A STOCHASTIC MODEL FOR COMPETING GROWTH ON $\mathbb{R}^d$

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March 2003

Abstract

A stochastic model, describing the growth of two competing infections on $\mathbb{R}^d$, is introduced. The growth is driven by outbursts in the infected region, an outburst in the type 1 (2) infected region transmitting the type 1 (2) infection to the previously uninfected parts of a ball with stochastic radius around the outburst point. The main result is that with the growth rate for one of the infection types fixed, mutual unbounded growth has probability zero for all but at most countably many values of the other infection rate. This is a continuum analog of a result of Häggström and Pemantle. We also extend a shape theorem of Deijfen for the corresponding model with just one type of infection.

Keywords: Spatial spread, Richardson's model, shape theorem, competing growth

AMS 2000 Subject Classification: Primary 60K35
Secondary 82B43.

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1 Introduction

In Deijfen (2002), a model is introduced that describes the random growth of an infected region in $\mathbb{R}^d$ by means of spherical outbursts in the infected region. The purpose of the present paper is to extend the work of Deijfen (2002) in two directions. First, we generalize the asymptotic shape theorem of that paper from bounded outburst radii to unbounded ones satisfying a certain moment condition; see Theorem 1.1. Second, we extend the model to encompass two competing types of infection, and prove a continuum analog of a result of Häggström and Pemantle (2000) concerning the impossibility of mutual unbounded growth; see Theorem 1.2.

The model in Deijfen (2002) can be viewed as a generalization to continuous space of the well-known Richardson model. The Richardson model, first introduced in Richardson (1973), describes growth on $\mathbb{Z}^d$. Sites can be either infected or uninfected: An uninfected site becomes infected at a rate proportional to the number of infected nearest neighbors and once infected it never recovers. In the continuum model the growth takes place by way of outbursts in the infected region, an outburst at an infected point causing a ball with stochastic radius around the outburst point to be infected. Hence, for all t the infected region at time t, denoted by S_t, is a union of randomly sized Euclidean balls. The dynamics is that, given S_t, the time until the next outburst occurs is exponentially distributed with expected value proportional to $|S_t|^{-1}$ and the outburst point is chosen uniformly in S_t. The main result in Deijfen (2002) is a shape theorem which asserts that if there is an upper bound for the radii of the outburst balls, then on the scale $1/t$ the set S_t has an asymptotic shape, which due to rotation invariance must be a Euclidean ball. The following result states that the conclusion of the shape theorem in Deijfen (2002) is still valid under the weaker assumption that the radius distribution, denoted by F, has a moment generating function. Here $B(x,r)$ is a closed ball with radius r centered at x.

Theorem 1.1 (Generalized shape theorem) *Fix $d \geq 1$ and consider the d-dimensional continuum growth model with rate λ. Assume that*

$$\int_0^\infty e^{-\varphi r} dF(r) < \infty \text{ for some } \varphi < 0 \tag{1}$$

and let $S_0 \subset \mathbb{R}^d$ be arbitrary but bounded with strictly positive Lebesgue measure. Then there is a real number $\mu > 0$ such that, for any $\varepsilon \in (0, \lambda\mu^{-1})$, almost surely

$$(1-\varepsilon)B\left(0, \lambda\mu^{-1}\right) \subset \frac{S_t}{t} \subset (1+\varepsilon)B\left(0, \lambda\mu^{-1}\right)$$

for all sufficiently large t.

In Häggström and Pemantle (1998) a two-type version of the Richardson model is introduced: Two particle types – type 1 and type 2 – compete for space on $\mathbb{Z}^d$, the dynamics being that an empty site becomes occupied by a type i

particle at a rate that is proportional to the number of nearest type i neighbors and an occupied site remains occupied forever. In this paper a similar two-type version of the continuum model is introduced. A careful description will be given in Section 3 but roughly the model is as follows: Two different non-reversible entities – which henceforth will be referred to as type 1 and type 2 infection – compete for space on $\mathbb{R}^d$. The infected region at time t can be divided in two disjoint sets S_t^1 and S_t^2 indicating the region occupied by type 1 infection and type 2 infection respectively. As in the one-type model the growth takes place by way of outbursts which infect the previously uninfected parts of a ball with random radius around the outburst point. The outbursts are of two types: Type 1 outbursts occur in the type 1 infected region and result in outburst balls of type 1, that is, the infection transmitted by the outburst is of type 1. Type 2 outbursts occur in the type 2 infected region and causes outburst balls loaded with type 2 infection. For $i = 1, 2$, given S_t^i the time until an outburst occurs in S_t^i is exponentially distributed with parameter $\lambda_i|S_t^i|$ and the outburst point is uniformly distributed over S_t^i. Note that if $\lambda_1 = \lambda_2$ and if we do not distinguish between the infection types this model is equivalent to the one-type model introduced in Deijfen (2002).

Consider the development of the infection in the two-type Richardson model. There are two possible scenarios:

1. One of the infection types is at some point surrounded by the other type, implying that only finitely many sites are ever occupied by the surrounded type.

2. Both infection types keep growing indefinitely.

Clearly the first scenario has positive probability but what about the second one? This issue is dealt with in Häggström and Pemantle (1998) and (2000). The main result in the first paper is that if $\lambda_1 = \lambda_2$ – that is, if the infection types are equal in power – and $d = 2$, then the event $G = \{$both infection types grow indefinitely$\}$ has positive probability. In the second paper the case $\lambda_1 \neq \lambda_2$ is considered and the main result is that if $\lambda_1 = 1$, then G has probability zero for all but at most countably many values of λ_2 (note that, by time-scaling, the assumption that $\lambda_1 = 1$ is no restriction). The main result in the present paper is that, for $d \geq 2$, this holds also in the two-type continuum model. This strongly suggests that G in fact has probability zero for *all* choices of $\lambda_2 \neq 1$. See Häggström and Pemantle (2000) for some intuitive reasoning behind this statement.

Before formulating the result we have to specify what the event "both infection types grow indefinitely" will mean in the continuum model. To this end, let

$$G_i = \{\text{the type } i \text{ infection reaches points arbitrarily far away from the origin}\}$$

and define $G = G_1 \cap G_2$ so that G hence is the event that both infection types reach arbitrarily far away from the origin simultaneously. If the outburst radius is bounded, the type 1 (2) infection is clearly prevented from growing any further if it is surrounded by a type 2 (1) layer whose thickness exceeds the upper bound for the size of an outburst. In this case the event G thus means that none of the infection types is enclosed by the other and so the concept of co-existence is similar to the lattice case. On the other hand, when the outburst radius is not bounded it is not possible to rule out G in finite time, indicating that we are dealing with a more subtle concept of co-existence.

It is not hard to show that the events G_1 and G_2 both have positive probability for all choices of λ_1 and λ_2 (see Proposition 5.1). The event $G = G_1 \cap G_2$ is more complicated to study. However, intuitively it is clear that G should not occur if $\lambda_1 \neq \lambda_2$: That both infection types reach arbitrarily far away from the origin means that some kind of balance of power reigns between the infection types and if one of them is more powerful than the other there is no reason to believe that such a balance should be possible. Our main result is a step on the way towards a formalization of this intuition. To formulate it some notation is needed. Let $P^{\lambda_1,\lambda_2}_{\Gamma_1,\Gamma_2}$ denote the probability law of the two-type growth process started at time zero from $S^1_0 = \Gamma_1$ and $S^2_0 = \Gamma_2$ and with infection rates λ_1 and λ_2 respectively. Our first result for the two-type model is that under the assumption that arbitrarily small outbursts are possible the particular choice of Γ_1 and Γ_2 is irrelevant in deciding whether the event G has positive probability or not.

Proposition 1.1 *Let (Γ_1,Γ_2) and (Γ'_1,Γ'_2) be two pairs of disjoint, bounded subsets of $\mathbb{R}^d$ with strictly positive Lebesgue measures. Furthermore, suppose that F has unbounded support and satisfies $F(\varepsilon) > 0$ for all $\varepsilon > 0$. Then*

$$P^{\lambda_1,\lambda_2}_{\Gamma_1,\Gamma_2}(G) > 0 \Rightarrow P^{\lambda_1,\lambda_2}_{\Gamma'_1,\Gamma'_2}(G) > 0.$$

Remark 1.1 The proof of Proposition 1.1 can easily be modified to cover the case with bounded support as well, provided the following (obviously necessary) condition on (Γ'_1,Γ'_2): If the radius distribution is bounded by r, then neither of Γ'_1 or Γ'_2 may contain an impenetrable layer of thickness r around the other.

In words, if we can find two sets Γ_1 and Γ_2 such that the event G has positive probability when starting from Γ_1 and Γ_2, then it follows that G has positive probability for all other initial sets Γ'_1 and Γ'_2 as well. Thus we may restrict our attention to the case when the process is started from two balls with radius given by the mean outburst radius, denoted by γ. The balls are placed next to each other, one of them centered at the point $-\mathbf{2}\gamma = (-2\gamma, 0, \ldots, 0)$ and the other at the origin. The notation for the probability law in this case is simplified by dropping the subscripts so that P^{λ_1,λ_2} hence denotes the law of the growth process started from $\Gamma_1 = B(-\mathbf{2}\gamma, \gamma)$ and $\Gamma_2 = B(0, \gamma)$. Furthermore, note that by time scaling we may assume that $\lambda_1 = 1$. The main result is as follows:

Theorem 1.2 *If F satisfies (1), then for $d \geq 2$ the set $\{\lambda_2;\ P^{1,\lambda_2}(G) > 0\}$ is countable.*

Thus $P^{1,\lambda_2}(G) = 0$ for all but at most countably many values of λ_2. As mentioned before, this strongly suggests that $P^{1,\lambda_2}(G) = 0$ for all $\lambda_2 \neq 1$. In the case $\lambda_2 = 1$ on the other hand, it seems reasonable to suspect that G has positive probability. That $\lambda_2 = 1$ means that the two infection types are equal in power and hence it should be possible for some kind of balance of power to arise. Let us summarize all this in the following conjecture:

Conjecture 1.1 *If (1) holds, we have* $\{\lambda_2;\ P^{1,\lambda_2}(G) > 0\} = \{1\}$ *for* $d \geq 2$.

The rest of the paper is organized as follows. Theorem 1.1 is proved in Section 2. Section 3 provides a closer description of the two-type continuum model and Section 4 contains a number of auxiliary results needed in the following sections. In Section 5 an analogue of the "key proposition" in Häggström and Pemantle (2000) is formulated and proved. This result will be of vital importance in the proof of Theorem 1.2 and due to the fact that the asymptotic shape for the continuum model is known to be a Euclidean ball, its proof is somewhat more appealing for the intuition compared to the lattice case. Theorem 1.2 is proved in Section 6 and Proposition 1.1 in Section 7.

2 The asymptotic shape

The aim in this section is to prove Theorem 1.1.

