Solvability estimates for the Dirichlet problem

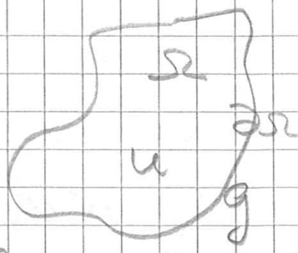
Andreas Rosén
April–May 2025

(A) Divergence form equations

Want to solve BVP = boundary value problem

$$\begin{cases} \operatorname{div} A(x)\nabla u(x) = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \quad (1) \\ u(x) = g(x) & \text{on } \partial\Omega \quad (2) \end{cases}$$

for u, given Dirichlet data g.



Consider first the PDE (1).
Want to allow as general coefficient matrices $A(x) = (A_{ij}(x))_{i,j=1}^n$ and domains $\Omega \subseteq \mathbb{R}^n$.

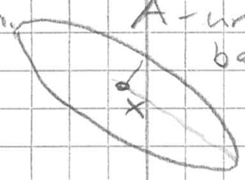
- Only $A \in L_\infty(\Omega)$ in general is assumed. (even degenerate A, where A is unbounded is sometimes considered.)
- Lower ellipticity / accretivity bound on A:
  $$\langle A(x)v, v\rangle \geq \kappa |v|^2, \quad \forall x \in \Omega, \ v \in \mathbb{R}^n$$
- We assume here that A_{ij} are real-valued, $A_{ij}(x) \in \mathbb{R}$. (Complex-valued A_{ij}, or more general systems can be considered.)
- In general, we do not assume A to be symmetric, i.e. $A_{ji} \neq A_{ij}$ may happen. (For constant A wlog $A_{ji} = A_{ij}$ can be assumed, but non-symmetric non-smooth A are challenging.)

Some motivations:

- Solving non-linear PDE using fixed-point iterations with linear PDEs typically requires estimates for potentially non-smooth $A = A(u)$. [GT:83]

- Div form elliptic PDEs $\mathrm{div}(A\nabla u) = 0$ are the most natural ones. Symmetric A can model an anisotropic heat conductivity/diffusion.

A-unit ball



- Div. form systems where $u : \Omega \to \mathbb{R}^N$ and $A : \mathbb{R}^n \otimes \mathbb{R}^N \to \mathbb{R}^n \otimes \mathbb{R}^N$ arise for example in elasticity theory. Here accretivity of A need to be weakened to a Gårding inequality
$$\int_\Omega \langle A\nabla f, \nabla f \rangle \, dx \geq \kappa \int_\Omega |\nabla f|^2 dx \quad , \forall f : \Omega \to \mathbb{R}^N,$$
which holds by Korn's inequality.
When $u : \Omega \to \mathbb{R}$ is scalar-valued, this is equivalent to accretivity.
- Equations arising in homogenization theory have often non-symmetric coeffs. Estimates that uniform in the scaling parameter are needed, amounting to non-smooth A quantitatively.
- The only known technique for obtaining sharp perturbation estimates $A \longmapsto u$, require estimates for complex (in particular non-symmetric) A.

Local/interior estimates

The classical de Giorgi - Nash - Moser interior estimates for solutions $u(x)$ to $\mathrm{div} A \nabla u = 0$ hold for general real-valued, possibly non-symmetric, accretive coefficients $A \in L_\infty(\Omega; \mathbb{R}^{n \times n})$.

See e.g. Moser [M:61]:

Harnack: $\max_K u \lesssim \min_K u$

if solution $u > 0$ in Ω, for any compact $K \subset \Omega$.

Proof is by reverse Hölder estimates of $\left(\int_K |u|^p dx\right)^{1/p}$, from $p = -\infty$ ($\min u$) to $p = +\infty$ ($\max u$).

Consequences:

- Interior Hölder regularity of u, some $\alpha > 0$.
- Interior Morrey regularity of ∇u [HK:07]

Estimates used: Caccioppoli, Poincaré, Sobolev embedding, John-Nirenberg.

Change of variables

Let $\rho : \Omega_1 \to \Omega_2$ be a Lipeomorphism (ρ and ρ^{-1} are Lipschitz)

Assume $\operatorname{div} A_1 \nabla u_1 = 0$ in Ω_1.

Define $u_2 = u_1 \circ \rho^{-1}$ and

$$A_2(\rho(x)) = \frac{1}{J_\rho(x)} \underline{\rho}_x A_1(x) \underline{\rho}_x^*$$

where $\underline{\rho}_x$ is Jacobian matrix of ρ, $\underline{\rho}_x^*$ = its transpose, and $J_\rho(x) = \det \underline{\rho}_x$.

Then $\operatorname{div} A_2 \nabla u_2 = 0$ in Ω_2.

Proof: It is always assumed that $\nabla u \in L_2^{loc.}(\Omega)$, and div is taken in distribution sense, so

$$\int_{\Omega_1} \langle A_1(x) \nabla u_1(x), \nabla \varphi_1(x) \rangle \, dx = 0$$

$\forall \varphi_1 \in H_0^1(\Omega)$ is assumed.

Changing variables $\Rightarrow$

$$\int_{\Omega_2} \langle A_1 \underline{g}^* \nabla u_2, \underline{g}^* \nabla \varphi_2 \rangle \frac{dy}{J_g} = 0$$

$\forall \varphi_2 = \varphi_1 \circ g^{-1} \in H_0^1(\Omega_2)$ $\square$

By changing of vars, we can "move the non-smoothness of Ω to A"

Ex 1: $h : \mathbb{R}^n \to \mathbb{R}$
Lipschitz
$\Omega_1 \subset \mathbb{R}^{1+n}$ domain above graf of h.

Assume $\Delta u_1 = \operatorname{div} I \nabla u_1 = 0$ in Ω_1 and map by $g(t,x) = (t - h(x), x)$ onto upper half space $\mathbb{R}^{1+n}_+$.

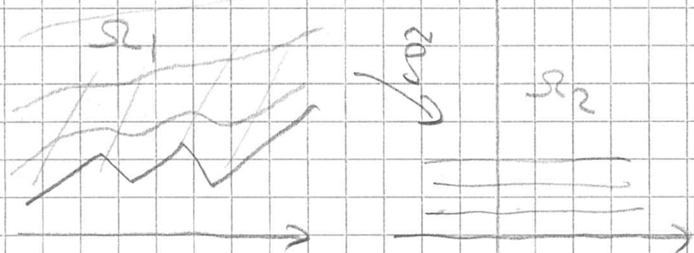
$$\underline{g} = \begin{bmatrix} 1 & -\nabla h \\ 0 & I \end{bmatrix} \Rightarrow A_2 = \begin{bmatrix} 1 & -\nabla h \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\nabla h^T & I \end{bmatrix}$$

$$= \begin{bmatrix} 1 + |\nabla h|^2 & -\nabla h \\ -\nabla h^T & I \end{bmatrix}$$

$u_2 = u_1 \circ g^{-1}$ solves $\operatorname{div} A_2 \nabla u_2 = 0$ in $\mathbb{R}^{1+n}_+$.

Note: A_2 is symmetric, L_∞ and accretive
<u>and</u> independent of vertical coord. t.

Ex 2: h and Ω_2 as in Ex 1



Instead of
$g^{-1}(t,x) = (t + h(x), x)$, we set
$\tilde{g}^{-1}(t,x) = (t + \eta_t * h(x), x)$
where $\eta(x) \in C_0^\infty(\mathbb{R}^n)$, $\int \eta = 1$, supp η small enough,
$\eta_t(x) = \frac{1}{t^n} \eta(\frac{x}{t})$ is a smooth mollifier.
This construction goes back to Dahlberg, Kenig, Nečas, Stein. [N:67] = very good book!

