Uniqueness in two-dimensional rigidity percolation

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November 9, 1999

Abstract

For bond percolation on the two-dimensional triangular lattice with arbitrary retention parameter $p \in [0,1]$, we show that the number of infinite rigid components is a.s. at most one. This proves a conjecture by Holroyd. Further results, concerning simultaneous uniqueness, and continuity (in p) of the probability that a given edge is in an infinite rigid component, are also obtained.

1 Introduction

We consider standard bond percolation on the triangular lattice $\mathbf{T} = (V, E)$ in two dimensions, defined as the graph with vertex set

$$V = \{i(1,0) + j(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) : i, j \in \mathbf{Z}\}$$

and edge set E consisting of all pairs of vertices $x, y \in V$ with $\|x - y\| = 1$, where $\|\cdot\|$ denotes Euclidean norm. Each edge of $\mathbf{T}$ is deleted independently with probability $1-p$, and thus kept with probability p. Retained and deleted edges are also called open and closed, and their status are represented by the symbols 1 and 0. Let $\mathbf{P}_p$ denote the resulting product probability measure on $\{0,1\}^E$ with marginal distributions $(1-p, p)$.

Of central interest in percolation theory is whether or not infinite connected components exist, and this typically depends on whether p is above or below a certain critical value p_c. For the triangular lattice $\mathbf{T}$, the critical value is known; this is due to Wierman [13] who showed that

$$\mathbf{P}_p(\exists \text{ infinite connected component}) = \begin{cases} 0 & \text{for } p \le p_c \\ 1 & \text{for } p > p_c \end{cases} \tag{1}$$

with $p_c = 2\sin(\frac{\pi}{18}) \approx 0.3473$. The next natural question is to ask for the number of infinite connected components when $p > p_c$. For the analogous problem for the cubic lattice $\mathbf{Z}^d$, it is well known that there is a.s. only one connected component. This result carries over to $\mathbf{T}$, and the easiest way to see this is to note that the famous Burton–Keane [3] encounter point argument works equally well on $\mathbf{T}$ as on $\mathbf{Z}^d$.

Originating in the physics literature, there has been some recent interest in studying percolation processes with focus on other aspects than connectivity. Instead of infinite *connected* components, one may consider infinite *rigid* components, or infinite *entangled* components. Here we shall prove the following rigidity analogue of the uniqueness of the infinite connected component result: Whenever an infinite rigid component exists for bond percolation on $\mathbf{T}$, it is a.s. unique (Theorem 3.4). This proves a conjecture by

Holroyd [11], who obtained a somewhat weaker result in the same direction (Theorem 3.1).

The corresponding uniqueness problem for so-called entanglement percolation has recently been treated in [7] and in [8].

It will become clear in Sections 4–7 that our arguments are specific to planar lattices in two dimensions, and this is why we do not consider more general lattices in higher dimensions. The reason for studying percolation on $\mathbf{T}$ rather than on the more usual square lattice $\mathbf{Z}^2$ is that the latter is uninteresting from the point of view of rigidity, because it does not contain any rigid subgraphs (except for some trivial examples with at most one edge).

A preliminary discussion of rigidity and rigidity percolation is given in Section 2. In Section 3 we discuss the issue of uniqueness of infinite rigid components and state our main uniqueness results. Proofs of these results are given in Sections 4 and 5. Some additional results, concerning continuity in p of the $\mathbf{P}_p$-probability that a given edge is in an infinite rigid component, and so-called "simultaneous uniqueness", will be obtained in Sections 6 and 7.

2 Rigidity percolation

Following is a brief recollection of (generic) two-dimensional rigidity of graphs, and of rigidity percolation on $\mathbf{T}$. We refer to Holroyd [11] for a more detailed and general account; in particular, rigidity is a dimension-dependent concept, and the definitions below have natural analogues in higher dimensions.

Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. Let ρ be an embedding of G in $\mathbf{R}^2$, i.e. ρ is an injective mapping from $V(G)$ to $\mathbf{R}^2$. The pair (G, ρ) is called a *framework*. A *motion* of a framework (G, ρ) is a differentiable family $(\rho_t : t \in [0,1])$ of embeddings of G in $\mathbf{R}^2$ with $\rho_0 = \rho$, satisfying

$$\|\rho_t(x) - \rho_t(y)\| = \|\rho(x) - \rho(y)\| \tag{2}$$

for all $t \in [0,1]$ and all vertices $x, y \in V(G)$ that share an edge. The motion is said to be *rigid* if (2) holds for *all* $x, y \in V(G)$. The embedding ρ and the framework (G, ρ) are said to be rigid if all their motions are rigid.

It is well known that for any finite G, either almost all (with respect to Lebesgue measure on $\mathbf{R}^{2|V(G)|}$) embeddings of G in $\mathbf{R}^2$ are rigid, or almost all such embeddings are not rigid. This makes the following definition natural.

