Uniqueness of the infinite entangled component in three-dimensional bond percolation

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Abstract

We prove uniqueness of the infinite entangled component for bond percolation on the three-dimensional cubic lattice above the entanglement critical probability. This improves earlier results by Grimmett and Holroyd.

1 Introduction

In standard bond percolation with retention parameter $p \in [0,1]$, one takes an infinite locally finite graph $G = (V, E)$, and deletes each edge independently with probability $1-p$, thus keeping it with probability p. Retained and deleted edges are also called open and closed, and their status are represented by the symbols 1 and 0. We write $\mathbf{P}_p$ for the resulting product probability measure on $\{0,1\}^E$ with marginal distributions $(1-p, p)$. Of central interest is the possible existence of infinite connected components, and it is well known (and easy to show) that there is a critical value $p_c = p_c(G) \in [0,1]$ such that

$$\mathbf{P}_p(\exists \text{ infinite connected component}) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c\,. \end{cases} \qquad (1)$$

G is often taken to be a periodic lattice in d-dimensional Euclidean space; the main example is the cubic lattice $\mathbf{Z}^d$ with edges between Euclidean nearest neighbors. In $d \geq 2$ dimensions, there is a nontrivial critical phenomenon, i.e. p_c is strictly between 0 and 1. One of the main results in percolation theory says that for lattices of this kind, there is $\mathbf{P}_p$-a.s. no more than one infinite connected component. This result is known as *uniqueness of the infinite cluster*; see e.g. [2], [5] and [6].

There has recently been some interest in studying percolation processes with focus on other aspects than connectivity. Instead of considering infinite *connected* components, one may look at infinite *entangled* components (appropriate definitions will be given in Section 2). A systematic mathematical study of entanglement percolation was initiated by Holroyd [11] and Grimmett and Holroyd [7]. The topic had earlier received attention in the physics literature; we refer to [11] and [7] for pointers to relevant papers.

The main result (Theorem 3.1) of the present paper establishes uniqueness of the infinite entangled component above the so-called entanglement critical probability. We thus provide an affirmative answer to a conjecture in [7] where weaker versions of our main result were obtained.

In the forthcoming paper [8], we shall treat the corresponding problem for so-called *rigidity* percolation (see also [10]), which is another alternative to the usual connectivity concept.

Like the theory of knots, the study of entanglements in bond percolation is a purely three-dimensional affair. Following [11] and [7], we shall restrict our study of entanglement percolation to $\mathbf{Z}^3$, although our methods certainly extend to other periodic lattices in three dimensions.

The next section contains a preliminary discussion of entanglements. In Section 3 we state our uniqueness result for infinite entangled components, which is then proved in Sections 4 and 5.

2 Entanglement percolation

The stage for the percolation process considered in this paper is the cubic lattice $\mathbf{Z}^3$, with edge set

$$E = \{\{x, y\} : x, y \in \mathbf{Z}^3, |x - y| = 1\}$$

where $|\cdot|$ is the Euclidean norm. By a subgraph of $\mathbf{Z}^3$, we mean a subset of E, and we write $\mathcal{G}$ for the family of all such subsets. We wish to have a definition of what it means for a subgraph of $\mathbf{Z}^3$ to be entangled. Our treatment of this issue will be somewhat condensed, and we refer to [11] and [7] for more detail.

By a *sphere* in three dimensions, we mean a closed two-dimensional simplicial complex in $\mathbf{R}^3$ which is homeomorphic to $\{x \in \mathbf{R}^3 : |x| = 1\}$ (a simplicial complex is, loosely speaking, a compact union of finitely many polyhedral pieces; see [14] for a precise definition). The complement of a sphere S has exactly two connected components, which are denoted the *inside* and the *outside* of S, in the obvious way. For any sphere S and any set $A \subset \mathbf{R}^3$, we say that S *separates* A if A intersects both the inside and the outside of S, but not S itself.

