Uniqueness of infinite rigid components in percolation models: the case of nonplanar lattices

Olle Häggström*†

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Abstract

We prove uniqueness of the infinite rigid component for standard bond percolation on periodic lattices in d-dimensional Euclidean space for arbitrary d, and more generally when the lattice is a quasi-transitive and amenable graph. Our approach to uniqueness of the infinite rigid component improves earlier ones, that were confined to planar settings.

1 Introduction

One of the most celebrated results in percolation theory is that for i.i.d. site or bond percolation on the cubic lattice $\mathbf{Z}^d$, there is a.s. at most one infinite connected component; see Harris [12], Aizenman, Kesten and Newman [1], Burton and Keane [5] and Grimmett [8]. As has been noted many times by various authors, the arguments of Burton and Keane extend easily to other periodic lattices in Euclidean space, as well as to amenable (quasi-)transitive graphs in general.

There has in recent years been some interest in studying not only *connected* components, but also *rigid* components, in percolation models. Jacobs and Thorpe [16] report numerical findings, while rigorous mathematical treatments can be found in the papers by Holroyd [13, 14, 15] and Häggström [9]. Here we will be interested in the rigidity analogue of the uniqueness of infinite connected components. This was studied for the case of bond percolation on the triangular lattice in two dimensions in [13] and [9] (the reason for studying the triangular rather than the more usual square lattice is that the latter contains no nontrivial rigid components). In [13], uniqueness of the infinite rigid component was shown to hold for all but at most countably many values of the retention parameter p. Then, in [9], this result was extended to all p. The techniques in [13] and [9] both rely heavily on planarity. The purpose of the present paper is to develop techniques for proving uniqueness of infinite rigid components in nonplanar settings. Our main result is the following; see Section 3 for careful definitions and (in Theorem 3.4) a more general version of the result.

Theorem 1.1 *Let G be a rigid d-dimensional periodic lattice with $d \geq 2$, and let $p_d(G)$ be the rigidity percolation critical value for G. Then, for bond percolation on G with retention parameter $p > p_d(G)$, there is a.s. a* unique *infinite rigid component.*

*Research supported by the Swedish Research Council

†Mathematical Statistics, Chalmers University of Technology, 412 96 Göteborg, Sweden, olleh@math.chalmers.se, http://www.math.chalmers.se/~olleh/

Some remarks:

1. The definition of infinite rigid components is not altogether straightforward. In [15], two natural (and non-equivalent) definitions are discussed, corresponding to the "free" and "wired" boundary conditions that are nowadays standard in models such as the random-cluster model [7] and uniform spanning forests [3]. As in [13] and [9], we shall employ the definition corresponding to free boudary. Under the alternative definition, corresponding to wired boundary, uniqueness of infinite rigid components becomes a trivial statement.

2. Rigidity is, as we shall see in the next section, a dimension-dependent concept. Mathematically, there is nothing strange about studying d'-dimensionally rigid components on a d-dimensional lattice with $d \neq d'$, but physically the interesting case is obviously $d = d'$. Theorem 3.4 below covers both cases, and in fact the possibility $d \neq d'$ does not induce any extra complications for the proof.

3. It is not known whether or not at the rigidity critical value $p_d(G)$ there is an infinite rigid component. One shortcoming of Theorem 1.1 (and of Theorem 3.4) is that it does not rule out the existence of more then one infinite rigid component at criticality. In contrast, the uniqueness result in [9] for the triangular lattice shows that the number of infinite rigid components at criticality must be 0 or 1.

The rest of this paper is organized as follows. In Section 2, we recall the definitions of rigidity and related concepts. In Section 3 we give some definitions pertaining to percolation and the lattice, and state the more general version of the main result. An outline of the proof of the main result is provided in Section 4. Section 5 deals with some general tools in percolation theory that will be of use to us in the actual proofs, which is finally given in Sections 6 and 7. Similarly as in the paper by Häggström [10] on uniqueness in so-called entanglement percolation, the proof splits naturally in two stages: the first stage (carried out in Section 6) is to obtain "uniqueness monotonicity" results in the spirit of Häggström and Peres [11] and Schonmann [19], while the second stage (carried out in Section 7) involves the combinatorial argument for uniqueness introduced by Burton and Keane [5]. We emphasize, however, that our proof is far from being a straightforward adaptation of the techniques in [10], as our focus on rigidity (as opposed to connectivity or entanglement) forces us to invoke several new ideas.

2 Rigidity

In this section, we recall the notion of (generic) d-dimensional rigidity of graphs. We shall essentially follow the exposition in [13], where more background and detail can be found.

Let $G = (V, E)$ be a finite graph, and let d be a positive integer. By a d-dimensional **embedding** of G, we mean a map $r : V \to \mathbf{R}^d$. The pair (G, r) is called a **framework**. Let $\|\cdot\|$ denote Euclidean norm on $\mathbf{R}^d$. A d-dimensional **motion** of a framework (G, r) is a differentiable family $\{r_t : t \in [0, 1]\}$ of embeddings of G such that for each edge $e = \langle x, y\rangle \in E$ and all $t \in [0, 1]$ we have

$$\|r_t(x) - r_t(y)\| = \|r(x) - r(y)\| \tag{1}$$

The motion is said to be **rigid** if (1) holds for all pairs of vertices $x, y \in V$ (and not just for those linked by an edge). The embedding r and the framework (G, r) are said to be rigid if all their d-dimensional motions are rigid.

Whether or not (G, r) is rigid turns out to depend in general not only on G but also on the embedding G. It is known, however, that for a given graph G, either almost all or almost no embedding of G in $\mathbf{R}^d$ are rigid (see, e.g., [6]); here "almost all" is with respect to $\mathbf{R}^{d|V|}$-dimensional Lebesgue measure. This makes the following definition natural.

Definition 2.1 *A finite graph $G = (V, E)$ is said to be d-***dimensionally rigid** *(or simply d-***rigid***) if Lebesgue-almost all embeddings of G in $\mathbf{R}^d$ are rigid.*

To make sense of the main question studied in this paper, we need to define rigidity also for infinite graphs:

Definition 2.2 *An infinite graph $G = (V, E)$ is said to be d-rigid if every finite subgraph of G is contained in some d-rigid finite subgraph of G.*

When considering infinite graphs in this paper, we shall always assume that they are countable and locally finite.

We shall be interested in d-rigid components of a graph G, by which we mean maximal d-rigid subgraphs of G. Such components are not necessarily disjoint, but the following weaker result holds. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, define their union $G_1 \cup G_2$ as the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

Proposition 2.3 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be (finite or infinite) d-rigid graphs. If $|V_1 \cap V_2| \geq d$, then $G_1 \cup G_2$ is d-rigid.*

Note also that $(d+1)$-rigidity of a graph implies d-rigidity, and that 1-rigidity is equivalent to connectivity.

For $d = 2$, it seems fairly clear that the notion of d-rigid components in a graph capture the physically relevant aspects of d-rigidity; see Holroyd [14, Sect. 4.8]. The situation for $d \geq 3$ is, however, more complicated: it is shown in [14] how to construct a graph $G = (V, E)$ with two vertices $x, y \in V$ such that x and y are not in the same 3-rigid component of G, while on the other hand Lebesgue-a.e. embedding r of G in $\mathbf{R}^3$ has the property that all 3-dimensional motions of (G, r) preserve the Euclidean distance between x and y. Be that as it may, our focus in this paper will be on d-rigid components.

3 Lattices, percolation and the main result

The percolation processes that we consider will be living on amenable quasi-transitive graphs, so we need to recall the notions of (quasi-)transitivity and amenability.

An automorphism of a graph $G = (V, E)$ is a bijective mapping $\gamma : V \to V$ such that $\gamma(x)$ and $\gamma(y)$ share an edge in G if and only if x and y do. We write $\mathrm{Aut}(G)$ for the group of automorphisms of G. For $\gamma \in \mathrm{Aut}(G)$, we will sometimes write γ also for the induced mapping from E to E.

Definition 3.1 *A graph $G = (V, E)$ is said to be* **transitive** *if for all $x, y \in V$, there exists a $\gamma \in \mathrm{Aut}(G)$ such that $\gamma(x) = y$. More generally, a graph $G = (V, E)$ is said* *to be* **quasi-transitive** *if V can be partitioned into a finite number of sets (orbits) $V_1, \ldots, V_k$ such that for any $i \in \{1, \ldots, k\}$ and any two vertices $x, y \in V_i$, there exists a $\gamma \in \mathrm{Aut}(G)$ such that $\gamma(x) = y$.*

For a graph $G = (V, E)$ and a subset $W \subset V$ of its vertex set, define the boundary ∂W of W as

$$\partial W = \{x \in V \setminus W : \exists y \in V \text{ such that } \langle x, y \rangle \in E\}.$$

Definition 3.2 *An infinite connected graph $G = (V, E)$ is said to be* **amenable** *if there exists a sequence of finite subsets $W_1, W_2, \ldots$ of V such that*

$$\lim_{n \to \infty} \frac{|\partial W_n|}{|W_n|} = 0.$$

Among the class of quasi-transitive amenable graphs, perhaps the physically most relevant are the periodic lattices in $\mathbf{R}^d$, defined as follows.