If $\Delta u_1 = 0$ in the Lipschitz domain Ω_1,
then $\operatorname{div} \tilde{A}_2 \nabla u_2 = 0$ in $\mathbb{R}^{1+n}_+$.
Now again $\tilde{A}_2$ is symmetric, L_∞, accretive.
☺ : $\tilde{A}_2 \in C^\infty_{loc.}$ ☹ : $\tilde{A}_2$ depends on t.
Qualitatively, $\nabla \tilde{A}_2$ satisfies a Carleson measure condition:

$$\int_{Q \times (0, \ell(Q))} |\nabla \tilde{A}_2(t,x)|^2 \, t \, dt \, dx \lesssim |Q|$$

$\forall$ cubes $Q \subseteq \mathbb{R}^n$.
(A "corner" on $\partial\Omega_1$ above x_0 typically gives $|\nabla A_2| \sim \dfrac{1}{t + |x - x_0|}$.)

(B) <u>Solvability estimates</u>

Recall: the PDE problem (1)+(2) is wellposed in the sense of Hadamard if

1. $\forall g$ $\exists$ solution u
2. the solution u is unique
3. u depends continuous on g

From Fredholm perspective, 1 & 2 are "algebraic" issues, while 3 is the real analysis problem.
Given a norm $\|\cdot\|$ on Dirichlet data g, and a norm $|||\cdot|||$ on solutions u, our <u>main problem</u> is to prove the <u>solvability estimate</u>

$$|||u||| \leq C \|g\|, \text{ for some } C < \infty.$$

<u>Energy solvability estimate</u>

The most fundamental / simplest / most natural solvability estimate is for

$$\|g\| = \|g\|_{H^{1/2}(\partial\Omega)} \text{ and } |||u||| = \|u\|_{H^1(\Omega)}.$$

This follows from Lax-Milgram's thm:

Fix an extension operator

$g \in H^{1/2}(\partial\Omega) \longmapsto f \in H^1(\Omega)$

so that $f|_\Omega = g$. With $u = v + f$

$$\begin{cases} \operatorname{div} A\nabla u = 0 \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases} \iff \begin{cases} \operatorname{div} A\nabla v = -\operatorname{div} A\nabla f \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}$$

Variational formulation for $v \in H_0^1(\Omega)$ is

$$\underbrace{\int_\Omega \langle A\nabla v, \nabla\varphi\rangle dx}_{= B(v,\varphi)} = \underbrace{-\int \langle A\nabla f, \nabla\varphi\rangle dx}_{= F(\varphi)}, \quad \forall\varphi \in H_0^1(\Omega)$$

Symmetry is not needed.

Lax-Milgram applies and give $v \in H_0^1(\Omega)$ with

$\|v\|_{H^1} \lesssim \|f\|_{H^1} \lesssim \|g\|_{H^{1/2}}$, and $\|u\|_{H^1} \lesssim \|g\|_{H^{1/2}}$,

whenever $A \in L_\infty(\Omega)$ is accretive. (For system, only the Gårding ineq. suffices.)

C^∞

Maximum principle:

This gives the solvability estimate for

$\|g\| = \|g\|_{L_\infty(\partial\Omega)}$ and $|||u||| = \|u\|_{L_\infty(\Omega)}$.

The following proof from [LSW: 63, thm 2.5]

only requires that A_{ij} be real coefficient.

Thm: Assume $A \in L_\infty(\Omega)$ are accretive and real.

Then $g \leq C$ on $\partial\Omega \Rightarrow u \leq C$ in Ω.

Proof: Let $\varepsilon > 0$,

$\Omega_0 := \{x \in \Omega;\ u(x) > C + \varepsilon\}$ and

$$\varphi(x) = \begin{cases} u(x) - C - \varepsilon, & x \in \Omega_0 \\ 0, & x \notin \Omega_0 \end{cases}$$

$\Omega_0 \subset \Omega$ is compact, so $\varphi \in H_0^1(\Omega)$ and

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle dx = 0.$$

But $\nabla\varphi(x) \neq 0 \Rightarrow x \in \Omega_0 \Rightarrow \nabla u(x) = \nabla\varphi(x)$, so

$$\int_\Omega |\nabla\varphi|^2 dx \underset{\substack{\uparrow \\ \text{accretivity}}}{\lesssim} \int_\Omega \langle A\nabla\varphi, \nabla\varphi\rangle dx = 0$$

$\therefore$ $\nabla\varphi = 0$ in Ω, so $\Omega_0 = \emptyset$ and $u \leq C+\varepsilon$, $\forall \varepsilon > 0$ follows, hence $u \leq C$. □

Higher regularity

If $A \in C^\infty$ and if Ω has a C^∞ boundary, then the solvability estimate

$$\|u\|_{H^{s+1/2}(\Omega)} \lesssim \|g\|_{H^s(\partial\Omega)}$$

can be proved for any Sobolev regularity $s \gtrsim \frac{1}{2}$.
See e.g. [T: 96, chapter 5.1, prop. 1.7].

The half space setup

From now on, we assume that $\Omega = \mathbb{R}^{1+n}_+ = \{(t,x) \,;\, t>0,\ x \in \mathbb{R}^n\}$. Write $\bar{x} = (t,x)$, $t = x_0$.

By examples 1 & 2, p. 4-5, a solvability estimate on $\mathbb{R}^{1+n}_+$ gives solvability results for general Lipschitz domains Ω, via a change of variables and partition of unity argument.

Note that it is natural to work with a scale-invariant domain $\mathbb{R}^{1+n}_+$ since the solvability estimates that we aim for are scale invariant:

If $|||u||| \lesssim \|g\|$ for solutions $\operatorname{div} A\nabla u = 0$ in Ω with $u = g$ on $\partial\Omega$, then

$|||u_R||| \lesssim \|g_R\|$ for solutions $\operatorname{div} A_R \nabla u_R = 0$ in $R\Omega$ with $u_R = g_R$ on $\partial\Omega_R$.

$u_R(x) = u(\frac{x}{R})$, $g_R(x) = g(\frac{x}{R})$, $A_R(x) = A(\frac{x}{R})$ for $x \in R\cdot\Omega = \{Ry;\ y\in\Omega\}$.

Note that $\|A_R\|_\infty = \|A\|_\infty$ and that accretivity constants are equal.

C.f. the case when Ω and A are smooth: here by localizing / rescaling, one can reduce to $\Omega = \mathbb{R}^{1+n}_+$ and A = constant, and apply Fourier transf.

A-harmonic measure

For $\bar{x} = (t,x) \in \mathbb{R}^{1+n}_+$, the functional

$$g \longmapsto u(t,x) = \text{solution}$$

is bounded on $C(\mathbb{R}^n)$ by max. principle, if A is real, so $\exists$ measure $\omega_{\bar{x}} = \omega^A_{\bar{x}}$ on $\mathbb{R}^n$ s.t.

$$u(\bar{x}) = \int_{\mathbb{R}^n} g(y)\, d\omega_{\bar{x}}(y)$$

$\omega_{\bar{x}}$ is called A-harmonic measure.

It is a probability measure: $\int_{\mathbb{R}^n} d\omega_{\bar{x}}(y) = 1$ & positive.

- By Harnacks ineq., $\omega_{\bar{x}}$ and $\omega_{\bar{y}}$ are mutually absolutely continuous, $\forall \bar{x}, \bar{y} \in \mathbb{R}^{1+n}_+$.

Examples by [CFK:81], using the theory for quasiconformal mappings, show that $\omega_{\bar{x}}$ may be completely singular with respect to Lebesgue meas. dx.

Heuristics: some regularity of $A(t,x)$ like in Ex 1 and Ex 2 in the transversal coordinate t is needed for $\omega_{\bar{x}}$ to be abs. cont. w.r.t. dx.