Definition 2.1 *A finite graph G is said to be rigid if (Lebesgue-)almost all embeddings of G in $\mathbf{R}^2$ are rigid.*

Holroyd [11] proposed a very natural extension to infinite graphs:

Definition 2.2 *An infinite locally finite graph G is said to be rigid if every finite subgraph of G is contained in some rigid finite subgraph of G.*

Next, we consider rigidity in bond percolation on the triangular lattice $\mathbf{T}$. For any configuration $\omega \in \{0,1\}^E$ of open and closed edges in $\mathbf{T}$, the set of open edges may be partitioned into (maximal) rigid components. Note, however, that a *vertex* may belong to more than one such component. What is the $\mathbf{P}_p$-probability of having some infinite rigid component? Analogously to (1), there exists a $p_r \in [0,1]$, called the *rigidity critical probability*, such that

$$\mathbf{P}_p(\exists \text{ some infinite rigid component}) = \begin{cases} 0 & \text{for } p < p_r \\ 1 & \text{for } p > p_r\,. \end{cases}$$

Unlike the connectivity critical probability p_c, the exact value of p_r is not known, although numerical findings in [12] suggest that $p_r \approx 0.6602$. One of the main results in Holroyd's paper [11] is the following:

Theorem 2.3 (Holroyd) *The rigidity critical probability p_r for bond percolation on $\mathbf{T}$ satisfies*

$$p_c < p_r < 1\,. \tag{3}$$

To show that $p_c \leq p_r \leq 1$ is a triviality, but the corresponding strict inequalities in (3) are not.

3 Uniqueness of the infinite rigid component

When an infinite rigid component exists, is it necessarily unique? Holroyd [11] came, in a sense, very close to an affirmative answer:

Theorem 3.1 (Holroyd) *Consider bond percolation on $\mathbf{T}$. For all $p \in (p_r, 1]$ with at most countably many exceptions, we have*

$$\mathbf{P}_p(\exists \text{ a unique infinite rigid component}) = 1\,.$$

In particular, it follows that uniqueness holds for (Lebesgue-)almost all $p \in (p_r, 1]$.

Our first result is the following.

Proposition 3.2 *Fix p_1 and p_2 such that $p_r < p_1 < p_2 \leq 1$. If*

$$\mathbf{P}_{p_1}(\exists \text{ a unique infinite rigid component}) = 1\,, \tag{4}$$

then

$$\mathbf{P}_{p_2}(\exists \text{ a unique infinite rigid component}) = 1\,.$$

By combining Theorem 3.1 and Proposition 3.2, we get the next result as an immediate consequence:

Corollary 3.3 *For bond percolation on $\mathbf{T}$ with retention parameter $p > p_r$, we have*

$$\mathbf{P}_p(\exists \text{ a unique infinite rigid component}) = 1\,. \tag{5}$$

Proof: Pick p as in the corollary. By Theorem 3.1, we can find a $p' \in (p_r, p)$ such that the $\mathbf{P}_{p'}$-probability of having a unique infinite rigid component is 1. By applying Proposition 3.2 with $p_1 = p'$ and $p_2 = p$, we obtain (5). $\square$

It remains to prove Proposition 3.2, and this will be done in Section 4. Our proof builds heavily on the work of Gandolfi, Keane and Russo [4] concerning uniqueness of the infinite connected component for certain dependent percolation processes having the so called *positive correlations* property (also known as the FKG property).

Corollary 3.3 has one shortcoming in that it says nothing about the *critical* case $p = p_r$. Ergodicity implies that

$$\mathbf{P}_{p_r}(\exists \text{ some infinite rigid component}) \in \{0, 1\}$$

but it remains an open problem to decide whether this probability is 0 or 1. If the probability is 1, then one would like to know how many infinite rigid components there are, but Corollary 3.3 does not answer this question. However, the following improvement does:

Theorem 3.4 *For bond percolation on* $\mathbf{T}$ *with retention parameter* p *satisfying*

$$\mathbf{P}_p(\exists \text{ some infinite rigid component}) = 1\,, \tag{6}$$

we have that this infinite rigid component is $\mathbf{P}_p$*-a.s. unique.*

This result will be proved in Section 5, by building further on the proof in Section 4 of Proposition 3.2. A main ingredient of this extended proof is the consideration of certain auxiliary percolation processes living on modified versions of $\mathbf{T}$.

The advantage of Theorem 3.4 over Proposition 3.2 and Corollary 3.3 is not only that it takes care of the critical case $p = p_r$, but also that its proof is more self-contained; in particular, it is independent of the long and complicated proof in [11] of Theorem 3.1. (On the other hand, our proof of Theorem 3.4 is also a bit complicated.)

4 Proof of uniqueness monotonicity

Fix $p \in [0, 1]$, and let X be a $\{0, 1\}^E$-valued random object with distribution $\mathbf{P}_p$. Also define a second $\{0, 1\}^E$-valued random object Y by setting

$$Y(e) = \begin{cases} 1 & \text{if } e \text{ is in an infinite rigid component of } X \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

for each $e \in E$. In other words, Y is obtained from X by deleting all edges that are not in an infinite rigid component.

An important step in our arguments consists of showing that if Y contains an infinite connected component, then this connected component is unique and contains circuits surrounding any finite part of $\mathbf{T}$. This is done by invoking a variant of a result of Gandolfi, Keane and Russo [4] (Theorem 4.1 below). Before stating that result, we need some more terminology:

- **Graph automorphisms.** A *graph automorphism* for $\mathbf{T}$ is a bijection $\gamma : V \to V$ with the property that for any $x, y \in V$, we have that $\gamma(x)$ and $\gamma(y)$ share an edge in E iff x and y share an edge in E. This induces a corresponding mapping $\gamma' : E \to E$ in the obvious way. Graph automorphisms of $\mathbf{T}$ are translations (of integer length and in directions that are multiples of $\frac{\pi}{3}$), rotations (around a given vertex, by angles that are multiples of $\frac{\pi}{3}$), reflections (in lines through vertices at directions that are again multiples of $\frac{\pi}{3}$), and compositions of these. A probability measure $\mathbf{Q}$ on $\{0, 1\}^E$, and the corresponding $\{0, 1\}^E$-valued random element Z, are said to be *automorphism invariant* if for any n, any $e_1, \ldots, e_n \in E$, any $i_1, \ldots, i_n \in \{0, 1\}$ and any graph automorphism γ we have

$$\mathbf{Q}(Z(e_1) = i_1, \ldots, Z(e_n) = i_n) = \mathbf{Q}(Z(\gamma'(e_1)) = i_1, \ldots, Z(\gamma'(e_n)) = i_n)\,.$$