Definition 3.3 *A graph $G = (V, E)$ is said to be a* **periodic lattice in $\mathbf{R}^d$** *if there is a finite set $\{v_1, \ldots, v_k\} \in \mathbf{R}^d$ such that*

$$V = \{v_i + z : i \in \{1, \ldots, k\}, z \in \mathbf{Z}^d\},$$

and another finite set $\{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \left(\mathbf{R}^d\right)^2$ such that

$$E = \{\langle x_i + z, y_i + z \rangle : i \in \{1, \ldots, n\}, z \in \mathbf{Z}^d\}.$$

Clearly, a periodic lattice in $\mathbf{R}^d$ is quasi-transitive with orbits $V_i = \{v_i + z : z \in \mathbf{Z}^d\}$. It is also amenable, as can be seen by taking $W_n = V \cap [-n, n]^d$ in Definition 3.2. As an example of a d-rigid d-dimensional periodic lattice, the reader may keep in mind the case $G = (V, E)$ where $V = \mathbf{Z}^d$ and E consists of all pairs of vertices at L_∞-distance 1 from each other. Note that already in the case $d = 2$, this example falls outside of the scope of the approaches of [13] and [9], because it fails to be planar.

For later purposes we introduce some more graph notation. Given $G = (V, E)$ and two vertices $x, y \in V$, we let $\mathrm{dist}(x, y)$ denote graph-theoretic distance between x and y in G. Given $x \in V$ and a positive integer n, we define the ball

$$B(x, n) = \{y \in V : \mathrm{dist}(x, y) \leq n\}.$$

When more than one graph is discussed, we may need to emphasize which graph G the graph-theoretic distance is with respect to, and will then write $\mathrm{dist}_G(x, y)$ for $\mathrm{dist}(x, y)$.

We next introduce percolation. In standard bond percolation, each edge of a graph $G = (V, E)$ is independently assigned value 0 or 1 with probabilities $1 - p$ and p, where $p \in [0, 1]$ is the so-called retention parameter. We write ψ_p for the induced probability measure on $\{0, 1\}^E$. The values 0 and 1 should be interpreted as "closed" and "open", respectively, and we can then go on to study connectivity and rigidity properties of the random subgraph of G consisting of all vertices, together with those edges that are assigned value 1.

By ergodicity of i.i.d. processes, the number of infinite d-rigid components in this random subgraph is, for any d and p, an a.s. constant $K = K(G, d, p)$. It is obvious (if not, see the simultaneous construction in the next section) that the probability of having *at least one* infinite rigid component is increasing in p. Hence, there exists a critical value $p_d = p_d(G) \in [0,1]$ such that

$$\psi_p(\exists \text{ some infinite } d\text{-rigid component}) = \begin{cases} 0 & \text{if } p < p_d \\ 1 & \text{if } p > p_d\,. \end{cases}$$

Holroyd [13] showed that if G is a connected d'-rigid d-dimensional periodic lattice with $d \geq 2$, then $p_{d'}(G) \in (0,1)$, i.e., that the threshold is nontrivial. Using the general stochastic domination result of Liggett, Schonmann and Stacey [17, Thm. 1.3], Holroyd's approach generalizes to prove that the threshold is nontrivial also in the more abstract setting of amenable quasi-transitive d-rigid graphs considered here (and in fact not even amenability is nedded for this). The aforementioned more general variant of our main result is the following.

Theorem 3.4 *Fix $d \in \{1,2,\ldots\}$, and let G be an infinite d-rigid amenable quasi-transitive graph satisfying $p_d(G) < 1$. Then, for all $p > p_d(G)$, the ψ_p-a.s. number $K(G,d,p)$ of infinite rigid components satisfies $K(G,d,p) = 1$.*

Clearly, this result implies Theorem 1.1, where "rigid" can be read as "d'-rigid" with arbitrary d'.

4 Proof outline

In this section we give an outline of the ideas behind the proof of Theorem 3.4, without entering too much detail. With the technology available in modern percolation theory, the natural first attempt when trying to prove a uniqueness (of some infinite component) result is the following two-step procedure.

1. Show, using the ideas of Newman and Schulman [18], that the number of infinite components must be a.s. either 0, 1 or ∞.

2. Rule out the case of infinitely many infinite components by adapting the encounter point arguments of Burton and Keane [5].

In our case, the first sted goes through via a careful but not particularly difficult adaptation of the Newman–Schulman technique; see Proposition 5.2. The second step turns out to be considerably more complicated. The following summary of how we handle the second step is given in a kind of reverse order compared to the detailed presentation in Sections 6 and 7. There, the structure is of the kind "first we carry out argument A, which allows us to prove B, putting us in a position to prove C" etc, whereas in this summary we stick closer to how the proof was actually discovered by explaining that "what we really need to do is C, which follows if we first establish B, which in turn requires A" and so on.

The original Burton–Keane argument (see [5] or [8]) proceeds by assuming (for contradiction) that there are infinitely many infinite connected components, and using local modification of configurations to show that so-called encounter points (see Section 7 for the definition) exist with positive probability. The existence of encounter points is known to be impossible for automorphism invariant percolation on amenable quasi-transitive graphs, and therefore the assumption of having infinitely many infinite components must be false.

Encounter points are defined in terms of connectivity rather than rigidity, and therefore we wish to define an auxiliary percolation process X^* whose connected components correspond to rigid components in the original process X. Trying to construct such a process X^* on G would be asking for trouble, because different rigid components in X will often intersect at a vertex. For this reason, X^* is defined on an auxiliary graph $G^*_{A,N}$ (see Definition 7.1) which (for some suitable A) has $A+1$ vertices corresponding to each vertex in G, and therefore enough space to encompass all rigid components without intersection.

Trying to create encounter points in X^* by means of local modification in X leads to difficulties because, due to the highly non-local character of rigidity observed in [14, 15], local changes in X may alter X^* in infinitely many places in ways that we do not know how to control. How, then, do we construct X^* in a way that globally reflects the rigidity structure of X, yet allows us to create encounter points by modifications that change X^* only locally?

The solution to this problem involves considering the original process X at two different parameter values $p_1 < p_2$ simultaneously. This gives two original processes X_{p_1} and X_{p_2} (coupled in the natural way; see Section 5.1). The auxiliary process X^* is first built up from the rigid components in X_{p_1}, whereas edges added in X_{p_2} are only allowed to be used in a manner which is more contrained: when a certain (finite but fairly large) neighborhood of a vertex x is intersected by three infinite rigid components in X_{p_1}, then we connect the corresponding components in X^* if *all* edges in a somewhat larger neighborhood of x are turned on in X_{p_2}. This is done in such a way that an encounter point is created unless the three infinite components in X^* connect to each other elsewhere.

To ensure that they avoid connecting to each other elsewhere (so that an encouter point is created) with positive probability, what we need to know is the following: there exist with positive probability three distinct infinite rigid components in X_{p_1}, none of which merge into a joint rigid component in X_{p_2}. That would follow if we could establish a rigidity analogue of the following **uniqueness monotonicity** result of Häggström and Peres [11] and Schonmann [19] for infinite connected components in percolation on quasi-transitive graphs:

> The set of values of p for which uniqueness holds a.s. in X_{p_1} is an interval containing $p = 1$. Moreover, the only changes in the set of infinite components that occur as p increases are enlargements and mergers of infinite components; no new infinite components are born.

Proposition 6.1 and Lemma 6.3 will together provide a statement for rigid components that is a tiny bit weaker than the above Häggström–Peres–Schonmann result, but good enough for our purposes. If we assume that there are at least two values p_1 and p_2 at which uniqueness of the infinite rigid component fails, then Proposition 6.1 and Lemma 6.3 in combination with the encounter point argument outlined above will together imply a contradiction. Thus, it reamins to rule out having non-uniqueness at exactly one value of p, and uniqueness monotonicity (Proposition 6.1) rules this out everywhere except at the critical value $p = p_r$ (which is why the critical case is the one that we are unable to handle).