For any edge $e \in E$, we write $\langle e \rangle$ for the closed line segment of unit length in $\mathbf{R}^3$ connecting the endpoints of e. For a subgraph $F \in \mathcal{G}$, we write $[F] = \bigcup_{e \in F} \langle e \rangle$.

The definition of a finite entangled graph is relatively undisputable: If $F \in \mathcal{G}$ is finite, then F is said to be entangled if there does not exist any sphere which separates $[F]$ (see [7, Proposition 2.1] for some equivalent conditions). Set

$$\mathcal{F} = \{F \in \mathcal{G} : F \text{ is finite and entangled}\}.$$

For infinite graphs, it turns out to be less clearcut how to define entanglement. As in [11] and [7], we shall therefore consider a class of possible definitions of entanglement:

Definition 2.1 *Let $\mathcal{E}$ be a class of subgraphs of $\mathbf{Z}^3$. We call $\mathcal{E}$ an* **entanglement system** *if the following conditions hold:*

(E1) *The intersection of $\mathcal{E}$ with the set of finite subgraphs of $\mathbf{Z}^3$ is exactly $\mathcal{F}$.*

(E2) *For any $A_1, A_2, \ldots \in \mathcal{E}$ such that any pairwise intersection of their vertex sets is nonempty, we have $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$.*

(E3) *If $A \in \mathcal{E}$, then there is no sphere which separates $[A]$.*

(E4) *$\mathcal{E}$ is translation invariant, i.e. for any $A \in \mathcal{E}$ and any $x \in \mathbf{Z}^3$, we have $A + x \in \mathcal{E}$.*

(E5) *For any $e \in E$, the event*

$$\{\text{the set of open edges contains some infinite subset } A \in \mathcal{E} \text{ containing } e\}$$

is measurable with respect to the usual product σ-field for $\{0,1\}^E$.

The translation invariance requirement in (E4) is not present in [7], although it does appear in [11]. Translation invariance is needed for our arguments in Sections 4 and 5, but we feel that (E4) is a very natural assumption, and that its removal is unlikely to be any interesting gain of generality.

For concreteness, we mention two particular (and distinct) entanglement systems:

$$\begin{aligned}\mathcal{E}_0 &= \{A \in \mathcal{G} : A \neq \emptyset \text{ and every finite subgraph of } A \text{ is contained} \\ &\qquad \text{in some } \mathcal{F}\text{-subgraph of } A\} \\ \mathcal{E}_1 &= \{A \in \mathcal{G} : A \neq \emptyset \text{ and } A \text{ is not separated by any sphere}\}.\end{aligned}$$

It turns out (see [7]) that $\mathcal{E}_0$ and $\mathcal{E}_1$ are "extremal", in the sense that $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}_1$ for any entanglement system $\mathcal{E}$.

For $\omega \in \{0,1\}^E$, let $K(\omega)$ be the set of edges that are open in ω. Also for $\omega \in \{0,1\}^E$, $x \in \mathbf{Z}^3$ and an entanglement system $\mathcal{E}$, define

$$C_x^{\mathcal{E}} = C_x^{\mathcal{E}}(\omega) = \bigcup \{A \subseteq K(\omega) : A \in \mathcal{E} \text{ and } A \text{ contains } x\}.$$

It is shown in [7] that if $C_x^{\mathcal{E}}$ is nonempty, then it is a member of $\mathcal{E}$ (note that since the union may be uncountable, this is not an immediate consequence of (E2)). By an *$\mathcal{E}$-component* (or an entangled component), we mean a maximal $\mathcal{E}$-subgraph of $K(\omega)$. The set of graphs $\{C_x^{\mathcal{E}} : x \in \mathbf{Z}^3\} \setminus \{\emptyset\}$ turns out to be precisely the set of $\mathcal{E}$-components, and these sets partition $K(\omega)$.