- Greens formula:

$$\iint_{\mathbb{R}^{1+n}_+} \Big((\underbrace{\operatorname{div} A^* \nabla G^*}_{\delta_{\bar{x}}})u - G^*(\underbrace{\operatorname{div} A \nabla u}_{0}) \Big) dt\, dx = \int_{\mathbb{R}^n} \Big((\nu \cdot A^* \nabla G^*) u - \underbrace{G^*}_{0}(\nu \cdot A \nabla u) \Big) dx$$

If a Greens function $G^*(\bar{y}, \bar{x})$

$\left(\operatorname{div}_{\bar y} A^*(\bar y) \nabla_{\bar y} G^*(\bar y, \bar x) = \delta_{\bar x}(\bar y) \;,\; G(y;\bar x) = 0 \;,\; y \in \mathbb{R}^n \right)$
is known, then

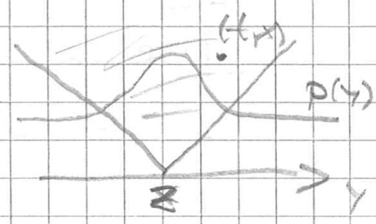
$$\frac{d\omega_{\bar x}(y)}{dy} = \underset{\substack{\parallel \\ (0,-1)}}{\nu(y)} \cdot A^*(y) \nabla_{\bar y} G^*(y, \bar x).$$

Non-tangential maximal function

For Δ, harm. measure is

$$d\omega_{(t,x)}(y) = c \cdot \frac{t}{(t^2 + |y-x|^2)^{\frac{n+1}{2}}} dy$$

Write $p(y) = c \dfrac{1}{(1+|y|^2)^{\frac{n+1}{2}}}$, so



$$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{t^n} p\left(\frac{x-y}{t}\right) g(y)\, dy.$$

Defn: The non-tangential max. function of u is

$$(N_* u)(z) = \sup_{t,x : |x-z|<t} |u(t,x)| \quad , z \in \mathbb{R}^n$$

(Hardy-Littlewood) max. function of g is

$$(Mg)(z) = \sup_{r>0} \fint_{|y-z|<r} |g(y)|\, dy$$

Subordination $\Rightarrow$

$$N_* u(z) \lesssim Mg(z) \quad , \forall z \in \mathbb{R}^n.$$

$M : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ is bounded if $1 < p \le \infty$.

We get a solvability estimate for Δ:

$$\underbrace{\| N_* u \|_p}_{= |||u|||} \lesssim \| g \|_p \quad \text{if} \quad p > 1.$$

A main problem: for which coeff. $A(t,x)$ and which $1 < p < \infty$ do we have the solvability estimate

$$\| N_* u \|_p \lesssim \| g \|_p \quad ?$$

Call this the p-solvability estimate.

It is known to hold iff

$d\omega = f\,dy$ is absolute cont. w.r.t. dy with reverse Hölder estimate $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$\left(\fint_Q f^q\,dy\right)^{1/q} \lesssim \fint_Q f\,dy \quad , \forall \text{ cubes } Q \subseteq \mathbb{R}^n.$$

See [KKPT:00, p. 241]. [K:91] for symmetric eqns.

Here $\omega = \omega_{\bar{x}_0}$ is harmonic measure for any fixed pole $\bar{x}_0 \in \mathbb{R}^{1+n}_+$.

The pioneer:

Thm: B. Dahlberg [D:77]

The 2-solvability est. holds for Δ on Lipschitz domains.

By change of vars as in Ex 1, this corresponds to a symmetric t-independent $A = A(x)$ on $\mathbb{R}^{1+n}_+$. We shall prove ~~this~~ 2-solvability for all such A.

BMO solvability

It is known that p-solvability $\Rightarrow$ q-solvability if $q > p$. At the endpoint $p = \infty$, there is a deeper solvability estimate than the maximum principle.

Consider Δ on $\mathbb{R}^{1+n}_+$.

Write $t\nabla_{t,x}\left(\frac{1}{t^n}p\left(\frac{x}{t}\right)\right) = \frac{1}{t^n}q\left(\frac{x}{t}\right)$,

so that $\int_{\mathbb{R}^n} q(x)\,dx = 0$ and

$$t\nabla_{t,x}u(t,x) = \int_{\mathbb{R}^n} \frac{1}{t^n}q\left(\frac{x-y}{t}\right) g(y)\,dy.$$

$$\Rightarrow \iint_{\mathbb{R}^{1+n}_+} |t\nabla_{t,x}u|^2 \frac{dt\,dx}{t} = \int_0^\infty \frac{dt}{t}\int dx \left|\frac{1}{t^n}q\left(\frac{x}{t}\right) * g(x)\right|^2$$

$$= /\text{Plancherel}/ = \int_0^\infty \frac{dt}{t}\int_{\mathbb{R}^n} d\xi\, \left|\hat{q}(t\xi)\,\hat{g}(\xi)\right|^2 = /\text{Fubini and } t \mapsto t/|\xi|\,/$$

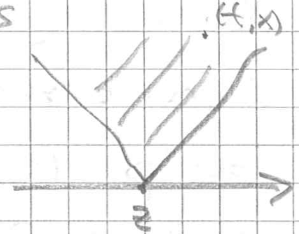
$$= \int_{\mathbb{R}^n} d\xi \left(\int_0^\infty \frac{dt}{t}\left|\hat{q}\left(t\frac{\xi}{|\xi|}\right)\right|^2\right)|\hat{g}(\xi)|^2$$

$$= c \cdot \int_{\mathbb{R}^n} |g(x)|^2\,dx \quad \text{since } \hat{q}(0) = 0 \text{ so that } \int_0^\infty \frac{dt}{t} \ldots \text{ is convergent.}$$

This proves a square function L_2-solvability estimate $\|Su\|_2 \lesssim \|g\|_2$.

Defn: The square function of u is

$$Su(z) = \left(\int_{|x-z|<t} |\nabla_{t,x} u|^2\, t\, \frac{dt\,dx}{t^n} \right)^{1/2}.$$



Now assume $g \in BMO(\mathbb{R}^n)$, i.e.,

$$⨍_Q |g(y) - g_Q|^2\,dy \leq C^2, \quad \forall \text{ cubes } Q \subset \mathbb{R}^n$$

g_Q: average $⨍_Q g$.

Best $C = \|g\|_{BMO(\mathbb{R}^n)}$. $BMO(\mathbb{R}^n) \supsetneq L_\infty(\mathbb{R}^n)$, $\ln|x| \in BMO(\mathbb{R}^n)$

Given cube $Q \subset \mathbb{R}^n$, consider $\iint_{\hat{Q}} |\nabla_{t,x} u|^2\, t\,dt\,dx$ over the Carleson box $\hat{Q} = (0, \underset{\uparrow \text{ side length}}{\ell(Q)}) \times Q$.

Set $h(y) = (g(y) - g_{3Q}) \cdot 1_{3Q}(y)$

$q(x) \sim \frac{1}{|x|^{n+1}}$ as $x \to \infty$, but pretend $\operatorname{supp} q \subset B(0,1)$ to avoid tail technicalities.

Let v be solution with h $v|_{\mathbb{R}^n} = h$.

$\Rightarrow \nabla v = \nabla u$ on $\hat{Q}$

L_2 estimate $\Rightarrow$

$$\iint_{\hat{Q}} |\nabla u|^2 t\,dt\,dx = \iint_{\hat{Q}} |\nabla v|^2 t\,dt\,dx \leq \iint_{\mathbb{R}^{1+n}_+} |\nabla v|^2 t\,dt\,dx$$

$$\lesssim \|h\|_2^2 = \int_Q |g - g_Q|^2\,dy$$

We get the solvability est.

$$\| |\nabla u|^2 t \|_C^{1/2} \lesssim \|g\|_{BMO},$$

where

the Carleson norm is

$$\|f\|_C = \sup_{\substack{\text{cube} \\ Q \subset \mathbb{R}^n}} \frac{1}{|Q|} \iint_{\hat{Q}} |f| \, dt \, dx.$$

We shall prove this solvability estimate for all <u>t-independent</u> real A, not necessarily symmetric, and that this entails p-solvability for p large enough depending on A.

