Two Badly Behaved Percolation Processes on a Nonunimodular Graph

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Abstract

We provide nonunimodular counterexamples to two properties that are well-known for automorphism invariant percolation on unimodular transitive graphs. The first property is that the number of encounter points in an infinite cluster is a.s. 0 or ∞. The second property is that speed of random walk on an infinite cluster is a.s. well-defined.

1 Introduction

In a highly influential 1996 paper, Benjamini and Schramm [5] proposed a systematic study of percolation beyond the usual $\mathbb{Z}^d$ setting. In particular, they suggested that the class of **transitive** graphs (or somewhat more generally the quasi-transitive ones) might provide a fruitful level of generality. A locally finite connected graph $G = (V, E)$ is said to be transitive if for any two vertices $v_1, v_2 \in V$ there exists a graph automorphism mapping v_1 on v_2; in other words, the graph "looks the same" from every vertex. Definitions of further concepts discussed in this section are deferred to Section 2. For surveys of the area that to a large extent was sparked by [5] and that is informally known as "percolation beyond $\mathbb{Z}^d$", see for instance Lyons [14], Häggström and Jonasson [11], and Häggström [10].

Benjamini and Schramm [5] focused on iid percolation on various transitive and other graphs G, and correctly identified amenability vs nonamenability of G as a relevant dichotomy. In a later paper, Benjamini, Lyons, Peres and Schramm [1] found that certain powerful techniques, in particular the so-called **mass-transport method**, extend far beyond the iid setting to automorphism invariant percolation processes, but require a property of G known as **unimodularity**. In many cases where results have been proved in the unimodular case, it turns out that it is not just the proof technique that breaks down, but also the results themselves. Hence, unimodularity vs nonunimodularity is a dichotomy whose substance goes beyond the mere applicability of certain mathematical tools.

The purpose of the present paper is to add further credibility to the view that the unimodularity vs nonunimodularity distinction is of substantial importance, by considering a couple of known properties of automorphism invariant percolation on unimodular graphs, and producing counterexamples to these properties for a particular nonunimodular graph known as **Trofimov's graph**.

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The first concrete property we consider is the number of so-called **encounter points** in infinite clusters. Encounter points is a geometric notion, which dates back to a classical paper of Burton and Keane [6], and which plays a role in percolation theory that is pivotal – both in the graph-theoretic and the metaphorical sense of the word. It is folklore knowledge in percolation theory that in automorphism invariant percolation on a unimodular transitive graph, a.s. every infinite cluster has either zero or infinitely many encounter points (Theorem 2.7). In Theorem 2.8, we establish that on Trofimov's graph, there exist percolation processes with arbitrary numbers of encounter points per infinite cluster.

The second property we consider concerns **simple random walk** on an infinite cluster of automorphism invariant percolation on G, in particular its so-called speed. If S_k denotes the distance between the random walker's position at time k and its starting point, then the speed S is defined as $S = \lim_{k\to\infty} S_k/k$, provided the limit exists. Benjamini, Lyons and Schramm [3] showed that for automorphism invariant percolation on a unimodular transitive graph, the speed exists a.s. (Theorem 2.9). In Theorem 2.10 we produce a counterexample to this behavior on Trofimov's graph.

The rest of the paper is organized as follows. In Section 2 we give the definitions and background missing in the present section, plus precise statements of the main results. In Section 3, which is included purely for expository purposes, we treat the case of unimodular graphs, recalling the mass-transport technique and showing how it leads to short proofs of Theorems 2.7 and 2.9. Finally, in Section 4, we introduce a technique, based on a kind of tree-indexed Markov chain, for generating counterexamples on Trofimov's graph, and apply it to prove Theorems 2.8 and 2.10.

2 Definitions and main results

We consider bond percolation on an infinite locally finite graph $G = (V, E)$, meaning that edges are assigned values 0 (closed) or 1 (open) at random. In other words, such a percolation process is a $\{0,1\}^E$-valued random object $X = \{X(e)\}_{e\in E}$. Percolation theorists are interested in connectivity and related properties of the subgraph of G obtained by throwing out all closed edges. In particular, focus tends to be on whether there exist infinite connected components, a.k.a. **infinite clusters**, and if so, how many there are and what are their properties. (Our choice to study bond rather than site percolation is essentially arbitrary. Everything in this paper translates, mutatis mutandis, to the site percolation setting where it is the vertices rather than the edges that are declared open or closed.) The special case where the $X(e)$'s are iid is called iid percolation.

To say much of interest, we need to impose some structure on G. We begin with transitivity.

Definition 2.1 *Let $G = (V, E)$ be an infinite locally finite connected graph. A bijective map $f : V \to V$ such that $\langle f(u), f(v)\rangle \in E$ if and only if $\langle u, v\rangle \in E$ is called a* **graph automorphism** *for G. The graph G is said to be* **transitive** *if for any $u, v \in V$ there exists a graph automorphism f such that $f(u) = v$. More generally, G is said to be* **quasi-transitive** *if there is a $k < \infty$ and a partitioning of V into k sets $V_1, \dots, V_k$ such that for $i = 1, \dots, k$ and any $u, v \in V_i$ there exists a graph automorphism f such that $f(u) = v$.*

A very natural class of percolation processes on a transitive (or quasi-transitive) graph G consists of the automorphism invariant ones.

Definition 2.2 *Let $G = (V, E)$ be an infinite locally finite connected graph. For a graph automorphism $f : V \to V$, write $f^* : E \to E$ for the induced mapping of edges, so that $f^*(\langle u, v\rangle) = \langle f(u), f(v)\rangle$. A $\{0,1\}^E$-valued bond percolation process is said to be* **automorphism invariant** *if for any n, any $e_1, \ldots, e_n \in E$, any $b_1, \ldots, b_n \in \{0,1\}$ and any $\gamma \in \mathrm{Aut}(G)$, we have*

$$\mathbb{P}(X(\gamma^* e_1) = b_1, \ldots, X(\gamma^* e_n) = b_n) = \mathbb{P}(X(e_1) = b_1, \ldots, X(e_n) = b_n)\,.$$

This encompasses many important examples: first and foremost iid percolation, but also things like uniform spanning forests (see, e.g., [2]), minimal spanning forests (see, e.g., [15]) and the random-cluster model (see, e.g., [7]).