For any entanglement system $\mathcal{E}$ and any $p \in [0,1]$, the $\mathbf{P}_p$-probability that there exists some infinite $\mathcal{E}$-component is 0 or 1 (simply by ergodicity of $\mathbf{P}_p$). This probability is furthermore increasing in p, and we may therefore define the *entanglement critical probability* as the number $p_{\mathrm{e}}^{\mathcal{E}} \in [0,1]$ satisfying

$$\mathbf{P}_p(\exists \text{ some infinite } \mathcal{E}\text{-component}) = \begin{cases} 0 & \text{for } p < p_{\mathrm{e}}^{\mathcal{E}} \\ 1 & \text{for } p > p_{\mathrm{e}}^{\mathcal{E}}, \end{cases} \tag{2}$$

analogously to the connectivity critical probability p_c in (1). It is a triviality to show that $0 \le p_{\mathrm{e}}^{\mathcal{E}} \le p_c$, but the corresponding strict inequalities, provided in the following result, are highly nontrivial:

Theorem 2.2 (Holroyd; Aizenman and Grimmett) *For percolation on $\mathbf{Z}^3$ and any entanglement system $\mathcal{E}$, we have*

$$0 < p_{\mathrm{e}}^{\mathcal{E}} < p_c. \tag{3}$$

The first inequality in (3) was proved in [11], where in fact it was shown that $p_{\mathrm{e}}^{\mathcal{E}} \ge 1/15616$; the second goes back to Aizenman and Grimmett [1].

3 Uniqueness of the infinite entangled component

When the probability in (2) is 1, it is natural to ask for the number of infinite entangled components. By ergodicity it is an a.s. constant. We shall prove the following:

Theorem 3.1 *For percolation on $\mathbf{Z}^3$ with any entanglement system $\mathcal{E}$ and any $p > p_{\mathrm{e}}^{\mathcal{E}}$, we have*

$$\mathbf{P}_p(\exists \text{ a } \textit{unique} \text{ infinite } \mathcal{E}\text{-component}) = 1. \tag{4}$$

This strengthens the uniqueness results in [7], where it was shown that (4) holds for p sufficiently close to 1, and furthermore that it holds when $\mathcal{E} = \mathcal{E}_0$ and p is greater than the connectivity critical probability p_c. (Observe also that (4) is trivial for $\mathcal{E} = \mathcal{E}_1$.)

We note one shortcoming of Theorem 3.1: It is conceivable that the probability in (2) might be 1 when p is *equal* to the entanglement critical probability $p_{\mathrm{e}}^{\mathcal{E}}$. Theorem 3.1 says nothing about the number of infinite entangled components in that case.

Our proof of Theorem 3.1 consists mainly of two parts, which will be handled in Sections 4 and 5, respectively. The first part is to prove that for $p_1 < p_2$ and the canonical coupling between $\mathbf{P}_{p_1}$ and $\mathbf{P}_{p_2}$, every infinite $\mathcal{E}$-component on level p_2 a.s. contains some infinite $\mathcal{E}$-component on level p_1. This part is based on the arguments used by Häggström and Peres [9] to prove monotonicity (in p) of uniqueness of the infinite cluster for percolation on nonamenable Cayley graphs. We thus note that the intensive recent efforts in percolation theory on "exotic" graph structures (see [4] and [12] for overviews) turns out to be useful also in the classical $\mathbf{Z}^d$ setting. The second part of our proof is a variant of the famous encounter point argument of Burton and Keane [5] for proving uniqueness of the infinite connected component.

We remark that it is very tempting to try to dispose of the first half of the proof, and instead approach the problem directly with Burton–Keane-type arguments. However, due to the rather "non-local" character of entanglement (compared to connectivity; see [11] or [7] for some striking examples), this appears to be difficult.