<u>Note</u>: Unlike the maximum principle, the BMO estimate relies on cancellations ($\int g \, dy = 0$)

- It is known that the BMO solvability estimate holds iff harmonic measure $\omega \in A_\infty(dy)$

  The Muckenhoupt class of weight A_∞ is defined as follows: $\exists \varepsilon, \delta \in (0,1)$:
  $E \subset Q$, Q = cube in $\mathbb{R}^n$, E any subset

  $$\frac{|E|}{|Q|} < \delta \implies \frac{\omega(E)}{\omega(Q)} < \varepsilon$$

  Lebesgue meas. A-harmonic meas. $\omega(E) = \int_E d\omega$

  See [DKP:10, Thm 2.1].

- It is known that

  $$A_\infty(dy) = \bigcup_{q>1} B_q(dy)$$

  where $B_q(dy)$ is the class of weights ω satisfying the <u>reverse Hölder est.</u>

  $$\forall \text{ cubes } Q \subset \mathbb{R}^n: \left(⨍_Q \left(\frac{d\omega}{dy}\right)^q dy \right)^{1/q} \lesssim ⨍_Q \frac{d\omega}{dy} \, dy.$$

  $\therefore$ For coefficient A, the BMO solv. est. holds iff the p-solvability est. holds for some $p < \infty$.

(C) L_2 theory for conjugate systems

Philosophy: analytic functions are more fundamental than harmonic functions.

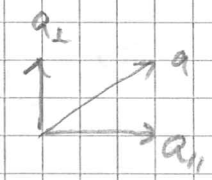
We generalize $\Delta u = 0$ in $\mathbb{R}^2 = \mathbb{C}$ to $\operatorname{div} A \nabla u = 0$ in $\mathbb{R}^{1+n}$, and seek generalized conjugate functions to u. Assume $\operatorname{div}_{t,x} A(x) \nabla_{t,x} u = 0$ in $\Omega = \mathbb{R}^{1+n}_+$.

Split matrix:

$$A(x) = \begin{array}{cc} \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} & \begin{array}{c} t \\ x \end{array} \\ \begin{array}{cc} \quad t \quad & \quad x \quad \end{array} & \end{array}$$

The natural "normal derivative" is the conormal derivative

$$\partial_{\nu_A} u = (A \nabla_{t,x} u)_\perp = a \partial_t u + b \nabla_x u$$



Notation: $a = \begin{bmatrix} a_\perp \\ a_\parallel \end{bmatrix} \in \mathbb{R}^{1+n}$

In $\mathbb{R}^{1+n}$, u will have n conjugate functions $V_\parallel(t,x) \in \mathbb{R}^n$, defined as follows:

Write $\partial_{\nu_A} u = \operatorname{div}_x V_\parallel$ at each fixed height t.

Possible since ∇_x is injective $\Leftrightarrow$ div_x is surjective.

Then insert into $\operatorname{div}_{t,x} A \nabla_{t,x} u = 0$:

$$\partial_t \underbrace{(A \nabla_{t,x} u)_\perp}_{= \operatorname{div}_x V_\parallel} + \operatorname{div}_x \underbrace{(A \nabla_{t,x} u)_\parallel}_{= c\partial_t u + d \nabla_x u} = 0$$

commute

We may, and will, choose $V_\parallel$ s.t.

$$\partial_t V_\parallel + c \partial_t u + d \nabla_x u = 0$$

Together with $a \partial_t u + b \nabla_x u = \operatorname{div}_x V_\parallel$, we have

$$\partial_t \underbrace{\begin{bmatrix} -u \\ V_\parallel \end{bmatrix}}_{V} + \underbrace{\begin{bmatrix} a^{-1} & -a^{-1} b \\ c a^{-1} & d - c a^{-1} b \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}}_{D} \begin{bmatrix} -u \\ V_\parallel \end{bmatrix} = 0$$

$\therefore$ We have a conjugate system V, containing $u = -v_\perp$, that solves a generalized Cauchy-Riemann equation

$$\partial_t v + BDv = 0.$$

We view B and D as multiplication / differential operators acting horizontally, in $L_2(\mathbb{R}^n; \mathbb{R}^{1+n}) = L_2(\mathbb{R}^n)$, for each fixed t.

Properties of B and D:

$D = \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}$ is self-adjoint, closed, densely defined unbounded in L_2.

Its range $R(D)$ is not closed. Its closure is $\mathcal{H} = \begin{bmatrix} L_2 \\ \mathcal{H}_{\parallel} \end{bmatrix}$ where $\mathcal{H}_{\parallel} = \{$curl-free vector fields$\}$.

By self-adjointness, its nullspace is $\mathcal{H}^{\perp} = \begin{bmatrix} 0 \\ \mathcal{H}_{\parallel}^{\perp} \end{bmatrix}$ div-free fields.

Prop: A is accretive $\Longleftrightarrow$ B is accretive.

(A is self-adjoint $\not\Leftrightarrow$ B is self-adjoint!)

Proof: For $v \in \mathbb{R}^{1+n}$, write $v_A = \begin{bmatrix} av_\perp + bv_\parallel \\ v_\parallel \end{bmatrix}$ and note $|v| \approx |v_A|$, if A bounded and accretive.

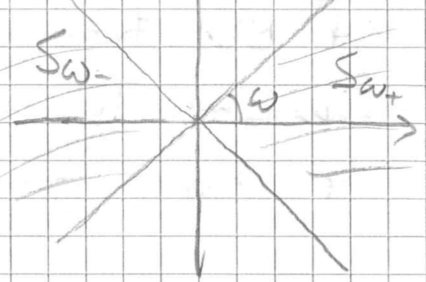
$$\langle Bv_A, v_A \rangle = \Big\langle \underbrace{\begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix} \begin{bmatrix} av_\perp + bv_\parallel \\ v_\parallel \end{bmatrix}}_{= \begin{bmatrix} v_\perp \\ cv_\perp + dv_\parallel \end{bmatrix}}, \begin{bmatrix} av_\perp + bv_\parallel \\ v_\parallel \end{bmatrix} \Big\rangle$$

$$= \Big\langle \begin{bmatrix} av_\perp + bv_\parallel \\ cv_\perp + dv_\parallel \end{bmatrix}, \begin{bmatrix} v_\perp \\ v_\parallel \end{bmatrix} \Big\rangle = \langle Av, v \rangle \gtrsim |v|^2 \approx |v_A|^2$$

$\Leftarrow$ follows since $A \longmapsto B = \hat{A}$ is self-inverse. $\square$

Remark: $d - ca^{-1}b$ appearing in B is a well-known construction in matrix theory called a Schur complement.

Prop: The spectrum $\sigma(BD)$ is contained in a bisector $S_\omega = S_{\omega+} \cup S_{\omega-}$, for $\omega < \frac{\pi}{2}$ depending on B.