Finally, our approach to proving the monotonicity results Proposition 6.1 and Lemma 6.3 is to mimic the ideas in [11] as closely as possible. What needs to be done is to rule out "new-born" infinite rigid components in the supercritical regime $p > p_r$. Such new-born components can be categorized according to whether they come within some fixed finite distance of the old infinite components in finitely many or infinitely many places. The former case is ruled out using mass-transport arguments (see Section 5.2) without any extra effort compared to [11]. The latter case can be reduced to the former using local modification, and here we need to do much more involved modifications than in [11] in order to avoid the global effects of rigidity. Whenever we turn a set of edges on in such a local modification, we are forced to turn certain edges off in a sufficiently thick layer around the turned-on edges to shield off infinite consequences, and it is partly this that makes the proofs of Proposition 6.1 and Lemma 6.3 long and complicated.

5 Some basic tools

In this section, we consider some general tools in percolation theory that we will need in the following sections for the proof of our main result.

5.1 The simultaneous construction

A standard construction useful for comparing percolation processes on $G = (V, E)$ at different values of the retention parameter p, is the following. Assign, to the edges of G, i.i.d. random variables $\{U(e)\}_{e \in E}$, each being uniformly distributed on the interval $[0, 1]$. Write Ψ for the induced probability measure on $[0, 1]^E$. For any $p \in [0, 1]$, define $\{X_p(e)\}_{e \in E}$ by setting

$$X_p(e) = \begin{cases} 1 & \text{if } U(e) < p \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for each $e \in E$. Clearly, $\{X_p(e)\}_{e \in E}$ has distribution ψ_p. Furthermore, for all $e \in E$, we have

$$X_p(e) \leq X_{p'}(e) \tag{3}$$

whenever $p \leq p'$, and this makes the simultaneous construction a very useful coupling of the measures ψ_p for all $p \in [0, 1]$. It is immediate from (3) that a quantity such as

$$\psi_p(\text{the vertex } x \text{ is in an infinite } d\text{-rigid component})$$

is increasing in p. More elaborate uses of the simultaneous construction can be found, e.g., in [4], [11], [19], [10] and in the following sections.

5.2 Mass transport

The mass-transport method was developed by Benjamini, Lyons, Peres and Schramm [2] in the setting of percolation on nonamenable lattices, where it replaces more classical density arguments that are confined to amenable lattices. In amenable settings, such as in the present paper, mass transport is less indispensable, but still a very convenient device.

Let $G = (V, E)$ be infinite and quasi-transitive. We identify $\gamma \in \text{Aut}(G)$ with a measure-preserving transformation of the probability space $([0, 1]^E, \Psi)$. Let $m(x, y, \omega)$ be a measurable nonnegative function of three variables: two vertices $x, y \in V$ and a realization $\omega \in [0, 1]^E$ of the simultaneous construction. Intuitively, $m(x, y, \omega)$ should be thought of as the amount of mass sent from x to y given the outcome ω of the simultaneous construction. We assume that $m(\cdot, \cdot, \cdot)$ satisfies

(i) $m(x, y, \omega) = 0$ whenever x and y are in different orbits of $\mathrm{Aut}(G)$, and

(ii) invariance under the diagonal action of $\mathrm{Aut}(G)$, i.e., $m(x, y, \omega) = m(\gamma(x), \gamma(y), \gamma(\omega))$ for all $x, y \in V$, all $\omega \in [0,1]^E$ and all $\gamma \in \mathrm{Aut}(G)$.

The following is a special case of the mass-transport principle in [2].

Theorem 5.1 *Let $G = (V, E)$ be quasi-transitive and amenable, let $m(\cdot, \cdot, \cdot)$ be as above, and define $M(x, y) = \int_{[0,1]^E} m(x, y, \omega) d\Psi$. Then, for any $x \in V$, we have*

$$\sum_{y \in V} M(x, y) = \sum_{y \in V} M(y, x) \, .$$

In other words, the expected mass sent out from a vertex x equals the expected mass received at x. (Interestingly, this symmetry fails for certain graphs when the amenability assumption is dropped. The precise notion for deciding when Theorem 5.1 holds turns out to be unimodularity of $\mathrm{Aut}(G)$; see [2].)

5.3 Local modification

Arguments based on local modifications of realizations are very common in percolation theory today; some influential early uses can be found in the papers by Newman and Schulman [18] and Burton and Keane [5]. In Sections 6 and 7 we shall apply this technique several times in a rather involved manner, and for that reason we choose to formalize it as an explicit coupling argument, as follows.

Write $\mathcal{E}$ for the set of all finite subsets of the edge set E of the graph $G = (V, E)$ that we are considering. Since E is countable, we have that $\mathcal{E}$ is countable as well. We can therefore find a probability measure on $\mathcal{E}$ that assigns positive probability to each $\eta \in \mathcal{E}$; let $\mathbf{Q}$ be any such probability measure. For $p \in [0, 1]$, consider now the coupling of two $\{0,1\}^E$-valued random objects, both with distribution ψ_p, obtained in the following way.

1. Pick $F \in \mathcal{E}$ according to $\mathbf{Q}$.

2. Conditionally on Step 1, and for each edge $e \in E \setminus F$ independently, set $X(e) = X'(e) = 0$ or 1 with respective probabilities $1 - p$ and p.

3. Conditionally on Steps 1 and 2, and for each edge $e \in F$ independently, set $X(e) = 0$ or 1, and independently $X'(e) = 0$ or 1, each with probability p of taking value 1.

We call this the local modification coupling, or **LM-coupling** for short. It is immediate that $\{X(e)\}_{e \in E}$ and $\{X'(e)\}_{e \in E}$ will, with probability 1, differ in at most finitely many edges. Furthermore – and this is the key property – the coupling admits conditional probabilities such that for any two edge configurations $\xi, \xi' \in \{0,1\}^E$ that differ only at finitely many edges, we have

$$\mathbf{P}(X' = \xi' \mid X = \xi) > 0 \, . \tag{4}$$

As a first illustration of how the LM-coupling works, we shall use it to prove the following weaker variant of our main result.

Proposition 5.2 *Fix d and p, and let $G = (V, E)$ be as in Theorem 3.4. Then the ψ_p-a.s. number $K = K(G, d, p)$ of infinite d-rigid components satisfies $K \in \{0, 1, \infty\}$.*

The overall structure of the proof will be quite close to that of the corresponding result for connectivity (in place of rigidity) in [18]. The details of the local modification are, however, more involved, due to the less "local" character of rigidity compared to connectivity. For instance, it is possible to add a single edge to an infinite graph consisting only of finite rigid components, and thereby create a single infinite rigid component; an example of this can be obtained by as minor modification of the graph in [15, Fig. 1, p. 1071]. Of course, no such monkey business can happen with connectivity.

Proof of Proposition 5.2: Recall that K is an a.s. constant. Assume for contradiction that $K \in \{2, 3, \ldots\}$. Fixing a vertex x, we can then find an n such that $\psi_p(A_n) > 0$, where A_n is the event that each of the K infinite d-rigid components intersect the ball $B(x, n)$ in at least d vertices. In the LM-coupling, this means that

$$\mathbf{P}(\{X(e)\}_{e \in E} \in A_n) > 0\,. \tag{5}$$

Now let $\tilde{B}$ denote the edge set of some finite rigid subgraph of G containing $B(x, n)$; such a $\tilde{B}$ can be found by the assumption that G is d-rigid. Also, let $\partial_d \tilde{B}$ denote the set of edges in $E \setminus \tilde{B}$ that sit within distance d from $\tilde{B}$.

Now imagine taking an edge consiguration $\xi \in A_n$, and modifying it by first giving value 0 to all edges in $\partial_d \tilde{B}$ that are not in any of the K infinite d-rigid components, and then giving value 1 to all edges in $\tilde{B}$. We claim that the resulting modified configuration ξ' has a unique infinite rigid component. To see this, note first that none of the K original infinite d-rigid components are hurt by the removal of edges in $\partial_d \tilde{B}$, and that by Proposition 2.3, the insertion of edges in $\tilde{B}$ causes them to merge into a single infinite d-rigid component. But we must check also that no additional infinite d-rigid components are created through this insertion. If such a component is created, it must intersect $\tilde{B}$, and therefore (since d-rigidity implies connectivity) it must intersect $\partial_d \tilde{B}$ in at least d vertices. But then it must intersect the union of the K original infinite d-rigid components in at least d vertices (because all edges in $\partial_d \tilde{B}$ that are not in that union, are removed), whence (again using Proposition 2.3) it will merge with them.