4 First part of the proof of Theorem 3.1: uniqueness monotonicity

We first recall the usual coupling between $\mathbf{P}_{p_1}$ and $\mathbf{P}_{p_2}$ for $p_1 < p_2$: Let $\mathbf{Q}_{p_1,p_2}$ be the probability measure on $\{0,1\}^E \times \{0,1\}^E$ corresponding to letting each edge $e \in E$ independently take value

$$\begin{cases} (0,0) & \text{w.p. } 1-p_2 \\ (0,1) & \text{w.p. } p_2 - p_1 \\ (1,1) & \text{w.p. } p_1\,. \end{cases}$$

We let $X = (X_1, X_2)$ be a $\{0,1\}^E \times \{0,1\}^E$-valued random element chosen according to $\mathbf{Q}_{p_1,p_2}$. It is clear that X_1 and X_2 have marginal distributions $\mathbf{P}_{p_1}$ and $\mathbf{P}_{p_2}$, and also that the set of open edges in X_1 is a.s. contained in the set of open edges in X_2.

The first part of the proof of Theorem 3.1 consists of proving the following result; an analogous result for infinite connected components in percolation on Cayley graphs was obtained in [9].

Proposition 4.1 *Let $\mathcal{E}$ be any entanglement system. Fix p_1 and p_2 such that $p_{\mathrm{e}}^{\mathcal{E}} < p_1 < p_2 < 1$, and pick $X_1, X_2 \in \{0,1\}^E$ according to the coupling $\mathbf{Q}_{p_1,p_2}$. Then, with probability 1, every infinite $\mathcal{E}$-component in X_2 contains some infinite $\mathcal{E}$-component in X_1.*

Proof: Fix a vertex $x \in \mathbf{Z}^3$, and let A_x denote the event that x is contained in some infinite $\mathcal{E}$-component of X_2 which does not contain any infinite $\mathcal{E}$-component of X_1. Equivalently, A_x can be described as the event that x is contained in some infinite $\mathcal{E}$-component of X_2 which does not even *intersect* any infinite $\mathcal{E}$-component of X_1. By translation invariance, it suffices to show that $\mathbf{Q}_{p_1,p_2}(A_x) = 0$. On the event A_x, define the random variable

$$K_x = \min_{y,z} \mathrm{dist}(y,z) \tag{5}$$

where y ranges over all vertices in the $\mathcal{E}$-component on level p_2 containing x, z ranges over all vertices contained in the union of all infinite $\mathcal{E}$-components on level p_1, and $\mathrm{dist}(\cdot,\cdot)$ is L^1-distance on $\mathbf{Z}^3$. The choice of p_1 ensures that K_x is finite, and it therefore suffices to show that

$$\mathbf{Q}_{p_1,p_2}(A_x, K_x = k) = 0 \tag{6}$$

for any $k \in \{1,2,\ldots\}$. On the event in (6), let N_x be the number of vertices y for which the minimum in (5) is attained. Fix $k \geq 1$. We will separately show that

$$\mathbf{Q}_{p_1,p_2}(A_x, K_x = k, N_x < \infty) = 0 \tag{7}$$

and that

$$\mathbf{Q}_{p_1,p_2}(A_x, K_x = k, N_x = \infty) = 0\,. \tag{8}$$

To show that the probability in (7) is 0, we use a so-called mass-transport argument. Imagine that all vertices of $\mathbf{Z}^3$ are equipped with the same amount of mass, and that each vertex sends some of its mass to other vertices according to some rule depending on the configuration (X_1, X_2). If this rule is translation invariant, then we get as a special case of the Mass-Transport Principle of Benjamini, Lyons, Peres and Schramm [3] that the expected mass sent from some (hence any) vertex equals the expected mass received at a vertex.

Consider the following mass-transport rule. If a vertex v sits in an infinite $\mathcal{E}$-component in X_2 whose minimum distance to the union of all infinite $\mathcal{E}$-components in X_1 is k, and the minimum is achieved for finitely many (say n) of the vertices in v's $\mathcal{E}$-component in X_2, then v sends mass $1/n$ to each of these n minimizers. Otherwise v sends no mass at all. This rule is translation invariant. Each vertex sends at most mass 1, so the expected mass sent from any vertex is at most 1. On the other hand, if the event in (7) had positive probability, then some vertices would receive infinite mass, so the expected mass received would be infinite. This would contradict the Mass-Transport Principle, and we therefore conclude that (7) holds.