We move on to the concept of amenability. Roughly speaking, amenability of a graph means the existence of finite subgraphs with arbitrarily small surface-to-volume ratio.

Definition 2.3 *The* **isoperimetric constant** *$h(G)$ of a graph $G = (V, E)$ is defined as*

$$h(G) \,=\, \inf_S \frac{|\partial S|}{|S|}\,,$$

where the infimum ranges over all finite nonempty subsets of V, and $\partial S = \{u \in V \setminus S : \exists v \in S \text{ such that } \langle u, v\rangle \in E\}$. The graph G is said to be **amenable** *if $h(G) = 0$; otherwise it is said to be* **nonamenable**.

The prototypical example of an amenable transitive graph is the $\mathbb{Z}^d$ lattice for any d, whereas a basic example of a nonamenable transitive graph is the infinite regular tree $\mathbb{T}_n$ (for $n \geq 2$) in which every vertex has exactly $n+1$ incident edges.

Next, unimodularity:

Definition 2.4 *Let $G = (V, E)$ be a transitive or quasi-transitive graph with automorphism group $\mathrm{Aut}(G)$. For $v \in V$, the* **stabilizer** *of v is defined as $\mathrm{Stab}(v) = \{\gamma \in \mathrm{Aut}(G) : \gamma v = v\}$. The graph G is said to be* **unimodular** *if for all $u, v \in V$ in the same orbit of $\mathrm{Aut}(G)$ we have the symmetry*

$$|\mathrm{Stab}(u)v| = |\mathrm{Stab}(v)u|\,. \tag{1}$$

Most examples of transitive graphs encountered in practice are unimodular. A basic counterexample, however, is Trofimov's graph, which we go on to define next. An **end** in a graph $G = (V, E)$ is an equivalence class of uni-infinite self-avoiding paths in X, with two paths equivalent if for all finite $W \subset V$ the paths are eventually in the same connected component of the graph obtained from G by deleting all $v \in W$.

Example 2.5 (Trofimov's graph [17]) *Consider the regular tree $\mathbb{T}_n$ with $n \geq 2$, and fix an end ξ in this tree. For each vertex v in the tree, there is a unique uni-infinite self-avoiding path from v that belongs to ξ. Call the first vertex after v on this path the* **ξ-parent** *of v, and call the other two neighbors of v its* **ξ-children**. *The* **ξ-grandparent** *of v is defined similarly in the obvious way. Let $G_{\mathcal{T},n} = (V_{\mathcal{T},n}, E_{\mathcal{T},n})$ be the graph that arises by taking $\mathbb{T}_n$ and adding, for each vertex v, an extra edge connecting v to its ξ-grandparent.*

Trofimov's graph $G_{\mathcal{T},n} = (V_{\mathcal{T},n}, E_{\mathcal{T},n})$ is obviously transitive, and furthermore it inherits the nonamenability property of $\mathbb{T}_n$. The key to seeing that it is also nonunimodular is the observation that given a vertex $v \in V_{\mathcal{T},n}$, the ξ-parent of v can be identified purely from the local graph structure, i.e. without any further knowledge of any labeling of the vertices. This can be seen as follows.

Among the n^2+n+2 neighbors of v, there are exactly n^2+1 that each share only one common neighbor with v; these are the ξ-grandchildren and the ξ-grandparent of v (the remaining $n+1$ neighbors of v each share exactly n neighbors with v). Among these n^2+1 vertices, there is exactly one that has no neighbor other than v in common with any of the others; this vertex is the ξ-grandparent of v. Finally, the ξ-parent of v is the unique joint neighbor of v and v's ξ-grandparent.

To complete the argument for nonunimodularity, let u be the ξ-parent of v and note that $|\mathrm{Stab}(u)v| = n$ whereas $|\mathrm{Stab}(v)u| = 1$ (each vertex has n ξ-children but only one ξ-parent), so (1) fails and G is nonunimodular.

Moving on to Theorems 2.7 and 2.8 about encounter points, we need first to recall from Burton and Keane [6] what an encounter point is.

Definition 2.6 *Given a graph $G = (V, E)$ and a percolation configuration $\eta \in \{0,1\}^E$, a vertex $v \in V$ is said to be an* **encounter point** *for v if the set of open edges in η contains at least three disjoint infinite self-avoiding paths starting at v with the further property that for any two of these paths, going from a vertex on one of them via open edges to a vertex on the other necessitates going through v.*

The following result is folklore, but will be proved in Section 3. See also Benjamini et al [1, Prop. 7.1] for a closely related result on the number of ends in infinite clusters, proved using the same idea.

Theorem 2.7 *Let G be a transitive unimodular graph, and let $X = \{X(e)\}_{e\in E}$ be an automorphism invariant bond percolation process on G. Then, a.s., any infinite cluster has either zero or infinitely many encounter points.*

In fact, extending the proof of this result to the case where transitivity is relaxed to quasi-transitivity is easy, but since we are interested in the result mainly as a contrast to Theorem 2.8 below, we are content with the transitive case. The same remark applies to Theorem 2.9 below concenring the speed of random walk on transitive unimodular graphs.

We further remark that in the case where G is amenable, a.s. every infinite cluster has zero encounter points; this was proved by Burton and Keane [6] although they stated their result more narrowly. Our main result concerning encounter points is the following.

Theorem 2.8 *For any $n \geq 2$ and $k \geq 1$, there exists an automorphism invariant bond percolation process X on Trofimov's graph $G_{\mathcal{T},n}$ with the property that it produces, a.s., infinitely many infinite clusters, all of which contain exactly k encounter points.*

Given the tree-indexed Markov chain technique to be outlined in Section 4, the proof of Theorem 2.8 is short and simple, and may be viewed as a kind of warm-up for the more involved construction to be used in proving Theorem 2.10 on random walk. A similar example on a different nonunimodular graph is given by Benjamini et al [1, Ex. 7.6].

