Home exam in **Randomized algorithms**, 2001.

The exam consists of five problems, each being worth four points, so that the total number of points is 20. Chalmers students need 10/20 to pass, whereas GU students need 12/20 and Ph.D. students need 14/20. **Deadline, for returning the solutions, is Wednesday, May 23, 2001.** For other rules concerning this exam, see `http://www.math.chalmers.se/~olleh/randomiserade_algoritmer.html`

1. Many problems take the shape of a question that just requires a simple YES/NO answer. Suppose that we have such a problem (whose answer depends on some input), and a randomized algorithm A with the properties that (i) the running time grows at most polynomially in the size of the input, and (ii) for any input, the probability of producing the correct answer is at least 0.51.

   (a) Show how to, given the algorithm A, construct another algorithm A^* which also runs in polynomial time, and which produces a correct answer with probability at least 0.99. (In case A^* involves running A many times, you should be explicit about *how* many times.)

   (b) Can a similar result be obtained if the probability in (ii) is 0.5 rather than 0.51?

2. This question concerns **reversibility vs nonreversibility** for Markov chains with very few states.

   (a) Show that any Markov chain with just two states is reversible.

   (b) Show that for three states, the above is no longer true. In other words, find an example of a three-state Markov chain which is not reversible.

3. **A Markov chain on permutations.** For a positive integer m, let S be the set of permutations of $(1, 2, \ldots, m)$. Consider the Markov chain with state space S and the following transition mechanism. At each integer time n, we pick $i, j \in \{1, 2, \ldots, m\}$ with $i < j$ uniformly (i.e., each pair has probability $\frac{2}{m(m-1)}$ of being picked). Then the subsequence of the permutation X_n starting in the i:th element and ending in the j:th element, is reversed. More precisely, if

$$X_n = (a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_{m-1}, a_m)\,,$$

   then

$$X_{n+1} = (a_1, a_2, \ldots, a_{i-1}, a_j, a_{j-1}, \ldots, a_{i+1}, a_i, a_{j+1}, \ldots, a_{m-1}, a_m)\,.$$

   (a) Show that this Markov chain is irreducible.

   (b) Discuss the relevance of the result in (a) to the simulated annealing algorithm in Example 12.3 of *"Finite Markov chains and algorithmic applications"*.

4. A simple version of the so-called **Widom–Rowlinson model** is defined as follows. Let G be a finite graph (for instance, a finite portion of the square lattice), with vertex set V. An assignment $\xi \in \{-1, 0, +1\}^V$ of -1's, 0's and $+1$'s to the vertices of G, is said to be **WR-feasible** if $\xi(u)\xi(v) \geq 0$ for all pairs of vertices that share an edge. In other words, ξ is WR-feasible if no edge has a -1 at one endpoint and a $+1$ at the other. The Widom–Rowlinson model now arises by picking a WR-feasible assignment at random, with equal probability for each such assignment. Write μ_G for the corresponding probability distribution on $\{-1, 0, +1\}^V$. (This can be seen as a model for matter and anti-matter, that cannot coexist at close distance.)

   Propose a sensible MCMC algorithm for simulating μ_G, and verify that it has the desired convergence property.

5. Consider the Gibbs sampler for the hard-core model, as described in Example 7.2 of *"Finite Markov chains and algorithmic applications"*. A couple of years ago, it was proved that if we restrict to graphs of degree at most 4 (i.e., every vertex has at most 4 neighbors), then the time taken for the distribution $\mu^{(n)}$ at time n to come within total variation distance ε from the target distribution μ_G, is at most $Ck^3 \log(\varepsilon^{-1})$. Here k is the number of vertices in the graph, and C is a constant (not depending on k or on ε).

   Use this result to show that there exists a randomized polynomial time approximation scheme for counting the number of feasible hard-core configurations of such graphs.