- **Positive correlations.** A function $f : \{0,1\}^E \to \mathbf{R}$ is said to be *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega \preceq \omega'$, where $\preceq$ is the usual coordinatewise partial order on $\{0,1\}^E$. A random element $Z \in \{0,1\}^E$ is said to have *positive correlations* if for all bounded increasing $f, g : \{0,1\}^E \to \mathbf{R}$ we have

$$\mathbf{E}[f(Z)g(Z)] \geq \mathbf{E}[f(Z)]\mathbf{E}[g(Z)]$$

where $\mathbf{E}$ denotes expectation.

- **Surrounding circuits.** Let H_n denote the closed convex hull in $\mathbf{R}^2$ of the hexagon that has its six corners in $(\pm n, 0)$ and $(\pm\frac{n}{2}, \pm\frac{n\sqrt{3}}{2})$. Let $V_n = V \cap H_n$ (so that V_n is the set of vertices at graph-theoretic distance at most n from the origin), and let E_n be the set of edges in E that have both endpoints in V_n. A **path** $r = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ in $\mathbf{T}$, with $v_0, \ldots, v_k \in V$, $e_0, \ldots, e_{k-1} \in E$ and e_i connecting v_i and v_{i+1} for each i, is identified in a natural way with a curve in $\mathbf{R}^2$, by identifying each edge with a unit length closed line segment connecting its two endvertices. The path r is said to be *self-avoiding* if the vertices $v_0, \ldots, v_k$ are all distinct, with the possible exception $v_k = v_0$. It is said to be a *circuit* if it is self-avoiding with $v_k = v_0$. Finally, it is said to be a *circuit surrounding* H_n if in addition $v_0, \ldots, v_{k-1} \in V \setminus V_n$ and every continuous curve $\pi : [0, \infty) \to \mathbf{R}^2$ with $\pi(0) \in H_n$ and $\lim_{t\to\infty} \|\pi(t)\| = \infty$ has to intersect r. (See [4] for an equivalent definition in terms of winding number.)

Theorem 4.1 *Let Z be a $\{0,1\}^E$-valued random object satisfying*

(i) *automorphism invariance,*

(ii) *ergodicity under each of the translations $x \mapsto x + (1,0)$, $x \mapsto x + (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $x \mapsto x + (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ separately, and*

(iii) *positive correlations.*

Suppose furthermore that Z has an infinite connected component a.s. Then the infinite connected component is a.s. unique, and moreover each H_n is a.s. surrounded by some circuit in Z.

Proof: An analogous result for site percolation on $\mathbf{Z}^2$ appears as the main result in Gandolfi, Keane and Russo [4], and the present result follows from a completely straightforward adaptation of their proof. $\Box$

Corollary 4.2 *Pick p in such a way that* (6) *holds, let X be a $\{0,1\}^E$-valued random object with distribution $\mathbf{P}_p$, and define the $\{0,1\}^E$-valued random object Y as in* (7)*. Then, with probability* 1*, Y has a unique connected component, and contains, for each n, a circuit surrounding H_n.*

Proof: A rigid component has to be connected (see e.g. [11, Proposition 6.6]), so with p as in the corollary we have that Y contains some infinite connected component a.s. Hence, it is sufficient to verify that Y satisfies conditions (i), (ii) and (iii) of Theorem 4.1.

The distribution of X is just i.i.d. measure, so X satisfies (i) and (ii), and it is easy to see that these properties are inherited by Y.

It remains to verify the positive correlations property (iii). Positive correlations holds for X; this is the well-known Harris–FKG inequality (see e.g. [5]). Now write $h : \{0,1\}^E \to \{0,1\}^E$ for the mapping defined in (7), and note that h is increasing, meaning that $\omega \preceq \omega'$ implies $h(\omega) \preceq h(\omega')$. Hence the compositions $f \circ h$ and $g \circ h$ are increasing whenever $f, g : \{0,1\}^E \to \mathbf{R}$ are, so that

$$\begin{aligned} \mathbf{E}[f(Y)g(Y)] &= \mathbf{E}[f(h(X))g(h(X))] \\ &\geq \mathbf{E}[f(h(X))]\mathbf{E}[g(h(X))] \\ &= \mathbf{E}[f(Y)]\mathbf{E}[g(Y)] \end{aligned}$$

where the inequality follows from the positive correlations property of X. $\square$

Corollary 4.2 seems to strongly suggest Theorem 3.4, but we are not quite there yet (see Remark 4.4 below). First we shall prove Proposition 3.2, for which the following lemma is useful.