It remains to prove (8). Given X_1, define the partition $\{B_1, B_2, B_3\}$ of E as

$$\begin{aligned}
B_1 &= \{e \in E : e \text{ is in some infinite } \mathcal{E}\text{-component of } X_1\}\,, \\
B_2 &= \{e \in E \setminus B_1 : \text{at least one endpoint of } e \text{ is within distance } k-1 \\
&\qquad \text{from some infinite } \mathcal{E}\text{-component of } X_1\}\,, \\
B_3 &= E \setminus (B_1 \cup B_2)\,.
\end{aligned}$$

Let D_x be the event that the set of open edges in B_3 on level p_2 contains some subgraph which

(i) contains x,

(ii) is in $\mathcal{E}$, and

(iii) comes within distance k from B_1 in infinitely many places.

Clearly, the event in (8) implies D_x, whence it suffices to show that

$$\mathbf{Q}_{p_1,p_2}(A_x \mid D_x) = 0\,. \tag{9}$$

On the event D_x, we can find infinitely many disjoint paths $\gamma_1, \gamma_2, \ldots$ of length k from B_1 to the $\mathcal{E}$-component of B_3 containing x. If we condition not only on D_x, but further on X_1 and $X_2(B_3)$, then each edge in B_2 is independently open on level p_2 with probability at least $(p_2 - p_1)/(1 - p_1)$. Hence each γ_i is independently open with probability at least

$$\left(\frac{p_2 - p_1}{1 - p_1}\right)^k$$

and by Borel–Cantelli, a.s. at least one of them will be open. But then the event A_x does not happen, so (9) is established, which in turn implies (8), and the proof is complete. $\Box$

5 Second part of the proof of Theorem 3.1: Burton–Keane combinatorics

Let the coupling $\mathbf{Q}_{p_1,p_2}$ and (X_1, X_2) be as in the previous section, and fix some entanglement system $\mathcal{E}$. Define the extended edge set $\tilde{E}$ for $\mathbf{Z}^3$ as

$$\tilde{E} = \{\{x, y\} : x, y \in \mathbf{Z}^3, x \neq y\},$$

and define the edge configuration $\tilde{X}_1 \in \{0,1\}^{\tilde{E}}$ by setting

$$\tilde{X}_1(e) = \begin{cases} 1 & \text{if the endpoints of } e \text{ are in the same } \mathcal{E}\text{-component of } X_1 \\ 0 & \text{otherwise} \end{cases}$$

for each $e \in \tilde{E}$. This gives a natural identification between $\mathcal{E}$-components in X_1 and connected components in $\tilde{X}_1$. The advantage of working with $\tilde{X}_1$ rather than X_1 is that connectivity is a simpler property to deal with then entanglement; this will allow us to invoke Burton–Keane-type combinatorics below. Also define the configuration $\tilde{X}_2 \in \{0,1\}^{\tilde{E}}$ by setting

$$\tilde{X}_2(e) = \begin{cases} \max(\tilde{X}_1(e), X_2(e)) & \text{for } e \in E, \\ \tilde{X}_1(e) & \text{for } e \in \tilde{E} \setminus E. \end{cases}$$

Lemma 5.1 *Fix p_1 and p_2 such that $p_e^{\mathcal{E}} < p_1 < p_2 < 1$. With $\mathbf{Q}_{p_1,p_2}$-probability 1, if $\tilde{X}_2$ has a unique infinite connected component, then X_2 has a unique infinite $\mathcal{E}$-component.*