As described in the introduction, the growth in the one-type process is generated by stochastically sized spherical outbursts in the infected region S_t. Given the development of the infection up to time t, the time until an outburst occurs somewhere in S_t is exponentially distributed with parameter $\lambda|S_t|$ and the location of the outburst is uniformly distributed over S_t. To formally construct the model in d dimensions, a $(d+1)$-dimensional Poisson process with rate λ is used, the extra dimension representing time. A bounded set $\Gamma \subset \mathbb{R}^d$ with strictly positive Lebesgue measure is picked to initiate the growth and also, to the points in the Poisson process i.i.d. radius variables with distribution F are attached. Starting at time zero the growth is then brought about by following the cylinder $\Gamma \times \mathbb{R}$ upwards along the time axis until a point in the Poisson process is found. An outburst then takes place at this point and an infection ball B_1 with radius given by the radius variable associated with this particular Poisson point is created around the outburst point. The infected region in now given by $\Gamma \cup B_1$. Scanning within the cylinder $(\Gamma \cup B_1) \times \mathbb{R}$ further upwards along the time axis a new Poisson point is eventually hit and a new infection ball B_2 arises around this point. And so on. For a more thorough description of the construction we refer to Deijfen (2002).

The main result in Deijfen (2002) is the asymptotic shape result in Theorem 1.1 under the stronger assumption that the outbursts radii are bounded. (The result was formulated for $\lambda = 1$ only, but the general result follows by a simple time scaling argument.)

Let $\tilde{T}(x)$ be the time when the entire ball with radius γ around the point x is infected in a unit rate process started from $S_0 = B(0, \gamma)$, that is,

$$\tilde{T}(x) = \inf\{t;\ B(x, \gamma) \subset S_t\}.$$

The time constant μ is given by

$$\mu = \lim_{n \to \infty} \frac{\mathrm{E}[\tilde{T}(\mathbf{n})]}{n} = \lim_{n \to \infty} \frac{\tilde{T}(\mathbf{n})}{n}, \tag{2}$$

where $\mathbf{n} = (n, 0, \ldots, 0)$. The existence of the limit in (2) and the fact that it is an almost sure constant is proved in Deijfen (2002). The proof does not use the assumption of bounded support for the radius distribution and hence it applies also if this assumption is dropped. In fact, the only part of the proof of the shape theorem in Deijfen (2002) that uses the assumption of bounded support for F is the one that establishes that $\mu > 0$, that is, that the infection does not grow faster than linearly in time. Hence a weaker condition that guarantees at most linear growth could replace the bounded support assumption without weakening the conclusion of the theorem. We will show that existence of the moment generating function of F is sufficient for the growth to be at most linear. More precisely, we will show:

Proposition 2.1 *If $\int_0^\infty e^{-\varphi r} dF(r) < \infty$ for some $\varphi < 0$, then $\mu > 0$.*

In view of the above discussion, Theorem 1.1 follows once we have proved Proposition 2.1. The main ingredient in the proof of Proposition 2.1 is a "larger" growth process in which the outbursts constitute a spatial branching process. This process will be referred to as the Branching Random Walk growth process. We will show that the BRW process grows at most linearly in time and since it can be shown that the original growth process is stochastically dominated by the BRW process, the proposition follows from this. The time constant μ is defined based on a unit rate process and hence we consider only unit rate processes for the remainder of this section.

The BRW growth process works in a similar way as the original process, with outbursts that infects a randomly sized shape around the outburst point. In the BRW process though, each outburst point is assigned its own independent Poisson process to generate new outbursts in the surrounding outburst shape. Furthermore, for technical reasons we will take the outbursts in the BRW process to be cubes rather than spheres.

To formally construct the BRW growth process, let $\{N_n\}$ be a sequence of independent unit rate Poisson processes on $\mathbb{R}^{d+1}$. The extra dimension represents time and the points in the n:th process are denoted (X_k^n, T_k^n) where $X_k^n \in \mathbb{R}^d$ and T_k^n gives the location on the time axis. Also, to each Poisson point, associate independently a variable R_k^n with distribution F. Finally, for $S \subset \mathbb{R}^d$, let $N_n(S \times \mathbb{R})$ denote the restriction of N_n to $S \times \mathbb{R}$. The process now evolves at time points $\{T_n\}$ by aid of cubic outbursts with side length $\{2R_n\}$ centered at points $\{X_n\}$ obtained inductively as follows:

1. Define $X_0 = 0$, $T_0 = 0$ and $R_0 = \gamma$ and let C_n denote a cube in $\mathbb{R}^d$ with side length $2R_n$ centered at X_n.

2. Given $\{X_i;\ i \leq n\}$, $\{T_i;\ i \leq n\}$ and $\{R_i;\ i \leq n\}$, for $i = 0, \ldots, n$, let

$$\acute{T}_i^n = \inf_k \{T_k^i;\ T_k^i > T_n \text{ and } (X_k^i, T_k^i) \in N_i(C_i \times \mathbb{R})\}$$

   and define $T_{n+1} = \min_i\{\acute{T}_i^n\}$. The point X_{n+1} is the (a.s. unique) point in $\mathbb{R}^d$ such that $(X_{n+1}, T_{n+1}) \in N_i$ for some i and R_{n+1} is the side length variable associated with (X_{n+1}, T_{n+1}).

The infected region after n outbursts is obtained as $\bar{S}_{(n)} = \cup_{i=0}^n C_i$ and for $t \in [T_n, T_{n+1})$ the infected region at time t is given by $\bar{S}_t = \bar{S}_{(n)}$.

Remark 2.1 In the above construction the initial set $\bar{S}_0$ is a cube with side length 2γ centered at the origin. The notation $\bar{S}_t$ is reserved for the infected region at time t starting from this particular configuration. Furthermore, in the original one-type model, it will often be convenient to take $S_0 = B(0, \gamma)$ and the notation S_t is henceforth used to represent the infected region at time t starting from this particular choice of S_0.

The first result is a lemma stating that $\{S_t\}$ is stochastically dominated by $\{\bar{S}_t\}$.

Lemma 2.1 *The processes S_t and $\bar{S}_t$ can be coupled in such a way that $S_t \subset \bar{S}_t$ for all t.*

Proof: First note that by definition we have $S_0 = B(0, \gamma)$ and $\bar{S}_0 = C(0, 2\gamma)$, where $C(0, 2\gamma)$ is a cube with side length 2γ centered at the origin. Hence $S_0 \subset \bar{S}_0$. Let N_0 be the Poisson process used to generate $\{S_t\}_{t>0}$ and let $\{N_k\}_{k\geq 1}$ be a sequence of independent Poisson processes on $\mathbb{R}^{d+1}$ that are independent also of N_0. A process with the same distribution as $\{\bar{S}_t\}$ is obtained by starting from $C(0, 2\gamma)$ and then using $\cup_{i=0}^{n-1} N_k$ to generate new outbursts in regions that previously has been exposed to n outbursts. That is, in intersections between n outburst balls we scan within n independent Poisson processes, always including N_0, upwards along the time axis to find new outburst points. Note that this is a different (but equivalent) way of constructing the process compared to the above definition. Some thought reveals that with this construction, given that $S_{(n)} \subset \bar{S}_{(n)}$, it will also hold that $S_{(n+1)} \subset \bar{S}_{(n+1)}$. It follows by induction over n that $S_{(n)} \subset \bar{S}_{(n)}$ for all n and the lemma is thereby proved. $\square$

The outbursts in the BRW process satisfy the independence structure usually assumed in branching processes, and is in fact a branching process with no deaths of the Crump-Mode-Jagers type, see e.g. Chapter 6 in Jagers (1975) for a description of the general process. Note that the reproduction of the ancestor at the origin is slightly different from the other individuals reproduction in that $R_0 \equiv \gamma$, that is, the side length of the infection cube surrounding the ancestor is deterministic.

Before proving Proposition 2.1 we state a theorem by Biggins that will play a key role in the proof. To formulate the result, consider a one-dimensional general spatial branching process in which all individuals are equal, that is, all individuals are of the same type and the distribution of its progeny in space and time is the same. The reproduction of an individual is described by a point process Z on $\mathbb{R}\times\mathbb{R}^+$ with each point corresponding to a child. Let the intensity measure of Z be denoted ν and let $m(\varphi,\phi)$ be its Laplace transform, that is,

$$m(\varphi,\phi)=\int e^{-\varphi x-\phi t}\nu(dx,dt).$$

Define

$$\alpha(\varphi)=\inf\{\phi;\, m(\varphi,\phi)\leq 1\}.$$

Finally, write H_t for the position of the rightmost individual at time t.

Theorem 2.1 (Biggins 1995) *Assume that $\alpha(\varphi)<\infty$ for some $\varphi<0$. Then there is a constant $\zeta<\infty$ such that almost surely $H_t/t\to\zeta$ as $t\to\infty$.*

With this result at hand we are ready to prove Proposition 2.1.

Proof of Proposition 2.1: We want to prove that the infected region in the original process grows at most linearly in time. By Lemma 2.1 this follows if we can show that the growth of the BRW process is at most linear. To do this we will assume that the ancestor in the BRW growth process has the same reproduction as the other individuals, that is, we will assume that the process is started from a cube with random side length distributed according to F. Linear growth for such a process establishes linear growth also for a process with an ancestor surrounded by a deterministic cube. This can be seen as follows: Let $\bar{S}_t^{\Gamma}$ denote the infected region at time t in a BRW process started from an arbitrary initial set Γ and let $\bar{S}_t^{[\Gamma,s]}$ denote the region infected at time $t\geq s$ in a BRW process started at time s emanating from Γ. If τ denotes the time when the cube $C(0,2\gamma)$ is infected in a BRW process started from a cube with random side length R, where $R\sim F$, then clearly

$$\bar{S}_t^{[C(0,2\gamma),\tau]}\subset\bar{S}_t^{C(0,2R)}$$

for $t\geq\tau$ and since $\bar{S}_t^{[C(0,2\gamma),\tau]}$ has the same distribution as $\bar{S}_{t-\tau}$, linear growth for $\bar{S}_t^{C(0,2R)}$ guarantees linear growth also for $\bar{S}_t$.

To prove linear growth in a BRW growth process with i.i.d. reproductions for all individuals, including the ancestor, consider the projection of such a process on the first coordinate axis. Some thought reveals that this projection is a one-dimensional branching process in which each individual gives birth to children according to a Poisson process in time with rate $(2R)^d$, where R is a random variable with distribution F. The children are distributed uniformly in an interval of length $2R$ centered at the parent. Given that $R=r$ in such a process, we have $\nu(dx,dt)=(2r)^{d-1}dxdt$ and the Laplace transform becomes

$$\begin{aligned} m_r(\varphi,\phi) &= \int_0^\infty \int_0^{2r} (2r)^{d-1} e^{-\varphi x-\phi t} dx dt \\ &= (\varphi\phi)^{-1}(2r)^{d-1}\left(1-e^{-2\varphi r}\right). \end{aligned}$$

Integrating over r yields

$$m(\varphi,\phi) = (\varphi\phi)^{-1}\int_0^\infty (2r)^{d-1}\left(1-e^{-2\varphi r}\right) dF(r),$$

which implies that

$$\alpha(\varphi) = \varphi^{-1}\int_0^\infty (2r)^{d-1}\left(1-e^{-2\varphi r}\right) dF(r).$$

Hence $\alpha(\varphi) < \infty$ iff $\int_0^\infty e^{-\varphi r} dF(r) < \infty$. Assume that this is the case for some $\varphi < 0$ and let H_t^1 denote the position of the rightmost individual in the projected process. Then, by Theorem 2.1, there is a constant ζ such that $H_t^1/t \to \zeta$ as $t \to \infty$. Thus, if (1) holds, the BRW process grows at most linearly in time in the direction of the first coordinate axis. But the same reasoning can be applied to all coordinate axes: If H_t^i denotes the position of the rightmost individual in the projection of the BRW process on the i:th coordinate axis ($i = 1,\ldots,d$), we have that $H_t^i/t \to \zeta$ for each i as $t \to \infty$. Hence, for any $\varepsilon > 0$, on the scale $1/t$ the set of outbursts in the BRW growth process is contained in a cube with side length $\zeta + \varepsilon$ centered at the origin if t is sufficiently large. This means that the process grows at most linearly in time and Lemma 2.1 completes the proof. $\Box$

3 Construction of the two-type model

In this section the two-type model is built up more formally by the construction of a Markov process whose state at time t is a subset of $\mathbb{R}^d$ and consists of two disjoint sets S_t^1 and S_t^2. The process may for example be thought of as describing the growth of two competing germ colonies and the set S_t^1 (S_t^2) will be referred to as the type 1 (2) infected region.