Lemma 4.3 *Let $A = (V_A, E_A)$ and $B = (V_B, E_B)$ be (finite or infinite) graphs, and let $A \cup B$ be the graph with vertex set $V_A \cup V_B$ and edge set $E_A \cup E_B$.*

(a) *If A and B are rigid, and $|V(A) \cap V(B)| \geq 2$, then $A \cup B$ is rigid.*

(b) *Conversely, if $|V(A) \cap V(B)| \leq 1$, then $A \cup B$ is not rigid.*

Proof: This result is stated and proved in [11, Propositions 6.8 and 6.7, and Metaproposition 6.10]. $\square$

Proof of Proposition 3.2: Fix p_1 and p_2 as in the proposition. Let A be the event that for each H_n, there exists an infinite rigid component containing a circuit surrounding H_n. By (4) and Corollary 4.2, we have

$$\mathbf{P}_{p_1}(A) = 1\,. \tag{8}$$

But A is an increasing event (meaning that for any $\omega, \omega' \in \{0,1\}^E$ such that $\omega \preceq \omega'$ and $\omega \in A$, we have $\omega' \in A$), which in conjunction with (8) implies

$$\mathbf{P}_{p_2}(A) = 1\,. \tag{9}$$

We wish to show that with $\mathbf{P}_{p_2}$-probability 1, any two edges $e_1, e_2 \in E$ that are in infinite rigid components, must in fact be in *the same* infinite rigid component. Since there are only countably many pairs of edges, it suffices to show this for fixed $e_1, e_2 \in E$. So fix e_1 and e_2, and pick n large enough so that $e_1, e_2 \in E_n$. By (9), we can then $\mathbf{P}_{p_2}$-a.s. find a circuit r surrounding H_n with the property that all edges in r are open and contained in the same rigid component. Lemma 4.3 (b) implies that if e_1 is contained in an infinite rigid component C, then this component must intersect r in at least two vertices, because otherwise C could be split in two parts (one "outside" and one "inside" r) with at most one vertex in common, and would thus not be rigid after all. But then Lemma 4.3 (a) guarantees that the rigid component containing e_1 is the same as the rigid component containing the edges of r. The same reasoning applies to e_2, so if e_1 and e_2 are both in infinite rigid components, then they are in the same rigid component. $\square$

Remark 4.4 Why doesn't the existence of surrounding circuits guaranteed by Corollary 4.2, combined with the reasoning in the proof of Proposition 3.2, imply Theorem 3.4? The answer is that the curcuit in Y surrounding H_n might (a priori) contain edges in more than one infinite rigid component. This possibility will be ruled out in the next section. $\square$

5 Proof of the full uniqueness result

In this section we will prove Theorem 3.4. We begin with a lemma about the way in which different rigid components can share a vertex. Fix a vertex x in $\mathbf{T}$. Denote the six edges coming out of x by $e_{x,0}, e_{x,1}, \ldots, e_{x,5}$, in such a way that, for $i = 0, \ldots, 5$, edge $e_{x,i}$ connects x with $x + (\cos(\frac{i\pi}{3}), \sin(\frac{i\pi}{3}))$. In other words, the edges incident to x are enumerated in counter-clockwise order, starting with the one connecting x and $x + (1, 0)$.

Lemma 5.1 *Let x be a vertex of $\mathbf{T}$, and let $i, j, k, l \in \{0, \ldots, 5\}$ be such that $i < j < k < l$. Suppose for a percolation configuration $\omega \in \{0,1\}^E$ that*

(i) *$e_{x,i}, e_{x,j}, e_{x,k}$ and $e_{x,l}$ are all open,*

(ii) *$e_{x,i}$ and $e_{x,k}$ are in the same rigid component $C_{i,k}$, and*

(iii) *$e_{x,j}$ and $e_{x,l}$ are in the same rigid component $C_{j,l}$.*

Then $C_{i,k} = C_{j,l}$.

Proof: Write y_i for the endpoint (other than x) of $e_{x,i}$, and define y_j, y_k and y_l similarly. We claim that there must exist a path $r_{i,k}$ in $C_{i,k}$ which starts at y_i, ends at y_k, and does not go through x. To see this, suppose for contradiction that there does *not* exist any such path. Then the edge set of $C_{i,k}$ can be partitioned into two sets according to whether x must be used when going (in $C_{i,k}$) from y_i to a given edge. These two edge sets have only a single vertex (x) in common, so Lemma 4.3 (b) shows that $C_{i,k}$ is not rigid, which is a contradiction.

Similarly, there must exist a path $r_{j,l}$ in the component $C_{j,l}$, which starts at y_j, ends at y_l, and does not go through x. By planarity, the paths $r_{i,k}$ and $r_{j,l}$ must intersect at some vertex z (different from x); see Figure 1. Hence $C_{i,k}$ and $C_{j,l}$ share two distinct vertices x and z, so by Lemma 4.3 (a) we have $C_{i,k} = C_{j,l}$. □

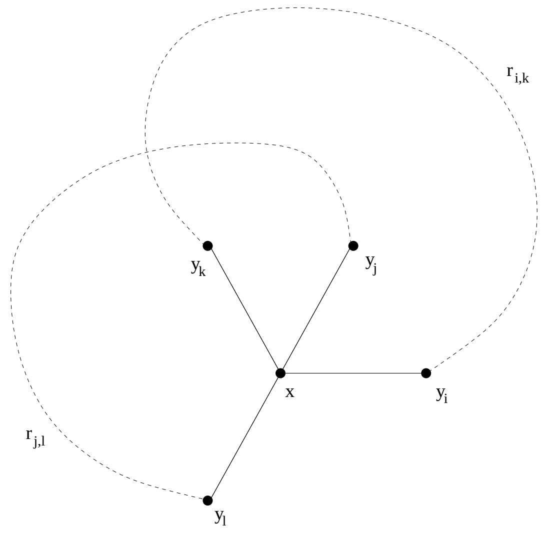

Figure 1: The paths $r_{i,k}$ and $r_{j,l}$ must intersect.