Now suppose that $\xi \in \{0, 1\}^E$ is some edge configuration for which A_n happens, and let ξ' be the modification of ξ outlined above. Using the LM-coupling and applying (4) for any such choice of ξ and ξ', we get

$$\mathbf{P}(\{X'(e)\}_{e \in E} \text{ has exactly one infinite } d\text{-rigid component} \,|\, \{X(e)\}_{e \in E} \in A_n) > 0$$

so that, using (5),

$$\begin{aligned} & \mathbf{P}(\{X'(e)\}_{e \in E} \text{ has exactly one infinite } d\text{-rigid component}) \\ \geq\ & \mathbf{P}(\{X(e)\}_{e \in E} \in A_n, \{X'(e)\}_{e \in E} \text{ has exactly one infinite } d\text{-rigid component}) > 0\,. \end{aligned}$$

But since $\{X'(e)\}_{e \in E}$ has distribution ψ_p, we have the desired contradiction to the ψ_p-a.s. constance of the number of infinite d-rigid components. □

The LM-coupling can also be combined with the simultaneous construction. We then pick two $[0, 1]^E$-valued random objects $\{U(e)\}_{e \in E}$ and $\{U'(e)\}_{e \in E}$, both with distribution Ψ, analogously to how $\{X(e)\}_{e \in E}$ and $\{X'(e)\}_{e \in E}$ were picked above: first $F \subset E$ is chosen according to $\mathbf{Q}$, and then $\{U(e)\}_{e\in E}$ and $\{U'(e)\}_{e\in E}$ are taken to be independent on F and identical on $E \setminus F$.

Note that if we pick $(\{U(e)\}_{e\in E}, \{U'(e)\}_{e\in E})$ according to this simultaneous LM-coupling, and then obtain $\{X_p(e)\}_{e\in E}$ from $\{U(e)\}_{e\in E}$ as in (2), and $\{X'_p(e)\}_{e\in E}$ from $\{U'(e)\}_{e\in E}$ in the analogous way, then the pair $(\{X_p(e)\}_{e\in E}, \{X'_p(e)\}_{e\in E})$ has exactly the same joint distribution as in the ordinary LM-coupling.

See (12) for a useful extension of the key property (4) to the setting of simultaneous LM-couplings.

6 Uniqueness monotonicity

This section is devoted to the first stage of the proof of our main result, which is to obtain rigidity analogues of the uniqueness monotonicity results obtained in [11] and [19] for connected components in percolation on nonamenable lattices. Our first such result is the following.

Proposition 6.1 *Let d and $G = (V, E)$ be as in Theorem 3.4, and fix p_1 and p_2 such that $p_d(G) < p_1 < p_2 < 1$. If the ψ_p-a.s. number of infinite d-rigid components $K = K(G, d, p)$ satisfies $K(G, d, p_1) = 1$, then $K(G, d, p_2) = 1$ holds as well.*

The proof will use the following (non-probabilistic) fact about rigidity.

Lemma 6.2 *Let d and r be positive integers, and let $G = (V, E)$ be a (possibly infinite) graph where each vertex has degree at most r. Then there exists a constant $A = A(d, r)$ such that the number of d-rigid components of G containing any given vertex x, is at most A.*

Proof: Call a d-rigid component of G containing x **small** if all its vertices are within distance d from x, and call it **large** otherwise. We shall bound the number of small and large such components separately. Note that the number of vertices at precisely a given distance l from x is at most r^l, so that the total number of vertices within distance d from x is at most $\sum_{l=0}^{d} r^l = \frac{r^{d+1}-1}{r-1}$. The number of small d-rigid components containing x is therefore certainly at most $2^{(r^{d+1}-1)/(r-1)}$.

To bound the number of large d-rigid components containing x, suppose for contradiction that there are at least

$$r^{\sum_{i=1}^{d} i} + 1 = r^{d(d+1)/2} + 1$$

such components. Since d-rigidity implies connectivity, each such component must, for each l, contain at least one vertex at distance exactly l from x. By the pigeonhole principle, at least two of these components must therefore intersect in at least d vertices (one at each of the distances $1, \ldots, d$ from x), and will therefore (due to Proposition 2.3) merge. Hence, an upper bound for the number of large d-rigid components containing x is $r^{d(d+1)/2}$. This completes the proof, with $A = 2^{(r^{d+1}-1)/(r-1)} + r^{d(d+1)/2}$ (although our bound for A is obviously very crude). □

Proof of Proposition 6.1: We shall consider the coupling of two $\{0,1\}^E$-valued random objects $\{X_{p_1}(e)\}_{e\in E}$ and $\{X_{p_2}(e)\}_{e\in E}$ with respective distributions ψ_1 and ψ_2, provided by the simultaneous construction in Section 5.1. By the assumption of the proposition, $\{X_{p_1}(e)\}_{e\in E}$ has, with probability 1, a unique infinite d-rigid component.

We write $\mathcal{C}_{p_1}$ for that component. Let us assume, for the purpose of deriving a contradiction, that $\{X_{p_2}(e)\}_{e\in E}$ has more than one infinite d-rigid component (in which case it has, in view of Proposition 5.2, infinitely many). Clearly, exactly one of these components will contain the vertex set of $\mathcal{C}_{p_1}$, so that there will also be infinite d-rigid components in $\{X_{p_2}(e)\}_{e\in E}$ that do not contain the vertex set of $\mathcal{C}_{p_1}$. We call these latter components **nasty**. Clearly, we are done if we can rule out the existence of nasty components.

We now claim that

$$\text{nasty components will not even } \textit{intersect } \mathcal{C}_{p_1}\,, \tag{6}$$

where intersection of components means intersection of their respective vertex sets. As a first step towards establishing (6), note that each nasty component will intersect $\mathcal{C}_{p_1}$ in at most $d-1$ vertices, because otherwise it would merge with $\mathcal{C}_{p_1}$ due to Proposition 2.3.

Next, consider the following mass transport. Each vertex $x \in V$ looks around to see whether it sits in a nasty component intersecting $\mathcal{C}_{p_1}$. If no, then no mass is sent from x, while if yes, then unit mass is sent from x and distributed equally among all vertices in the same orbit of x that

(i) are in the same nasty component $\mathcal{C}$ as x, and

(ii) minimizes, among all such vertices, the (graph-theoretic) distance from the set of vertices that are both in $\mathcal{C}$ and in $\mathcal{C}_{p_1}$.

(Clearly, any nasty component $\mathcal{C}$ intersecting $\mathcal{C}_{p_1}$ in at most $d-1$ vertices, has only finitely many such minimizers.) If x happens to sit in several nasty components $\mathcal{C}_1, \ldots, \mathcal{C}_k$, then it sends mass 1 as above separately for each such component.

Since G is quasi-transitive and thus of bounded degree, we have from Lemma 6.2 that the number of nasty components intersecting a vertex $x \in V$ is bounded. Therefore, the expected mass sent from each vertex is finite. On the other hand, if there are nasty components intersecting $\mathcal{C}_{p_1}$, then, clearly, some vertices will receive infinite mass. Hence the expected mass received at some vertices will be infinite, and we have a contradiction to the mass-transport principle (Theorem 5.1), so that (6) is established.

Given a nasty component $\mathcal{C}$, define its distance to $\mathcal{C}_{p_1}$ as

$$d_{min}(\mathcal{C}) = \min\{\mathrm{dist}(x,y) : x, y \in V,\ x \text{ is in } \mathcal{C} \text{ and } y \text{ is in } \mathcal{C}_{p_1}\}\,, \tag{7}$$

and let $n_{d_{min}}(\mathcal{C})$ denote the number of such distance-minimizing vertices in $\mathcal{C}$, i.e., $n_{d_{min}}(\mathcal{C})$ is the cardinality of the set

$$\{x \in V : x \text{ is in } \mathcal{C}, \text{ and } \exists y \in \mathcal{C}_{p_1} \text{ such that } \mathrm{dist}(x,y) = d_{min}(\mathcal{C})\}\,. \tag{8}$$

By (6), each nasty component $\mathcal{C}$ must have $d_{min}(\mathcal{C}) \geq 1$. Furthermore, it must satisfy $n_{d_{min}}(\mathcal{C}) = \infty$, because the event of having a finite value of $n_{d_{min}}(\mathcal{C})$ for some nasty component $\mathcal{C}$ is ruled out by a minor modification of the mass-transport argument used in establishing (6). We will now go on to show that for each $k \in \{1, 2, \ldots\}$, we have that

$$\Psi(\exists \text{ a nasty component } \mathcal{C} \text{ with } d_{min}(\mathcal{C}) = k \text{ and } n_{d_{min}}(\mathcal{C}) = \infty) = 0\,. \tag{9}$$

Once we have shown this, we have ruled out the existence of nasty components altogether, thereby completing the proof of the proposition.