For $G = (V, E)$ and a percolation process $X = \{X(e)\}_{e\in E}$ on G, simple random walk $Z = (Z_0, Z_1, \ldots)$ on X starting at a fixed vertex $v \in V$ is defined as follows. First let $Z_0 = v$. Given Z_0, pick Z_1 according to uniform distribution among all vertices that share an open edge in X with Z_0. (This is ill-defined in the boring case in which v has no adjacent open edges, in which case we take the random walk to stay put: $Z_0 = Z_1 = Z_2 = \cdots = v$.) We then proceed iteratively: given $(Z_0, Z_1, \ldots, Z_k)$, we pick Z_{k+1} according to uniform distribution on the set of vertices sharing an open edge with Z_k.

Various properties of this random walk and its asymptotic behavior are of interest. Here we focus on the distance

$$S_k = \mathrm{dist}_G(Z_0, Z_k)$$

between the random walker and its starting point, and its asymptotics as $k \to \infty$, where dist_G denotes graph-theoretic distance in the graph G. We define the asymptotic speed $S = \lim_{k\to\infty} \frac{S_k}{k}$, provided the limit exists. If the random walk happens to start in a finite cluster, then of course S_k is bounded, so $S = 0$. When it starts in an infinite cluster, things are less obvious, but it turns out that we have the following.

Theorem 2.9 (Benjamini, Lyons and Schramm [3]) *Let G be a transitive unimodular graph, and let $X = \{X(e)\}_{e\in E}$ be an automorphism invariant bond percolation process on G. Then the asymptotic speed S of simple random walk on X exists a.s.*

We emphasize that this is not the main result in [3] – the authors go on to establish a variety of fairly general conditions on X beyond automorphism invariance, for guaranteeing that S is strictly positive as long as the walk starts in an infinite cluster. The unimodularity condition in Theorem 2.9 cannot be dropped:

Theorem 2.10 *For any $n \geq 2$, there exists an automorphism invariant bond percolation process X on Trofimov's graph $G_{\mathcal{T},n}$ with the property that for simple random walk on X, the asymptotic speed S a.s. does not exist.*

Rather than using graph-theoretic distance with respect to G, we could have chosen to look at graph-theoretic distance with respect to the percolation configuration. This may affect the value of the asymptotic speed, but does not change whether or not it exists in Theorems 2.9 and 2.10 whose proofs go through in that setting with very small changes.

3 On unimodular graphs

As mentioned in Section 1, the so-called mass-transport technique has turned out to be important in the study of percolation on nonamenable (quasi-)transitive graphs in the unimodular case. Here we explain it in the transitive setting.

For an automorphism invariant bond percolation process X on a transitive graph $G = (V, E)$, let μ be the corresponding probability measure on $\{0,1\}^E$. Consider a nonnegative function $m(u, v, \omega)$ of three variables: two vertices $u, v \in V$ and the percolation configuration ω taking values in $\Omega = \{0,1\}^E$. The way to think of $m(u, v, \omega)$ is as the mass transported from u to v given the configuration ω. We assume, crucially, that $m(\cdot,\cdot,\cdot)$ is invariant under the diagonal action of $\mathrm{Aut}(G)$, meaning that $m(u, v, \omega) = m(\gamma u, \gamma v, \gamma\omega)$ for all u, v, ω and $\gamma \in \mathrm{Aut}(G)$. The following result is due to Benjamini, Lyons, Peres and Schramm [1], who were inspired by an embryo for the technique in [9].

Theorem 3.1 (The Mass-Transport Principle, [1]) *Given G, μ and $m(\cdot,\cdot,\cdot)$ as above, let*

$$M(u, v) = \int_\Omega m(u, v, \omega)d\mu(\omega)$$

for any $u, v \in V$. If G is unimodular, then the expected total mass transported out of any vertex v equals the expected mass transported into v, i.e.,

$$\sum_{u \in V} M(v,u) \,=\, \sum_{u \in V} M(u,v)\,. \tag{2}$$

The Mass-Transport Principle as stated here fails if G is not unimodular. To see this for Trofimov's graph $G_{\mathcal{T},n}$, we may consider the mass transport in which each vertex simply sends unit mass to its ξ-parent, regardless of the percolation configuration. Then each vertex sends mass 1 but receives mass n, thus violating (2). On the other hand, Benjamini et al [1] do have a (somewhat less powerful) version of the Mass-Transport Principle that holds also in the nonunimodular case, involving a reweighting of the mass sent from u to v by a factor $\frac{|\mathrm{Stab}(u)v|}{|\mathrm{Stab}(v)u|}$.

The proof of the Mass-Transport Principle is particularly simple in the case where G is the Cayley graph of a finitely generated group H, so here we settle for that, referring to [1] for the general case.

Proof of Theorem 3.1 for Cayley graphs. For $u, v \in V$, u and v are group elements of H, and there is a unique element $h = uv^{-1} \in H$ such that $u = hv$. This gives

$$\begin{aligned}\sum_{u \in V} M(v,u) &= \sum_{h \in H} M(v,hv) \,=\, \sum_{h \in H} M(h^{-1}v,v) \\ &= \sum_{h' \in H} M(h'v,v) \,=\, \sum_{u \in V} M(u,v)\,.\end{aligned}$$

$\square$

The proof of Theorem 2.7 exemplifies the power and simplicity of the mass-transport technique.

Proof of Theorem 2.7. Assume for contradiction that there exist, with positive probability, infinite clusters with a finite nonzero number of encounter points, and consider the following mass transport. Each vertex that sits in such an infinite cluster sends unit mass, and distributes it equally among the encounter points of its cluster. All other vertices send zero mass. Since no vertex sends more than unit mass, we have $\sum_{u \in V} M(v,u) \le 1$. On the other hand, any encounter point in an infinite cluster with finitely many encounter points receives infinite mass, so $\sum_{u \in V} M(u,v) = \infty$. Hence $\sum_{u \in V} M(v,u) < \sum_{u \in V} M(u,v)$, contradicting (2) as desired. $\square$

Moving on to the task of proving Theorem 2.9 on simple random walk, it turns out useful to first consider a **delayed simple random walk** $(Z'_0, Z'_1, \ldots)$, defined by taking $Z'_0 = v$ and then proceeding iteratively: given $(Z'_0, Z'_1, \ldots, Z'_k)$, select Z'' at random (uniform distribution) among all the G-neighbors of Z'_k, and let

$$Z'_{k+1} = \begin{cases} Z'' & \text{if } X(\langle Z'_k, Z'' \rangle) = 1 \\ Z'_k & \text{if } X(\langle Z'_k, Z'' \rangle) = 0\,. \end{cases}$$

The ordinary simple random walk $(Z_0, Z_1, \ldots)$ with distribution as defined in Section 2 can then be recovered by sampling $(Z'_0, Z'_1, \ldots)$ at the times of its jumps, i.e., by setting

$$(Z_0, Z_1, \ldots) \,=\, (Z'_{J_0}, Z'_{J_1}, \ldots)$$

where $J_0 = 0$, $J_1 = \min\{i > J_0 : Z'_i \neq Z'_{J_0}\}$ and, iteratively, $J_{k+1} = \min\{i > J_k : Z'_i \neq Z'_{J_k}\}$.