The above argument was inspired by a similar use of planarity by Alm [2] in the context of first-passage percolation.

We now come to the main additional ingredient (compared to Section 4) in the proof of Theorem 3.4, which is the introduction of some "decorated" variants of the lattice $\mathbf{T}$. The main shortcoming of representing infinite rigid components by the process Y is that an infinite connected component in Y may contain more than one infinite rigid component (although the event that this happens will be shown to have probability 0, it is easy to construct deterministic configurations where it happens). The following modified lattice $\mathbf{T}' = (V', E')$ is tailored to take care of this problem.

Fix a small $\varepsilon > 0$ (for concreteness, we may take $\varepsilon = 0.1$). $\mathbf{T}'$ is obtained from $\mathbf{T}$ by the following three step procedure (see the left and the middle part of Figure 2):

1. Replace each $x \in V$ by six vertices $x'_0, x'_1, \ldots, x'_5 \in V'$, where, for $i = 0, \ldots, 5$, the vertex x'_i is located at $x + (\varepsilon \cos(\frac{i\pi}{3}), \varepsilon \sin(\frac{i\pi}{3}))$. We call $x'_0 \ldots, x'_5$ the *satellites* of x.

2. For each $x \in V$ and each $i, j \in \{0, \ldots, 5\}$ with $i \neq j$, connect x'_i and x'_j by an edge in E', so that in other words the six satellites of x are connected as in the complete graph K_6. These edges in E' are said to be *local*.

3. Replace each edge $e \in E$ by an edge $e' \in E'$ as follows. If e has endpoints x and y, then we let e' connect the satellites x'_i and y'_j, where $i, j \in \{0, \ldots, 5\}$ are chosen in the unique way to make $\|x'_i - y'_j\| = 1 - 2\varepsilon$. These edges in E' are said to be *regional*. Note that each vertex in V' is incident to exactly one regional edge.

Given the percolation realization $X \in \{0,1\}^E$, we define the configuration $X' \in \{0,1\}^{E'}$ as follows. If $e' \in E'$ is a regional edge, then we let $X'(e') = X(e)$, where $e \in E$ is the edge in $\mathbf{T}$ corresponding to e'. If on the other hand $e' \in E'$ is a local edge connecting two satellites x'_i and x'_j, then we let

$$X'(e') = \begin{cases} 1 & \text{if } X(e_{x,i}) = X(e_{x,j}) = 1, \text{ and } e_{x,i} \text{ and } e_{x,j} \text{ are} \\ & \text{in the same rigid component of } X \\ 0 & \text{otherwise} \end{cases}$$

($e_{x,i}$ and $e_{x,j}$ are defined as in Lemma 5.1).

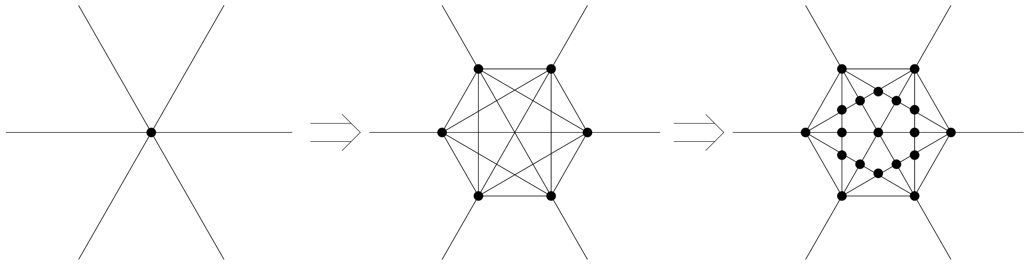

Figure 2: The local modification (around a vertex $x \in V$) of $\mathbf{T}$ via $\mathbf{T}'$ into $\mathbf{T}''$.

Lemma 5.2 *Two regional edges $e', \tilde{e}' \in E'$ (with $X'(e') = X'(\tilde{e}') = 1$) are in the same connected component of X' if and only if the corresponding edges $e, \tilde{e} \in E$ are in the same rigid component of X.*

Proof: Suppose that e and $\tilde{e}$ are in the same rigid component R of X. Since rigid components are connected, we can find a sequence of edges $e_1, \ldots, e_n \in R$ with $e_1 = e$

and $e_n = \tilde{e}$, such that e_i and e_{i+1} share a vertex in $\mathbf{T}$ for $i = 1, \ldots, n+1$. It is immediate from the costruction of X' that, for each such i, the edges e'_i and e'_{i+1} (defined in the obvious way) are in the same connected component of X', and the same must then hold for e' and $\tilde{e}'$ as well.

For the converse, suppose that e' and $\tilde{e}'$ are in the same connected component C of X'. Then we can find a sequence of edges $e'_1, \ldots, e'_n \in C$ with $e'_1 = e'$ and $e'_n = \tilde{e}'$, such that e'_i and e'_{i+1} share a vertex in $\mathbf{T}'$ for $i = 1, \ldots, n-1$. Now thin this sequence into another sequence $e'_{(1)}, \ldots, e'_{(k)}$ of only regional edges with $e'_{(1)} = e'$ and $e'_{(k)} = \tilde{e}'$, by simply removing each local edge from the sequence. By the construction of X', we have for $i = 1, \ldots, k-1$ that $e_{(i)}$ and $e_{(i+1)}$ are in the same rigid component of X, and it follows that the same thing holds also for e and $\tilde{e}$. $\Box$

Lemma 5.2 implies in particular that if X' has a unique infinite connected component, then X has a unique infinite rigid component. Theorem 4.1 would therefore follow if we could adapt the Gandolfi–Keane–Russo theorem (see Theorem 4.1) to show a.s. uniqueness of the infinite connected component in X'. The problem with this idea is that the Gandolfi–Keane–Russo technique builds heavily on planarity, whereas the lattice $\mathbf{T}'$ is not planar. To solve this problem, we shall modify the lattice one step further, to get a lattice $\mathbf{T}'' = (E'', V'')$ which is planar and which at the same time has connectivity properties that (as far as the process X' is concerned) are essentially the same as for $\mathbf{T}'$.