To construct the model, let N_1 and N_2 be two independent Poisson processes on $\mathbb{R}^{d+1}$ with intensities λ_1 and λ_2 respectively. The extra dimension represents time and the points in N_i ($i = 1, 2$) are denoted (X_k^i, T_k^i), where $X_k^i \in \mathbb{R}^d$ and T_k^i gives the location on the time axis. Furthermore, to each point in the Poisson processes a random radius is associated. The radius variables are assumed to be i.i.d. with expected value γ. We will use the processes N_1 and N_2 together with the attached radius variables to construct three sequences $\{T_n\}$, $\{X_n\}$ and $\{R_n\}$ indicating the time points, locations and radii respectively of the outbursts, and two sequences $\{S_{(n)}^i\}$ ($i = 1, 2$) specifying the type i infected region after n outbursts. The intuition is as follows: At time zero a ball with radius γ around the point $-\mathbf{2}\gamma = (-2\gamma, 0, \ldots, 0)$ is infected with type 1 infection and a ball with radius γ around the origin is infected with type 2 infection so that $S^1_{(0)} = B(-\mathbf{2}\gamma, \gamma)$ and $S^2_{(0)} = B(0, \gamma)$. The growth is then brought about by scanning within the set $(S^1_{(0)} \cup S^2_{(0)}) \times \mathbb{R}$ upwards along the time axis until either $S^1_{(0)}$ hits a point in N_1 or $S^2_{(0)}$ hits a point in N_2. An outburst then takes place at this location infecting all points within some random distance from the outburst point. The type of the infection is determined by the region in which the outburst occurs: An outburst in the type i infected region generates outburst balls of type i. After the outburst the new infected region is given by $S^1_{(1)} \cup S^2_{(1)}$ where the infected region with the same infection type as the outburst might be enlarged compared to before the outburst and the other region is unchanged. Next we follow the set $(S^1_{(1)} \cup S^2_{(1)}) \times \mathbb{R}$ further upwards along the time axis and eventually one of the regions $S^1_{(1)}$ and $S^2_{(1)}$ hits a new point in N_1 or N_2 respectively. This causes a new outburst and the infected region is enlarged in the same way as described above. And so on.

Formally the sequences $\{T_n\}$ (time points for the outbursts), $\{X_n\}$ (locations of the outbursts), $\{R_n\}$ (radii of the outburst balls) and $\{S^i_{(n)}\}$ are constructed inductively as follows:

1. Define $T_0 = 0$, $S^1_{(0)} = B(-\mathbf{2}\gamma, \gamma)$ and $S^2_{(0)} = B(0, \gamma)$. Also, for $S \subset \mathbb{R}^d$, let $N_i(S \times \mathbb{R})$ denote the restriction of N_i to $S \times \mathbb{R}$.

2. Given T_n and $S^i_{(n)}$ $(i = 1, 2)$, define $T_{n+1} = \min\{\acute{T}^1_{n+1}, \acute{T}^2_{n+1}\}$, where

$$\acute{T}^i_{n+1} = \inf_k \{T^i_k;\ T^i_k > T_n \text{ and } (X^i_k, T^i_k) \in N_i(S^i_{(n)} \times \mathbb{R})\}.$$

   The point X_{n+1} is the (a.s. unique) point in $\mathbb{R}^d$ such that $(X_{n+1}, T_{n+1}) \in N_i$ for some i and R_{n+1} is the radius variable associated with the point (X_{n+1}, T_{n+1}).

3. Once the points T_{n+1}, X_{n+1} and R_{n+1} are specified, the infected regions $S^1_{(n)}$ and $S^2_{(n)}$ are updated as follows: If $(X_{n+1}, T_{n+1}) \in N_1$, that is, if the outburst is of type 1, then

$$\begin{cases} S^1_{(n+1)} = S^1_{(n)} \cup [B(X_{n+1}, R_{n+1}) \cap (S^1_{(n)} \cup S^2_{(n)})^c] \\ S^2_{(n+1)} = S^2_{(n)} \end{cases}$$

   so that, in words, the type 1 infected region is enlarged by the previously uninfected parts of the outburst ball $B(X_{n+1}, R_{n+1})$ and the type 2 infected region remains unchanged. If the outburst is of type 2, that is, if $(X_{n+1}, T_{n+1}) \in N_2$, then the type 2 infected region is updated analogously while the type 1 infected region is left unchanged.

For $t \in [T_n, T_{n+1})$ the type i infected region at time t is given by $S^i_t = S^i_{(n)}$ and the total infected region at time t is $S^1_t \cup S^2_t$.

Write Δ_{n+1}^i for the time counting from T_n until an outburst of type i occurs. By standard properties of the Poisson process, we have

$$\Delta_{n+1}^i|\mathcal{F}_n^i \sim \text{Exp}\{\lambda_i|S_{(n)}^i|\},$$

where $\mathcal{F}_n^i = \sigma(S_{(0)}^i, \ldots, S_{(n)}^i)$. Furthermore, given S_t^i the memoryless property of the exponential distribution implies that the time until an outburst occurs somewhere in S_t^i is exponentially distributed with parameter $\lambda_i|S_t^i|$ and also the location of the outburst is uniformly distributed over S_t^i.

To make sure that the model does not explode by generating infinitely many outbursts in finite time, we have to show that the sequence $\{T_n\}$ does not have a finite limit point. This is done in the following proposition, which is the analogue of Proposition 2.1 in Deijfen (2002):

Proposition 3.1 *If the radius distribution has finite moment of order d, then almost surely $T_n \to \infty$ as $n \to \infty$.*

Proof: Let $\{\Delta_n\}$ denote the increments of the process $\{T_n\}$, that is, $\Delta_n := T_n - T_{n-1}$ is the time between two successive outbursts regardless of type. Since $T_n = \sum_{k=1}^n \Delta_k$ the proposition follows if we can show that $\sum_{k=1}^\infty \Delta_k = \infty$ almost surely. To do this, note that

$$\Delta_k = \min\{\Delta_k^1, \Delta_k^2\}.$$

Since $\Delta_k^i|\mathcal{F}_{k-1}^i \sim \text{Exp}\{\lambda_i|S_{(k-1)}^i|\}$ it follows that

$$\Delta_k|\mathcal{F}_{k-1} \sim \text{Exp}\{\lambda_1|S_{(k-1)}^1| + \lambda_2|S_{(k-1)}^2|\},$$

where $\mathcal{F}_k = \sigma(S_{(0)}^1, \ldots S_{(k)}^1, S_{(0)}^2, \ldots S_{(k)}^2)$. Furthermore, by properties of the Poisson process, given $\mathcal{F}_{k-1}$ we can write

$$\Delta_k = \frac{k}{\lambda_1|S_{(k-1)}^1| + \lambda_2|S_{(k-1)}^2|} \cdot E_k$$

where $\{E_k\}$ are independent, $E_k \sim \text{Exp}(k)$. A trivial upper bound for $|S_{(k-1)}^i|$ $(i = 1, 2)$ is given by

$$|S_{(k-1)}^i| \leq v_0 + \sum_{n=1}^{k-1} V_n,$$

where v_0 is the volume of the initial type i γ-ball in $\mathbb{R}^d$ and V_n denotes the volume of a d-dimensional ball with radius R_n. Let $v = \text{E}[V_n]$, which is finite by the assumption of the proposition. By the strong law of large numbers,

$$\frac{1}{k}\sum_{n=1}^{k-1} V_n \to v \quad \text{as } k \to \infty$$

and hence, if k is large,

$$\frac{1}{k}|S^i_{(k-1)}| \leq 2v.$$

Thus, for large k,

$$\Delta_k \geq \frac{1}{(\lambda_1 + \lambda_2)2v} \cdot E_k$$

and we are done if we can show that $\sum_{k=1}^{\infty} E_k = \infty$. But this is an easy consequence of Kolmogorov's three series theorem: Let $\tilde{E}_k = E_k - \mathrm{E}[E_k] = E_k - 1/k$, so that $E_k = \tilde{E}_k + 1/k$. Since $\sum_{k=1}^{\infty} 1/k = \infty$ it suffices to show that $\sum_{k=1}^{\infty} \tilde{E}_k$ converges almost surely. Using the fact that $\sum_{k=1}^{\infty} \mathrm{E}[\tilde{E}_k^2] = \sum_{k=1}^{\infty} 1/k^2 < \infty$ this follows from the three series theorem. $\square$

4 Preliminaries

In this section we prove a number of auxiliary results needed in the later sections. The first two lemmas concern the relation between the one-type model and the two-type model.

Lemma 4.1 *Let S^2_t denote the type 2 infected region at time t in a two-type process with distribution $P^{1,\lambda}$ and let S_t denote the infected region at time t in a one-type process with rate λ. The two-type process and the one-type process can be coupled in such a way that $S^2_t \subset S_t$ for all t.*

Proof: Couple the two processes by letting the one-type process and the type 2 outbursts in the two-type process be generated by the same rate λ Poisson process with the same radius variables attached. If $S^2_{(n)} \subset S_{(n)}$, it then also holds that $S^2_{(n+1)} \subset S_{(n+1)}$. Since $S^2_0 = S_0$, the lemma follows by induction over n. $\square$

Lemma 4.2 *Consider the one-type process $\{S_t\}_{t\geq 0}$ with rate $\lambda \leq 1$ and the two-type process $\{S^1_t \cup S^2_t\}_{t\geq 0}$ with distribution $P^{1,\lambda}$. These can be coupled in such a way that*

$$S_t \subset S^1_t \cup S^2_t \tag{3}$$

for all t.

