Probability on bunkbed graphs

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Abstract

We discuss various stochastic models (random walk, percolation, the Ising and random-cluster models) on so-called bunkbed graphs, i.e., on any graph that is obtained as a Cartesian product of the complete graphs on two vertices and another graph. A new correlation inequality for the Ising model on bunkbed graphs is presented.

1 Introduction

A major area of probability theory today is the study of stochastic models on graphs, and in particular of how properties of the graph are reflected in the behavior of the stochastic models; see, e.g., Aldous [1], Woess [15], Häggström [10], Lyons [13], and Lyons and Peres [14]. The graphs are usually finite or countably infinite. Here, we shall for simplicity focus on finite graphs.

We shall also specialize to so-called **bunkbed graphs**. Given any graph $G = (V, E)$, we can define its corresponding bunkbed graph $G_2 = (V_2, E_2)$ as follows. Imagine G drawn in the plane, with an exact copy of it positioned straight above it in the third coordinate direction, and with, for each $v \in V$, an edge connecting v to its copy right above it. In more precise language, $G_2 = (V_2, E_2)$ is obtained by taking

$$V_2 = V \times \{0, 1\}$$

and

$$E_2 = \{\langle (u,i), (v,i) \rangle : \langle u, v \rangle \in E, i \in \{0,1\}\} \cup \{\langle (u,0), (u,1) \rangle : u \in V\}.$$

An edge $e \in E_2$ will be called horizontal if it is of the form $\langle (u,i), (v,i) \rangle$, and vertical if it is of the form $\langle (u,0), (u,1) \rangle$. We shall always assume that G is connected, so that G_2 is connected as well.

Consider now two vertices $(u,0)$ and $(v,0)$ downstairs in G_2, together with the node $(v,1)$ sitting upstairs, right above $(v,0)$. Is $(u,0)$ "closer" to $(v,0)$ than to $(v,1)$? If we measure "closeness" in the usual graph distance, then the answer is yes: the shortest path from $(u,0)$ to $(v,0)$ is precisely one step shorter than the shortest path from $(u,0)$ to $(v,1)$. Other definitions of closeness in graphs are possible, but it is tempting to suggest that any reasonable (whatever that means!) such definition should tell us $(u,0)$ and $(v,0)$ are closer to each other than are $(u,0)$ and $(v,1)$.

In this paper, we will discuss how this philosophy is reflected in various stochastic models on bunkbed graphs. Most of this discussion has appeared earlier, but only in Swedish; see Häggström [11].

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It is reasonable to expect, given a nice stochastic model on G_2, that the relevant two-point quantities such as correlations or connectivity probabilities should exhibit stronger relations between $(u,0)$ and $(v,0)$ than between $(u,0)$ and $(v,1)$. The (somewhat vague) hope is that if we can prove such things for bunkbed graphs, then this may of lead to a better understanding of "closeness" on more general graphs.

In the next section, we shall recall three of the most basic and important stochastic models on graphs – random walks, percolation, and the Ising model – with particular focus on how they behave on bunkbed graphs. A recent result (Theorem 2.4) concerning the Ising model on bunkbed graphs is presented in the end of that section. In Section 3, we shall recall the intimate relation between the Ising model and a certain dependent percolation model known as the random-cluster model. Finally, in Section 4, we prove Theorem 2.4.

2 Some stochastic models on graphs

2.1 Random walk

A random walk on a graph $G = (V, E)$ is a Markov chain $(X(0), X(1), X(2), \ldots)$ with state space V and the following update mechanism. If $X(n) = v$, and $v \in V$ has d neighbors $w_1, \ldots, w_d$ in G, then $X(n+1)$ is chosen according to uniform distribution on $\{w_1, \ldots, w_d\}$. In other words, at each integer time the random walker stands at a vertex, and chooses at random (uniformly) one of the edges emanating from that vertex, in order to decide where to go next.