Proof: Suppose $\tilde{X}_2$ has a unique infinite connected component. Then this connected component contains all infinite $\mathcal{E}$-components of X_1. Hence, by Proposition 4.1, we have $\mathbf{Q}_{p_1,p_2}$-a.s. that every infinite $\mathcal{E}$-component of X_2 intersects the infinite connected component of $\tilde{X}_2$. But by repeated use of property (E2) of entanglement systems, we see that any two vertices in the infinite connected component of $\tilde{X}_2$ are in the same $\mathcal{E}$-component of X_2. The lemma follows. $\Box$

We shall stick to the assumption that $p_e^{\mathcal{E}} < p_1 < p_2 < 1$ throughout the rest of this section. In view of Lemma 5.1, our task is to show that $\tilde{X}_2$ has $\mathbf{Q}_{p_1,p_2}$-a.s. a unique infinite connected component. The first step in this direction is to prove the following.

Lemma 5.2 *The number of infinite connected components in $\tilde{X}_2$ is a $\mathbf{Q}_{p_1,p_2}$-a.s. constant, which equals either 1 or ∞.*

Proof: This follows by a standard Newman–Schulman [13] argument. First, the a.s. constance follows by ergodicity of $\mathbf{Q}_{p_1,p_2}$. Second, suppose for contradiction that

$$\mathbf{Q}_{p_1,p_2}(\tilde{X}_2 \text{ has exactly } k \text{ infinite connected components}) > 0$$

for some $k \in \{2, 3, \ldots\}$. Any configuration $\omega \in \{0,1\}^{\tilde{E}}$ which has exactly k infinite connected components can be turned into one with strictly fewer infinite connected components by turning on finitely many edges in E. This, in combination with the fact that each $e \in E$ is present in $\tilde{X}_2$ with conditional probability at least $\frac{p_2-p_1}{1-p_1}$ given the status of all other edges, easily implies that

$$\mathbf{Q}_{p_1,p_2}(\tilde{X}_2 \text{ has at most } k-1 \text{ infinite connected components}) > 0$$

which contradicts the a.s. constance of the number of infinite connected components. $\Box$

Next, for $N \in \{1, 2, \ldots\}$ and $x \in \mathbf{Z}^3$, let $\Lambda_{N,x}$ denote the set of edges in E that have both endpoints in $[-N, N]^3 + x$. Also let $\partial\Lambda_{N,x}$ denote the set of edges that have *exactly one* of its endpoints in $[-N, N]^3 + x$.

Lemma 5.3 *For any positive integer N and any $x \in \mathbf{Z}^3$, we have that every infinite connected component of $\tilde{X}_2$ which contains some vertex in $[-N, N]^3 + x$, also contains some edge in $\partial\Lambda_{N,x}$.*

Proof: Assume that the vertex $y \in [-N, N]^3 + x$ is in an infinite connected component of $\tilde{X}_2$. Then there is an infinite self-avoiding path in $\tilde{X}_2$ starting at y. This path must contain some edge e with one endpoint in $[-N, N]^3+x$ and the other in $\mathbf{Z}^3 \setminus [-N, N]^3+x$. If $e \in E$, then $e \in \partial\Lambda_{N,x}$ and we are done. On the other hand, if $e \in \tilde{E} \setminus E$, then its endpoints are in the same $\mathcal{E}$-component of X_1. This $\mathcal{E}$-component of X_1 must then contain some edge e' in $\partial\Lambda_{N,x}$ because otherwise that $\mathcal{E}$-component would be separated by a sphere (for instance, the sphere given by the boundary of the set $[-(N + \frac{1}{2}), N + \frac{1}{2}]^3 + x$), contradicting property (E3) of entanglement systems. But e' is clearly in the same connected component of $\tilde{X}_2$ as y, so the proof is complete. $\Box$