Proof: For $t = 0$, (3) is trivial. To couple the one-type and the two-type process, let N_1 and N_2 be two independent Poisson processes with rate $1 - \lambda$ and λ respectively. Use $N_1 \cup N_2$ to generate the type 1 outbursts in the two-type process and use N_2 to generate all outbursts in the one-type process and the type 2 outbursts in the two-type process. Note that the two-type process is obtained in a different way here compared to in Section 3, but it is easy to see that it gets the correct distribution. Also, it is easy to see that (3) is preserved for $t > 0$. $\square$

The next two lemmas are needed in the proof of Proposition 5.2. To formulate them, introduce a new, hampered version of the one-type process by placing "ceilings" and "floors" in $\mathbb{R}^d$ restricting the growth in all directions but one: Write $(x_1, \ldots, x_d)$ for the coordinates of a point $x \in \mathbb{R}^d$ and let S_t^b denote the infected region at time t in a one-type process where all infection outside the stripe $\Omega_b = \{x \in \mathbb{R}^d;\ |x_i| \leq b \text{ for all } i \geq 2\}$ is ignored. The process $\{S_t^b\}$ hence works exactly like the original one-type process except that points with $|x_i| \geq b$ for some $i \geq 2$ are immune to the infection. The following lemma says that Ω_b is filled with infection linearly in time.

Lemma 4.3 *Consider a hampered one-type process with unit rate. Assume that (1) holds and let $S_0^b \subset \Omega_b$ be bounded with strictly positive Lebesgue measure. Then, for any dimension d there is a real number $\mu_b > 0$ such that, for any $\varepsilon \in (0, \mu_b^{-1})$, almost surely*

$$(1-\varepsilon)\left\{x \in \Omega_b;\ |x_1| \leq t\mu_b^{-1}\right\} \subset S_t^b \subset (1+\varepsilon)\left\{x \in \Omega_b;\ |x_1| \leq t\mu_b^{-1}\right\}$$

for all sufficiently large t.

The proof of the lemma for the case of bounded outburst radii is a straightforward but tedious adaptation of the proof of the shape theorem in Deijfen (2002), and the general case follows as in Section 2. We therefore omit the proof.

Let $\tilde{T}^b(x)$ be the analogue of $\tilde{T}(x)$ in the process S_t^b, that is, $\tilde{T}^b(x)$ is the time when the γ-ball around the point x is infected in a unit rate hampered process started from $S_0 = B(0,\gamma)$. The time-constant μ_b is defined analogously to the time-constant for the unhampered process, that is,

$$\mu_b := \lim_{n\to\infty} \frac{\mathrm{E}[\tilde{T}^b(\mathbf{n})]}{n} = \lim_{n\to\infty} \frac{\tilde{T}^b(\mathbf{n})}{n},$$

where $\mathbf{n} = (n, 0, \ldots, 0)$. The following lemma states that as b becomes large the speed of the growth in the hampered process approaches the speed in the unhampered process.

Lemma 4.4 *As $b \to \infty$ we have $\mu_b \to \mu$.*

Proof: Trivially $\mu_b \geq \mu$ so it suffices to show that $\lim_{b\to\infty} \mu_b \leq \mu$. To this end, consider a one-type process with unit rate and pick $\delta > 0$ and $p \in (0,1)$. We will show that

$$P\left(\tilde{T}^b(k\mathbf{n}) > (1+\delta)\mu nk\right) \leq p \tag{4}$$

if n, k and b are large. Since $p > 0$ was arbitrary this implies that almost surely

$$\lim_{n\to\infty} \frac{\tilde{T}^b(\mathbf{n})}{n} \leq (1+\delta)\mu$$

for large b and since also $\delta > 0$ was arbitrary the proposition follows. To prove (4), first note that by Theorem 1.1 and (2) we have

$$\mathrm{E}[\tilde{T}(\mathbf{n})] \leq (1+\delta/3)\mu n, \tag{5}$$

if n is large. Define $D_n^b = \tilde{T}^b(\mathbf{n}) - \tilde{T}(\mathbf{n})$ and let F_n^b be the event that the hampered process S_t^b reaches $\partial\Omega_b$ before time $\tilde{T}^b(\mathbf{n})$. We will show that

(i) $P(F_n^b) \to 0$ as $b \to \infty$;

(ii) $\mathrm{E}[D_n^b | F_n^b] \leq cn$ for some constant $c \in \mathbb{R}$.

The claim (i) follows easily by noting that $P(F_n^b) \leq P(\|S_{\tilde{T}(\mathbf{n})}\| > b)$. Since almost surely $\tilde{T}(\mathbf{n}) < \infty$, Proposition 3.1 gives that $P(\|S_{\tilde{T}(\mathbf{n})}\| < \infty) = 1$ and hence $P(\|S_{\tilde{T}(\mathbf{n})}\| > b) \to 0$ as $b \to \infty$.

To establish (ii), write τ_b for the time when the infection reaches $\partial\Omega_b$ and note that

$$\mathrm{E}[D_n^b | F_n^b] \leq \mathrm{E}\left[\tilde{T}^b(\mathbf{n}) - \tau_b | F_n^b\right].$$

Now imagine that at time τ_b a new process is started from the origin using only outbursts that touch the x-axis, that is, at time τ_b all infection except a ball with radius γ around the origin is erased and the infection then evolves in time along the x-axis using the same $d+1$-dimensional Poisson process as the original process. Let $\tau_{\mathbf{n}}$ denote the time counting from τ_b when the γ-ball around the point $\mathbf{n}$ is infected in this new process. Since $\tilde{T}^b(\mathbf{n}) \leq \tau_b + \tau_{\mathbf{n}}$ we have

$$\mathrm{E}[D_n^b | F_n^b] \leq \mathrm{E}[\tau_{\mathbf{n}}]. \tag{6}$$

Using the same technique as in the proof of Lemma 3.1 in Deijfen (2002) it follows that the time until the point $\mathbf{n}$ is infected in the x-axis process can be bounded by a sum of $n\lceil 2\gamma^{-1}\rceil$ independent exponential variables with mean $\eta = \eta(d)$. Furthermore, it is not hard to see that the time from when the point $\mathbf{n}$ is infected until the entire γ-ball around $\mathbf{n}$ is infected can be bounded by a sum of $m = m(d)$ exponential variables, which may be defined so that their mean equals η. Hence $\mathrm{E}[\tau_{\mathbf{n}}] \leq cn$, where c can be taken as $\eta\lceil m + 2\gamma^{-1}\rceil$. The statement (ii) now follows from (6).

By (i) we can pick b large so that $P(F_n^b) \leq \mu\delta/3c$. Using (ii) and the fact that

$$\mathrm{E}[D_n^b] \leq P(F_n^b)\mathrm{E}[D_n^b | F_n^b],$$

it follows that, for such b, we have

$$\mathrm{E}[D_n^b] \leq \delta\mu n/3. \tag{7}$$

Now, if n is chosen large enough to ensure (5) and b large enough to ensure (7), then

$$\begin{aligned} \mathrm{E}[\tilde{T}^b(\mathbf{n})] &= \mathrm{E}[\tilde{T}(\mathbf{n}) + D_n^b] \\ &\leq (1+\delta/3)\mu n + \delta\mu n/3 \\ &= (1+2\delta/3)\mu n. \end{aligned} \tag{8}$$

It remains to show that this implies (4). To this end, let $\tilde{T}^b((j-1)\mathbf{n}, j\mathbf{n})$ denote the time it takes for the infection to invade the entire γ-ball around the point $j\mathbf{n}$ in a process started at time $\tilde{T}^b((j-1)\mathbf{n})$ emanating from the point $(j-1)\mathbf{n}$. The variables $\{\tilde{T}^b((j-1)\mathbf{n}, j\mathbf{n});\ j \geq 1\}$ are iid with expected value $\mathrm{E}[\tilde{T}^b(\mathbf{n})]$ and hence, by the strong law of large numbers, almost surely

$$\frac{1}{k}\sum_{j=1}^{k} \tilde{T}^b((j-1)\mathbf{n}, j\mathbf{n}) \to \mathrm{E}[\tilde{T}^b(\mathbf{n})] \quad \text{as } k \to \infty.$$

Using (8) this implies that

$$P\left(\sum_{j=1}^{k} \tilde{T}^b((j-1)\mathbf{n}, j\mathbf{n}) > (1+\delta)\mu n k\right) \leq p$$

if k is large. Since

$$\tilde{T}^b(k\mathbf{n}) \leq \sum_{j=1}^{k} \tilde{T}^b((j-1)\mathbf{n}, j\mathbf{n})$$

this proves (4). □

The next lemma is needed to prove Proposition 1.1. It involves the concept of *effective* outbursts: An outburst is said to be effective if it causes previously uninfected regions to be infected, that is, if it reaches outside the boundary of the infected region.

Lemma 4.5 *Assume that F satisfies (1) and let Λ be a bounded subset of $\mathbb{R}^d$.*

(a) *The number of effective outbursts that occur in Λ during the progress of the growth in a two-type process is almost surely finite.*

(b) *If in addition F has unbounded support and if $\Lambda^c \cap [S_0^1 \cup S_0^2]$ has positive Lebesgue measure, there is a positive probability that no effective outbursts ever occur in Λ.*

Remark 4.1 In analogy with Remark 1.1, Lemma 4.5(b) extends to the case with bounded radii provided that $\Lambda^c \cap [S_0^1 \cup S_0^2]$ is not "strangled" by $\Lambda \cap [S_0^1 \cup S_0^2]$ in the sense of Remark 1.1.

Proof of Lemma 4.5: By time-scaling and symmetry it is enough to consider a process with infection rates $(1, \lambda)$, where $\lambda \leq 1$. Furthermore, the choice of initial sets does not affect the arguments in the proof. Hence we may restrict our attention to a process with distribution $P^{1,\lambda}$.

(a) Write N_Λ for the number of effective outbursts in Λ. Lemma 4.2 and Theorem 1.1 implies that almost surely

$$S_t^1 \cup S_t^2 \supset B\left(0, \frac{\lambda\mu^{-1}t}{2}\right)$$

for all sufficiently large t. Hence the minimal distance between points in Λ and points in $(S_t^1 \cup S_t^2)^c$ is at least $\lambda\mu^{-1}t/4$ for all sufficiently large t. It follows that

$$\begin{aligned}
\mathrm{E}[N_\Lambda] &\leq |\Lambda| \int_0^\infty \left(\int_{\lambda\mu^{-1}t/4}^\infty dF(r) \right) dt \\
&= |\Lambda| \int_0^\infty \left(\int_0^{4r/\lambda\mu^{-1}} dt \right) dF(r) \\
&= \frac{4|\Lambda|}{\lambda\mu^{-1}} \int_0^\infty r dF(r).
\end{aligned}$$

The last integral is finite by the assumption on F and hence N_Λ is finite almost surely.

(b) The calculation in (a) shows that there is an r (depending on Λ) such that if $S_t^1 \cup S_t^2$ contains the ball $B(0,r)$, then the conditional expectation of the number of effective outbursts in Λ after time t is at most $1/2$. Let A denote the event that the ball $B(0,r)$ is fully infected before the first outburst in Λ. Then

$$\begin{aligned}
P(N_\Lambda = 0) &\geq P(N_\Lambda = 0|A)P(A) \\
&\geq \frac{1}{2}P(A),
\end{aligned}$$

which is clearly positive. $\square$

We will later on need the following refinement of Lemma 4.5.