The point of first considering the delayed simple random walk is that it satisfies a rather strong stationarity property. The following result, which goes back to Häggström and Peres [12] and Lyons and Schramm [16], says that the percolation configuration "as seen from the point of view of the delayed random walker" is stationary in time.

Lemma 3.2 *Let $G = (V, E)$ be transitive and unimodular. Fix $v \in V$, and for each $u \in V$, define γ_u to be some G-automorphism mapping u on v. Let $X \in \{0,1\}^E$ be an $\mathrm{Aut}(G)$-invariant bond percolation process, and define a Borel measurable $F : \{0,1\}^E \to [0,1]$ that is $\mathrm{Stab}(v)$-invariant in the sense that $F(X) = F(\gamma X)$ for all $\gamma \in \mathrm{Stab}(v)$. Then*

$$(F(\gamma_{Z'_0} X), F(\gamma_{Z'_1} X), F(\gamma_{Z'_2} X), \ldots)$$

is a stationary sequence.

Proof. Note that for any F as in the lemma, any n and any $a_0, \ldots, a_n \in [0,1]$ we may form another function

$$G(X) \;=\; \mathbb{P}(F(\gamma_{Z'_0} X) \leq a_0, F(\gamma_{Z'_1} X) \leq a_1, \ldots, F(\gamma_{Z'_n} X) \leq a_n \,|\, X)$$

which itself satisfies the requirements on F. Hence, in order to prove the lemma, it suffices to show that

$$\mathbb{E}[F(\gamma_{Z'_0} X)] \;=\; \mathbb{E}[F(\gamma_{Z'_1} X)] \tag{3}$$

for any F as in the lemma. Write d for the degree of (i.e., common to all vertices in) G, and for each $u \in V$, write D_u for the number of edges e incident to u satisfying $X(e) = 1$. Consider the mass transport where, for any $u_1, u_2 \in V$, the mass sent from u_1 to u_2 is

$$\begin{cases} \frac{1}{d} F(\gamma_{u_2} X) & \text{if } \langle u_1, u_2 \rangle \in E \text{ with } X(\langle u_1, u_2 \rangle) = 1 \\ \frac{d - D_{u_1}}{d} F(\gamma_{u_1} X) & \text{if } u_2 = u_1 \\ 0 & \text{otherwise.} \end{cases}$$

The point of this choice is that

$$\sum_{v \in V} M(u, v) \;=\; \mathbb{E}[F(\gamma_{Z'_1} X)] \tag{4}$$

and

$$\sum_{v \in V} M(v, u) \;=\; \mathbb{E}[F(\gamma_{Z'_0} X)]\,. \tag{5}$$

Due to $\mathrm{Stab}(v)$-invariance of F, the Mass-Transport Principle applies, so combining (4) and (5) with (2) yields (3), and we are done. $\Box$

Proof of Theorem 2.9. We wish to apply the pointwise Subadditive Ergodic Theorem (see, e.g., [13, Thm. 9.14]) to the array $\{\mathrm{dist}_G(Z'_k, Z'_l\}_{k,l \geq 0}$. Note first that the subadditive relation $\mathrm{dist}_G(Z'_i, Z'_k) \leq \mathrm{dist}_G(Z'_i, Z'_j) + \mathrm{dist}_G(Z'_j, Z'_k)$ holds for any $i, j, k \geq 0$. Furthermore, for any $m, n \geq 0$ and any $i_1 \ldots, i_n$, the functions

$$F \;=\; \mathbb{P}(\mathrm{dist}_G(Z'_0, Z'_m) \leq i_1, \mathrm{dist}_G(Z'_m, Z'_{2m}) \leq i_2, \ldots, \mathrm{dist}_G(Z'_{(n-1)m}, Z'_{nm}) \leq i_n)$$

and

$$G = \mathbb{P}(\mathrm{dist}_G(Z_0', Z_1') \leq i_1, \mathrm{dist}_G(Z_0', Z_2') \leq i_2, \ldots, \mathrm{dist}_G(Z_0', Z_n') \leq i_n)$$

are Stab(v)-invariant, so Lemma 3.2 applies to show that the stationarity assumptions of the Subadditive Ergodic Theorem are in force, so that, a.s., the limit

$$S' = \lim_{n\to\infty} \frac{1}{n} \mathrm{dist}_G(Z_0', Z_n')$$

exists.

Another application of Lemma 3.2 gives that $(\mathbb{I}_{\{Z_1' \neq Z_0'\}}, \mathbb{I}_{\{Z_2' \neq Z_1'\}}, \ldots)$ is a stationary sequence, so the ordinary pointwise Ergodic Theorem ensures that its (possibly random) ergodic average $T = \lim_{n\to\infty} \frac{1}{n} \mathbb{I}_{\{Z_k' \neq Z_{k-1}'\}}$ exists. The combined existence of S' and T ensures the existence of

$$S = \lim_{n\to\infty} \frac{\mathrm{dist}_G(Z_0, Z_n)}{n} = \frac{S'}{T}\,.$$

$\square$

4 On Trofimov's graph

In this section, we devise a general scheme for constructing a certain kind $\mathrm{Aut}(G_{\mathcal{T},n})$-invariant bond percolation processes on Trofimov's graph $G_{\mathcal{T},n} = (V_{\mathcal{T},n}, E_{\mathcal{T},n})$. This scheme will then be applied to produce two concrete counterexamples: first a fairly straightforward one the establishes Theorem 2.8, and then a somewhat more complicated one that establishes Theorem 2.10.