Fix k, and consider the following sequential way of revealing the pair of edge configurations $(\{X_{p_1}(e)\}_{e\in E}, \{X_{p_2}(e)\}_{e\in E})$.

1. Reveal $\{X_{p_1}(e)\}_{e \in E}$.
2. Define (based on what we saw in stage 1) the edge set

$$E^* = \{\langle x, y \rangle \in E : \text{for all vertices } z \text{ in } \mathcal{C}_{p_1} \text{ we have } \mathrm{dist}(x, z) \geq k \text{ and } \mathrm{dist}(y, z) \geq k\}, \quad (10)$$

and then reveal $\{X_{p_2}(e)\}_{e \in E^*}$.
3. Finally, reveal $\{X_{p_2}(e)\}_{e \in E \setminus E^*}$.

Define the $\{0,1\}^E$-valued random configuration $\{\tilde{X}(e)\}_{e \in E}$ by setting

$$\tilde{X}(e) = \begin{cases} X_{p_2}(e) & \text{if } e \in E^* \\ X_{p_1}(e) & \text{if } e \in E \setminus E^* . \end{cases} \quad (11)$$

Another way to say this is that $\tilde{X}(e) = 1$ exactly for those edges $e \in E$ for which we know that $X_{p_2}(e) = 1$ already after the second stage of the three-stage revelation procedure above.

A crucial observation now is that any nasty component $\mathcal{C}$ in $\{X_{p_2}(e)\}_{e \in E}$ such that $d_{min}(\mathcal{C}) = k$, is an infinite d-rigid component already in $\{\tilde{X}(e)\}_{e \in E}$. This is because the only difference between $\{X_{p_2}(e)\}_{e \in E}$ and $\{\tilde{X}(e)\}_{e \in E}$ is that some of the edges in $E \setminus E^*$ are open in $\{X_{p_2}(e)\}_{e \in E}$ but closed in $\{\tilde{X}(e)\}_{e \in E}$, while such a $\mathcal{C}$ does not (by the definition of $d_{min}(\mathcal{C})$) contain any edges in $E \setminus E^*$. We call an infinite d-rigid component of $\{\tilde{X}(e)\}_{e \in E}$ **potentially nasty**, if it has $d_{min}(\mathcal{C}) = k$ and contains infinitely many vertices at distance k from $\mathcal{C}_{p_1}$. Since the possibility $n_{d_{min}}(\mathcal{C}) < \infty$ has been ruled out, we have that any nasty component seen in $\{X_{p_2}(e)\}_{e \in E}$ must be identical to a potentially nasty component seen in $\{\tilde{X}(e)\}_{e \in E}$.

The idea now is that after the second stage of the revelation procedure, each potentially nasty component comes close to $\mathcal{C}_{p_1}$ in so many places (infinitely many!) that with probability one there must be some place where the third stage will open up sufficiently many new edges to make the potentially nasty component merge with $\mathcal{C}_{p_1}$. If we had been dealing with connectivity rather than d-rigidity, then things would have been easy from here, following arguments of Häggström and Peres [11]. However, due to the more complicated nature of d-rigidity, we are forced to go through a few more technical arguments.

Fix $x \in V$, and consider the set $B(x, k+d)$ of vertices in G within distance $k+d$ from x. By the assumption that G is d-rigid, we can find a finite rigid subgraph of G containing $B(x, k+d)$, and then an integer $m < \infty$ such that $B(x, m)$ contains all vertices of that finite rigid subgraph. The smallest value of m needed to make this work depends on the choice of $x \in V$, but since G is quasi-transitive we can find an m that works for any x; let us fix such an m.

Call a vertex x in a potentially nasty component $\mathcal{C}$ of $\{\tilde{X}(e)\}_{e \in E}$ **pivotal** if

(i) it has distance k to $\mathcal{C}_{p_1}$, and

(ii) all edges $e \in E^*$ that have both endpoints in $B(x, m)$ satisfy $\tilde{X}(e) = 1$.

Note that pivotality of a vertex can be determined just from the first two stages of the above revelation process. If a vertex x is pivotal, and stage 3 of the revelation process causes all edges in $E \setminus E^*$ within distance m from x to become open in $\{X_{p_2}(e)\}_{e \in E}$, then in fact *all* edges within distance m from x will be open in $\{X_{p_2}(e)\}_{e \in E}$. When this happens to a pivotal vertex x, we say that x **experiences a jackpot**. By the choice of m, a jackpot at x creates a d-rigid component that intersects $\mathcal{C}$ and $\mathcal{C}_{p_1}$ in at least d vertices each, thus (by repeated use of Proposition 2.3) merging them into a single d-rigid component.

From the construction of the pair $(\{X_{p_1}(e)\}_{e\in E}, \{X_{p_2}(e)\}_{e\in E})$, we see that in stage 3 of the revelation procedure, each edge $e \in E \setminus E^*$ that is closed in $\{\tilde{X}(e)\}_{e\in E}$ becomes open in $\{X_{p_2}(e)\}_{e\in E}$ independently with probability $\frac{p_2-p_1}{1-p_1}$. Hence we can find an $\varepsilon > 0$ such that each pivotal vertex x in $\mathcal{C}$ has probability at least ε of experiencing a jackpot, thus causing $\mathcal{C}$ and $\mathcal{C}_{p_1}$ to merge. as described above. If now $\mathcal{C}$ has infinitely many pivotal vertices, then we can find an infinite subset A of the set of pivotal vertices in $\mathcal{C}$, such that no two vertices in A are within distance $2m$ from each other. For different vertices in A, the events that they experience jackpots are conditionally independent (given stages 1 and 2), and by the second Borel–Cantelli lemma we therefore have with probability 1 that at least one jackpot will occur, causing $\mathcal{C}$ to merge with $\mathcal{C}_1$.

Hence, if we can show that with probability 1 each potentially nasty component has infinitely many pivotal vertices, then (9) is established and the proof will be complete. So we need to rule out having a potentially nasty component $\mathcal{C}$ with at most finitely many pivotal vertices. There are two cases to take care of:

CASE 1. $\mathcal{C}$ has a finite nonzero number of pivotal vertices.

CASE 2. $\mathcal{C}$ has no pivotal vertices.

A mass-transport argument takes care of CASE 1: Imagine the mass transport where each vertex x sitting in a potentially nasty component with a finite nonzero number of pivotal vertices sends unit mass, and distributes it equally among those pivotal vertices (or – if they are in the wrong orbit – the vertices in the $\mathrm{Aut}(G)$-orbit of x that are closest to the pivotal vertices); all other vertices send no mass at all. If the scenario in CASE 1 happens with positive probability, then the expected mass received is infinite (because some vertices will receive infinite mass), while the expected mass sent from a vertex is finite, contradicting the mass-transport principle (Theorem 5.1).

In order to take care of CASE 2, we will invoke an argument based on local modification. To this end, we have to insist that the $[0,1]^E$-valued random object $\{U(e)\}_{e\in E}$ underlying the construction of $\{X_{p_1}(e)\}_{e\in E}$ and $\{X_{p_2}(e)\}_{e\in E}$, is jointly constructed with another $[0,1]^E$-valued random object $\{U'(e)\}_{e\in E}$ as in the simultaneous LM-coupling introduced in the last two paragraphs of Section 5.3. Then we obtain $\{X'_{p_1}(e)\}_{e\in E}$ and $\{X'_{p_2}(e)\}_{e\in E}$ from $\{U'(e)\}_{e\in E}$ in the same way that $\{X_{p_1}(e)\}_{e\in E}$ and $\{X_{p_2}(e)\}_{e\in E}$ were obtained from $\{U(e)\}_{e\in E}$. Finally, $\{\tilde{X}'\}_{e\in E}$ is constructed from $\{X'_{p_1}(e)\}_{e\in E}$ and $\{X'_{p_2}(e)\}_{e\in E}$ according to the obvious analogue of (11).