Lemma 4.6 *Consider a two-type process with F satisfying (1). For any $\delta, \xi > 0$, there exists an $r_0^* < \infty$ such that for all $r^* \geq r_0^*$ we get that if the process is started with (S_0^1, S_0^2) satisfying*

$$S_0^2 \subset B(0, r^*) \quad \textit{and} \quad S_0^1 \cup S_0^2 \supset B(0, r^*(1+\xi)),$$

then

$$P(\textit{infection 2 never makes an effective outburst}) > 1 - \delta.$$

To prove Lemma 4.6 we need the following auxiliary result, which asserts that if the initial set in a two-type process is large, then the infection will continue to grow at least with the speed stipulated by the shape theorem for the weaker infection type.

Lemma 4.7 *For $\lambda \leq 1$ and any $\delta, \varepsilon \in (0,1)$, there is an $s < \infty$ such that if a two-type process with infection rates $(1, \lambda)$ is started in such a way that $S_0^1 \cup S_0^2 \supset B(0, s\lambda\mu^{-1})$, then*

$$P\left(S_t^1 \cup S_t^2 \supset B\left(0, (1-\varepsilon)(s+t)\lambda\mu^{-1}\right) \;\; \forall t \geq 0\right) > 1 - \delta.$$

Proof of Lemma 4.7: In view of Lemma 4.2 it is enough to prove the corresponding statement for the one-type process, that is, it is enough to prove that there is an $s < \infty$ such that if a one-type process with parameter λ is started with $S_0 \supset B(0, s\lambda\mu^{-1})$, then

$$P\left(S_t \supset B\left(0, (1-\varepsilon)(s+t)\lambda\mu^{-1}\right) \ \ \forall t \geq 0\right) > 1 - \delta. \tag{9}$$

To do this, consider a one-type process $\{S_t^*\}_{t\geq 0}$ with parameter λ and initial condition, say, $S_0 = B(0, \gamma)$. Define the event

$$A_t = \left\{B\Big(0, \Big(1 - \frac{\varepsilon}{2}\Big)t\lambda\mu^{-1}\Big) \subset S_t^* \subset B\Big(0, \Big(1 + \frac{\varepsilon}{2}\Big)t\lambda\mu^{-1}\Big)\right\},$$

and note that, by Theorem 1.1, there exists an $s < \infty$ such that

$$P\bigg(A_t \text{ holds for all } t \geq s\Big(1 - \frac{\varepsilon}{2}\Big)\bigg) > 1 - \delta. \tag{10}$$

Now couple the processes $\{S_t\}_{t\geq 0}$ and $\{S_t^*\}_{t\geq 0}$ in such a way that the latter is generated by the former's underlying Poisson process delayed by time $s(1-\varepsilon/2)$. If $S_0 \supset B(0, s\lambda\mu^{-1})$, on the event in (10) we get that $S_0 \supset S^*_{s(1-\varepsilon/2)}$ and, by the choice of the coupling, $S_t \supset S^*_{s(1-\varepsilon/2)+t}$ for all $t \geq 0$. This implies (9). $\Box$

Proof of Lemma 4.6: By time-scaling and symmetry it suffices to consider a process with infection rates $(1, \lambda)$, where $\lambda \leq 1$. For such a process it follows from Lemma 4.7 (with $\varepsilon = \min\{\frac{\xi}{4}, \frac{1}{4}\}$) that if r^* is taken to be sufficiently large, then

$$P\left(S_t^1 \cup S_t^2 \supset B\Big(0, r^*\Big(1 + \frac{\xi}{2}\Big) + (1-\varepsilon)t\lambda\mu^{-1}\Big) \ \forall t \geq 0\right) > 1 - \frac{\delta}{2}. \tag{11}$$

Let π_d be such that $\pi_d(r^*)^d$ is the volume of a d-dimensional ball with radius r^* and write N_{r^*} for the number of effective outbursts in $B(0, r^*)$. On the event in (11), we have that an effective outburst inside $B(0, r^*)$ at time t has to have radius at least $r^*\xi/2 + (1-\varepsilon)t\lambda\mu^{-1}$. Hence on the event in (11) we have

$$\begin{aligned}
\mathrm{E}[N_{r^*}] &\leq \pi_d(r^*)^d \int_0^\infty \left(\int_{r^*\xi/2+(1-\varepsilon)t\lambda\mu^{-1}}^\infty dF(r)\right) dt \\
&\leq \pi_d(r^*)^d \int_{r^*\xi/2}^\infty \left(\int_0^{r/(1-\varepsilon)\lambda\mu^{-1}} dt\right) dF(r) \\
&= \frac{\pi_d(r^*)^d}{(1-\varepsilon)\lambda\mu^{-1}} \int_{r^*\xi/2}^\infty r^{d+1} dF(r).
\end{aligned}$$

When $r \geq r^*\xi/2$, we have $(r^*)^d \leq 2^d r^d/\xi^d$. Thus

$$\mathrm{E}[N_{r^*}] \leq \frac{\pi_d 2^d}{(1-\varepsilon)\lambda\mu^{-1}\xi^d} \int_{r^*\xi/2}^\infty r^{d+1} dF(r).$$

The last integral is finite by the assumption on F and hence $\mathrm{E}[N_{r^*}]$ can be made arbitrarily small by taking r^* large. Take r^* large enough so that $\mathrm{E}[N_{r^*}]$ is at most 1/2 and such that (11) holds. Then with probability at least $1-\delta$ the type 2 infection never makes an effective outburst, as desired. $\Box$

5 A key proposition

In this section we formulate and prove an analogue of Proposition 2.2 in Häggström and Pemantle (2000). The proposition will play a key role in the proof of Theorem 1.2 and as in Häggström and Pemantle (2000), the proof is rather lengthy and technical; this appears to be unavoidable. We will opt for a geometrical argument that is a bit different from the one of Häggström and Pemantle. The unboundedness of the outbursts radii causes some extra complications in our case, but on the other hand the fact that the asymptotic shape is a sphere makes the geometric intuition a bit more accessible.

We begin by observing that the events G_1 and G_2 have positive probability.

Proposition 5.1 *If F has unbounded support, then, for all infection rates (λ_1, λ_2) and all choices of initial sets (Γ_1, Γ_2) which are bounded and have positive Lebesgue measure, we have $P^{\lambda_1,\lambda_2}_{\Gamma_1,\Gamma_2}(G_i) > 0$ for $i = 1, 2$.*

Remark 5.1 Proposition 5.1 extends to the case of bounded radii as well, provided that neither Γ_1 nor Γ_2 surrounds the other in the sense of Remark 1.1.

Proof of Proposition 5.1: Since the total infected region increases to cover all of $\mathbb{R}^d$ we have $P^{\lambda_1,\lambda_2}_{\Gamma_1,\Gamma_2}(G_1 \cup G_2) = 1$. Furthermore, by Lemma 4.5(b), there is a positive probability that no effective outbursts ever occur in S^1_0. Hence $P^{\lambda_1,\lambda_2}_{\Gamma_1,\Gamma_2}(G_2) > 0$. Similarly it can be seen that $P^{\lambda_1,\lambda_2}_{\Gamma_1,\Gamma_2}(G_1) > 0$. $\Box$

The proof of Theorem 1.2 is based on the fact that if both infection types will survive in the long run they have to grow equally fast, that is, the asymptotic speed of the growth for the type 1 and the type 2 infection have to be the same. This is formulated in Lemma 6.1, which says that on the event of indefinite growth for the weaker infection type, the size of the asymptotic shape of the total infected region is determined by the weaker infection type. This means that if the weaker infection type grows indefinitely, then the asymptotic speed of the growth for the stronger infection type can not exceed the speed of the weaker type. The key step in proving Lemma 6.1 is to show that if the stronger infection type gets a large enough lead over the weaker infection type infinitely often – which indeed will be the case if the asymptotic shape is larger than the capacity of the weaker infection type allows for – then almost surely the stronger infection type will eventually eradicate the weaker one. To formulate this key result, assume that $\lambda_1 = 1$ (note that by time-scaling this is no restriction) and write $\lambda_2 = \lambda$. Furthermore, for an arbitrary set $\Gamma \subset \mathbb{R}^d$, let

$$\|\Gamma\| = \sup\{|x|;\ x \in \Gamma\}.$$

The proposition now runs as follows:

Proposition 5.2 *Assume that F satisfies (1). Also, fix $\lambda < 1$ and $\varepsilon > 0$ and let*

$$D = \left\{ \|S_t^1\| \geq (1+3\varepsilon)tB\left(0, \lambda\mu^{-1}\right) \text{ for arbitrarily large } t \right\}.$$

Then $P^{1,\lambda}(G_2|D) = 0$.

Proof: Write $\tilde{S}_t^i$ for the set of points whose entire γ-ball is contained in the type i infected area at time t. Points in $\tilde{S}_t^i$ will be referred to as *strongly* type i infected at time t. Let

$$Q = (1+3\varepsilon)B(0,\lambda\mu^{-1}) \backslash (1+2\varepsilon)B(0,\lambda\mu^{-1})$$

and introduce the event

$$E_t = \{\tilde{S}_t^1 \cap tQ \neq \emptyset\}.$$

Note that almost surely

$$D \Rightarrow \{E_t \text{ occurs for arbitrarily large } t\}. \tag{12}$$

To see this, write Π for the set of type 1 outbursts that occur in tQ for some t during the progress of the growth and let Π_γ be those outbursts in Π whose radius is at least γ. If D occurs then $|\Pi| = \infty$ and, since each outburst in Π has radius greater than γ with some probability $p > 0$, it follows from Levy's version of the Borel-Cantelli lemma (see Williams (1991), section 12.15) that $|\Pi_\gamma| = \infty$ as well. But if $|\Pi_\gamma| = \infty$ the region tQ must contain strongly type 1 infected points infinitely often and (12) is verified.

Now fix $\varepsilon > 0$ and $\lambda < 1$. We want to pick $\delta > 0$ and $\alpha \in (0, \varepsilon]$ so that $(1+\delta)^{-1}\mu^{-1} > (1+\alpha)\lambda\mu^{-1}$ (here the left-hand side should be thought of as a lower bound for the speed of a hampered unit rate process and the right-hand side as an upper bound for the speed of an unhampered process with rate λ). Hence we define

$$\delta = \frac{1-\lambda}{2\lambda}$$

and

$$\alpha = \min\left\{\frac{1-\lambda}{2(1+\lambda)}, \varepsilon\right\}.$$

Let

$$F_t = \{\|S_s^2\| \subset (1+\alpha)sB(0,\lambda\mu^{-1}) \text{ for all } s \geq t\},$$

and write $\mathcal{F}_t = \sigma(S_s^1 \cup S_s^2;\ s \leq t)$. We will show that, for some fixed $c > 0$, we have almost surely on the event E_t that

$$P^{1,\lambda}(G_2^c|\mathcal{F}_t, F_t) > c \quad \text{if } t \text{ is large.} \tag{13}$$

Using Levy's 0-1 law the proposition follows from this: By Theorem 1.1 and Lemma 4.1, $P^{1,\lambda}(F_t) \to 1$ as $t \to \infty$. Together with (12) and (13) this implies that, almost surely on the event D,

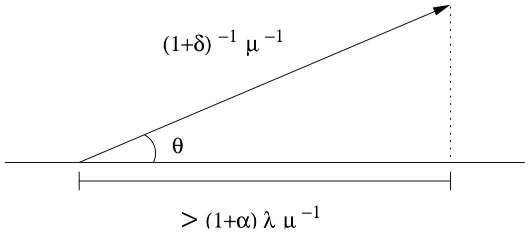


Figure 1: Choice of θ.

$$P^{1,\lambda}(G_2^c|\mathcal{F}_t) > c/2 \quad \text{infinitely often.} \tag{14}$$

Levy's 0-1 law tells us that almost surely $P^{1,\lambda}(G_2^c|\mathcal{F}_t)$ tends to the indicator function of G_2^c and (14) prevents $P^{1,\lambda}(G_2^c|\mathcal{F}_t)$ from converging to 0 on D. Hence $P^{1,\lambda}(G_2^c|\mathcal{F}_t) \to 1$ on D, implying that $P^{1,\lambda}(G_2|D) = 0$, as desired.