Proof: Let $\lambda u - BDu = f$.

$$\Rightarrow \underset{\mathrm{Im}}{\langle} B^{-1}(\lambda u - BDu), u\rangle \underset{\mathrm{Im}}{=} \langle B^{-1}f, u\rangle$$

$$D^* = D \Rightarrow \mathrm{Im}\big(\lambda \underbrace{\langle B^{-1}u, u\rangle}_{\substack{\text{lies in compact}\\ \text{subset of right}\\ \text{half-plane}}}\big) = \mathrm{Im}\langle B^{-1}f, u\rangle$$

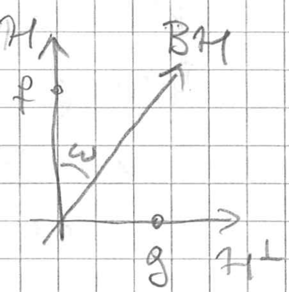
$$\lambda \notin S_\omega \Rightarrow \|u\|^2 \lesssim \|f\| \cdot \cancel{\|u\|}$$

$$\therefore \|(\lambda - BD)^{-1}f\| \lesssim \|f\|$$

(A duality argument shows $\lambda - BD$ surjective.) $\square$

Prop: We have a topological (not orth.) "perturbed Hodge" splitting

$$L_2 = \underset{\overline{R(D)}}{\underset{\|}{B\mathcal{H}}} \oplus \underset{N(D)}{\underset{\|}{\mathcal{H}^\perp}} = \overline{R(BD)} \oplus N(BD).$$



Proof: Let $f \in \mathcal{H}$, $g \in \mathcal{H}^\perp$.

Accretivity $\Rightarrow$

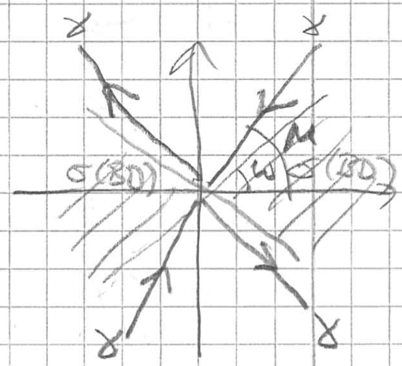
$$\|Bf\| \approx \|f\| \lesssim \langle Bf, f\rangle$$
$$= \langle Bf + g, f\rangle \lesssim \|Bf + g\| \cdot \|f\|$$

$\Rightarrow \|Bf\| \lesssim \|Bf + g\|$, so

$\|Bf\| + \|g\| \approx \|Bf + g\|$, proving that $B\mathcal{H}$ and $\mathcal{H}^\perp$ are transversal.

(A duality argument shows that $B\mathcal{H}$ and $\mathcal{H}^\perp$ span L_2.) $\square$

We synthesize more general functions $\varphi(BD)$ of BD from the resolvents $\frac{1}{\lambda - DB}$, by Dunford functional calculus:



$$\varphi(BD) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\lambda)}{\lambda - BD}\, d\lambda$$

with a curve $\gamma = \partial S_\mu$, for some $\omega < \mu < \frac{\pi}{2}$, positively oriented around $\sigma(BD) \subset S_\omega$.

The integral is convergent if $\varphi(\lambda) \to 0$ as $\lambda \to 0$ and $\lambda \to \infty$, not too slowly, since $\left\| \frac{1}{\lambda - BD} \right\| \lesssim \frac{1}{|\lambda|}$ on γ.

Since BD is not self-adjoint, the symbol $\varphi(\lambda)$ must be holomorphic on some neighbourhood of S_μ.

Thm: By strong continuity, the Dunford functional calculus extend to a bounded homomorphism

$$\varphi(\lambda) \in H^\infty(S_\mu) \longrightarrow \mathcal{L}(L_2(\mathbb{R}^n)) \ni \varphi(BD)$$

i.e., $\exists\, C < \infty$:

$$\|\varphi(BD)\|_{L_2 \to L_2} \leq C \|\varphi\|_{L_\infty(S_\mu)}$$

This is a very deep result, essentially equivalent to the Kato square root ~~problem~~ estimate

$$\|\sqrt{-\operatorname{div} A \nabla}\, u\|_{L_2} \approx \|\nabla u\|_{L_2}.$$

See [AAMc: 10a], [AAMc: 10b].

Ex: $\varphi(\lambda) = 1_{S_{\mu+}}(\lambda) = \begin{cases} 1, & \lambda \in S_{\mu+} \\ 0, & \text{else} \end{cases}$

gives a bounded projection

$E^+ = 1_{S_{\mu+}}(BD)$ = spectral projection for $S_{\mu+}$.

The subspace $E^+ L_2 = R(E^+)$ is a generalized Hardy subspace for $\partial_t v + BDv = 0$ in $\mathbb{R}^{1+n}_+$.

- If $v_0(x) \in E^+ L_2$, define

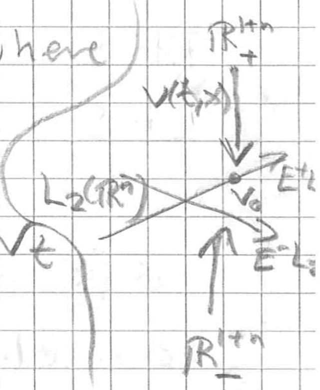
  $v_t(\cdot) = v(t, \cdot) = \varphi_t(BD) v_0$, where

  $\varphi_t(\lambda) = \begin{cases} e^{-t\lambda}, & \lambda \in S_{\mu+} \\ 0, & \text{else.} \end{cases}$

  $\Rightarrow \partial_t v_t = \partial_t e^{-tBD} v_0 = -BD\, v_t$

  and $v_t \xrightarrow{L_2} v_0$ as $t \to 0^+$,

  $v_t \xrightarrow{L_2} 0$ as $t \to \infty$.



- By an integrating factor argument, all such solutions v to $\partial_t v_t + BD v_t = 0$ in $\mathbb{R}^{1+n}_+$ is the "Cauchy integral" $v_t = e^{-tBD} v_0$ of $v_0 = \lim_{t \to 0^+} v_t \in E^+ L_2$.

Square function / quadratic estimates

Let $\psi(\lambda) = \dfrac{\lambda}{1+\lambda^2}$ (or use any other $\psi \in H^\infty(S_\mu)$ with decay at 0 and ∞).

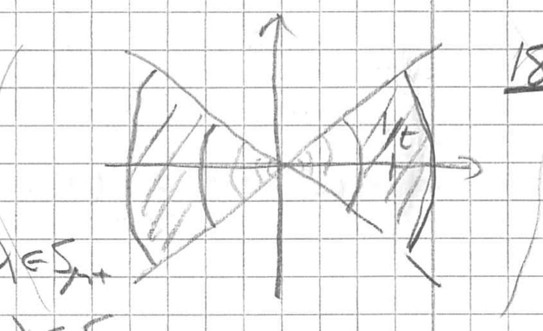
Set $\psi_t(\lambda) = \psi(t\lambda)$, where $t > 0$ is scaling parameter. We have quadratic estimates

$$\int_0^\infty \| \psi_t(BD) f \|^2_{L_2(\mathbb{R}^n)} \frac{dt}{t} \approx \| f \|^2_{L_2}, \quad \forall f \in B\mathcal{H}$$

This is another deep result, equivalent to Thm, and is a "Littlewood-Paley decomposition" for the operator BD.

Interpretation: $\psi_t(BD) \approx$ spectral projection for the dyadic band

$\{\frac{1}{2t} < |\lambda| < \frac{2}{t}\} \cap S_\mu$.



<u>Ex</u>: Apply Q.E. with

$$\psi(\lambda) = \lambda e^{-|\lambda|} = \begin{cases} \lambda e^{-\lambda}, & \lambda \in S_{\mu+} \\ \lambda e^{+\lambda}, & \lambda \in S_{\mu-} \end{cases}$$

and $f = V_0 \in E^+L_2 \subset B\mathcal{H}$

RHS $= \|V_0\|_2^2 \approx \|u(0,\cdot)\|_2^2 + \|V_\parallel(0,\cdot)\|_2^2$.

