447}$$

Thus we obtain Sturm-Liouville problems, with respect to both x and t. We start with the Sturm-Liouville problem in x:

$$\begin{cases} X'' = \lambda X, \\ X(0) = X(2) = 0. \end{cases} \tag{11.0.448}$$

For $\lambda < 0$ the general solution of this problem is given by

$$X(x) = A\cos\sqrt{-\lambda}\,x + = B\sin\sqrt{-\lambda}\,x. \tag{11.0.449}$$

The boundary data yields

$$\begin{cases} X(0) = A = 0, \\ X(2) = B\sin 2\sqrt{-\lambda} = 0 \quad \Rightarrow \quad 2\sqrt{-\lambda} = n\pi, \ (B \neq 0). \end{cases} \tag{11.0.450}$$

Hence the eigenvalues and eigenfunctions for (11.0.448) are given by

$$\lambda_n = -\frac{n^2\pi^2}{4}, \qquad \text{and} \quad X_n(x) = \sin\frac{n\pi}{2}x, \quad n = 1, 2, \ldots \tag{11.0.451}$$

As for the t dependent differential equation we get, for every n, that

$$T_n'' - 4\lambda_n T_n = 0, \tag{11.0.452}$$

with the general solution

$$T_n(t) = C_n \cos n\pi t + D_n \sin n\pi t. \tag{11.0.453}$$

To compute the coefficients we use the derivate of $T_n(t)$:

$$T_n'(t) = -n\pi C_n \sin n\pi t + n\pi D_n \cos n\pi t \Rightarrow T_n'(0) = n\pi D_n. \tag{11.0.454}$$

where the initial data

$$w_t(x,0) = 0 \quad \Rightarrow \quad T_n'(t) = n\pi D_n = 0 \quad \Rightarrow \quad D_n = 0. \tag{11.0.455}$$

Since now both the differential equation and the boundary conditions are homogeneous for w, thus

$$w(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} C_n \cos n\pi t \cdot \sin \frac{n\pi}{2} x. \tag{11.0.456}$$

is the general solution for (11.0.446). To compute C_n we use the initial data to get

$$w(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{2} x = x - \frac{x^2}{2}. \tag{11.0.457}$$

The equation (11.0.457) is valid through choosing C_n as the Fourier coefficients for the function $x - x^2/2$ with respect to the complete orthogonal system $X_n(x) = \{\sin \frac{n\pi}{2} x\}_{n=1}^{\infty}$. Thus

$$C_n = \frac{1}{M_n} \int_0^2 \left(x - \frac{x^2}{2} \right) \sin \frac{n\pi}{2} x \, dx, \tag{11.0.458}$$

where

$$M_n = \int_0^2 \sin \frac{n\pi}{2} x \, dx = 1. \tag{11.0.459}$$

Hence using partial integration we get

$$\begin{aligned}
C_n &= \int_0^2 \Big(x - \frac{x^2}{2}\Big) \sin \frac{n\pi}{2} x \, dx \\
&= \underbrace{\Big[- \Big(x - \frac{x^2}{2}\Big) \frac{\cos \frac{n\pi}{2} x}{\frac{n\pi}{2}} \Big]_0^2}_{=0} + \int_0^2 (1-x) \frac{\cos \frac{n\pi}{2} x}{\frac{n\pi}{2}} \, dx \\
&= \underbrace{\Big[- (1-x) \frac{\sin \frac{n\pi}{2} x}{\frac{n^2\pi^2}{4}} \Big]_0^2}_{=0} + \int_0^2 \frac{\sin \frac{n\pi}{2} x}{\frac{n^2\pi^2}{4}} = \Big[- \frac{1}{\frac{n^3\pi^3}{8}} \cos \frac{n\pi}{2} x \Big]_0^2 \qquad (11.0.460) \\
&= -\frac{(-1)^n}{\frac{n^3\pi^3}{8}} + \frac{1}{\frac{n^3\pi^3}{8}} = \frac{-(-1)^n + 1}{\Big(\frac{n\pi}{2}\Big)^3} = \begin{cases} 0, & n = 2k \\ \frac{16}{n^3\pi^3}, & n = 2k+1 \end{cases}
\end{aligned}$$