Similarly to (4), it is easy to see that this simultaneous LM-coupling admits conditional probabilities with the following property: for any four edge configurations $\xi_{p_1}, \xi_{p_2}, \xi'_{p_1}, \xi'_{p_2} \in \{0,1\}^E$ satisfying

(i) $\xi_{p_1}(e) \le \xi_{p_2}(e)$ for all $e \in E$,

(ii) $\xi'_{p_1}(e) \le \xi'_{p_2}(e)$ for all $e \in E$, and

(iii) $(\xi_{p_1}(e), \xi_{p_2}(e)) = (\xi'_{p_1}(e), \xi'_{p_2}(e))$ for all but at most finitely many e,

we have

$$\mathbf{P}(X'_{p_1} = \xi'_{p_1}, X'_{p_2} = \xi'_{p_2} \mid X_{p_1} = \xi_{p_1}, X_{p_2} = \xi_{p_2}) > 0\,. \tag{12}$$

Now imagine a realization $X_{p_1} = \xi_{p_1}$ and $X_{p_2} = \xi_{p_2}$ where a vertex $x \in V$ sits at distance k from $\mathcal{C}_{p_1}$, in a potentially nasty component $\mathcal{C}$ that has no pivotal vertices. (This is the scenario of CASE 2.) Define E^* as in (10), and construct the configurations $\xi'_{p_1}, \xi'_{p_2} \in \{0,1\}^E$ as follows.

- For all edges $e \in E$ that have at least one endpoint in $V \setminus B(x, m+d)$, set
$$\xi'_{p_1}(e) = \xi_{p_1}(e) \tag{13}$$
and
$$\xi'_{p_2}(e) = \xi_{p_2}(e)\,. \tag{14}$$

- For all edges $e \in E$ that have both endpoints in $B(x, m+d)$, set
$$\xi'_{p_1}(e) = \begin{cases} 1 & \text{if } e \text{ is a part of } \mathcal{C}_{p_1} \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$
and
$$\xi'_{p_2}(e) = \begin{cases} 1 & \text{if } e \text{ is a part of } \mathcal{C}_{p_1} \text{ or of } \mathcal{C} \\ & \text{ or if } e \in E^* \text{ and has both endpoints in } B(x,m) \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Suppose that $X'_{p_1} = \xi'_{p_1}$ and $X'_{p_2} = \xi'_{p_2}$. It follows from (13) and (15) that the infinite d-rigid component $\mathcal{C}'_{p_1}$ of $\{X'_{p_1}(e)\}_{e\in E}$ is identical to $\mathcal{C}_{p_1}$. Next, consider the infinite rigid component $\mathcal{C}'$ of $\{X'_{p_2}(e)\}_{e\in E}$ containing $\mathcal{C}$. From (14), (16) and the definition of E^*, we get that $\mathcal{C}'$ does not intersect $\mathcal{C}_{p_1}$. Furthermore, since $\mathcal{C}$ is potentially nasty, $\mathcal{C}'$ must be potentially nasty as well, and the assignments of 1's in (16) guarantees that x is a pivotal vertex in $\mathcal{C}'$. No extra pivotal vertices in $\mathcal{C}'$ can be created, except possibly some in the finite region $B(x, m)$. Hence, $\mathcal{C}'$ will have a finite nonzero number of pivotal vertices. This, using (12), shows that if the scenario in CASE 2 has positive probability, then so does the scenario in CASE 1. But CASE 1 has already been ruled out, so CASE 2 is ruled out as well. Hence, any potentially nasty component must (with probability 1) have infinitely many pivotal vertices, and this is what we needed to establish (9), so the proof is complete. $\square$

In the uniqueness monotonicity results in [11] and [19], some information in the case of having multiple infinite components at level p_1 was obtained at no extra charge: by letting the union of infinite connected components at level p_1 play the role of $\mathcal{C}_{p_1}$ above, it was shown that every infinite p_2-component had to contain some infinite p_1-component. One might think that the analogous fact could be extracted from our proof of Proposition 6.1. It seems, however, that (6) is more difficult to establish when $\mathcal{C}_{p_1}$ is the union of infinitely many infinite d-rigid components. The following result is a weaker variant of the desired "nesting" result.

Lemma 6.3 *Let d and $G = (V, E)$ be as in Theorem 3.4, and suppose that $p_1 \in (p_d(G), 1)$ is such that $K(G, d, p_1) = \infty$. Let Ψ be as in Section 5.1. Then, for all $p_2 \in (p_1, 1)$ except possibly for a countable set of exceptional values, we have Ψ-a.s. that every infinite d-rigid component of $\{X_{p_2}(e)\}_{e\in E}$ contains some infinite d-rigid component of $\{X_{p_1}(e)\}_{e\in E}$.*

Before the reader starts to think too hard about how to strengthen this result by removing the qualification *"except possibly for a countable set of exceptional values"*, we note that once our main result (Theorem 3.4) is established, we will know that the desired strengthening holds vacuously.

Proof of Lemma 6.3: Let p_1 be as in the lemma, and pick p_2 and p_3 in such a way that $p_1 < p_2 < p_3 < 1$. Suppose that p_2 is such that it is **not** the case that Ψ-a.s. all infninte d-rigid components in $\{X_{p_2}(e)\}_{e \in E}$ contains some infinite d-rigid component of $\{X_{p_1}(e)\}_{e \in E}$. Mimicking the notation and terminology of the proof of Proposition 6.1, we write $\mathcal{C}_{p_1}$ for the union of infinite d-rigid components in $\{X_{p_1}(e)\}_{e \in E}$, and we call an infinite d-rigid component $\mathcal{C}$ in $\{X_{p_2}(e)\}_{e \in E}$ **nasty** if it does not contain an infinite d-rigid component of $\{X_{p_1}(e)\}_{e \in E}$. For a nasty component $\mathcal{C}$, define $d_{min}(\mathcal{C})$ and $n_{d_{min}}(\mathcal{C})$ as in (7) and (8). (Note that the arguments in the proof of Proposition 6.1 to show that $d_{min} \geq 1$ are invalid in the setting of Lemma 6.3, so here we need to again take into account the possibility that $d_{min} = 0$.)

We now claim that with probability 1,

$$\text{all nasty components } \mathcal{C} \text{ satisfy } n_{d_{min}}(\mathcal{C}) = \infty. \tag{17}$$

To see this, consider the following mass transport. Every vertex in a nasty component $\mathcal{C}$ with $n_{d_{min}}(\mathcal{C}) < \infty$ sends away unit mass, which is distributed equally among the $\mathrm{dist}(x, \mathcal{C}_1)$-minimizers of $\mathcal{C}$ (with the usual modification if these sit in the wrong $\mathrm{Aut}(G)$-orbit). If, with positive probability, there exists a nasty component $\mathcal{C}$ with $n_{d_{min}}(\mathcal{C}) < \infty$, then the expected mass received at a vertex is infinite, while the expected mass sent is finite, giving the usual contradiction to Theorem 5.1. Hence (17) is established.

Fix a nonnegative integer k; we shall now rule out the possibility of having a nasty component $\mathcal{C}$ with $n_{d_{min}}(\mathcal{C}) = k$.

Consider first the conditional distribution of $\{X_{p_3}(e)\}_{e \in E}$ given $\{X_{p_2}(e)\}_{e \in E}$, which is simply as follows: all edges that are open in $\{X_{p_2}(e)\}_{e \in E}$ are open in $\{X_{p_3}(e)\}_{e \in E}$ as well, while those that are closed in $\{X_{p_2}(e)\}_{e \in E}$ become open in $\{X_{p_3}(e)\}_{e \in E}$ independently, each with probability $\frac{p_3 - p_2}{1 - p_2}$. Letting m be as in the proof of Proposition 6.1, we can therefore find an $\varepsilon > 0$ such that for any $x \in V$, the conditional probability of D_x given $\{X_{p_2}(e)\}_{e \in E}$ is always at least ε; here D_x is defined as the event that all edges within distance m from x are open in $\{X_{p_3}(e)\}_{e \in E}$. Furthermore, if $\mathrm{dist}(x, y) \geq 2m$, then the events D_x and D_y are conditionally independent given $\{X_{p_2}(e)\}_{e \in E}$.

If we now see a nasty component $\mathcal{C}$ in $\{X_{p_2}(e)\}_{e \in E}$, then we can, due to (17), find an infinite set $A_{\mathcal{C}}$ of vertices in $\mathcal{C}$ that are all at distance k from $\mathcal{C}_{p_1}$, and with the property that $\mathrm{dist}(x, y) \geq 2m$ for all $x, y \in A_{\mathcal{C}}$. By applying the second Borel–Cantelli lemma to the events $\{D_x : x \in A_{\mathcal{C}}\}$, we see that a.s. at least one of them will happen. Due to our choice of m, this implies that $\mathcal{C}$ will a.s. merge with one of the infinite d-rigid components of $\{X_{p_1}(e)\}_{e \in E}$. Such a merger is bound to happen for any nasty component $\mathcal{C}$ with $d_{min}(\mathcal{C}) = k$, but since k was arbitrary we have it for all nasty components.