A much-studied class of problems for random walks concern hitting probabilities. Given the initial value $X(0) = u$ and two other vertices $v, w \in V$, what is the probability that the random walk hits v before hitting w? In other words, letting $T_v = \min\{n : X(n) = v\}$ and $T_w = \min\{n : X(n) = w\}$, we are interested in $\mathbf{P}(T_v < T_w)$ where, as usual, $\mathbf{P}$ denotes probability. Calculating such probabilities turns out to be equivalent to calculating voltages in a certain electric network on G, and this has turned out to be an extremely useful observation as it allows results for electric networks, such as Rayleigh's monotonicity principle, to come into play in the study of random walks; see, e.g., the wonderful monograph by Doyle and Snell [3]. For bunkbed graphs, Bollobás and Brightwell [2] exploited electrical network arguments in order to show that a random walk starting at $(u,0)$ satisfies

$$\mathbf{P}(T_{(v,0)} < T_{(v,1)}) \geq \frac{1}{2},$$

in agreement with our general philosophy about bunkbed graphs.

A similar question is whether we for a fixed initial value $X(0) = (u,0)$ and a fixed time n have

$$\mathbf{P}(T_{(v,0)} \leq n) \geq \mathbf{P}(T_{(v,1)} \leq n). \tag{1}$$

Somewhat surprisingly, Bollobás and Brightwell found a counterexample demonstrating that (1) is not true in general. If we take $G = (V, E)$ with $V = \{u, v, w\}$ and $E = \{\langle u, v\rangle, \langle v, w\rangle\}$, and start random walk on the corresponding bunkbed graph G_2 at $X_0 = (u,0)$, then the reader will quickly be able to verify that $\mathbf{P}(T_{(w,0)} \leq 3) < \mathbf{P}(T_{(w,1)} \leq 3)$, ruling out the hope that (1) might hold in general. This example appears to be related to some sort of preiodicity of the random walk: for instance, the random walk can only reach $(w,0)$ at even time points. Such periodicities do not occur if the random walk is defined in continuous time (which can be done in a natural way). Bollobás and Brightwell therefore conjectured that an analogue of (1) would hold for continuous time random walk. In Häggström [9], this conjecture was proved.

2.2 Percolation

In standard bond percolation on a graph $G = (V, E)$, we fix the so-called retention parameter $p \in [0, 1]$, and let $\{X(e)\}_{e \in E}$ be i.i.d. (independent and identically distributed) random variables with $\mathbf{P}(X(e) = 1) = p$ and $\mathbf{P}(X(e) = 0) = 1 - p$. The values 0 and 1 should be thought of as "absent" and "present", respectively, and we consider the random subgraph of G obtained by throwing out all edges e with $X(e) = 0$.

Percolation theory deals mainly with connectivity properties of this random subgraph. By far the most studied cases are to take G to be either a complete graph (see Janson et al. [12]) or a regular lattice in two or more dimensions (see Grimmett [8]). In the case with the complete graph, the theory deals mainly with asymptotics when the number of vertices tends to ∞, whereas in the lattice case, the graph is usually taken to be infinte to begin with. In both cases, much of the interest in the models arises from the fact that they exhibit a threshold phenomenon: the probability of getting a very large connected component (one which contains a nontrivial fraction of the original graph) is 0 or 1 depending on whether p is above or below some critical value.

The random subgraphs we are considering are identified in the obvious way with random elements of $\{0, 1\}^E$. Let ϕ_p^G denote the probability measure on $\{0, 1\}^E$ corresponding to the percolation model described above. For two vertices $u, v \in V$, and a $\{0, 1\}^E$-valued random element X, we write $u \stackrel{X}{\longleftrightarrow} v$ for the event that there exists a path from u to v in the subgraph of G corresponding to X. The following conjecture concerning percolation on bunkbed graphs is along the lines of our general intuition for stochastic models on such graphs.

Conjecture 2.1 *Let $G = (V, E)$ be a graph, and let $G_2 = (V_2, E_2)$ be the corresponding bunkbed graph. Let $p \in [0, 1]$, and pick a $\{0, 1\}^{E_2}$-valued random element X according to $\phi_p^{G_2}$. We then have, for any $u, v \in V$, that*

$$\mathbf{P}((u, 0) \stackrel{X}{\longleftrightarrow} (v, 0)) \geq \mathbf{P}((u, 0) \stackrel{X}{\longleftrightarrow} (v, 1)) \,. \tag{2}$$

I have been aware of this conjecture since the mid-1990's, but have not been able to trace its roots.

So far, we have only discussed the most basic percolation model, exhibiting independence between edges. A wide variety of dependent models have been studied as well. The most studied and perhaps also the most important one is the so-called random-cluster model (also known as the Fortuin–Kasteleyn or FK model) which is defined as follows.