What remains in order to prove uniqueness of the infinite connected component in $\tilde{X}_2$ (Lemma 5.5 below), is to invoke a reasonably straightforward Burton–Keane-type argument. For a finite set Z whose cardinality $|Z|$ is at least 3, we define a 3-*partition* of Z to be a set $H = \{H_1, H_2, H_3\}$ of disjoint nonempty subsets of Z with $H_1 \cup H_2 \cup H_3 = Z$. Burton and Keane [5] provide the following combinatorial lemma:

Lemma 5.4 (Burton and Keane) *Let $\mathbf{H}$ be a nonempty set of 3-partitions of a set Z, such that for any $H, H' \in \mathbf{H}$ with $H \neq H'$ we have*

$$H_i \cup H_j \subseteq H'_k \text{ for some } i, j, k \in \{1, 2, 3\} \text{ with } i \neq j\,.$$

Then $|Z| \geq |\mathbf{H}| + 2$.

Lemma 5.5 *$\tilde{X}_2$ has $\mathbf{Q}_{p_1,p_2}$-a.s. a unique infinite connected component.*

Proof: Assume for contradiction that $\tilde{X}_2$ has infinitely many infinite connected components $\mathbf{Q}_{p_1,p_2}$-a.s.; by Lemma 5.2, this is the only case that needs to be ruled out. We can then find an $n < \infty$ such that with positive probability, $\Lambda_{n,x}$ is intersected by at least 3 infinite connected components (this probability clearly does not depend on the choice of $x \in \mathbf{Z}^3$). This implies, by the same arguments as in the proof of Lemma 5.2, that

$$\mathbf{Q}_{p_1,p_2}(L_{n,x}) > 0 \tag{10}$$

where $L_{n,x}$ is the event that

(i) all edges in $\Lambda_{n,x}$ are open,

(ii) some infinite connected component intersects $\Lambda_{n,x}$, and

(iii) the (unique) connected component intersecting $\Lambda_{n,x}$ would split into at least 3 infinite connected components (plus possibly some finite connected components) if all edges in $\Lambda_{n,x}$ were removed.

Call $x \in \mathbf{Z}^3$ an *encounter point* if the event $L_{n,x}$ happens (note that we are using a definition of encounter point which differs from the standard one introduced in [5]); equation (10) then asserts that there exist encounter points with positive probability. Furthermore, call $x \in \mathbf{Z}^3$ a *special encounter point* if it is an encounter point that has no other encounter points within distance $n+1$. We claim that special encounter points exist with positive probability. To see this, note that if x and y are encounter points, then we may remove an edge from $\Lambda_{n,y} \setminus \Lambda_{n,x}$ so that y is no longer an encounter point while x still is. This "killing" of encounter points in the vicinity of x may be repeated until x is a special encounter point.

Let $\varepsilon > 0$ be the probability that a given vertex $x \in \mathbf{Z}^3$ is a special encounter point. Let $\mathbf{0}$ denote the origin in $\mathbf{Z}^3$. Pick an integer N large enough so that

$$\varepsilon(2(N-n)+1)^3 > 6(2N+1)^2 \tag{11}$$

(the significance of this choice is that the left-hand side of (11) is the expected number of special encounter points in $[-(N-n), N-n]^3$, and the right-hand side is the number of edges in $\partial\Lambda_{N,\mathbf{0}}$).