Up to here, the proof has been concerned with fixed p_2 and p_3. Note, however, that the conclusion that

> any infinite d-rigid component on level p_2 will on level p_3 have merged with som infinite d-rigid component on level p_1

can be applied to all rational $p_2, p_3 \in (p_1, 1)$ such that $p_2 < p_3$, and the conclusion will hold for all such p_2, p_3 simultaneously. Consequently, if we view the configurations

$\{X_p(e)\}_{e \in E}$ for all $p \in (p_1, 1)$ as a stochastic process where p plays the role of time, then no infinite d-rigid component will avoid being absorbed by $\mathcal{C}_{p_1}$ for any nontrivial interval of "time". That is, any vertex $x \in V$ will sit in an infinite d-rigid component not containing one of the infinite d-rigid components of $\mathcal{C}_{p_1}$, for at most one "timepoint" p. But since V is countable, this means that the existence of nasty components in $\{X_p(e)\}_{e \in E}$ has positive probability for at most countably many $p \in (p_1, 1)$, and the proof is complete. $\square$

7 Encounter points

In this section, we shall combine the results of the previous section with an encounter point argument à la Burton and Keane [5], in order to finally prove Theorem 3.4. The encounter point argument will be applied not to the original percolation process on G, but to a derived process on a modified graph $G^*_{A,N}$ which is defined as follows.

Definition 7.1 *Given a graph $G = (V, E)$ together with two positive integers A and N, we define its (A, N)-**boosted** graph $G^*_{A,N} = (V^*_{A,N}, E^*_{A,N})$ as follows. We set*

$$V^*_{A,N} = V \times \{0, 1, \dots, A\}$$

and

$$E^*_{A,N} = E_1 \cup E_2$$

where

$$E_1 = \{\langle (x,i), (y,j) \rangle : i, j \in \{1, 2, \dots, A\}, \, x, y \in V, \, \langle x, y \rangle \in E\}$$

and

$$E_2 = \{\langle (x,0), (y,j) \rangle : j \in \{1, 2, \dots, A\}, \, x, y \in V, \, \mathrm{dist}_G(x,y) \le N\} \, .$$

The point of considering $G^*_{A,N}$ rather than G directly is that we will be able to translate rigid components in percolation on G into connected components in a certain derived percolation process on $G^*_{A,N}$, to which we will then be able to apply a Burton–Keane argument. Each open edge $\langle (x,i), (y,j) \rangle \in E_1$ will correspond to an open edge $\langle x, y \rangle \in E$, and the "extra space" created by the multiplication of the vertex set allows us to avoid intersection in $G^*_{A,N}$ of distinct rigid components that intersect in G. Vertices of the form $\langle x, 0 \rangle$ and E_2 are designed specifically to create encounter points.

It is easy to see that if G is quasi-transitive, then so is $G^*_{A,N}$. Another graph property that plays a key role when applying encounter point arguments is amenability, and therefore the following lemma is important.

Lemma 7.2 *Let A and N be positive integers, let $G = (V, E)$ be an amenable graph of bounded degree, and let $G^*_{A,N}$ be its (A, N)-boosted graph. Then $G^*_{A,N}$ is amenable.*

Proof: Fix $\varepsilon > 0$, and let W be a finite subset of V such that $\frac{|\partial W|}{|W|} < \varepsilon$; the existence of such a subset is immediate from amenability of G. Define the corresponding subset W^* of $V^*_{A,N}$ as

$$W^* = \{(x,i) : x \in W, i \in \{0, \dots, A\}\} \, .$$

We have $|W^*| = (A+1)|W|$, and our next job is to estimate $|\partial W^*|$. From the construction of $G^*_{A,N}$, we see that if $(x,i) \in \partial W^*$, then $\mathrm{dist}_G(x, W) \le N$. Let r denote the maximum degree in G. Clearly, for any n, there are at most $r^{n-1}|\partial W|$ vertices $x \in V$

such that $\mathrm{dist}_G(x, W) = n$. The number of vertices $x \in V$ with $\mathrm{dist}_G(x, W) \leq N$ is therefore at most

$$|\partial W| \sum_{n=1}^{N} r^{N-1} = \frac{|\partial W|(r^N - 1)}{r - 1}.$$

Hence $|\partial W^*| \leq \frac{(A+1)|\partial W|(r^N-1)}{r-1}$, so that

$$\begin{aligned} \frac{|\partial W^*|}{|W^*|} &\leq \frac{|\partial W|(r^N - 1)}{|W|(r-1)} \\ &\leq \frac{\varepsilon(r^N - 1)}{r - 1}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this can be made arbitrarily small, so $G^*_{A,N}$ is amenable. □

We shall be considering a certain subgroup of the automorphism group $\mathrm{Aut}(G^*_{A,N})$. Each $\gamma \in \mathrm{Aut}(G)$ induces an automorphism $\gamma^* \in \mathrm{Aut}(G^*_{A,N})$ by taking $\gamma^*((x,i)) = (\gamma(x), i)$ for each $x \in V$ and $i \in \{0, 1 \ldots, A\}$. Write $\mathrm{Aut}^*(G^*_{A,N})$ for the class of automorphisms of $G^*_{A,N}$ that arise in such a way from automorphisms of G.

Let us next recall the notion of encounter points. Given a graph $G = (V, E)$ and a configuration $\xi \in \{0,1\}^E$ of open and closed edges in G, we say that the vertex $x \in V$ is an **encounter point** for the configuration ξ if

(i) ξ has three (but not four) edge-disjoint infinite open paths starting at x, and

(ii) ξ is such that if it is modified by setting $\xi(e) = 0$ for all edges e incident to x, then the three edge-disjoint paths (minus their respective first edges) in (i) end up in different connected components of the modified configuration.

Using Lemma 7.2, the following result is easily extracted from the arguments of Burton and Keane [5].

Proposition 7.3 *Fix positive integers A and N, suppose that $G = (V, E)$ is a quasi-transitive and amenable graph, and let $G^*_{A,N}$ be its (A, N)-boosted graph. Suppose furthermore that the random edge configuration $X \in \{0,1\}^{E^*_{A,N}}$ is picked according to an $\mathrm{Aut}^*(G^*_{A,N})$-invariant probability measure. Then, with probability 1, there are no encounter points for X.*

Equipped with this result, we are now, finally, in a position to give the proof of our main theorem (Theorem 3.4).

Let G be as in the theorem. The overall structure of the proof will be as follows: First we assume, for contradiction, that there exists a $p > p_d$ such that the a.s. number $K(G, d, p)$ of infinite d-rigid components in $\{X_p(e)\}_{e \in E}$ satisfies $K(G, d, p) > 1$. From that, we show (for a suitable choice of A and N) how to construct an $\mathrm{Aut}(G^*_{A,N})$-invariant probability measure on $\{0,1\}^{E^*_{A,N}}$ with the property that it yields encounter points with positive probability (this will build on the conclusions of Proposition 6.1 and Lemma 6.3). Since $G^*_{A,N}$ is quasi-transitive and amenable, this contradicts Proposition 7.3, whence we must have $K(G, d, p) = 1$ for all $p > p_d$, completing the proof.

The substantial part of the proof is of course how to construct the probability measure on $\{0,1\}^{E^*_{A,N}}$ that has the desired properties. Let us go ahead with the details.

Proof of Theorem 3.4: Suppose that $K(G, d, p) > 1$ for some $p > p_d$. Then, by Proposition 6.1, we have $K(G, d, p_1) > 1$ for all $p_1 \in (p_d, p)$. Proposition 5.2 then ensures that $K(G, d, p_1) = \infty$ for all such p_1. Fix $p_1 \in (p_d, p)$. Due to Lemma 6.3, we now have in the simultaneous coupling that for all but at most countably $p_2 \in (p_1, p)$,

$$\Psi(\text{each of the infinitely many infinite } d\text{-rigid components of } \{X_{p_2}(e)\}_{e\in E} \text{ contains an infinite } d\text{-rigid component of } \{X_{p_1}(e)\}_{e\in E}) = 1, \tag{18}$$

so let us fix a $p_2 \in (p_1, p)$ such that this holds.

We shall now go on to construct a dependent percolation process $\{X^*(e)\}_{e\in E^*_{A,N}}$ for the graph $G^*_{A,N}$; here A is the maximal number of d-rigid components of $\{X_{p_1}(e)\}_{e\in E}$ that can contain the same vertex in G (Lemma 6.2 guarantees that A is finite) while N is specified as follows. Due to our choice of p_2, we can a.s. find infinitely many infinite d-rigid components in $\{X_{p_1}(e)\}_{e\in E}$ that are all contained in distinct d-rigid components of $\{X_{p_2}(e)\}_{e\in E}$. For a given vertex $x \in V$, we can therefore find an $N < \infty$ such that with positive probability, at least three such infinite d-rigid components of $\{X_{p_1}(e)\}_{e\in E}$ intersect the ball $B(x, N)$. Furthermore, since G is quasi-transitive, we can (and do) choose N in such a way that this holds uniformly for all x.