To this end, we represent each edge in $\mathbf{T}'$ by a straight line in $\mathbf{R}^2$ connecting its two endvertices, and let $\mathbf{T}''$ have vertex set

$$V'' = V' \cup V^*$$

where V^* is the set of points in $\mathbf{R}^2$ where two or more edges in E' cross each other. Clearly, only local edges cross each other.

Some local edges will then pass through three vertices in V^* (see Figure 2). E'' is obtained from E' by replacing each such edge e' (connecting, say, x'_i and x'_j, and passing through $z''_1, z''_2, z''_3 \in V^*$, in that order), by four edges: one from x'_i to z''_1, one from z''_1 to z''_2, one from z''_2 to z''_3, and one from z''_3 to x'_j. We call $e' \in E'$ the *parent* of these four edges in E''. The lattice $\mathbf{T}''$ obtained in this way is planar (see Figure 2 again), and it turns out that the Gandolfi–Keane–Russo theorem can be adapted in the same straightforward manner as for Theorem 4.1, to obtain the following result.

Proposition 5.3 *Let Z'' ba a $\{0,1\}^{E''}$-valued random object satisfying assumptions* (i), (ii) *and* (iii) *of Theorem* 4.1. *Suppose furthermore that Z'' has an infinite connected component a.s. Then this infinite connected component is a.s. unique, and each H_n is a.s. surrounded by some circuit in Z''.*

Proof: See the proof of Theorem 4.1. $\Box$

Now define $X'' \in \{0,1\}^{E''}$ as follows. If $e'' \in E''$ is also on E', then we let $X''(e'') = X'(e'')$; otherwise we let $X''(e'') = X'(e')$, where e' is the parent of e''. (This means e.g. that the same line segments are open and closed in the right part of Figure 2 as in the middle part.)

Lemma 5.4 *For any two regional edges e' and $\tilde{e}' \in E'$ (with $X'(e') = X'(\tilde{e}') = 1$) we have that e' and $\tilde{e}'$ are in the same connected component of X' if and only if they are in the same connected component of X''.*

Proof: Lemma 5.1 implies that if two local edges in E' cross each other and are both open (in X'), then they are also in the same connected component of X'. Hence the additional connectivity of $\mathbf{T}''$ caused by the vertices in V^* does not make any difference to the connected components in X' and X''. □

Proof of Theorem 3.4: Fix p such that (6) holds, let X be a $\{0,1\}^E$-valued random object with distribution $\mathbf{P}'_p$, and define $X' \in \{0,1\}^{E'}$ and $X'' \in \{0,1\}^{E''}$ as above. Then X has a.s. at least one infinite rigid component. Hence, X' has a.s. at least one infnite connected component by Lemma 5.2, so that X'' has a.s. at least one infinite connected component by Lemma 5.4. By the arguments in the proof of Corollary 3.3, we furthermore have that X'' satisfies the assumptions (i), (ii) and (iii) of Theorem 4.1. Thus, Proposition 5.3 tells us that X'' has a.s. a unique infinite connected component. Using Lemma 5.4, we deduce that the same thing holds for X'. Finally, this implies, by Lemma 5.2, that X has a unique rigid component a.s. □

Remark 5.5 Recall that Proposition 5.3 guarantees not only that X'' has a unique infinite connected component, but also, for each n, that this infinite connected component contains a circuit surrounding H_n. By arguing as in the above proof, we therefore get the following result: *For any p such that* (6) *holds, we have $\mathbf{P}_p$-a.s. that the unique rigid component contains, for each n, a circuit surrounding H_n.* (The usefulness of this fact will be demonstrated in Sections 6 and 7.) Alternatively, we could have deduced the same result by combining Theorem 3.4 and Corollary 4.2. □

Remark 5.6 An inspection of our arguments for Theorem 3.4 reveals that the only properties of the product probability measure $\mathbf{P}_p$ that are used (besides (6)), are that the assumptions (i), (ii) and (iii) of Theorem 4.1 hold for the corresponding $\{0,1\}^E$-valued random object X. This means that we have in fact proved the following generalization of Theorem 3.4 and Remark 5.5: *Let X be any $\{0,1\}^E$-valued random object satisfying assumptions* (i), (ii), *and* (iii) *of Theorem* 4.1, *that has a.s. some infinite rigid component. Then X has a.s. a unique infinite rigid component, and for each n this infinite rigid component contains a.s. a circuit surrounding H_n.* Examples to which this result applies to show that the number of infinite rigid components is a.s. at most one, are:

- The so-called free and wired random-cluster measures for the random-cluster model on $\mathbf{T}$ with cluster parameter $q \geq 1$ (see e.g. [6]).
- The analogous free and wired Gibbs measures for the random triangle model (see [9]).
- Site percolation: Declare each vertex $x \in V$ idependently to be open or closed with probabilities p and $1-p$, and for each $e \in E$ let $X(e) = 1$ if and only if both endpoints of e are declared open.