Suppose for a given edge configuration that $x_1, \ldots, x_l$ with $l \geq 1$ are special encounter points in $[-(N-n), N-n]^3$, that are all in the same infinite connected component. Let Z be the set of edges in $\partial\Lambda_N$ that are in the same connected component as $x_1, \ldots, x_l$. For each $m \in \{1, \ldots, l\}$, we can partition Z into $r \geq 3$ nonempty sets $H_1^m, \ldots, H_r^m$, in such a way that two edges in Z fall in the same set H_i^m if and only if they would fall in the same connected component if all edges in Λ_{n,x_m} were removed (that $r \geq 3$ follows from Lemma 5.3 and the definition of encounterpoints). Next set $\bar{H}_1^m = H_1^m$, $\bar{H}_2^m = H_2^m$ and $\bar{H}_3^m = Z \backslash (H_1^m \cup H_2^m)$, so that $(\bar{H}_1^m, \bar{H}_2^m, \bar{H}_3^m)$ is a 3-partition of Z. A moment's thought reveals that the set of 3-partitions $\{(\bar{H}_1^m, \bar{H}_2^m, \bar{H}_3^m)\}_{m=1,\ldots,l}$ satisfies the combinatorics of Lemma 5.4 (this is where it is essential to consider special encounter points rather than simply all encounter points). Hence $|Z| \geq l+2$. By summing over all connected components that contain special encounter points in $[-(N-n), N-n]^3$, we get that the number of open edges in $\partial\Lambda_{N,\mathbf{0}}$ must be at least the number of special encounter points in $[-(N-n), N-n]^3$. By taking expectations (denoted $\mathbf{E}$), we get that

$$\begin{aligned}
&\mathbf{E}(\text{number of open edges in } \partial\Lambda_{N,\mathbf{0}}) \geq \\
&\quad \geq \mathbf{E}(\text{number of special encounter points in } [-(N-n), N-n]^3) \\
&\quad = \varepsilon(2(N-n)+1)^3 \\
&\quad > 6(2N+1)^2 \\
&\quad = |\partial\Lambda_{N,\mathbf{0}}|
\end{aligned}$$

(the last inequality is by the choice of N), and we have the desired contradiction. $\square$

Proof of Theorem 3.1: Immediate by applying Lemmas 5.5 and 5.1, with $p_2 = p$ and $p_1 = \frac{p_e^{\mathcal{E}}+p}{2}$. $\square$

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References

[1] Aizenman, M. and Grimmett, G.R. (1991) Strict inequalities for critical points in percolation and ferromagnetic models, *J. Statist. Phys.* **63**, 817–835.

[2] Aizenman, M., Kesten H. and Newman, C.M. (1987) Uniqueness of the infinite cluster and continuity of connectivity functions for short- and long-range percolation, *Commun. Math. Phys.* **111**, 505–532.

[3] Benjamini, I., Lyons, R., Peres, Y. and Schramm, O. (1999) Group-invariant percolation on graphs, *Geom. Funct. Anal.* **9**, 29–66.

[4] Benjamini, I. and Schramm, O. (1999) Recent progress on percolation beyond $\mathbf{Z}^d$, `http://www.wisdom.weizmann.ac.il/~schramm/papers/pyond-rep/index.html`

[5] Burton, R. and Keane, M. (1989) Density and uniqueness in percolation, *Commun. Math. Phys.* **121**, 501–505.

[6] Grimmett, G.R. (1997) Percolation and disordered systems, *Lectures on Probability Theory and Statistics (Saint-Flour 1996)*, 153–300, *Lecture Notes in Math.* **1665**, Springer, Berlin.

[7] Grimmett, G.R. and Holroyd, A.E. (1999) Entanglement in percolation, *J. London Math. Soc.*, to appear.

[8] Häggström, O. (1999) Uniqueness in two-dimensional rigidity percolation, in preparation.

[9] Häggström, O. and Peres, Y. (1999) Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously, *Probab. Th. Rel. Fields* **113**, 273–285.

[10] Holroyd, A.E. (1998) Existence and uniqueness of infinite components in generic rigidity percolation, *Ann. Appl. Probab.* **8**, 944–973.

[11] Holroyd, A.E. (1998) Existence of a phase transition in entanglement percolation, *Math. Proc. Cambridge Phil. Soc.*, to appear.

[12] Lyons, R. (1999) Phase transitions on nonamenable graphs, preprint.

[13] Newman, C.M. and Schulman, L.S. (1981) Infinite clusters in percolation models, *J. Statist. Phys.* **26**, 613–628.

[14] Rourke, C.P. and Sanderson, B.J. (1972) *Introduction to Piecewise-Linear Topology*, Springer, New York.

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