Mathematics
Chalmers University of Technology
and Göteborgs University

TMA371 PARTIAL DIFFERENTIAL EQUATIONS, ASSIGNMENT 2A

1. Consider the heat equation:

$$\begin{aligned} u_t - \Delta u =& 0, \qquad x \in \Omega, \quad t > 0, \\ u =& 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) =& u_0, \qquad x \in \Omega. \end{aligned}$$

a) Show the following stability estimates:

$$\|u(t)\|^2 + \int_0^t \|\nabla u\|^2\, ds \leq \|u_0\|^2, \qquad t > 0,$$

$$\|\Delta u\| \leq \frac{1}{t}\|u_0\|, \qquad t > 0.$$

The latter estimate is the so called parabolic smoothing estimate (or strong stability), that describes the fact that the solution is smmother than the initial data (it gains regularity).
b) How do these estimates change if you substitute $u_{xx} + 4u_{yy}$ for Δu ?
c) Solve the problem with $\Omega = [0,1]$ using a Fourier series and study how fast the coefficients for the different Fourier modes decay. Prove the smoothing estimate by using this Fourier series representation of the exact solution.

2. Solve Problems 17.27 and 17.35 in the book.