As a preliminary step in constructing the $\mathrm{Aut}(G_{\mathcal{T},n})$-invariant bond percolation processes, we first devise the following scheme for attaching values to the vertices of $G_{\mathcal{T},n}$ in an $\mathrm{Aut}(G_{\mathcal{T},n})$-invariant way. The construction is reminiscent of tree-indexed Markov chains (see, e.g., [4]) but allows for a kind of sibling dependencies that has previously been exploited in the construction of random-cluster measures for trees [8].

Scheme I. Let $\mathcal{S} = (s_1, s_2, \ldots)$ be a finite or countably infinite set, fix $n \geq 2$, and consider the following class of procedures for picking an $\mathcal{S}^{V_{\mathcal{T},n}}$-valued random object. For each $s \in \mathcal{S}$, let Q_s be a probability measure on $\mathcal{S}^n$. For each such Q_s, we let $\hat{Q}_s$ be a symmetrized version corresponding to permuting the coordinates $1, \ldots, n$ at random. In other words, $\hat{Q}_s$ is the distribution of the $\mathcal{S}^n$-valued random element $(X_{\pi_1}, \ldots, X_{\pi_n})$ where $X = (X_1, \ldots, X_n) \in \mathcal{S}^n$ is chosen according to Q_s, and independently $\pi = (\pi_1, \ldots, \pi_n)$ is chosen according to uniform distribution on the set of permutations of $(1, \ldots, n)$. With the same notation, let Q_s^{proj} be the distribution of X_{π_1}. Note that $\{Q_s^{proj}(t)\}_{s,t\in\mathcal{S}}$ is the transition matrix of a discrete time Markov chain on $\mathcal{S}$. Assume that this Markov chain has a unique stationary distribution, and denote the distribution by ρ.

Next define, for any $v \in V_{\mathcal{T},n}$, the set $D(v) \subset V_{\mathcal{T},n}$ as consisting of v, its ξ-children, its ξ-grandchildren, its ξ-great grandchildren, and so on. Given a fixed vertex $v_1 \in V_{\mathcal{T},n}$, we now proceed to define a configuration $Y^1 = \{Y_v^1\}_{v\in D(v_1)} \in \mathcal{S}^{D(v_1)}$, i.e., Y^1 will be an assignment of values from $\mathcal{S}$ to the vertices in $D(v_1)$. First, assign v_1 value $Y^1(v_1)$ chosen according to the distribution ρ on $\mathcal{S}$. Next, assign the n children of v_1 values according to the joint distribution $\hat{Q}_{Y^1(v_1)}$. Then continue inductively, by letting the ξ-children of a vertex v that has been assigned value $Y^1(v)$ attain values according to the joint distribution $\hat{Q}_{Y^1(v)}$, with the obvious conditional independence structure. This defines the random configuration $\{Y^1(v)\}_{v\in D(v_1)}$. Note that if we follow a fixed line of ξ-descendants downwards, what we see is a Markov chain with transition matrix $\{Q_s^{proj}(t)\}_{s,t\in\mathcal{S}}$. Hence, and by the choice of $Y^1(v_1)$, we get for every $v \in D(v_1)$ that $Y^1(v)$ gets distribution ρ.

We go on to define $v_2 \in V_{\mathcal{T},n}$ to be the ξ-parent of v_1, and continue inductively for each i by letting v_{i+1} be the ξ-parent of v_i. Note that $D(v_1) \subset D(v_2) \subset \cdots$ and that the sequence exhausts $V_{\mathcal{T},n}$, i.e., $\cup_{i=1}^{\infty} D(v_i) = V_{\mathcal{T},n}$. For each i, we can now define the configuration $Y^i(D(v_i)) \in \mathcal{S}^{D(v_i)}$ by the same procedure as for the case $i = 1$. A key point now is that, by the homogeneity of the construction and the fact that the distribution ρ is preserved as we move down the chain $v_i, v_{i-1}, \ldots, v_1$, we get for $j < i$ that $Y^i(D(v_j))$ has the same distribution as $Y^j(D(v_j))$. Hence, by Kolmogorov's extension theorem, we can send $i \to \infty$ and obtain a probability measure μ on $\mathcal{S}^{V_{\mathcal{T},n}}$ such that for any i, the projection of μ on $\mathcal{S}^{D(v_i)}$ agrees with the distribution of $Y^i(D(v_i))$.

Scheme I is now defined as the random assignment of $\mathcal{S}$-values to $V_{\mathcal{T},n}$ obtained by choosing $Y \in \mathcal{S}^{V_{\mathcal{T},n}}$ according to μ. $\Box$

Lemma 4.1 *With $\mathcal{S}$, Q, ρ, μ and $Y \in \mathcal{S}^{V_{\mathcal{T},n}}$ as in Scheme I, the distribution of Y is* $\mathrm{Aut}(G_{\mathcal{T},n})$*-invariant.*

Proof. What we need to show is that for any finite $W = \{w_1, \ldots, w_n\} \subset V_{\mathcal{T},n}$, any $s_1, \ldots, s_n \in \mathcal{S}$ and any $\gamma \in \mathrm{Aut}(G_{\mathcal{T},n})$ we have

$$\mathbb{P}(Y(w_1) = s_1, \ldots, Y(w_n) = s_n) \,=\, \mathbb{P}(Y(\gamma w_1) = s_1, \ldots, Y(\gamma w_n) = s_n)\,. \tag{6}$$

We will assume that W consists of a single vertex w plus, for some $m \geq 1$, all the ξ-descendants of w down to the m'th generation starting from w. There is no loss of generality in doing so, because any finite vertex set in $G_{\mathcal{T},n}$ is contained in such a W. But for such W, the desired relation (6) is obvious due to the facts (a) that $Y(w)$ and $Y(\gamma w)$ both have distribution ρ, and (b) that the descendants of w and the descendants of γw obtain their Y-values from those of their ancestors via identical mechanisms. $\Box$

Given Scheme I, we construct automorphism invariant percolation processes on $G_{\mathcal{T},n}$ as follow.