It remains to prove (13). To describe the idea in the proof, note that on E_tF_t we have $\|S_t^2\| \leq (1+\varepsilon)t\lambda\mu^{-1}$ and $\|\tilde{S}_t^1\| \geq (1+2\varepsilon)t\lambda\mu^{-1}$, that is, the strongly type 1 infected region at time t has a lead of at least $\varepsilon t\lambda\mu^{-1}$ units of length as compared to the type 2 infected region. We will show that if t is large, then with large probability this lead gives the type 1 infection time to create a layer that completely surrounds the type 2 infection. Moreover, if this layer is sufficiently thick – which it will indeed be if t is large – then Lemma 4.6 gives a lower bound for the probability that no type 2 outbursts that reach outside the layer ever occur. The proof is to a large extent based on a geometrical construction, which is easiest to picture in two dimensions. Hence we give the details for $d = 2$ and indicate at the end of the proof how the geometrical arguments can be generalized to higher dimensions.

To describe the geometrical construction, first define an angle $\theta \in (0, \pi/2)$ such that if a vector of length $(1+\delta)^{-1}\mu^{-1}$ that forms the angle θ with a given line is projected on that same line, then the length of this projection is strictly greater than $(1+\alpha)\lambda\mu^{-1}$; se Figure 1. Since

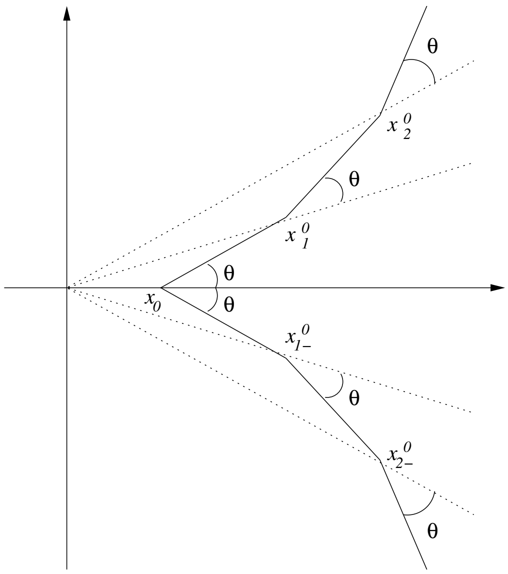
$$(1+\delta)^{-1}\mu^{-1} - (1+\alpha)\lambda\mu^{-1} \geq \mu^{-1}/2,$$

we can for example pick θ such that

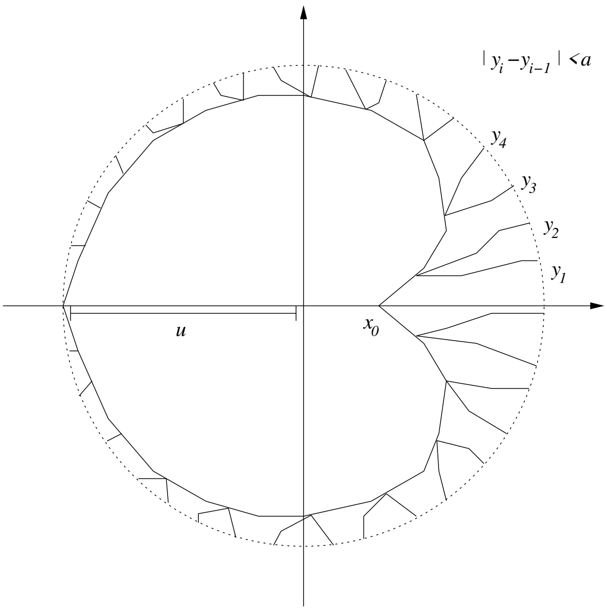
$$\begin{aligned} \cos\theta &= \frac{(1+\alpha)\lambda\mu^{-1} + \mu^{-1}/4}{(1+\delta)^{-1}\mu^{-1}} \\ &= (1+\delta)[(1+\alpha)\lambda + 1/4]. \end{aligned}$$

Now fix a point $x_0 \in Q$ located on the positive x-axis and draw two line segments starting from x_0 with angle θ and $-\theta$ respectively to the x-axis; see Figure 2(a). The length of the segments is taken to be

$$l = \frac{\varepsilon\lambda\mu^{-1}}{2(1+\alpha)}$$



(a) The inner spiral.



(b) The branches hit $\partial B(0, u)$ at points $\{y_i\}$.

Figure 2: Geometrical construction

(the reason for this particular choice of l will be clear later on in the proof). Let l_1^0 and l_{1-}^0 denote the two segments and write x_1^0 and x_{1-}^0 for the terminal points (the zeroes in the superscripts will be explained later). From the points x_1^0 and x_{1-}^0 we draw two new line segments l_2^0 and l_{2-}^0 of length l. The segment l_2^0 (l_{2-}^0) should form the angle θ ($-\theta$) with an imaginary line through the origin and x_1^0 (x_{1-}^0). We continue to draw line segments like this in an outward spiral running in both directions. The segments should all be of length l and depending on the sign of the x-coordinate of its starting point it should form the angle θ or $-\theta$ with an imaginary line through the origin and its starting point. Eventually the two spiral arms will meet at a point $(-u, 0)$ on the negative x-axis. Let $2n$ denote the number of segments needed to achieve this. We then have two sets of line segments, $\{l_k^0\}_1^n$ and $\{l_{k-}^0\}_1^n$, constituting the upper and lower spiral arm respectively, and two sets of terminal points for the line segments, $\{x_k^0\}_1^n$ and $\{x_{k-}^0\}_1^n$. Note for the future that $|x_k^0| - |x_{k-1}^0| \geq l\cos\theta$, which implies that

$$|x_k^0| \geq (1+2\varepsilon)\lambda\mu^{-1} + kl\cos\theta. \tag{15}$$

Now extend the construction by adding more edges, still of length l, branching out from the points $\{x_k^0\}$ and $\{x_{k-}^0\}$ towards the boundary of the circle with radius u around the origin; see Figure 2(b). The branches should be built up so that edges hit $\partial B(0, u)$ at points $\{y_j\}$ located not more than a distance a from each other, where

$$a = \frac{1-\alpha}{1+\alpha} \cdot \frac{l}{4}$$

(the choice of a is motivated later). Furthermore, the number of segments used to join a point x_k^0 (or x_{k-}^0) to a point on the circle boundary $\partial B(0, u)$ should not exceed $n-k$. We group the line segments in *generations* depending on how many links away from x_0 they are: An edge whose starting point is linked to x_0 using $k-1$ other edges is placed in generation k. If l_k^i (l_{k-}^i) denotes segment number i in generation k in the upper (lower) half plane we thus have n generations $\{l_k^i, l_{k-}^i\}_{i\geq 0}$, where the edges with $i = 0$ belongs to the inner spiral. Let $\{x_k^i\}$ and $\{x_{k-}^i\}$ denote the terminal points of the segments l_k^i and l_{k-}^i respectively. The last demand on the construction is that (15) should hold for all terminal points in generation k, that is,

$$\min_i |x_k^i| \geq (1+2\varepsilon)\lambda\mu^{-1} + kl\cos\theta. \tag{16}$$

When the line segments are arranged, we complete the construction by forming channels of width, say, $l/100$ around all segments.

Write $m(x_0)$ for the total number of channels required in the above construction. As indicated this number depends on the choice of the starting point x_0. Let m denote the largest value for $m(x_0)$ when $x_0 \in Q$ and pick a time point t_0 that fulfills the conditions (i)-(iii) described below. Some of these conditions might seem awkward at first, but their purpose will gradually become clear.

(i) Combining Lemma 4.3 and Lemma 4.4 yields that

$$\lim_{b\to\infty}\lim_{t\to\infty}\frac{T^b(\mathbf{t})}{t}=\mu \quad \text{a.s.}$$

Hence for each $p>0$ and $\delta>0$ we have

$$P\left(T^b(\mathbf{t})\geq(1+\delta)\mu t\right)\leq p \tag{17}$$

if b and t are large. Let t_0 be large enough to ensure that (17) holds for $b\geq t_0l/100$ and $t\geq t_0l$ when $\delta=(1-\lambda)/2\lambda$ and $p=1/2m$.

(ii) Write S_t for the infected area at time t in a one-type process with unit rate and let t_0 be large enough to guarantee that

$$P\left((1-\alpha)B\left(0,\mu^{-1}\right)\subset\frac{S_t}{t}\subset(1+\alpha)B\left(0,\mu^{-1}\right)\right)>1-\frac{1}{2m}$$

for $t\geq t_0\varepsilon\lambda/8(1+\alpha)^2$.

(iii) Let $\xi=a/4u$ and $\delta=1/2$ in Lemma 4.6 and pick t_0 so that $t_0u\geq r_0^*$.

We will show that (13) holds for such a choice of t_0. To this end, fix $t\geq t_0$ and note that on E_t we can pick a point $x_0\in(\tilde{S}_t^2\cap tQ)/t$ to serve as starting point for the geometrical construction described above (by rotation invariance we may assume that x_0 is located on the positive x-axis). At time t we then have a strongly type 1 infected point tx_0 with $|x_0|\geq(1+2\varepsilon)\lambda\mu^{-1}$. Also, since $\alpha\leq\varepsilon$, on F_t the type 2 infected area at time t does not reach further than $(1+\varepsilon)t\lambda\mu^{-1}$ from the origin. Now define

$$t'=(1+\delta)\mu tl$$

and consider the state of the infection at time $t+t'$. For the type 2 infection, by the choice of l and α we have on F_t that

$$\begin{aligned}\|S_{t+t'}^2\| &\leq (1+\alpha)(t+t')\lambda\mu^{-1}\\ &\leq \left(1+\frac{3}{2}\varepsilon\right)t\lambda\mu^{-1}.\end{aligned}\tag{18}$$

To deal with the type 1 infection, let $\tilde{T}_k^i$ ($\tilde{T}_{k-}^i$), $k\geq1$, denote the time counting from $t+(k-1)t'$ until the terminal point of the segment l_k^i (l_{k-}^i) is strongly type 1 infected assuming that at time $t+(k-1)t'$ all type 1 infection is erased and replaced by a γ-ball around the starting point of l_k^i (l_{k-}^i) while the type 2 infection is left as in the original process. By (18), on F_t the type 2 infection has not yet reached any parts of the channels in the first generation at time $t+t'$. Hence up to time $t+t'$ the spread of the type 1 infection inside the first generation channels behaves like hampered one type processes with $b=tl/100$.