Definition 2.2 *Given $G = (V, E)$ and $\xi \in \{0, 1\}^E$, define $k(\xi)$ as the number of connected components (including isolated vertices) in ξ. For $p \in [0, 1]$ och $q > 0$, we define the random-cluster measure $\phi_{p,q}^G$ as the probability measure on $\{0, 1\}^E$ which to each $\xi \in \{0, 1\}^E$ assigns probability*

$$\phi_{p,q}^G(\xi) = \frac{q^{k(\xi)}}{Z_{p,q}^G} \prod_{e \in E} p^{\xi(e)} (1 - p)^{1 - \xi(e)} \,,$$

where $Z_{p,q}^G = \sum_{\xi' \in \{0,1\}^E} q^{k(\xi')} \prod_{e \in E} p^{\xi'(e)} (1 - p)^{1 - \xi'(e)}$ is a normalizing constant.

Note that taking $q = 1$ yields standard bond percolation. Other choices of q yield intricate dependencies between edges. The main reason why this particular generalization has received so much attention is the intimate relation between the case $q = 2$ and the Ising model (to be outlined in Section 3) and the corresponding ralationship between the cases $q = 3, 4, \ldots$ and the so-called Potts model. See Georgii et al. [7] for more on these relationships, and how they are exploited to obtain results concerning the phase transition behavior in Ising and Potts models.

For the random-cluster model on bunkbed graphs, it is reasonable to conjecture that the inequality (2) is valid not only for $\phi_p^{G_2}$ but also for $\phi_{p,q}^{G_2}$ with arbitrary choice of p and q. A priori, this seems harder for dependent cases $q \neq 1$ than for the i.i.d. case $q = 1$, and it may therefore seem strange that we have a proof of the conjecture for $q = 2$ (see Corollary 4.1 below) while not for $q = 1$. The explanation for this surprising situation lies in the aforementioned connection between the $q = 2$ random-cluster model and the Ising model.

2.3 The Ising model

The Ising model gives a way of assigning the values -1 and 1 (called spin values) to the vertices of a graph $G = (V, E)$, in a random but correlated manner. The corresponding probability measures on $\{-1, 1\}^V$ are called Gibbs measures, and their definition is as follows.

Definition 2.3 *For a graph $G = (V, E)$, we define the Gibbs measure μ_β^G for the Ising model on G at inverse temperature $\beta \geq 0$ as the measure on $\{-1, 1\}^V$ which to each configuration $\eta \in \{-1, 1\}^V$ assigns probability*

$$\mu_\beta^G(\eta) = \frac{1}{\tilde{Z}_\beta^G} \exp\left(\beta \sum_{\langle x,y \rangle \in E} \eta(x)\eta(y)\right)$$

where $\tilde{Z}_\beta^G$ is a normalizing constant making μ_β^G a probability measure.

Note that we could alternatively write

$$\mu_\beta^G(\eta) = \frac{1}{\hat{Z}_\beta^G} \exp\left(-2\beta \sum_{\langle x,y \rangle \in E} I_{\{\eta(x) \neq \eta(y)\}}\right) \tag{3}$$

with a different normalizing constant $\hat{Z}_\beta^G$.

In the case $\beta = 0$, the values at different vertices become i.i.d. with distribution $(\frac{1}{2}, \frac{1}{2})$ on $\{-1, 1\}$. Taking $\beta > 0$ retains the marginal distribution $(\frac{1}{2}, \frac{1}{2})$ at a single vertex (because the model is invariant under global interchange of -1's and 1's), but the values at different vertices become dependent, due to fact that μ_β^G favors configurations with agreement along many nearest-neighbor pairs in the graph. In the extreme situation where we let $\beta \to \infty$, we will in the limit have probability $\frac{1}{2}$ for the two configurations -1^V and 1^V in which all vertices agree about their spin value.