Scheme II. There are two kinds of edges in $G_{\mathcal{T},n}$: those that connect ξ-parent to ξ-child, and those that connect ξ-grandparent to ξ-grandchild. Corresponding to these, define two functions $g_1 : \mathcal{S}^2 \to \{0,1\}$ and $g_2 : \mathcal{S}^2 \to \{0,1\}$. Pick a configuration $Y \in \mathcal{S}^{V_{\mathcal{T},n}}$ as in Scheme I, and then generate a percolation process $X \in \{0,1\}^{E_{\mathcal{T},n}}$ from Y in the following manner. For each edge $e = \langle u, v\rangle$ connecting a ξ-parent u to a ξ-child v, let $X(e) = g_1(Y(u), Y(v))$, whereas for each edge e connecting a ξ-grandparent u to a ξ-grandchild v, let $X(e) = g_2(Y(u), Y(v))$. $\Box$

Lemma 4.2 *The percolation process X obtained as in Scheme II is* $\mathrm{Aut}(G_{\mathcal{T},n})$*-invariant.*

Proof. It is clear from the construction that X inherits $\mathrm{Aut}(G_{\mathcal{T},n})$-invariance from Y, whose $\mathrm{Aut}(G_{\mathcal{T},n})$-invariance is ensured by Lemma 4.1. $\Box$

Remark. A possible generalization of Scheme II would be to allow randomization by letting the functions g_1 and g_2 be $[0,1]$- rather than $\{0,1\}$-valued, and to obtain X from Y by letting each $e \in \langle u, v\rangle$ independently attain value 1 with probability

$$\begin{cases} g_1(u,v) & \text{if } u \text{ is the } \xi\text{-parent of } v \\ g_2(u,v) & \text{if } u \text{ is the } \xi\text{-grandparent of } v. \end{cases}$$

It turns out, however, that this does not lead to a more general class of $\mathrm{Aut}(G_{\mathcal{T},n})$-invariant bond percolation processes, because it is possible to encode all the necessary randomness using a larger state space in Scheme I. We omit the details. □

The specific implementations of this construction for proving Theorems 2.8 and 2.10 will have in common that $g_2(s,t) = 0$ for all $s, t \in \mathcal{S}$, i.e., the grandparent-grandchild edges are never open in the percolation processes. This may at first sight appear paradoxical, as it suggests that these edges are superfluous and that it would be possible to construct automorphism invariant percolation processes with the desired properties already on the regular tree $\mathbb{T}_n$, which would contradict Theorems 2.7 and 2.9. However, the grandparent-grandchild edges *are* essential, because they restrict the class of graph automorphisms, thereby expanding the class of automorphism invariant percolation processes (even if we restrict to those that live only on the parent-child edges).

The lack of open grandparent-grandchild edges in our constructions implies that the connected components will be trees. Each such tree will have a ξ-topmost vertex. The specific constructions will be all about controlling the branching behavior of these trees as we move downwards (away from ξ) in the tree. In the construction for proving Theorem 2.8, a finite number of encounter points will be obtained by letting the tree branch a finite number of times, below which there are only naked branches downwards to ∞. In the construction for Theorem 2.10, we exploit the fact that the speed at which simple random walk on a tree moves away from its starting point depends on how fast the tree branches, and we will make the branching of the tree oscillate between binary branching (corresponding to speed $\frac{1}{3}$) and trinary branching (speed $\frac{1}{2}$) over suitably increasing spatial scales.

Proof of Theorem 2.8. Take $\mathcal{S} = \{1, 2, \ldots, k, k+1, k+2\}$. The construction will be such that $Y(v) = i$ corresponds to the event that from v there are exactly $k+2$ paths to ∞ going "downwards" (i.e., away from ξ) from v – these paths are self-avoiding but not necessarily disjoint. The transition mechanisms $Q_1, \ldots, Q_{k+2}$ will all be deterministic (though randomness will automatically be introduced when passing from Q_i to the permutation invariant version $\hat{Q}_i$), and defined as follows:

$$\begin{cases} \text{For } i = 1 & Q_1((1, k+2, k+2, \ldots, k+2)) = 1 \\ \text{For } i = 2, \ldots, k+2 & Q_i((i-1, 1, k+2, k+2, \ldots, k+2)) = 1. \end{cases}$$

The corresponding Markov chain on $\mathcal{S}$ is irreducible and aperiodic, so it has a unique stationary distribution ρ. This specifies the first stage (Scheme I) of the construction.

The second stage (Scheme II) is as follows. As already declared, the grandparent-grandchild edge function will be $g_2(i,j) = 0$ for all $i, j \in \mathcal{S}$. The parent-child function will be given by

$$g_1(i,j) = \begin{cases} 0 & \text{if } j = k+2 \\ 1 & \text{otherwise.} \end{cases}$$

This defines the percolation process. The interpretation of $Y(v) = i$ as the number of infinite paths downwards from v is easy to check, first for $i = 1$, and then, inductively, for $i = 2, 3, \ldots, k+2$. It furthermore follows from the construction that every infinite cluster has exactly $k+2$ paths to infinity, and that the number of vertices in the cluster with $X(v) = i$ is

$$\begin{cases} \infty & \text{for } i = 1 \\ 1 & \text{for } i = 2, \ldots, k+2\,. \end{cases} \tag{7}$$

Again from the construction we have that the degree of a vertex v with $Y(v) = i$ is 2 if $i = 1$ or $i = k+2$, while if $i \in \{2, 3, \ldots, k+1\}$ then the degree is 3. In this latter case v is also an encounter point (which obviously it cannot be in case the degree is 2). From (7) we know there are exactly k vertices v in each infinite cluster with $Y(v) \in \{2, 3, \ldots, k+1\}$, so there are exactly k encounter points, as desired. $\Box$

Remark. The construction proving Theorem 2.8 can easily be modified to obtain automorphism invariant percolation processes on $G_{\mathcal{T},n}$ in which infinite clusters with different numbers of encounter points coexist. Any subset of $\{0, 1, 2, \ldots\} \cup \infty$ can be obtained. We omit the details. $\Box$

Moving on to the proof of Theorem 2.10 on the existence of percolation processes on $G_{\mathcal{T},n}$ with badly behaved random walk properties, it simplifies matters to restrict to $n \geq 4$. Arguably, the cases $n = 2, 3$ are not terribly important, as the point here is to obtain a nonunimodular counterexample to Theorem 2.9, but will be mentioned for completeness at the very end.