□

6 A continuity result

A well-known result for standard bond percolation on $\mathbf{Z}^d$ is that the probability that a given vertex x is in an infinite connected component, is a continuous function of p

above the connectivity critical probability p_c. The usual proof of this result (see e.g. [5]) works equally well for $\mathbf{T}$.

In this section we consider the analogous problem for rigidity percolation. For $p \in [0,1]$, we define $\phi(p)$ to be the $\mathbf{P}_p$-probability that a given edge e in $\mathbf{T}$ is in an infinite rigid component (note that this definition is independent of the choice of $e \in E$). The function $\phi(p) : [0,1] \to [0,1]$ is obviously increasing. We shall prove the following result.

Theorem 6.1 *The function $\phi(p)$ is continuous on $(p_r, 1]$.*

Remark 6.2 Obviously, $\phi(p) = 0$ for $p < p_r$, so the only remaining possible point of discontinuity for $\phi(p)$ is at the rigidity critical value $p = p_r$. To decide whether we have continuity at $p = p_r$ seems to be a challenging open problem. For a small partial result on this problem, see Proposition 6.4 below. □

The first ingredient in our proof of Theorem 6.1 is the following lemma about rigid components of finite subgraphs of $\mathbf{T}$.

Lemma 6.3 *Fix a positive integer n, an edge $e^* \in E_n$, and two edge configurations $\omega, \omega' \in \{0,1\}^E$ satisfying*

(i) $\omega \preceq \omega'$,

(ii) *ω contains a rigid component C which in turn contains a circuit r surrounding H_n,*

(iii) *$\omega(e) = \omega'(e)$ for all edges e inside the circuit r,*

(vi) $\omega(e^*) = 1$,

(v) *e^* is not in the rigid component of ω containing r.*

Then e^* is not in the rigid component of ω' containing r.

In other words, what the lemma says is that if e^* is not in the same rigid component as the edges of r, then this situation cannot be changed by turning on edges outside r only. In the proof, we shall allow the following slip of notation: an edge set $A \subset E$ is identified with the graph with edge set A and vertex set V_A consisting of those vertices $x \in V$ that are incident to at least one edge in A. The notation $A \cup B$ is used as in Lemma 4.3.

Proof of Lemma 6.3: Assume first that, in addition to conditions (i)–(v), we also have

(vi) ω and ω' have only finitely many open edges.

To prove the lemma under the additional assumption (vi), it suffices to show that for any set $A \in E \setminus C$ of edges inside r such that $e^* \in A$ and $\omega(e) = 1$ for all $e \in A$, and any finite set $B \in E \setminus C$ of edges outside r, we have

$$A \cup C \cup B \ \text{ is not rigid.} \tag{10}$$

Fix two such edge sets A and B. By assumption (v), we have that

$$A \cup C \ \text{ is not rigid.}$$

By the definition of rigidity, we therefore have for (Lebesgue-)a.e. embedding ρ of $A \cup C$ in $\mathbf{R}^2$ that there exists a non-rigid motion

$$\rho_t : [0,1] \to (\mathbf{R}^2)^{V_A \cup V_B}$$

of $(A \cup C, \rho)$. By considering this motion relative to two fixed vertices $x, y \in V_C$, we see that there exists another non-rigid motion $\hat{\rho}_t$ of $(A \cup C, \rho)$ such that $\hat{\rho}_t(z) \equiv \rho(z)$ for all $z \in V_C$. Now let $\hat{\rho}^*$ be any embedding of $A \cup C \cup B$ in $\mathbf{R}^2$ whose projection on $(\mathbf{R}^2)^{V_A \cup V_C}$ equals $\hat{\rho}$. Consider the function

$$\hat{\rho}_t^* : [0,1] \to (\mathbf{R}^2)^{V_A \cup V_C \cup V_B}$$

defined by

$$\hat{\rho}_t^*(x) = \begin{cases} \hat{\rho}_t(x) & \text{for } x \in V_A \cup V_C \\ \hat{\rho}^*(x) & \text{for } x \in V_B \setminus (V_A \cup V_C). \end{cases}$$

Since C contains the circuit ρ, we have by planarity that there are no edges in $A \cup C \cup B$ linking a vertex $x \in V_B \setminus (V_A \cup V_C)$ to a vertex $y \in V_A \setminus V_C$. It is therefore clear that $\hat{\rho}_t^*$ is a motion of $A \cup C \cup B$. On the other hand, since $\hat{\rho}_t$ is a non-rigid motion, we have that $\hat{\rho}_t^*$ is also non-rigid, so (10) is established.

It remains to show that the assumption (vi) may be removed, but this is just a straightforward application of Definition 2.2. □

As the next ingredient in the proof of Theorem 6.1, we consider the so-called *standard coupling* of the percolation processes given by $\mathbf{P}_p$ for all p simultaneously: Let $\mathbf{Q}$ be the product probability measure on $[0,1]^E$ whose marginals are uniform distribution on $[0,1]$. Let U be the corresponding $[0,1]^E$-valued random object, so that the variables $U(e)$ are i.i.d. and uniformly distributed on $[0,1]$. For each $p \in [0,1]$, we may define the $\{0,1\}^E$-valued random object X_p by letting

$$X_p(e) = \begin{cases} 1 & \text{if } U(e) < p \\ 0 & \text{otherwise.} \end{cases}$$

for each $e \in E$. Clearly, X_p has distribution $\mathbf{P}_p$ for each p.