LHS: $\psi_t(BD)V_0 = tBDe^{-tBD}V_0 = tBDV_t$
$= -t\partial_t V_t$

But since $\partial_t V_\parallel + c\partial_t u + d\nabla_x u = 0$, we can eliminate $V_\parallel$:

$$\|\partial_t V\|_2^2 = \|\partial_t u\|_2^2 + \|\partial_t V_\parallel\|^2 \approx \|\nabla_{t,x} u\|^2$$

Q.E $\Leftrightarrow$

$$\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x} u|^2 \, t\,dt\,dx \approx \|u_0\|_2^2 + \|V_{\parallel,0}\|_2^2$$

Compare to square fcn est. for Δ on p. 10-11:

Only assuming $A \in L_\infty$ accretive and t-independent (the BD L_2-theory works even for systems) the <u>square function L_2-solvability</u> estimate

$$\|Su\|_2 \approx \|u\|_2 \quad \text{hold}$$

if at $\mathbb{R}^n$, we can prove the <u>Hilbert transform estimate</u>

$$\|V_{\parallel,0}\|_2 \lesssim \|u_0\|_2.$$

Also note that $\|Su\|_2 \gtrsim \|u\|_2$ always holds.

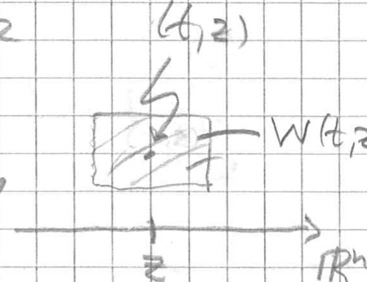
Non-tangential maximal estimates

Next we prove:

Thm: For a conjugate system $v = \begin{bmatrix} -u \\ v_{\parallel} \end{bmatrix}$ in $\mathbb{R}^{1+n}_+$, and $v_t = e^{-tBD} v_0$, we have
$$\| N_*(u) \|_2 \lesssim \| v_0 \|_2 \approx \| u_0 \|_2 + \| v_{\parallel,0} \|_2$$

It suffices to bound a modified NT-max fcn

$$\tilde{N}_*(v)(z) = \sup_{t>0} \underbrace{\left(\iint_{W(t,z)} |v|^2 \right)^{1/2}}_{L_2 \text{ Whitney average}}$$



The Whitney regions $W(t,z)$:

- "ball/cube"-like around (t,z)
- diameter $\approx$ distance to $\mathbb{R}^n$ of $W(t,z)$

The precise geometry is unimportant: give equivalent norms $\| \tilde{N}_*(v) \|_2$.

By interior regularity:
$$\| N_*(u) \|_2 \lesssim \| \tilde{N}_*(u) \|_2 \leq \| \tilde{N}_*(v) \|_2$$
if $\operatorname{div} A \nabla u = 0$ and A is <u>real</u>.

To show
$$\| \tilde{N}_*(v) \|_2 \lesssim \| v_0 \|_2$$
we use quadratic estimates to replace e^{-tBD} by $\frac{1}{1+itBD}$. Let

$$\psi(\lambda) = \left(e^{-\lambda} - \frac{1}{1+i\lambda} \right) \cdot 1_{S_{\mu+}}(\lambda), \text{ so}$$

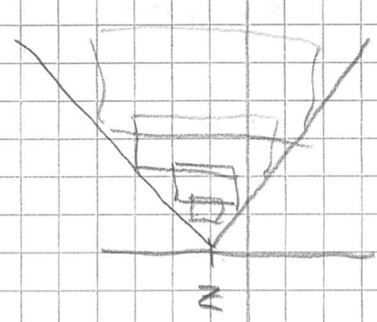
$$v_t = (1+itBD)^{-1} v_0 + \psi(tBD) v_0 = I + II$$

II: $\psi \to 0$ when $\lambda \to 0$ and when $\lambda \to \infty$.

so $\int_0^\infty \|\mathcal{Y}_t(BD)v_0\|_2^2 \frac{dt}{t} \lesssim \|v_0\|_2^2$.

Now consider NT-max of $\boxed{II}$: $v_2 = \mathcal{Y}_t(BD)v_0 \Rightarrow$

$$\tilde{N}_*(v_2)(z)^2 = \sup_{t>0} \frac{1}{t^{n+1}} \iint_{W(t,z)} |v_2|^2 \, ds\, dy$$



$$\lesssim \iint_{|y-z|<c\cdot s} |v_2|^2 \frac{ds\,dy}{s^{n+1}}$$

$$\Rightarrow \|\tilde{N}_*(v_2)\|_2^2 \lesssim \int_{\mathbb{R}^n} dz \iint_{|y-z|<c\cdot s} |v_2(s,y)|^2 \frac{ds\,dy}{s^{n+1}}$$

$$\lesssim \iint_{\mathbb{R}^{1+n}_+} |v_2(s,y)|^2 \frac{ds\,dy}{s} \lesssim \|v_0\|^2$$

$\underline{I}$: This term cannot be bounded by quadratic ests., since $(1+itBD)^{-1}v_0 \not\to 0$, $t \to 0$.

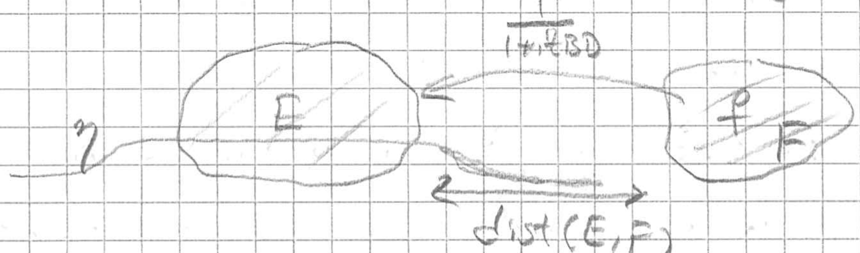
Instead a max. fcn argument based on the following is used.

<u>Lemma</u>: $\forall m \in \mathbb{Z}_+ \; \exists C_m < \infty \; \exists \delta > 0 \; \forall q \in (2-\delta, 2+\delta) \; \forall t > 0$:

$$\|(1+itBD)^{-1} f\|_{L_q(E)} \le \frac{C_m}{\left(1+\frac{\operatorname{dist}(E,F)}{t}\right)^m} \|f\|_{L_q(F)}$$

if $\operatorname{supp} f \subset F$

$E, F \subset \mathbb{R}^n$



<u>Proof</u>:

- <u>First $q=2$</u>. We have proved $m=0$.

  Proceed by induction, using commutator estimates:

  $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta = 0$ on F, $\eta = 1$ on E

  $\|\nabla \eta\|_\infty \lesssim \frac{1}{\operatorname{dist}(E,F)}$ is used.

$$\|(1+itBD)^{-1}f\|_{L_2(E)} \leq \|\eta(1+itBD)^{-1}f\|_{L_2(\mathbb{R}^n)}$$
$$= \|\underbrace{[\eta,(1+itBD)^{-1}]}\, f\|_{L_2(\mathbb{R}^n)}$$

$$= -(1+itBD)^{-1}\underbrace{[\eta, 1+itBD]}_{= itB\underbrace{[\eta,D]}_{\|\cdot\| \lesssim \|\nabla\eta\|_\infty}}(1+itBD)^{-1}$$

$$\lesssim t\frac{1}{\mathrm{dist}(E,F)}\|(1+itBD)^{-1}f\|_{L_2(\mathrm{supp}\nabla\eta)}$$

Iterate using $\mathrm{dist}(\mathrm{supp}(\nabla\eta), F) \approx \mathrm{dist}(E,F)$.

- $\underline{q \approx 2}$: Interpolation with $q=2$ $\Rightarrow$
  suffices to consider $m=0$ when $q\neq 2$.