The graph G is most often taken to be (a large portion of) some periodic lattice in 2 or 3 dimensions. In the asymptotics as the lattice size tends to ∞, the following very interesting threshold or phase transition phenomenon happens, which is related to the threshold phenomenon for percolation mentioned in the previous subsection. There is a critical value $\beta_c \in (0, \infty)$ (depending on the lattice) such that for $\beta < \beta_c$, the fraction of vertices with spin value 1 will tend to $\frac{1}{2}$ in probability, while for $\beta > \beta_c$ the "desire" to have aligned values at nearest-neighbour pairs is strong enough to break this global symmetry, and yields a substantial majority of either -1's or 1's. This phase transition phenomenon is the main reason for the Ising model being a much-studied object in probability and statistical mechanics; see e.g. [7] for an introduction. Physically, the vertices may be thought of as atoms in a ferromagnetic material, the values -1 and 1 are two possible spin orientations of the atoms, while $\frac{1}{\beta}$ is the temperature, and the phase transition phenomenon means that the material is spontaneously magnetized at low but not at high temperatures.

Another interesting property of the Ising model is that if $Y \in \{-1,1\}^V$ is chosen according to the Gibbs measure μ_β^G, then, for any $u, v \in V$, we have that $Y(u)$ and $Y(v)$ are positively correlated. Because of the ± 1-symmetry of the model, this is the same as saying that

$$\mathbf{E}[Y(u)Y(v)] \geq 0\,. \tag{4}$$

This is a special case of the famous FKG inequality, proved by Fortuin et al. [6], and we shall see a simple proof in the next section, Corollary 3.3.

It is reasonable to expect that, for fixed β , the correlation between $Y(u)$ and $Y(v)$ should be greater, the"closer" u and v are to each other. For bunkbed graphs, we have the following result, first proved in [11].

Theorem 2.4 *Let $G = (V, E)$ be a graph, and let $G_2 = (V_2, E_2)$ be the corresponding bunkbed graph. Fix $\beta \geq 0$, and pick $Y \in \{-1,1\}^{V_2}$ according to the Gibbs measure $\mu_\beta^{G_2}$. We then have, for any $u, v \in V$, that*

$$\mathbf{E}[Y((u,0))Y((v,0))] \geq \mathbf{E}[Y((u,0))Y((v,1))]\,.$$

See Section 4 for the proof.

3 Ising and random-cluster models: the connection

That the random-cluster model has intimate connections to the Ising model was realized already by its inventors Fortuin and Kasteleyn [5]. Today, the best way to understand this connection is via the following explicit coupling.

Theorem 3.1 *Fix $p \in [0,1)$, let $G = (V, E)$ be a graph, and pick $X \in \{0,1\}^E$ according to the random-cluster measure $\phi_{p,2}^G$. Then pick $Y \in \{-1,1\}^V$ as follows. For each connected component of X, let all vertices in the component take the same value, -1 or 1, determined by a fair coin toss. Do this independently for different connected components. Then Y is distributed according to the Ising model Gibbs measure μ_β^G, with $\beta = -\frac{1}{2}\log(1-p)$.*

This was proved by Edwards and Sokal [4] by means of a simple but ingeneous counting argument. The reader will be able to reconstruct it by working out the sum in the right hand side of

$$\mathbf{P}(Y = \eta) = \sum_{\xi \in \{0,1\}^E} \mathbf{P}(X = \xi,\, Y = \eta)$$

where the first thing to do is to check which summands are nonvanishing for a given η. Alternatively, consult the literature, for instance [7].

A useful consequence of Theorem 3.1 is the following.

Corollary 3.2 *Let $G = (V, E)$ be a graph, fix $\beta \geq 0$, and pick $Y \in \{-1, 1\}^V$ according to the Ising model Gibbs measure μ_β^G. Also pick $X \in \{0, 1\}^E$ according to the random-cluster measure $\phi_{p,2}^G$ with $p = 1 - e^{-2\beta}$. Then, for arbitrary $u, v \in V$, we have*

$$\mathbf{E}[Y(u)Y(v)] = \mathbf{P}(u \overset{X}{\longleftrightarrow} v)\,. \tag{5}$$

Proof: We may assume that Y was obtained as in Theorem 3.1. The event $(u \overset{X}{\longleftrightarrow} v)$ then implies that $Y(u)Y(v) = 1$. Conditional on $\neg(u \overset{X}{\longleftrightarrow} v)$, we have that $Y(u)Y(v)$ is -1 and 1 with probability $\frac{1}{2}$ each. Hence,

$$\begin{aligned} &\mathbf{E}[Y(u)Y(v)] \\ &= \mathbf{P}(u \overset{X}{\longleftrightarrow} v)\mathbf{E}[Y(u)Y(v) \mid (u \overset{X}{\longleftrightarrow} v)] + \mathbf{P}(\neg(u \overset{X}{\longleftrightarrow} v))\mathbf{E}[Y(u)Y(v) \mid \neg(u \overset{X}{\longleftrightarrow} v)] \\ &= \mathbf{P}(u \overset{X}{\longleftrightarrow} v)\,. \end{aligned}$$