Thus the solution to the equation system (11.0.462) is given by

$$w(x,t) = \frac{16}{\pi^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \cos\Big((2k+1)\pi t\Big) \cdot \sin\Big(\frac{(2k+1)\pi}{2} x\Big). \qquad (11.0.461)$$

and finally we have the solution for the original problem (11.0.462), viz

$$u(x,t) = -\frac{x^2}{2} + \frac{16}{\pi^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \cos\Big((2k+1)\pi t\Big) \cdot \sin\Big(\frac{(2k+1)\pi}{2} x\Big).$$

Example 52. Solve the following Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{for} \quad r = \sqrt{x^2+y^2} < 1, \quad (u\text{bounded}) \\ u(1,\theta) := f(\theta) = \sin^2\theta + \cos\theta, & \text{in polar coordinates} \\ \quad u \text{ is bounded} & \text{as} \quad r \to 0. \end{cases} \qquad (11.0.462)$$

Solution: Let $u = u(r,\theta)$ in the spherical coordinates. Set the solution

$$u(r,\theta) = \sum_{n=0}^{\infty} \Big(A_n r^n + B_n r^{-n-1}\Big) P_n(\cos\theta). \qquad (11.0.463)$$

That u is bounded for $r = 0$ implies $B_n = 0$. Further

$$u(1,\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) = \sin^2\theta + \cos\theta = 1 + \cos\theta - \cos^2\theta. \qquad (11.0.464)$$

Thus with $s = \cos\theta$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_n P_n(s) &= 1 + s - s^2 = P_0(s) + P_1(s) - \frac{1}{3}\Big(P_0(s) + 2P_2(s)\Big) \\ &= \frac{2}{3}P_0(s) + P_1(s) - \frac{2}{3}P_2(s).\end{aligned} \tag{11.0.465}$$

Identifying the coefficients we get

$$A_0 = \frac{2}{3}, \qquad A_1 = 1, \qquad A_2 = -\frac{2}{3}, \qquad A_n = 0, \quad n > 2. \tag{11.0.466}$$

Consequently the solution is

$$\begin{aligned}u(r,\theta) &= A_0 P_0(\cos\theta) + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) \\ &= \frac{2}{3} + r\cos\theta - \frac{2}{3}r^2\frac{1}{2}(3\cos^2\theta - 1) \\ &= \frac{1}{3}(2 + r^2) + r\cos\theta - r^2\cos^2\theta.\end{aligned} \tag{11.0.467}$$

Example 53. The function $f(x)$ is continuous and has the Fourier transform

$$\hat{f}(\xi) = \frac{\ln(1+\xi^2)}{\xi^2}. \tag{11.0.468}$$

a) Determine the mass of f : $\int_{-\infty}^{\infty} f(x)\,dx$ **b)** Compute $f(0)$.

Solution: **a)** Note that

$$\hat{f}(\xi) = \frac{\ln(1+\xi^2)}{\xi^2} \quad \Rightarrow \quad \hat{f}(0) = \lim_{\xi\to 0}\frac{\ln(1+\xi^2)}{\xi^2} = 1. \tag{11.0.469}$$

Using the definition of the Fourier transform and (11.0.469) yields

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx \quad \Rightarrow \quad \hat{f}(0) = \int_{-\infty}^{\infty} f(x)\,dx = 1. \tag{11.0.470}$$

b) Recall the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\,d\xi. \tag{11.0.471}$$

Then for $x = 0$ we have, using partial integration, that

$$\begin{aligned} f(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\ln(1+\xi^2)}{\xi^2}\,d\xi = \frac{1}{\pi}\int_{0}^{\infty} \frac{\ln(1+\xi^2)}{\xi^2}\,d\xi \\ &= \frac{1}{\pi}\Big[-\frac{1}{\xi}\ln(1+\xi^2)\Big]_0^\infty + \frac{1}{\pi}\int_0^\infty \frac{1}{\xi}\cdot\frac{2\xi}{1+\xi^2}\,d\xi \\ &= \frac{1}{\pi}\Big[-\underbrace{\lim_{\xi\to\infty}\frac{\ln(1+\xi^2)}{\xi}}_{=0} + \lim_{\xi\to 0^+}\underbrace{\frac{\ln(1+\xi^2)}{\xi}}_{\xi\hat{f}(\xi)}\Big] + \frac{2}{\pi}\Big[\arctan\xi\Big]_0^\infty \\ &= \frac{1}{\pi}\lim_{\xi\to 0^+}\xi\hat{f}(\xi) + \frac{2}{\pi}\cdot\frac{\pi}{2} = 1. \end{aligned}$$

Thus we have $f(0) = 1$.