Consider $(1+itBD)v = f$
$$\overset{\shortparallel}{\begin{bmatrix}-u\\ v_\parallel\end{bmatrix}}$$

Need $\|v\|_{L_q(\mathbb{R}^n)} \lesssim \|f\|_{L_q(\mathbb{R}^n)}$

The algebra on p.13, replacing ∂_t by $\frac{1}{it}$, shows

$$\underbrace{\begin{bmatrix}1 & it\,\mathrm{div}_x\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\ it\nabla_x\end{bmatrix}}_{=L} u = \begin{bmatrix}1 & it\,\mathrm{div}_x\end{bmatrix}\begin{bmatrix}-af_\perp\\ f_\parallel - cf_\perp\end{bmatrix}$$

$L: W^{1,q}(\mathbb{R}^n) \longrightarrow W^{-1,q}(\mathbb{R}^n)$, $\forall q$.

Sneiberg extrapolation from $q=2$ shows invertibility when $|q-2|<\delta$. See [S:74].

$\Rightarrow$ $\|u\|_q + t\|\nabla u\|_q \lesssim \|f\|_q$

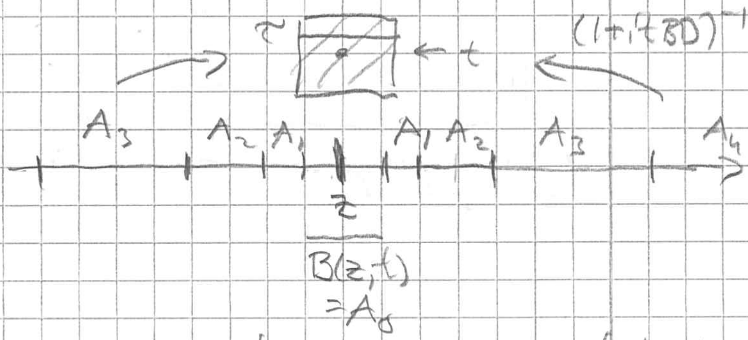
$\Rightarrow$ $\|v_\parallel\|_q \lesssim \|u\|_q + \|t\nabla u\|_q + \|f\|_q \lesssim \|f\|_q$.

$\square$

Back to I:

Again by interior regularity, it suffices to bound $\|\underbrace{\tilde N_*^q}_{q\text{-Whitney averages}}(\underbrace{v_1}_{(1+itBD)^{-1}v_0})\|_2$ for some $q<2$.

Fix Whitney region $W(t,z)$ and $\tau \approx t$.



Split $\mathbb{R}^n = \bigcup_{j=0}^{\infty} A_j$ into dyadic annuli, $A_j = B(z, 2^j t) \setminus B(z, 2^{j-1} t)$.

Lemma $\Rightarrow$

$$\left(\int_{B(z,t)} |R_\tau f|^q \right)^{1/q} \lesssim \sum_{j=0}^{\infty} \underbrace{\left(\frac{2^j t}{\tau} \right)^{-m}}_{\approx 2^{-jm}} \underbrace{\left(\int_{A_j} |f|^q \right)^{1/q}}_{\lesssim (2^j t)^n M(f^q)}$$

$$\lesssim \left(t^n M(f^q) \right)^{1/q} \quad \text{if } m > n/q.$$

$$\Rightarrow \| \tilde{N}_*^q (v_1) \|_2 \lesssim \| M(f^q)^{1/q} \|_2$$

$$= \Big(\underbrace{\int (M(f^q))^{2/q}}_{\lesssim \int f^{q \cdot \frac{2}{q}} \text{ if } q<2} \Big)^{1/2} \lesssim \|f\|_2 .$$

$\square$

Summary of part C

A solution to $\mathrm{div}_{t,x} A(x) \nabla_{t,x} u = 0$ in $\mathbb{R}^{1+n}_+$ has n conjugate functions $v_\parallel$.

- For any accretive t-independent $A \in L_\infty$ (even for systems, non-real A) we have a priori estimates

$$\|N_* u\|_2 \lesssim \|u_0\|_2 + \|v_{\parallel,0}\|_2 \approx \|Su\|_2$$

For real A, we have $N_* u \approx \tilde{N}_* u$

- When L_2-solvability holds in the sense that the Hilbert transform bound

$$\|v_{\parallel,0}\|_2 \lesssim \|u\|_2$$ holds, then

we have L_2 solvability estimates

$$\|\tilde{N}_* u\|_2 \lesssim \|u_0\|_2 \quad \text{and}$$

$$\|Su\|_2 \lesssim \|u_0\|_2$$

In fact $\|\tilde{N}_* u\|_2 \approx \|Su\|_2 \approx \|u_0\|_2$.

(D) $\mathbb{P}$-solvability estimates

We shall prove such by proving $\|v_{\parallel,0}\|_2 \lesssim \|u_0\|_2$ assume certain algebraic structure of A, using integration by parts.

Symmetric A:

Consider the generalized CR-system

$$\partial_t v + BDv = 0.$$

Recall $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow B = \hat{A} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}$

and note

$$B^* = \begin{bmatrix} a^{*-1} & a^{*-1}c^* \\ -b^* a^{*-1} & d^* - b^* a^{*-1} c^* \end{bmatrix} \neq \widehat{A^*}$$

Rather $B^* = N \widehat{A^*} N$, where $N = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\therefore A^* = A \not\Rightarrow B^* = B$, but instead

$$B^* = NBN.$$

Prop: $A^* = A \Rightarrow \|v_{\parallel,0}\|_2 \lesssim \|u_0\|_2$.

Proof:

$$\int_{\mathbb{R}^n} \langle NB^{-1} v_0, v_0 \rangle \, dx$$

$$= \iint_{\mathbb{R}^{1+n}_+} \underbrace{-\partial_t \langle NB^{-1} v, v \rangle}\, dt\, dx$$

$$= \langle N\cancel{B^{-1}}(\cancel{B}Dv), v \rangle + \langle NB^{-1}v, (BDv) \rangle$$

$$= \langle (ND + DN)v, v \rangle = 0.$$

$$\Rightarrow \int_{\mathbb{R}^n} \langle B^{-1}(v_\perp + v_\parallel), -v_\perp + v_\parallel \rangle \, dx = 0$$

$$\Rightarrow \int |v_\parallel|^2 dx \underset{B^{-1}\text{ accr.}}{\lesssim} \int \langle B^{-1} v_\parallel, v_\parallel \rangle dx + \underbrace{\|u\|_2 \|v_\parallel\|_2}_{\leq \frac{\varepsilon}{2}\|v_\parallel\|_2^2 + \frac{1}{2\varepsilon}\|u\|_2^2} + \|u\|_2^2$$

ε small $\Rightarrow$

$$\|v_\parallel\|_2^2 \lesssim \|u\|_2^2 \quad \square$$

Note: this proves in particular Dahlberg's thm on L_2 solvability for Δ on Lipschitz domain. (p. 10). Indeed, parametrizing as in Ex. 1 (p. 4), this amounts to symmetric coefficient of the special form

$$A = \begin{bmatrix} 1+|Dh|^2 & -Dh \\ -Dh & I \end{bmatrix} \quad \text{in } \mathbb{R}^{1+n}_+ .$$

Triangular A:

A second class of coefficients, for which L_2-solvability holds is the following triangular matrices.