$\square$

Since the right hand side in (5) is nonnegative, the well-known correlation inequality (4) pops out as an immediate consequence of Corollary 3.2:

Corollary 3.3 *Let $G = (V, E)$ be a graph, fix $\beta \geq 0$, and pick $Y \in \{-1, 1\}^V$ according to the Ising model Gibbs measure μ_β^G. Then, for any $u, v \in V$, we have $\mathbf{E}[Y(u)Y(v)] \geq 0$.*

4 Proof of the new correlation inequality

In this section we give the proof of Theorem 2.4, and finish with a simple corollary for the random-cluster model.

Proof of Theorem 2.4: We wish to show that

$$\mathbf{E}[Y((u,0))(Y((v,0)) - Y((v,1)))] \geq 0\,.$$

For obvious symmetry reasons, we have that

$$\mathbf{E}[Y((u,1))(Y((v,1)) - Y((v,0)))] = \mathbf{E}[Y((u,0))(Y((v,0)) - Y((v,1)))]$$

whence

$$\mathbf{E}[(Y((u,0)) - Y((u,1)))(Y((v,0)) - Y((v,1)))] = 2\mathbf{E}[Y((u,0))(Y((v,0)) - Y((v,1)))]\,.$$

We are therefore done if we can show that

$$\mathbf{E}[(Y((u,0)) - Y((u,1)))(Y((v,0)) - Y((v,1)))] \geq 0\,. \tag{6}$$

Given $Y \in \{-1, 1\}^{V_2}$, define $W \in \{-1, 0, 1\}^V$ by letting $W(w) = \frac{1}{2}(Y((w,0)) - Y((w,1)))$ for each $w \in V$. Define also the (random) graph $G_Y = (V_Y, E_Y)$ by setting

$$V_Y = \{w \in V : W(w) \in \{-1, 1\}\}$$

and

$$E_Y = \{e = \langle w, z \rangle \in E : w, z \in V_Y\}\,.$$

Write $W(V_Y)$ for the spin configuration $\{W(v)\}_{v \in V_Y}$, and note that the triple

$$\{G_Y, \{Y(w,i)\}_{w \notin V_Y, i \in \{0,1\}}, W(V_Y)\}$$

uniquely determines Y. We now make the following claim:

Claim A: Given G_Y and $\{Y(w,i)\}_{w\notin V_Y, i\in\{0,1\}}$, the conditional distribution of $W(V_Y)$ is given by the Gibbs measure $\mu_{2\beta}^{G_Y}$.

For the purpose of proving this, we make the following definitions. For $Y \in \{-1,1\}^{V_2}$, take $n(Y)$ to be the number of "unhappy" edges, i.e.,

$$n(Y) = |\{\langle u,v\rangle \in E_2 : Y(u) \neq Y(v)\}| .$$

Then decompose $n(Y)$ into

$$n(Y) = n_a(Y) + n_b(Y) + n_c(Y)$$

where $n_a(Y)$ is the number of vertical unhappy edges, $n_b(Y)$ is the number of horizontal unhappy edges with at least one endpoint incident to a happy vertical edge, and $n_c(Y)$ is the number of horizontal unhappy edges with both enpoints incident to an unhappy vertical edge. Also let $n^*(W(V_Y))$ denote the number of unhappy edges in G_Y. A moment of thought reveals that if we fix G_Y and $\{Y(w,i)\}_{w\notin V_Y, i\in\{0,1\}}$, then $n_a(Y)$ and $n_b(Y)$ are also fixed. Furthermore, each unhappy edge in G_Y corresponds to a pair of unhappy edges in G_2 (one upstairs and one downstairs) in the category counted by $n_c(Y)$, so that $n_c(Y) = 2n^*(W(V_Y))$. Together with (3), this implies that for fixed G_Y and $\{Y(w,i)\}_{w\notin V_Y, i\in\{0,1\}}$ and two different configurations $\eta, \eta' \in \{-1,1\}^{V_Y}$ we get

$$\frac{\mathbf{P}(W(V_Y) = \eta \,|\, G_Y, \{Y(w,i) : w \notin V_Y, i \in \{0,1\}\})}{\mathbf{P}(W(V_Y) = \eta' \,|\, G_Y, \{Y(w,i) : w \notin V_Y, i \in \{0,1\}\})} \quad = \quad e^{4\beta(n^*(\eta') - n^*(\eta))} .$$

This, in view of (3) and normalization, proves Claim A.