Proof of Theorem 6.1: Fix $p > p_r$. Holroyd [11, Proposition 8.2] observed that left-continuity of $\phi(p)$ at p follows from the $\mathbf{P}_p$-a.s. uniqueness of the infinite rigid component. Hence, and in view of our uniqueness results in Section 3, we have left-continuity, so it only remains to show right-continuity, i.e. that

$$\lim_{h \downarrow 0} \phi(p+h) - \phi(p) = 0\,. \tag{11}$$

Fix an arbitrary edge $e \in E$, and note that

$$\begin{aligned} &\lim_{h \downarrow 0} \phi(p+h) - \phi(p) = \\ &\quad = \lim_{h \downarrow 0} \mathbf{P}_{p+h}(e \text{ is in an infinite rigid component}) \\ &\qquad\quad - \mathbf{P}_p(e \text{ is in an infinite rigid component}) \\ &\quad = \lim_{h \downarrow 0} \mathbf{Q}(e \text{ is in an infinite rigid component of } X_{p+h}) \\ &\qquad\quad - \mathbf{Q}(e \text{ is in an infinite rigid component of } X_p) \\ &\quad = \lim_{h \downarrow 0} \mathbf{Q}(e \text{ is in an infinite rigid component of } X_{p+h} \text{ but not of } X_p) \\ &\quad = \mathbf{Q}(D_p)\,, \end{aligned} \tag{12}$$

with the event D_p defined as

$$D_p = \{e \text{ is in an infinite rigid component of } X_{p+h} \text{ for all } h > 0, \text{ but not of } X_p.\}$$

Fix n such that $e \in E_n$. By Remark 5.5, we have with $\mathbf{Q}$-probability 1 that the unique rigid component in X_p contains a circuit r surrounding H_n. Lemma 6.3 tells us that if e is not in the infinite rigid component (i.e. not in the rigid component containing r) in X_p, and yet belongs to the infinite rigid component in X_{p+h}, then

$$\begin{array}{c}\text{at least one edge inside the circuit } r \text{ is}\\ \text{closed in } X_p \text{ but open in } X_{p+h}.\end{array} \tag{13}$$

On the event D_p, we have that the event (13) happens for all $h > 0$, and since there are only finitely many edges inside r, it follows that some edge $\tilde{e}$ inside r has to be closed on level p but open on level $p+h$ for all $h > 0$. This means that $U(\tilde{e}) = p$, which has $\mathbf{Q}$-probability 0. Now sum over all possible circuits r and all $\tilde{e}$ inside r (this is a countable sum), to deduce that $\mathbf{Q}(D_p) = 0$, which (using (12)) implies (11). □

We next state our partial result concerning continuity of $\phi(p)$ at the rigidity critical value p_r.

Proposition 6.4 *The function $\phi(p)$ is either left- or right-continuous at $p = p_r$.*

Proof: Suppose that $\phi(p)$ is not left-continuous at $p = p_r$. Since $\lim_{h\uparrow 0} \phi(p_r+h) = 0$, we then have $\phi(p_r) > 0$. By Remark 5.5, we thus have $\mathbf{P}_{p_r}$-a.s. that there is a unique rigid component which contains, for each n, some circuit surrounding H_n. Right-continuity of $\phi(p)$ at $p = p_r$ then follows by arguing as in the proof of Theorem 6.1. □

7 Simultaneous uniqueness

Consider the standard coupling $\mathbf{Q}$ of the percolation processes X_p for all $p \in [0,1]$ simultaneously, introduced in Section 6. By Corollary 3.3 in combination with Fubini's Theorem, we have

$$\mathbf{Q}(X_p \text{ contains a unique infinite rigid component for Lebesgue-a.e. } p \in (p_r, 1]) = 1\,.$$

It is natural to ask whether "Lebesgue-a.e. p" can be replaced by "every p" in the above statement; the analogous problem with infinite connected components in place of infinite rigid components has been treated e.g. in [1] and [10]. We have the following answer.

Theorem 7.1 *The standard coupling $\mathbf{Q}$ satisfies*

$$\mathbf{Q}(X_p \textit{ contains a unique infinite rigid component for all } p \in (p_r, 1]) = 1\,.$$

Proof: It is enough to show that for any $\varepsilon > 0$ we have

$$\mathbf{Q}(X_p \text{ contains a unique infinite rigid component for all } p \in (p_r + \varepsilon, 1]) = 1\,.$$

For this, it is enough to show that for any $e_1, e_2 \in E$ we have

$$\mathbf{Q}(\exists p \in (p_r + \varepsilon, 1] \text{ such that } e_1 \text{ and } e_2 \text{ are in different infinite rigid components of } X_p) = 0\,. \tag{14}$$

Fix $e_1, e_2 \in E$ and $\varepsilon > 0$, and write $A^{\varepsilon}_{e_1,e_2}$ for the event in (14). Pick n large enough so that $e_1, e_2 \in E_n$. By Remark 5.5, we have $\mathbf{Q}$-a.s. that the unique infinite rigid component in $X_{p_r+\varepsilon}$ contains a circuit r surrounding E_n. By arguing as in the proof of Theorem 3.2, we see that for all $p \geq p_r + \varepsilon$ we have the following: if e_1 is in an infinite rigid component, then e_1 is also in the same rigid component as the circuit r. The same reasoning applies to the edge e_2, and (14) follows. $\square$

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