Invariant percolation on trees and the mass-transport method

Olle Häggström
Mathematical Statistics, Chalmers University of Technology
412 96 Göteborg, Sweden
olleh@math.chalmers.se

1. Introduction

In bond percolation on an infinite locally finite graph $G = (V, E)$, each edge is randomly assigned value 0 (absent) or 1 (present) according to some probability measure on $\{0,1\}^E$. One then studies connectivity properties of the random subgraph of G which arises by removing each edge with value 0. Maximal connected components of that subgraph are called clusters, and of particular interest is the possible existence of infinite clusters.

Here we focus on the case where G is the regular tree $\mathbf{T}_n$ of order $n \geq 2$. That is, $\mathbf{T}_n$ is the (unique) infinite connected graph that has no circuits and in which there are exactly $n+1$ edges emanating from each vertex. We write $\mathbf{E}_n$ and $\mathbf{V}_n$ for the edge and vertex sets of $\mathbf{T}_n$.

The most studied choice of probability measure on $\{0,1\}^E$ is i.i.d. measure. When $G = \mathbf{T}_n$, this reduces to the study of Galton–Watson branching processes with binomial offspring distribution. Here we consider the more general class of *automorphism invariant* probability measures on $\{0,1\}^{\mathbf{E}_n}$, i.e. measures that are invariant under graph automorphisms of $\mathbf{T}_n$; a graph automorphism for $\mathbf{T}_n$ is a bijection $\gamma : \mathbf{V}_n \to \mathbf{V}_n$ that preserves adjacency, together with the induced mapping $\gamma' : \mathbf{E}_n \to \mathbf{E}_n$.

There are several interesting examples (besides i.i.d. measure) of such probability measures, including the random-cluster model and uniform spanning forests; see e.g. Häggström (1996, 1997, 1998).

In Häggström (1997), we showed that any automorphism invariant probability measure on $\{0,1\}^{\mathbf{E}_n}$ whose marginal probability that an edge is present is at least $2/(n+1)$, produces infinite clusters with positive probability (this bound was also shown to be sharp). The proof involved a mass-transport argument, which was extended and exploited with great success by Benjamini, Lyons, Peres and Schramm (1998a). The mass-transport method is discussed in Section 2.

Several results concerning the number and topological structure of infinite clusters were also given in Häggström (1997). For space reasons, some of the proofs were omitted in the final version of that paper. In Section 3 we shall recall these results and give new proofs, which are short and simple, based on the mass-transport method.

2. The mass-transport method

Let $\mathrm{Aut}(\mathbf{T}_n)$ denote the set of graph automorphisms for $\mathbf{T}_n$. If μ is an automorphism invariant probability measure on $\{0,1\}^{\mathbf{E}_n}$, then every $\gamma \in \mathrm{Aut}(\mathbf{T}_n)$ acts as a measure-preserving transformation on the probability space $(\{0,1\}^{\mathbf{E}_n}, \mu)$. Let $m(x, y, \omega)$ be a nonnegative function of three variables: two vertices $x, y \in \mathbf{V}_n$, and $\omega \in \{0,1\}^{\mathbf{E}_n}$. Intuitively, $m(x, y, \omega)$ should be thought of as the mass transported from x to y given the configuration ω. We assume that $m(\cdot,\cdot,\cdot)$ is invariant under the diagonal action of $\mathrm{Aut}(G)$, i.e., $m(x, y, \omega) = m(\gamma x, \gamma y, \gamma\omega)$ for all x, y, ω and $\gamma \in \mathrm{Aut}(\mathbf{T}_n)$.

Theorem 2.1 (The mass-transport principle for $\mathbf{T}_n$) *Given $m(\cdot,\cdot,\cdot)$ as above, let*

$$M(x,y) = \int_{\{0,1\}^{\mathbf{E}_n}} m(x,y,\omega)\, d\mu(\omega)\,.$$

Then the expected total mass transported out of any vertex x equals the expected total mass transported into x, i.e.,

$$\sum_{y \in \mathbf{V}_n} M(x,y) = \sum_{y \in \mathbf{V}_n} M(y,x)\,.$$

This is a special case of the mass-transport principle proved by Benjamini et al. (1998a). Their result extends to the case where $\mathbf{T}_n$ is replaced by an arbitrary Cayley graph, or more generally by a so called unimodular transitive graph (without unimodularity, the result fails). The mass-transport method has turned out to be extremely useful in the study of percolation on nonamenable graph structures, where it replaces density arguments which are available only in amenable settings; see e.g. Benjamini et al. (1998a, 1998b), Häggström and Peres (1999), Häggström, Peres and Schonmann (1998) and Lyons and Schramm (1998) for numerous interesting applications. The usefulness and simplicity of the mass-transport method will also be exemplified in the next section.

3. Some results on infinite clusters

It is natural to ask for the number of infinite clusters produced by a percolation process. Theorem 3.1 below says that the number of infinite clusters in automorphism invariant percolation on $\mathbf{T}_n$ is a.s. either 0 or ∞, except in the trivial case where all edges are present. To avoid this triviality, we call a probability measure μ on $\{0,1\}^{\mathbf{E}_n}$ *nice* if it assigns probability 0 to the configuration where all edges are present, and we write $\mathcal{A}_n$ for the class of all nice automorphism invariant probability measures on $\{0,1\}^{\mathbf{E}_n}$. For $\omega \in \{0,1\}^{\mathbf{E}_n}$, write $K(\omega)$ for the number of infinite clusters in ω.