Construction for Theorem 2.10 with $n \geq 2$. This time, we let the state space $\mathcal{S}$ for the vertex variables be $\mathcal{S} = \{0, 1, 2, \ldots\}$, which will be given the following interpretation. $Y(v) = 0$ indicates that v is the "topmost" vertex of its connected component: it has no open edge to its ξ-parent or its ξ-grandparent, implying that the component contains only v and (some subset of) its descendants. More generally, $Y(v) = i$ denotes how far (i.e. how many generations up) the component's topmost vertex is from v.

Let $M_1 < N_1 < M_2 < N_2 < M_3 \cdots$ be a rapidly increasing sequence of positive integers; exactly how rapidly will be made more specific later. For $s \in \mathcal{S}$, we set Q_s to be deterministic: $Q_s(R_s) = 1$ with

$$R_s = \begin{cases} (s+1, s+1, 0, 0, \ldots, 0) & \text{if } s \in [0, M_1) \cup [N_1, M_2) \cup [N_2, M_3) \cup \cdots \\ (s+1, s+1, s+1, 0, 0, \ldots, 0) & \text{if } s \in [M_1, N_1) \cup [M_2, N_2) \cup [M_3, N_3) \cup \cdots \end{cases} \tag{8}$$

The corresponding Markov chain has the property that at each step, it has probability at least $\frac{n-3}{n}$ of jumping to state 0. Hence it is positive recurrent with a unique stationary distribution ρ. This completes Scheme I and defines the vertex process $\{Y(v)\}_{v \in V_{\mathcal{T},n}}$. Scheme II consists in setting $g_2(s,t) = 0$ for all $s, t \in \mathcal{S}$ (as usual, no grandparent-grandchild edges) and

$$g_1(s,t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{otherwise.} \end{cases}$$

The resulting percolation process $\{X(e)\}_{e \in E_{\mathcal{T},n}}$ consists of infinite trees each containing a root (the topmost vertex) followed by binary splitting down to generation M_1, then trinary splitting down to generation N_1, then binary splitting again down to generation M_2, and so forth alternating between binary and trinary splitting. $\Box$

Consider now random walk $Z_0, Z_1, \ldots$ on an infinite cluster of the percolation process just defined, and note that $Y(Z_0), Y(Z_1), \ldots$ is a discrete time birth-and-death process, i.e. a Markov chain on $\mathcal{S} = \{0, 1, \ldots\}$ with transition matrix $\{P_{i,j}\}_{i,j \in \mathcal{S}}$ such that $P_{i,j} = 0$ unless $|i - j| = 1$. The transition probabilities when $|i - j| = 1$ in this particular birth-and-death process are given by $P_{0,1} = 1$, and, for $i \geq 1$,

$$P_{i,i+1} = 1 - P_{i,i-1} = \begin{cases} \frac{2}{3} & \text{if } i \in [0, M_1) \cup [N_1, M_2) \cup [N_2, M_3) \cup \cdots \\ \frac{3}{4} & \text{if } i \in [M_1, N_1) \cup [M_2, N_2) \cup [M_3, N_3) \cup \cdots \end{cases} \tag{9}$$

Lemma 4.3 *There exists a choice of* $(M_1, N_1, M_2, N_2, M_3, \ldots)$ *such that the birth-and-death process* $Y(Z_0), Y(Z_1), \ldots$ *satisfies*

$$\mathbb{P}\left(\liminf_{k \to \infty} \frac{Y(Z_k)}{k} = \frac{1}{3} \right) = 1 \tag{10}$$

and

$$\mathbb{P}\left(\limsup_{k \to \infty} \frac{Y(Z_k)}{k} = \frac{1}{2} \right) = 1 \, . \tag{11}$$

Proof. Assume first that the initial value $Y(Z_0)$ is 0. The birth-and-death process stochastically dominates one in which $P_{i,i+1} = \frac{2}{3}$ for every $i \geq 1$ and for which $\frac{Y(Z_k)}{k}$ would tend to $\frac{1}{3}$ a.s.; hence the actual birth-and-death process satisfies $\liminf_{k \to \infty} \frac{Y(Z_k)}{k} \geq \frac{1}{3}$ a.s., regardless of the choice of $(M_1, N_1, M_2, N_2, M_3, \ldots)$. We similarly obtain $\limsup_{k \to \infty} \frac{Y(Z_k)}{k} \leq \frac{1}{2}$ regardless of $(M_1, N_1, M_2, N_2, M_3, \ldots)$. For the slightly more intricate conclusions $\liminf_{k \to \infty} \frac{Y(Z_k)}{k} \leq \frac{1}{3}$ and $\limsup_{k \to \infty} \frac{Y(Z_k)}{k} \geq \frac{1}{2}$ we need to make a judicious choice of $(M_1, N_1, M_2, N_2, M_3, \ldots)$.

Fix a decreasing sequence $\varepsilon_1, \varepsilon_1', \varepsilon_2, \varepsilon_2', \varepsilon_3, \ldots$ tending to 0. For any $m \geq 0$, define the random hitting time $T_m = \inf\{k : Y(Z_k) = m\}$. Note that a birth-and-death process whose "jump to the right" probabilities $P_{i,i+1}$ exceed and are bounded away from $\frac{1}{2}$ is transient, so that in particular $\mathbb{P}(T_m < \infty) = 1$ for any m.

If, hypothetically, we had $P_{i,i+1} = \frac{2}{3}$ for every $i \geq 1$, then $\frac{Y(Z_k)}{k}$ would tend to $\frac{1}{3}$ a.s. as $k \to \infty$. Hence, by choosing M_1 sufficiently large, we can ensure that

$$\mathbb{P}\left(\frac{Y(Z_{T_{M_1}})}{T_{M_1}} \leq \frac{1}{3} + \varepsilon_1 \right) \geq 1 - \varepsilon_1 \, .$$

Fix such an M_1.