On the scale t the channels has length tl and since $t \geq t_0$ it follows from the condition (i) in the choice of t_0 that on E_t we have

$$P^{1,\lambda}\left(\tilde{T}_1^i \geq t' | \mathcal{F}_t, F_t\right) \leq \frac{1}{2m} \quad \text{for all } i, \tag{19}$$

where $\tilde{T}_1^i$ can also be replaced by $\tilde{T}_{1-}^i$. For the state of the infection at time $t+2t'$, a similar reasoning as for the time $t+t'$ yields that

$$\|S_{t+2t'}^2\| \leq (1+2\varepsilon)t\lambda\mu^{-1}, \tag{20}$$

implying that the analog of (19) holds also for the second generation passage times $\{\tilde{T}_2^i\}$ and $\{\tilde{T}_{2-}^i\}$. Now note that

$$\begin{aligned}(1+\alpha)t'\lambda\mu^{-1} &= (1+\delta)(1+\alpha)\lambda tl \\ &\leq (1+\delta)[(1+\alpha)\lambda + 1/4]tl \\ &= tl\cos\theta.\end{aligned}$$

Hence on F_t we have

$$\begin{aligned}\|S_{t+kt'}^2\| &\leq \|S_{t+(k-1)t'}^2\| + (1+\alpha)t'\lambda\mu^{-1} \\ &\leq \|S_{t+(k-1)t'}^2\| + tl\cos\theta.\end{aligned}$$

Using (20) this implies that

$$\|S_{t+kt'}^2\| \leq (1+2\varepsilon)t\lambda\mu^{-1} + (k-2)tl\cos\theta \tag{21}$$

for $k \geq 2$ and thus, by (16), at time $t+kt'$ the type 2 infection has not yet reached any parts of the channels surrounding the line segments in the k:th generation. Up to time $t+kt'$, the spread of the type 1 infection inside the k:th generation channels hence behaves like hampered one type processes with $b = tl/100$. It follows from the condition (i) in the choice of t_0 that on E_t the bound in (19) holds also for $\{\tilde{T}_k^i\}$ and $\{\tilde{T}_{k-}^i\}$, that is,

$$P^{1,\lambda}\left(\tilde{T}_k^i > t' | \mathcal{F}_t, F_t\right) \leq \frac{1}{2m} \quad \text{for all } k \geq 1 \text{ and } i \geq 0,$$

where $\tilde{T}_k^i$ can also be replaced by $\tilde{T}_{k-}^i$. Let C_t denote the event that no passage time in the system exceed t', that is,

$$C_t = \bigcap_{k,i}\{\tilde{T}_k^i \leq t' \cap \tilde{T}_{k-}^i \leq t'\}.$$

Since there are at most m channels in the system, on E_t we obtain

$$P^{1,\lambda}(C_t | \mathcal{F}_t, F_t) > \frac{1}{2}. \tag{22}$$

Note that on $E_t F_t C_t$ all points $\{ty_j\}$ on the boundary of the circle $B(0, tu)$ are strongly type 1 infected at time $t + nt'$. Furthermore, the distance from the circle boundary $\partial B(0, tu)$ to the type 2 infected region at time $t+nt'$ is at least $tl \cos\theta$, that is, $\|S^2_{t+nt'}\| \leq u - tl\cos\theta$. This follows from (21) combined with the fact that $u \geq (1+2\varepsilon)t\lambda\mu^{-1} + (n-1)tl\cos\theta$, which is a consequence of (16). It can be seen that $\cos\theta \geq 1/2$ and hence we have

$$\|S^2_{t+nt'}\| \leq u - \frac{tl}{2}. \tag{23}$$

The next step is to use the one-type shape theorem to show that with large probability the strong type 1 infection at the points $\{ty_j\}$ will expand and create a connected type 1 layer around the type 2 infection. To this end, define

$$t'' = \frac{\varepsilon\lambda t}{8(1+\alpha)^2}$$

and note that (23) combined with the fact that $(1+\alpha)t''\lambda\mu^{-1} \leq tl/4$, gives that on F_t we have

$$\|S^2_{t+nt'+t''}\| \leq u - \frac{tl}{4}. \tag{24}$$

Now, for each j, assume that at time $t+nt'$ a new process is started by reducing the type 1 infection to the γ-ball around the point ty_j. More precisely, at time $t+nt'$ all type 1 infection except the one in $B(ty_j, \gamma)$ is erased while the type 2 infection is left unchanged. For $s \geq t + nt'$, let $S^{1(j)}_s$ denote the type 1 infected region at time s in such a process and for $s \geq 0$ define

$$A^j_s = \left\{ B\left(ty_j, (1-\alpha)s\mu^{-1}\right) \subset S^{1(j)}_{t+nt'+s} \subset B\left(ty_j, (1+\alpha)s\mu^{-1}\right) \right\}.$$

Since $(1+\alpha)t''\mu^{-1} = tl/4$, the event $A^j_{t''}$ does not depend on the state of the infection outside $B(ty_j, tl/4)$ and by (24), $B(ty_j, tl/4)$ does not contain any type 2 infection at time $t+nt'+t''$. Hence the one-type shape theorem can be applied to estimate the probability for the event $A^j_{t''}$ and since $t \geq t_0$ it follows from the condition (ii) in the choice of t_0 that

$$P^{1,\lambda}\left(A^j_{t''}\right) \geq 1 - \frac{1}{2m}$$

on E_t. Let

$$A_t = \bigcap_j A^j_{t''}.$$

The number of points $\{ty_j\}$ on $\partial B(0, tu)$ is clearly bounded by m implying that on E_t we have

$$P^{1,\lambda}(A_t | \mathcal{F}_t, F_t, C_t) > \frac{1}{2}. \tag{25}$$

In words A_t is the event that all circles with radius $(1-\alpha)t''\mu^{-1}$ around the points $\{ty_j\}$ are type 1 infected at time $t+nt'+t''$. Since $(1-\alpha)t''\mu^{-1}=ta$, where ta is recognized as the distance between the points $\{ty_j\}$, the circles overlap each other so that a layer of type 1 infection with thickness at least $ta/2$ concentrated around $\partial B(0,tu)$ is created.

Remember that the aim is to establish (13). Trivially

$$P^{1,\lambda}(G_2^c|\mathcal{F}_t,F_t)\geq P^{1,\lambda}(G_2^c|\mathcal{F}_t,F_t,C_t,A_t)\cdot P^{1,\lambda}(A_t|\mathcal{F}_t,F_t,C_t)\cdot P^{1,\lambda}(C_t|\mathcal{F}_t,F_t)$$

and using (22) and (25) it follows that on E_t we have

$$P^{1,\lambda}(G_2^c|\mathcal{F}_t,F_t)>P^{1,\lambda}(G_2^c|\mathcal{F}_t,F_t,C_t,A_t)\cdot 2^{-2}. \tag{26}$$

What remains is to bound the probability that G_2^c occurs on $E_tF_tC_tA_t$ from below. To do this, note that on $E_tF_tC_tA_t$, at time $t+nt'+t''$ the type 2 infection is contained in $B(0,tu)$ and the annulus $B(0,tu+ta/4)\backslash B(0,tu)$ is filled with type 1 infection. In between the type 1 layer and the type 2 infection there might however still be uninfected regions. Clearly we are done if we can find a lower bound for the probability that G_2^c occurs when these regions are assumed to be occupied by type 2 infection. Hence consider a two-type growth process with infection rates $(1,\lambda)$ started from a connected configuration without holes such that $S_0^2\subset B(0,tu)$ and $S_t^1\cup S_t^2\supset B(0,tu+ta/4)$. It follows from Lemma 4.6 and the condition (iii) in the choice of t that the probability that the type 2 infection never makes an effective outbursts in such a process is at least 1/2. Combining this with (26) yields

$$P^{1,\lambda}(G_2^c|\mathcal{F}_t,F_t)>\frac{1}{2^3}.$$

Hence (13) is established for $d=2$ and the proposition is proved.

For $d\geq 3$, the geometrical construction is obtained by first rotating the two-dimensional inner spiral around the x-axis a finite number of times and then add branches – emanating from the rotated spiral arms – that hit the surface of the ball $B(0,u)$ closely enough. As in the two-dimensional case, Lemma 4.3 and Lemma 4.4 can be combined to show that, if t is large, then with large probability the type 1 infection travels fast enough through the channels to reach the points on $\partial B(0,u)$ in time to create a thick layer around the type 2 infection. The rest of the proof is analogous. □

6 Proof of Theorem 1.2

In this section we prove Theorem 1.2 using arguments similar to those used for the main result of Häggström and Pemantle (2000), but with a different twist at the end, which is needed because of the unboundedness of the outbursts radii. The proof is based on Proposition 5.2 and a coupling of the two-type processes with distributions $\{P^{1,\lambda}\}_{\lambda\geq 0}$ valid for all $\lambda\in[0,1]$ simultaneously.

The part of the proof where Proposition 5.2 comes into play is formulated separately in the following lemma, which says roughly that the shape theorem holds on the event of unbounded growth for the weaker infection type and that the radius of the asymptotic shape in this case is determined by the weaker infection type.

Lemma 6.1 *Let $S_t^1 \cup S_t^2$ be the region infected at time t in a two-type process with distribution $P^{1,\lambda}$, where $\lambda \in [0,1]$ and assume that (1) holds. Then, for any $\varepsilon \in (0, \lambda\mu^{-1})$, we have $P^{1,\lambda}$-a.s. on the event G_2 that*

$$(1-\varepsilon)B\left(0, \lambda\mu^{-1}\right) \subset \frac{S_t^1 \cup S_t^2}{t} \subset (1+\varepsilon)B\left(0, \lambda\mu^{-1}\right)$$

for all sufficiently large t.