Example 54. Determine the solution $f(t)$, $t > 0$, for the integral equation

$$\begin{cases} f''(t) - 4f'(t) + f(t) + 6\displaystyle\int_0^t f(\tau)\,d\tau = 2e^t, \\ f(0_-) = 1, \quad f'(0) = 0. \end{cases} \tag{11.0.472}$$

Solution: We Laplace transform the equation to get

$$s^2F(s) - sf(0_-) - f'(0) - 4sF(s) + 4f(0_-) + F(s) + 6\,\frac{F(s)}{s} = \frac{2}{s-1}. \tag{11.0.473}$$

Then using the initial data yields

$$\Big(s^2 - 4s + 1 + \frac{6}{5}\Big)F(s) = s - 4 + \frac{2}{s-1}. \tag{11.0.474}$$

Thus we have

$$\begin{aligned} F(s) &= \frac{s-4+\frac{2}{s-1}}{s^2-4s+1+\frac{6}{5}} = \frac{s}{s-1}\cdot\frac{2+(s-1)(s-4)}{s^2-4s^2+s+6} \\ &= \frac{s}{s-1}\cdot\frac{(s^2-5s+6)}{(s+1)(s^2-5s+6)} = \frac{s}{(s-1)(s+1)} \\ &= \frac{1}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s+1}. \end{aligned} \tag{11.0.475}$$

Hence by the inverse Laplace transform we get

$$f(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t), \qquad t \geq 0. \tag{11.0.476}$$

Example 55. Let $u(x,t)$ be the solution for the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0, \qquad 0 < x < \pi, \qquad (DE) \\ u(0,t) = 0, & u(\pi,t) = 0, \qquad (BC) \\ u(x,0) = 0, & u_t(x,0) = g(x), \qquad (IC). \end{cases} \tag{11.0.477}$$

Show that for $t > 0$ we have

$$\int_0^\pi |u_t(x,t)|^2\,dx \le \int_0^\pi |g(x)|^2\,dx. \tag{11.0.478}$$

Solution: By the separation of variables, $u(x,t) = X(x)T(t) \neq 0$, the (DE) yields

$$\frac{X''}{X} = \frac{T''}{c^2T} = \lambda = -\mu^2, \quad \mu > 0. \tag{11.0.479}$$

($\lambda \geq 0$ gives trivial solutions). The Sturm-Liuville problem for X is

$$\begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(\pi) = 0, \end{cases} \tag{11.0.480}$$

with the general solution $X(x) = A\sin\mu x + B\cos\mu x$. Using the boundary data we get $X(0) = B = 0$ and $X(\pi) = A\sin\mu\pi = 0$. For a non-trivial solution we have $A \neq 0$ and thus $\mu = 1, 2, \ldots$, i.e., $\mu = n > 0$. Thus the eigenvalues and eigenfunctions for the problem (11.0.480) are

$$\lambda_n = -n^2 \quad \text{and} \quad X_n(x) = \sin nx, \quad n \geq 1. \tag{11.0.481}$$

Similarly, the Sturm-Liuoville problem for T would be

$$\begin{cases} T'' + c^2\mu^2 T = 0 \\ T(0) = 0, \quad T'(0) = g(x), \end{cases} \tag{11.0.482}$$

with the general solution $T_n n(t) = P_n \sin nct + Q_n \cos nct$. Using the initial data we get $T_n(0) = Q_n = 0$. Thus $T_n(t) = P_n \sin nct$ and by superpositioning

$$u(x,t) = \sum_{n=1}^{\infty} P_n \sin nct \cdot \sin nx \quad \Rightarrow \quad u_t(x,t) = \sum_{n=1}^{\infty} ncP_n \cos nct \cdot \sin nx.$$

Then invoking the second initial condition yields

$$u_t(x,0) = \sum_{n=1}^{\infty} ncP_n \sin nx = g(x). \tag{11.0.483}$$

Finally using Parseval's formula

$$\begin{aligned}\int_0^\pi |u_t(x,t)|^2\,dx &\le \sum_{n=1}^\infty |ncP_n|^2 \cos^2 nct \cdot \frac{\pi}{2} \le \sum_{n=1}^\infty (ncP_n)^2 \frac{\pi}{2} \\ &= \int_0^\pi |u_t(x,0)|^2\,dx = \int_0^\pi |g(x)|^2\,dx.\end{aligned} \tag{11.0.484}$$