For later purposes, let M be large enough so that $B(x, M)$ contains the vertex set of a d-rigid subgraph of G containing $B(x, N + d)$; such an M can be found due to the assumption that G is d-rigid. Furthermore, we can (again due to quasi-transitivity of G) take M to be such that this holds uniformly in $x \in V$. Let us fix such an M.

Fix a vertex $x \in V$. For each d-rigid component $\mathcal{C}$ of $\{X_{p_1}(e)\}_{e\in E}$ containing x, we pick a number $I_{x,\mathcal{C}} \in \{1, \dots, A\}$ at random, uniformly. This is done without replacement, so that for no two d-rigid components $\mathcal{C}, \mathcal{C}'$ containing x we have $I_{x,\mathcal{C}} \neq I_{x,\mathcal{C}'}$. We then go on to do this sort of assignment for all vertices $x \in V$ (independently for different vertices).

For each edge $e = \langle (x, i), (y, j)\rangle \in E^*_{A,N}$, we set

$$X^*(e) = \begin{cases} 1 & \text{if there exists a } d\text{-rigid component } \mathcal{C} \text{ of } \{X_{p_1}(e)\}_{e\in E} \text{ such that} \\ & \quad \text{(a) } \mathcal{C} \text{ contains the edge } \langle x, y\rangle \\ & \quad \text{(b) } I_{x,\mathcal{C}} = i, \text{ and} \\ & \quad \text{(c) } I_{y,\mathcal{C}} = j. \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the distribution of $\{X^*(e)\}_{e\in E^*_{A,N}}$ is $\mathrm{Aut}^*(G^*_{A,N})$-invariant. Furthermore, note that if we forget about isolated vertices in $\{X_{p_1}(e)\}_{e\in E}$ and $\{X^*(e)\}_{e\in E^*_{A,N}}$, then there is a natural one-to-one correspondence between on one hand d-rigid components in $\{X_{p_1}(e)\}_{e\in E}$, and on the other hand connected components in $\{X^*(e)\}_{e\in E^*_{A,N}}$.

So far, no edge $e = \langle (x, 0), (y, j)\rangle \in E^*_{A,N}$ takes value 1 in $\{X^*(e)\}_{e\in E^*_{A,N}}$, but we shall soon go on to define a modified edge configuration $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$ where some of these edges are turned on.

Fix $x \in V$, and let ξ_{p_1} and ξ_{p_2} be two $\{0,1\}^E$-valued configurations such that

(i) $\xi_{p_1}(e) \le \xi_{p_2}(e)$ for all $e \in E$, and

(ii) $B(x, N)$ is intersected by (at least) three infinite d-rigid components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of ξ_{p_1}, that are contained in three distinct d-rigid components of ξ_{p_2}.

Then obtain ξ'_{p_1} from ξ_{p_1} by setting

$$\xi'_{p_1}(e) = \begin{cases} 0 & \text{if both endpoints of } e \text{ are within distance } N + d \text{ from } x, \\ & \text{and } e \text{ is not part of any of } \mathcal{C}_1, \mathcal{C}_2 \text{ or } \mathcal{C}_3, \\ \xi_{p_1}(e) & \text{otherwise,} \end{cases}$$

and set $\xi'_{p_2}(e) = \xi_{p_2}(e)$ for all $e \in E$. By applying the simultaneous LM-coupling and (12) to this choice of $(\xi_{p_1}, \xi_{p_2}, \xi'_{p_1}, \xi'_{p_2})$, and by our choice of N, we can deduce that

$$\Psi(D_x) > 0 \tag{19}$$

where the event D_x is defined as

$$\begin{aligned} D_x \;=\; & \{B(x,N) \text{ is intersected by exactly three infinite } d\text{-rigid components in } \{X_{p_1}(e)\}_{e\in E}, \\ & \text{and these are all in different } d\text{-rigid components of } \{X_{p_2}(e)\}_{e\in E}\}\,. \end{aligned}$$

Vertices x such that D_x happens can via a certain local modification (namely, by turning on all edges within distance M from x in X_2) be turned into so-called trifurcators, which we go on to define next. These will be used to create encounter points in the auxiliary percolation process $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$ to be defined a few lines later.

Call $x \in V$ a **trifurcator** if

(a) $B(x,N)$ is intersected by exactly three infinite d-rigid components in $\{X_{p_1}(e)\}_{e\in E}$, and

(b) $X_{p_2}(e) = 1$ for all edges e that have both endpoints within distance M from x.

Next enlarge the probability space $([0,1]^E, \Psi)$ by including an independent fair coin toss for each $x \in V$. Call $x \in V$ a **lucky trifurcator** if it is a trifurcator whose coin toss comes up heads. Finally, call $x \in V$ an **exclusive lucky trifurcator** if it is the only lucky trifurcator in $B(x, 2M)$.

We now obtain $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$ from $\{X^*(e)\}_{e\in E^*_{A,N}}$ by turning on edges of the form $\langle (x,0),(y,j)\rangle \in E^*_{A,N}$ as follows. If $x \in V$ is an exclusive lucky trifurcator, we write $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ for its three defining infinite d-rigid components of $\{X_{p_1}(e)\}_{e\in E}$. For each $i \in \{1,2,3\}$, a vertex y_i in $\mathcal{C}_i$ is chosen at random (uniformly) among those that minimize the distance to x, and we set

$$\tilde{X}^*(\langle (x,0),(y_i, I_{y_i,\mathcal{C}_i})\rangle) = 1\,.$$

For all other edges $e \in E^*_{A,N}$, we set $\tilde{X}^*(e) = X^*(e)$. Note that it is immediate from the construction that $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$ is $\mathrm{Aut}^*(G^*_{A,N})$-invariant.

Now assume, for a given vertex $x \in V$, that $\{X_{p_1}(e)\}_{e\in E}$ and $\{X_{p_2}(e)\}_{e\in E}$ are such that the event D_x happens, and write $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ for the three infinite d-rigid components of $\{X_{p_1}(e)\}_{e\in E}$ intersecting $B(x,N)$. We noted above that each rigid component in $\{X_{p_1}(e)\}_{e\in E}$ forms a distinct rigid component in $\{X^*(e)\}_{e\in E^*_{A,N}}$, and we now make the stronger claim regarding $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ that

$$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \text{ are in three distinct connected components of } \{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}\,. \tag{20}$$

To see this, note first that if an exclusive lucky trifurcator connects up two connected components $\mathcal{C}$ and $\mathcal{C}'$ of $\{X^*(e)\}_{e\in E^*_{A,N}}$, then the corresponding d-rigid components in $\{X_{p_1}(e)\}_{e\in E}$ will merge in $\{X_{p_2}(e)\}_{e\in E}$; this follows from the definition of trifurcators in conjunction with our choice of M. So if, for some $\{i,j\} \subset \{1,2,3\}$, we have that $\mathcal{C}_i$ and $\mathcal{C}_j$ are in the same connected component of $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$, then the alternating chain of exclusive lucky trifurcators and connected components of $\{X^*(e)\}_{e\in E^*_{A,N}}$ that connect $\mathcal{C}_i$ and $\mathcal{C}_j$ in $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$, will cause the d-rigid components $\mathcal{C}_i$ and $\mathcal{C}_j$ to merge in $\{X_{p_2}(e)\}_{e\in E}$. But this would contradict the assumption that the event D_x happens, so we have established the claim that D_x implies (20).

Now imagine again that $\{X_{p_1}(e)\}_{e\in E}$ and $\{X_{p_2}(e)\}_{e\in E}$ are such that the event D_x happens (so that in particular (20) holds), but then modify $\{X_{p_2}(e)\}_{e\in E}$ by turning on all edges within distance M from x. If furthermore the coin tosses introduced in the definition of a lucky trifurcator yield "heads" at x but "tails" at all other vertices in $B(x, 2M)$ (an event of positive conditional probability given everything else), then x is an exclusive lucky trifurcator, and furthermore the only difference that this modification makes to $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$ is the inclusion of the three edges incident to $(x,0)\in V^*_{A,N}$ that connect up $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. This causes $(x,0)$ to be an encounter point for $\{\tilde{X}^*(e)\}_{e\in E^*_{A,N}}$. Hence, using (19) and the property (12) of the simultaneous LM-coupling, we get that $(x,0)$ has positive probability of being such an encounter point. This contradicts Proposition 7.3, and the proof is complete. $\square$

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