Proof: Let $\|S_t^1 \cup S_t^2\|_*$ denote the minimum distance from the origin to $(S_t^1 \cup S_t^2)^c$, that is,

$$\|S_t^1 \cup S_t^2\|_* = \sup\{s;\ B(0,s) \subset S_t^1 \cup S_t^2\}.$$

The lemma follows if we can show that

$$\frac{\|S_t^1 \cup S_t^2\|}{t} \to \lambda\mu^{-1} \quad \text{and} \quad \frac{\|S_t^1 \cup S_t^2\|_*}{t} \to \lambda\mu^{-1}$$

$P^{1,\lambda}$-a.s. on the event G_2. Since $\|S_t^1 \cup S_t^2\|_* \le \|S_t^1 \cup S_t^2\|$ it suffices to prove that $P^{1,\lambda}$-a.s. on G_2 we have

$$\limsup_{t\to\infty} \frac{\|S_t^1 \cup S_t^2\|}{t} \le \lambda\mu^{-1} \tag{27}$$

and

$$\liminf_{t\to\infty} \frac{\|S_t^1 \cup S_t^2\|_*}{t} \ge \lambda\mu^{-1}. \tag{28}$$

The lower bound (28) follows immediately from Lemma 4.2 and Theorem 1.1. To establish (27), note that by Lemma 4.1 and Theorem 1.1

$$\limsup_{t\to\infty} \frac{\|S_t^2\|}{t} \le \lambda\mu^{-1}.$$

We are done if we can show that S_t^2 can also be replaced by S_t^1 here, that is, if we can show that

$$\limsup_{t\to\infty} \frac{\|S_t^1\|}{t} \le \lambda\mu^{-1}. \tag{29}$$

But this is a consequence of Proposition 5.2, since if (29) fails there is an $\varepsilon > 0$ such that the type 1 infected region reaches outside $(1+3\varepsilon)tB(0,\lambda\mu^{-1})$ for arbitrarily large t and by Proposition 5.2 this prevents the event G_2. $\square$

Moving on to the aforementioned simultaneous coupling of the two-type processes with distributions $\{P^{1,\lambda}\}_{\lambda\in[0,1]}$, let N_1 and N_2 be two independent unit rate Poisson processes. We will couple the growth processes by successively thinning the Poisson process N_2 and then use it to generate the type 2 outbursts. This is done as follows: Associate independently to each point in N_2 a random variable uniformly distributed over $[0,1]$, and let λN_2 be the set of points in N_2 whose attached uniform variable is smaller than or equal to λ. Then λN_2 is a Poisson process with rate λ and hence, for each $\lambda \in [0,1]$ a two-type process $\{S_t^1(\lambda) \cup S_t^2(\lambda)\}_{t\geq 0}$ with distribution $P^{1,\lambda}$ is obtained by starting from $B(-\mathbf{2}\gamma,\gamma)$ and $B(0,\gamma)$ at time 0 and then using N_1 to generate the type 1 outbursts and λN_2 to generate the type 2 outbursts. Write Q for the probability measure underlying this coupling and let G_i^λ denote the event that the type i infection grows indefinitely at parameter value λ.

Proof of Theorem 1.2: By time-scaling and symmetry we have

$$P^{1,\lambda}(G) = P^{1,1/\lambda}(G)$$

and hence it is enough to prove that $P^{1,\lambda}(G) = 0$ for all but at most countably many $\lambda \in [0,1]$. To this end, we will show that for any $\lambda' < \lambda \in [0,1]$ we have

$$Q(G_1^\lambda \cap G_2^{\lambda'}) = 0. \tag{30}$$

This implies that with Q-probability 1 the event $G_1^\lambda \cap G_2^\lambda$ occurs for at most one $\lambda \in [0,1]$: By construction of the probability measure Q, the event G_1^λ is decreasing in λ – that is, if G_1^λ occurs then $G_1^{\lambda'}$ occurs for all $\lambda' < \lambda$ – and the event G_2^λ is increasing in λ. Hence the set of lambdas for which the event $G_1^\lambda \cap G_2^\lambda$ occurs is Q-a.s. an interval. If with positive Q-probability the interval were non-degenerated there would be $\lambda' < \lambda$ in $[0,1]$ such that the event $G_1^\lambda \cap G_2^\lambda \cap G_1^{\lambda'} \cap G_2^{\lambda'}$ has positive Q-probability. This however contradicts (30). Thus with Q-probability 1 the interval consists of a single point, implying that Q-a.s. the event $G_1^\lambda \cap G_2^\lambda$ occurs for at most one $\lambda \in [0,1]$. Clearly

$$P^{1,\lambda}(G) = Q(G_1^\lambda \cap G_2^\lambda)$$

and hence it follows that $\{\lambda \in [0,1];\ P^{1,\lambda}(G) > 0\}$ is countable.

To establish (30), fix $\lambda' < \lambda \in [0,1]$ and assume that $G_2^{\lambda'}$ occurs. By Lemma 6.1 we then have

$$\limsup_{t\to\infty} \frac{\|S_t^1(\lambda') \cup S_t^2(\lambda')\|}{t} \leq \lambda' \mu^{-1}$$

so that in particular

$$\limsup_{t\to\infty} \frac{\|S_t^1(\lambda')\|}{t} \leq \lambda' \mu^{-1}.$$

From the construction of the Q-coupling it is clear that $\|S_t^1(\lambda)\| \leq \|S_t^1(\lambda')\|$ and hence it follows that

$$\limsup_{t\to\infty} \frac{\|S_t^1(\lambda)\|}{t} \leq \lambda' \mu^{-1}. \tag{31}$$

Furthermore, by Lemma 6.1, if t is large

$$\limsup_{t\to\infty} \frac{\|S_t^1(\lambda)\cup S_t^2(\lambda)\|_*}{t} \geq \lambda\mu^{-1}. \tag{32}$$

Now pick $\varepsilon > 0$ such that $(1+\varepsilon)\lambda' < (1-\varepsilon)\lambda$. Combining (31) and (32) yields that there is a time T such that for $t \geq T$ we have

$$S_t^1(\lambda) \subset (1+\varepsilon)tB(0,\lambda'\mu^{-1}) \tag{33}$$

and

$$(1-\varepsilon)tB(0,\lambda\mu^{-1}) \subset S_t^1(\lambda)\cup S_t^2(\lambda). \tag{34}$$

By (34), an outburst that occurs at a time point $t \geq T$ must reach outside $(1-\varepsilon)tB(0,\lambda\mu)$ to be effective and by the choice of ε we have

$$(1+\varepsilon)B(0,\lambda'\mu^{-1}) \subset (1-\varepsilon)B(0,\lambda\mu^{-1}).$$

Thus an effective type 1 outburst at a time point $t \geq T$ would cause the type 1 infected region to reach outside $(1+\varepsilon)B(0,\lambda'\mu^{-1})$. This conflicts with (33) and hence no effective type 1 outbursts can occur after time T. Clearly this prevents the event G_1^λ. □

7 Proof of Proposition 1.1

This section is devoted to the proof of Proposition 1.1.

Proof of Proposition 1.1: Pick bounded sets $\Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2'$ of positive Lebesgue measure such that Γ_1 and Γ_2 and also Γ_1' and Γ_2' are disjoint. We will show that if G has positive probability in the process started from (Γ_1,Γ_2), then G occurs with positive probability in the process started from (Γ_1',Γ_2') as well. To this end, first consider the process started from (Γ_1,Γ_2). By Lemma 4.5(a) almost surely only finitely many effective outbursts occur in the set $\Gamma_1'\cup\Gamma_2'$ during the progress of the growth in this process and hence there is a time $t<\infty$ such that with probability, say, 1/2 no effective outbursts occur in $\Gamma_1'\cup\Gamma_2'$ after time t. Let U_i denote the set of effective type i outbursts that occur in the set $S_t^1\cup S_t^2$ after time t. A second application of Lemma 4.5(a) yields that the sets U_i are almost surely finite.

Now consider a process started from (Γ_1',Γ_2'), coupled with the one started from (Γ_1,Γ_2) in such a way that the same Poisson processes are used to generate the outbursts after time t. Before time t the process evolves independently of the one started from (Γ_1,Γ_2). We will describe a scenario for this process that causes the infection to develop in the same way as in the process started from (Γ_1,Γ_2) after time t. To prepare for this, join each point in U_i by a curve with a point in the interior of Γ_i'. The connections are made by aid of concatenations of straight line segments and the restrictions on a connection joining a point in U_1 (U_2) with a point in Γ_1' (Γ_2') are:

- It is not allowed to cross any part of Γ_2' (Γ_1').
- It should stay within the region infected at time t in the process started from (Γ_1, Γ_2).
- It can not pass through points in U_2 (U_1).

The first restriction might not be possible to fulfill if the set Γ_1' (Γ_2') is enclosed by Γ_2' (Γ_1'). However if this should be the case we condition on a large outburst occurring in Γ_1' (Γ_2') at some early time point transmitting the type 1 (2) infection past Γ_2' (Γ_1'). Then we use the outer parts of the type 1 (2) infected region as terminal for the connections. When the connections are made we let each one of them be surrounded by a path of width 2ε, where $\varepsilon > 0$ is chosen small enough to guarantee that the paths are disjoint. (In two dimensions it is sometimes impossible to avoid paths from crossing each other and hence we have to allow overlapping paths at crossing points.) Let $\{P_i^k;\ i = 1, 2 \text{ and } k \geq 1\}$ denote the paths and write $T(P_i^k)$ for the time it takes for the infection to wander along P_i^k from Γ_i' to its terminal point in U_i by aid of ε-small outbursts not reaching outside the path (in the two-dimensional case we allow for outbursts with radius 2ε at the possible crossings). Furthermore, define τ to be the time when all points in $U_1 \cup U_2$ are reached by the infection using the paths, that is,

$$\tau = \max_{i,k}\{T(P_i^k)\}.$$

The desired scenario for the process started from (Γ_1', Γ_2') is now obtained as follows:

1. Assume that the infection wander along the paths from Γ_1' and Γ_2' to the points in U_1 and U_2 by aid of ε-small outbursts.
2. Suppose that $\tau \leq t$. Also assume that no outbursts except for the ones on the paths occur before time τ and that no outbursts at all occur in the time interval (τ, t). At time t then, the infected region consists of the initial sets Γ_1' and Γ_2' together with fine infected strings linking these sets to the points in U_1 and U_2.
3. After time t the same Poisson processes as in the process started from (Γ_1, Γ_2) are used to generate the outbursts. Hence we know that effective outbursts of the same type as in the process started from (Γ_1, Γ_2) will take place at the points in U_1 and U_2. During the progress of the growth it might happen that some parts of the region that is infected at time t in the process started from (Γ_1, Γ_2) are infected by another infection type. Assume that no effective outbursts take place in those regions. By Lemma 4.5(b) this event has positive probability.

Write G_{Γ_1,Γ_2} for the event that both infection types grow indefinitely in the process started from (Γ_1, Γ_2) and write $\hat{G}_{\Gamma_1',\Gamma_2'}$ for the same event in the coupled process started from (Γ_1', Γ_2'). Trivially

$$P(\hat{G}_{\Gamma_1',\Gamma_2'}) \geq P(\hat{G}_{\Gamma_1',\Gamma_2'} | G_{\Gamma_1,\Gamma_2}) P(G_{\Gamma_1,\Gamma_2}).$$

The second factor on the right-hand side is positive by assumption. As for the first factor, note that if both infection types grow indefinitely in the process started from (Γ_1, Γ_2), the above scenario guarantees mutual unbounded growth also in the coupled process started from (Γ_1', Γ_2'), since in both processes the only outbursts that will reach outside the region infected at time t in the process started from (Γ_1, Γ_2) are the ones in U_1 and U_2. Furthermore, the above scenario clearly has positive probability because it only depends on finitely many outbursts. Hence also the first factor is positive and it follows that $P(\hat{G}_{\Gamma_1', \Gamma_2'}) > 0$. Since $P(\hat{G}_{\Gamma_1', \Gamma_2'}) = P^{\lambda_1, \lambda_2}_{\Gamma_1', \Gamma_2'}(G)$, we are done. $\square$

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