Next note, again hypothetically, that if we had $P_{i,i+1} = \frac{3}{4}$ for all $i \geq M_1$, then we would get $\lim_{k \to \infty} \frac{Y(Z_k)}{k} = \frac{1}{2}$ a.s., because by transience the process will hit values to the left of M_1 only finitely often, which is not enough to influence $\lim_{k \to \infty} \frac{Y(Z_k)}{k}$. Hence, by picking N_1 large enough, we can make sure that

$$\mathbb{P}\left(\frac{Y(Z_{T_{N_1}})}{T_{N_1}} \geq \frac{1}{2} - \varepsilon_1' \right) \geq 1 - \varepsilon_1' \, .$$

Fix such an N_1.

We then continue iteratively. Given $(M_1, N_1, M_2, \ldots, N_{i-1})$, we pick M_i large enough to ensure that

$$\mathbb{P}\left(\frac{Y(Z_{T_{M_i}})}{T_{M_i}} \leq \frac{1}{3} + \varepsilon_i \right) \geq 1 - \varepsilon_i \, , \tag{12}$$

and then N_i large enough so that

$$\mathbb{P}\left(\frac{Y(Z_{T_{N_i}})}{T_{N_i}} \geq \frac{1}{2} - \varepsilon_i' \right) \geq 1 - \varepsilon_i' \, . \tag{13}$$

We thus get a sequence $(M_1, N_1, M_2, N_2, M_3, \ldots)$ ensuring that (12) and (13) hold for any i. Since $\lim_{i\to\infty} \varepsilon_i = 0$ and $\lim_{i\to\infty} \varepsilon_i' = 0$, we get the inequalities

$$\liminf_{k\to\infty} \frac{Y(Z_k)}{k} \le \frac{1}{3} \text{ and } \limsup_{k\to\infty} \frac{Y(Z_k)}{k} \ge \frac{1}{2}$$

needed to deduce (10) and (11). Hence the lemma is established for the case $Y(Z_0) = 0$.

For the more general case $Y(Z_0) = i$, note that we can couple $(Y(Z_0), Y(Z_1), \ldots)$ with another birth-and-death process $(Y(Z_0^*), Y(Z_1^*), \ldots)$ starting with $Y(Z_0^*) = 0$ in such a way that from the first time T_i^* that the latter process hits state i, it acts as a time-delayed version of the former:

$$(Y(Z^*_{T_i^*}), Y(Z^*_{T_i^*+1}), Y(Z^*_{T_i^*+2}), \ldots) = (Y(Z_0), Y(Z_1), Y(Z_2), \ldots).$$

Since $\lim_{k\to\infty} \frac{T_i^*+k}{k} = 1$, and since we already know that $\liminf_{k\to\infty} \frac{Y(Z_k^*)}{k} = \frac{1}{3}$, we get

$$\begin{aligned}
\liminf_{k\to\infty} \frac{Y(Z_k)}{k} &= \liminf_{k\to\infty} \frac{Y(Z^*_{T_i^*+k})}{k} = \liminf_{k\to\infty} \frac{T_i^*+k}{k} \frac{Y(Z^*_{T_i^*+k})}{T_i^*+k} \\
&= \liminf_{k\to\infty} \frac{Y(Z^*_{T_i^*+k})}{T_i^*+k} = \liminf_{l\to\infty} \frac{Y(Z_l^*)}{l} = \frac{1}{3}.
\end{aligned}$$

Similarly, $\limsup_{k\to\infty} \frac{Y(Z_k)}{k} = \limsup_{l\to\infty} \frac{Y(Z_l^*)}{l} = \frac{1}{2}$, so (10) and (11) are established for arbitrary starting values, and the proof is complete. $\square$

Proof of Theorem 2.10. Consider first the case $n \ge 4$. We then define the percolation process as above, and note that what we need to show is that

$$\lim_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(Z_0, Z_k)}{k} \text{ fails to exist a.s.} \tag{14}$$

Define w as the topmost vertex (in the direction of ξ) in the infinite cluster containing Z_0, and note that since, for any k, Z_k is a ξ-descendant of w, we get the following. If $Y(Z_k)$ is even, then the shortest path from w to Z_k in G consists $\frac{Y(Z_k)}{2}$ grandparent-grandchild edges, whereas if $Y(Z_k)$ is odd, this shortest path consists of $\frac{Y(Z_k)-1}{2}$ grandparent-grandchild edges plus one parent-child edge. Hence,

$$\mathrm{dist}_{G_{\mathcal{T},n}}(w, Z_k) = \begin{cases} \frac{Y(Z_k)}{2} & \text{if } Y(Z_k) \text{ is even} \\ \frac{Y(Z_k)+1}{2} & \text{if } Y(Z_k) \text{ is odd.} \end{cases} \tag{15}$$

Transience yields $\lim_{k\to\infty} Y(Z_k) = \infty$, so that $\lim_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(w,Z_k)}{Y(Z_k)} = \frac{1}{2}$. In combination with Lemma 4.3, this gives

$$\liminf_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(w, Z_k)}{k} = \frac{1}{6} \text{ and } \limsup_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(w, Z_k)}{k} = \frac{1}{4} \text{ a.s.} \tag{16}$$

Next note that $\lim_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(w,Z_0)}{k}$ because $\mathrm{dist}_{G_{\mathcal{T},n}}(w, Z_0)$ is a constant (random, but independent of k). This, together with the triangle inequality for $\mathrm{dist}_{G_{\mathcal{T},n}}$, allows us to go from (16) to

$$\liminf_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(Z_0, Z_k)}{k} = \frac{1}{6} \text{ and } \limsup_{k\to\infty} \frac{\mathrm{dist}_{G_{\mathcal{T},n}}(Z_0, Z_k)}{k} = \frac{1}{4} \text{ a.s.,}$$

so (14) follows and the proof for $n \geq 4$ is complete.

For the cases $n = 2$ and $n = 3$, the above construction doesn't work (for $n = 2$ the defining equation (8) fails to make sense, whereas for $n = 3$ we run into problems when trying to establish positive recurrence of the corresponding Markov chain). There are various ways to modify it. For instance, we can switch from letting the percolation process live on parent-child edges to letting it live on grandparent-grandchild edges; this leaves plenty of room for alternating between binary and trinary splitting already for $n = 2$. Adapting the proof for $n \geq 4$ to such a construction is tedious but straightforward, so we omit the details. $\square$

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