Theorem 3.1 *For $\mu \in \mathcal{A}_n$, we have*

$$\mu(\omega:\ K(\omega) \in \{0,\infty\}) = 1\,.$$

Proof. Let X be a $\{0,1\}^{\mathbf{E}_n}$-valued random element with distribution μ. Obtain $X^* \in \{0,1\}^{\mathbf{E}_n}$ from X as follows. If $K(X) \in \{0,\infty\}$, then let $X^* = X$. Otherwise pick one of the infinite clusters of X uniformly at random, and delete all edges that are not in that infinite cluster. Hence, if $K(X) \in \{1,2,\ldots\}$, then $K(X^*) = 1$. Furthermore, if $\mu \in \mathcal{A}_n$, then the distribution μ^* of X^* is clearly in $\mathcal{A}_n$, and we have therefore reduced the problem to showing that

$$\mu(\omega : K(\omega) = 1) = 0 \tag{1}$$

for all $\mu \in \mathcal{A}_n$.

Assume for contradiction that $\mu \in \mathcal{A}_n$ and (1) fails. Consider the mass-transport where if $K(\omega) \neq 1$, no mass at all is sent, while if $K(\omega) = 1$, then each vertex x which is not in the infinite cluster sends unit mass to the (unique) vertex y in the infinite cluster which is closest to x. Then the expected mass sent from any vertex is at most 1. On the other hand, each vertex on the boundary of a unique infinite cluster receives infinite mass, so that the expected mass received at any vertex is infinite. This contradicts the mass-transport principle (Theorem 2.1). QED

The number of *ends* of an infinite cluster C in $\mathbf{T}_n$, is defined as the number of different (but not necessarily disjoint) infinite self-avoiding paths in C leading out of a given vertex x in C. Note that this definition is independent of the choice of x. Say that an infinite cluster is of *type* j if it has exactly j ends. For $j \in \{1, 2, \ldots\} \cup \{\infty\}$ and $\omega \in \{0,1\}^{\mathbf{E}_n}$, write $K_j(\omega)$ for the number of infinite clusters of type j in ω.

Theorem 3.2 *For $\mu \in \mathcal{A}_n$, we have*

$$\mu(\omega : \textit{ all infinite clusters are of type } 1,\ 2 \textit{ or } \infty) = 1\,. \tag{2}$$

Moreover, for $j = 1, 2, \infty$, we have $\mu(K_j(X) = 0$ or $K_j(X) = \infty) = 1$.

Proof. Call $x \in \mathbf{V}_n$ in an infinite cluster C an *encounter point*, if there are at least three disjoint infinite self-avoiding paths from x in C. It is easy too see that if C is of type $j \in \{3, 4, \ldots\}$, then C contains a finite nonzero number of encounter points.

To prove the first part of the theorem, assume for contradiction that $\mu \in \mathcal{A}_n$ and (2) fails. Consider the mass transport where each vertex sitting in an infinite cluster C containing a finite nonzero number of encounter points sends away unit mass, and distributes it equally among all encounter points in C. Then any encounter point in an infinite cluster of type $j \in \{3, 4, \ldots\}$ receives infinite mass. Hence the expected mass received at a vertex is ∞, while the expected mass sent is at most 1. This contradicts the mass-transport principle.

To prove the second part of the theorem, fix $j \in \{1, 2, \infty\}$, let X have distribution $\mu \in \mathcal{A}_n$, obtain X^* from X by deleting all edges that are not in infinite clusters of type j, note that the distribution of X^* is in $\mathcal{A}_n$, and apply Theorem 3.1. QED

Call an end of an infinite cluster C *isolated* if the corresponding infinite self-avoiding open path starting at a given vertex x in C eventually does not intersect any other infinite self-avoiding open path in C starting at x, and note that this definition is independent of the choice of x.

Theorem 3.3 *For $\mu \in \mathcal{A}_n$, we have that μ assigns zero probability to the existence of isolated ends in infinite clusters of type ∞.*

Proof. Infinite clusters of type ∞ contain encounter points, and therefore each isolated end in an infinite cluster has a "last" enounter point, which we call the *shoulder* of that end. Assume for contradiction that $\mu \in \mathcal{A}_n$ assigns positive probability to the existence of isolated ends in infinite clusters of type ∞. Consider the mass-transport where each vertex sitting "beyond" the shoulder in such an end sends unit mass to its shoulder. Then a shoulder receives infinite mass. Hence, the expected mass sent from a vertex is at most 1, and the expected mass received is ∞, giving the usual contradiction. QED

Suppose now that measure $\mu \in \mathcal{A}_n$ satisfies the *finite energy* condition, i.e. the conditional probability of an edge being present given all other edges is always strictly between 0 and 1. Suppose also that μ assigns positive probability to the existence of infinite clusters of type 1 or 2 (in which case there are infinitely many such clusters, by Theorem 3.2). By using the (by now standard) arguments of Newman and Schulman (1981) involving finite modifications of configurations, we see that this implies that the existence of infinite clusters of type 3 also has positive μ-probability. But this contradicts Theorem 3.2, so we have proved the following.

Theorem 3.4 *If $\mu \in \mathcal{A}_n$ satisfies the finite energy condition, then μ-a.s. each infinite cluster has infinitely many ends.*

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SUMMARY

Some results concerning infinite clusters in automorphism invariant percolation on a regular tree, are recalled from a 1997 paper by the same author. New simple proofs, using the mass-transport method, are presented.

RÉSUMÉ

Nous rappelons certains résultats d'un article de 1997 par le même auteur concernant les amas infinis dans la percolation invariante à l'automorphisme sur des arbres réguliers. Pour ces résultats, nous présentons des nouvelles preuves simples basées sur la méthode de transport de masse.