Now, Claim A in conjunction with Corollary 3.3 implies that

$$\mathbf{E}[W(x)W(y) \,|\, G_Y, \{Y(w,i)\}_{w\notin V_Y, i\in\{0,1\}}] \geq 0 \tag{7}$$

whenever $x, y \in V_Y$. If, on the other hand, at least one of the vertices x and y is not in V_Y, then the left hand side in (7) is obviously 0. Integrating over all possible outcomes of G_Y och $\{Y(w,i) : w \notin V_Y, i \in \{0,1\}\}$ gives

$$\mathbf{E}[W(x)W(y)] \geq 0$$

for any $x, y \in V$. Recalling the definition of W, we get

$$\mathbf{E}[(Y((u,0)) - Y((u,1)))(Y((v,0)) - Y((v,1)))] = 4\mathbf{E}[W(x)W(y)] \geq 0 ,$$

and (6) is established, which is what we needed. □

The following result for the random-cluster model follows immediately from Theorem 2.4 combined with Corollary 3.2.

Corollary 4.1 *Let $G = (V,E)$ be a graph, and $G_2 = (V_2, E_2)$ the corresponding bunkbed graph. Let $p \in [0,1]$, and pick $X \in \{0,1\}^{E_2}$ according to the random-cluster measure $\phi_{p,2}^{G_2}$. Then*

$$\mathbf{P}((u,0) \overset{X}{\longleftrightarrow} (v,0)) \geq \mathbf{P}((u,0) \overset{X}{\longleftrightarrow} (v,1))$$

for any $u, v \in V$.

References

[1] Aldous, D. (1991) Applications of random walks on finite graphs, *Selected Proceedings of the Sheffield Symposium on Applied Probability* (ed. I. Basawa and R. Taylor), 12–26, IMS, Hayward, California.

[2] Bollobás, B. and Brightwell, G. (1997) Random walks and electrical resistances in products of graphs, *Discrete Applied Mathematics* **73**, 69–79.

[3] Doyle, P. and Snell, J.L. (1984) *Random Walks and Electric Networks*, Mathematical Monograph **22**, Mathematical Association of America.

[4] Edwards, R.G. and Sokal, A.D. (1988) Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm, *Physical Review D* **38** 2009–2012.

[5] Fortuin, C.M. and Kasteleyn, P.W. (1972) On the random-cluster model. I. Introduction and relation to other models, *Physica* **57**, 536–564.

[6] Fortuin, C.M., Kasteleyn, P.W. and Ginibre, J. (1971) Correlation inequalities on some partially ordered sets, *Communications in Mathematical Physics* **22**, 89–103.

[7] Georgii, H.-O., Häggström, O. and Maes, C. (2001) The random geometry of equilibrium phases, *Phase Transitions and Critical Phenomena, Volume 18* (C. Domb and J.L. Lebowitz, eds), pp 1–142, Academic Press, London.

[8] Grimmett, G.R. (1999) *Percolation* (2nd edition), Springer, New York.

[9] Häggström, O. (1998) On a conjecture of Bollobás and Brightwell concerning random walks on product graphs, *Combinatorics, Probability and Computing* **7**, 397–401.

[10] Häggström, O. (2000) Markov random fields and percolation on general graphs, *Advances in Applied Probability* **32**, 39–66.

[11] Häggström, O. (2003) Sannolikhetsteori på tvåvåningsgrafer, *NORMAT*, to appear.

[12] Janson, S. Łuczak, T. and Ruciński, A. (2000) *Random Graphs*, Wiley, New York.

[13] Lyons, R. (2000) Phase transitions on nonamenable graphs, *Journal of Mathematical Physics* **41**, 1099–1146.

[14] Lyons, R. and Peres, Y. (2003) *Probability on Trees and Networks*, Cambridge University Press, to appear. Preliminary version available at
`http://php.indiana.edu/~rdlyons/`

[15] Woess, W. (2000) *Random Walks on Infinite Graphs and Groups*, Cambridge University Press.