Prop: If $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, that is, if $b = 0$, then

$$\|v_{\parallel,0}\|_2 \lesssim \|u\|_2 .$$

Proof: Since $\|Su\|_2 \approx \|v_{\parallel,0}\|_2 + \|u\|_2$, it suffices to show

$\|Su\|_2 \lesssim \|u\|_2$, or explicitly

$$\iint_{\mathbb{R}^{1+n}_+} |Du|^2 t \, dt \, dx \lesssim \int_{\mathbb{R}^n} |u|^2 dx .$$

Calculate:

$$\int_{\mathbb{R}^n} |u|^2 dx \underset{\text{accretivity}}{\approx} \int_{\mathbb{R}^n} a|u|^2 dx = - \iint_{\mathbb{R}^{1+n}_+} \overset{a(x)}{a}\, \partial_t(u^2) \, \overset{1}{dt} \, dx$$

$$= \iint a\, \partial_t^2(u^2)\, t \, dt\, dx$$

integration by parts in t.

$$= a\,\partial_t(2u\,\partial_t u) = 2a(\partial_t u)^2 + 2u\,\partial_t(a\,\partial_t u)$$

$$/\ \operatorname{div} A\nabla u = 0 \iff \partial_t(a\,\partial_t u) + \operatorname{div}_x(c\,\partial_t u + d\nabla_x u)\ /$$

$$= \iint 2a(\partial_t u)^2\, t\, dt\, dx - \iint 2u \operatorname{div}_x(c\,\partial_t u + d\nabla_x u)\, t\, dt\, dx$$

$$= \iint \Big(2a(\partial_t u)^2 + 2\langle \nabla_x u,\ c\,\partial_t u + d\nabla_x u\rangle\Big)\, t\, dt\, dx$$

int. by parts in x.

$$= 2\iint \langle \nabla_{t,x} u,\ A\nabla_{t,x} u\rangle\, t\, dt\, dx \gtrsim \iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x} u|^2\, t\, dt\, dx$$

accretivity.

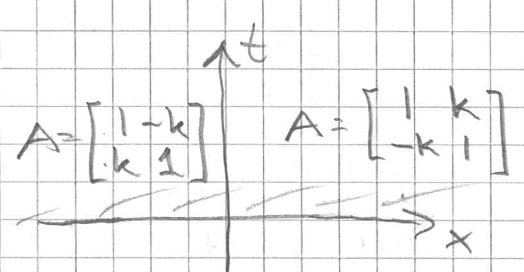
$\to (*)$

The Koch counterexample

We have seen that the 2-solvability estimate holds for real t-independent coefficients $A(x)$, provided A is symmetric or lower triangular. A concrete example where it fails is provided by

$$A(x) = \begin{bmatrix} 1 & k\cdot \operatorname{sgn}(x) \\ -k\operatorname{sgn}(x) & 1 \end{bmatrix}$$

in $\mathbb{R}^2_+$, where $k \in \mathbb{R}$.



Concretely, the PDE

$\operatorname{div} A\nabla u = 0$ for u in $\mathbb{R}^2_+$

means that

$\Delta u = 0$ in quadrants 1 and 2, and for $x = 0$, $t > 0$ u is continuous but its gradient has jumps

$$\partial_x u(0+,t) - \partial_x u(0-,t) = 2k\,\partial_t u(0,t) \quad \text{at } x = 0.$$

(*) See [AMM: 12] for the connection to the Kato estimate and interpolation theory.

Indeed, the jump on the positive t-axis amounts to $(A\nabla u)\cdot \underbrace{e_1}_{\text{horizontal}} = 0$, which means that $\operatorname{div} A\nabla u = 0$ holds in distribution sense there. Note that $\operatorname{div}\begin{bmatrix} 1 & \pm k \\ \mp k & 1 \end{bmatrix}\nabla u = \Delta u$ inside the quadrants.

For these equations $\operatorname{div} A\nabla u = 0$, the harmonic measure is explicitly known:

$$u(t,x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{2xty + |y|^{-\alpha}\operatorname{Im}\left\{(|x|+it)^{\alpha+1}\left(y^2-(|x|-it)^2\right)\right\}}{\left(t^2+(x-y)^2\right)\left(t^2+(x+y)^2\right)}}_{=\frac{d\omega_{(t,x)}(y)}{dy} = f(y)} g(y)\,dy$$

where $\tan(\frac{\pi}{2}\alpha) = k$, $|\alpha|<1$

Let $p\in(1,\infty)$ and consider the p-solvability estimate, see p 9-10. We need to verify the reverse Hölder estimate

$$\left(⨍_Q f^q\,dy\right)^{1/q} \lesssim ⨍_Q f\,dy, \quad \forall Q.$$

where $\frac{1}{p}+\frac{1}{q}=1$. In particular $f\in L^q(\mathbb{R})$.

We note $f(y)\sim |y|^{-\alpha}$ around $y=0$.

Hence $f\notin L^q(\mathbb{R})$ if $\alpha q > 1$.

$\Leftrightarrow k = \tan(\frac{\pi}{2}\alpha) > 1/\tan(\frac{\pi}{2p})$

$\therefore$ 2-solvability fails if $k>1$.

Moreover, $\forall p<\infty$: p-solvability fails if k is large enough.

See [KKPT:00, p. 251-253] and [A:09, eqn. (3)].

BMO solvability

We end these lectures by discussing the proof of the following state-of-the-art solvability estimate for the Dirichlet problem

<u>Thm</u>: If $A(x)$ are real and t-independent coefficients, not necessarily symmetric or triangular, then the BMO solvability estimate (p. 11-12) holds. Equivalently, A-harmonic measure $\omega \in A_\infty(dy)$.

<u>Cor</u>: For real and t-independent $A(x)$, the p-solvability estimate holds $\forall p > p_0$, for some $p_0 < \infty$ depending on A.

This was proved in [HKMP:15, thm 1.23], building on the 2-dimensional ($n=1$) proof from [KKPT:00, thm 3.1]. These proofs contained 3 main steps.

(A) Proving equivalence between the non-tang. maximal function and square function:

$$\|Su\|_p \approx \|N_* u\|_p \quad , \quad \forall \text{ solutions } u.$$

It is known that if this holds for some p, then it holds for all p.

(B) Proving the ε-approximability property:

Fix $\varepsilon > 0$. $\forall$ solutions u with $\|u\|_\infty \leq 1$

$\exists \varphi : \mathbb{R}^{1+n}_+ \to \mathbb{R}$ s.t.

$$\|u - \varphi\|_\infty \leq \varepsilon$$

$$\sup_{\substack{Q \subset \mathbb{R}^n \\ \text{cube}}} \frac{1}{|Q|} \iint_{\hat{Q}} |\nabla \varphi| \, dt\, dx \leq C_\varepsilon \qquad (*)$$

Recall that we aim for the BMO solvability estimate

$$\sup_Q \frac{1}{|Q|} \iint_{\hat{Q}} |\nabla u|^2 \, t \, dt \, dx \lesssim \|g\|_{BMO}^2 .$$

For bounded u we expect $|\nabla u| \lesssim \frac{1}{t}$, so $|\nabla u|^2 t \lesssim |\nabla u|$. However, (*) is known to fail for $\varphi = u$. This is why we need to allow for $\varphi \neq u$.

(C) Proving $\omega \in A_\infty(dy)$.

(A): $N < S$ we know from part C of these notes. $S < N$ was proved in [HKMP:15, thm 1,7].

(A) ⇒ (B): This involves a stopping time argument related to Carleson's corona theorem and the H^1-BMO duality. See [G:81, sec. VIII.6], [D:81].

(B) ⇒ (C): This builds on Garnett's quantitative Fatou result [G:81, cor. 6.2, p 339]. See [KKPT:00, thm 2.3, p.242-249].

In fact (A)-(C) are all equivalent, since the implication (C) ⇒ (A) was proved in [DJK:84].

There is a simpler proof of BMO-solvability due to [KKPT:16], which avoids step (B) and the use of ε-approx., which we now outline.

Recall that $L_\infty \subseteq BMO$, so $\|g\|_{BMO} \lesssim \|g\|_\infty$.

One first proves the following weaker version of the BMO solvability estimate.

Thm [HKMP:15, cor. 1.10]

$$\sup_Q \frac{1}{|Q|} \iint_{\hat{Q}} |\nabla u|^2 t \, dt dx \lesssim \|u\|_\infty^2 \quad (*)$$

holds for bounded solutions u.

Proof-idea:

- If $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is triangular, we proved (p. 24-25) that

$$\iint_{\mathbb{R}^{1+n}_+} |\nabla u|^2 t \, dt dx \lesssim \int_{\mathbb{R}^n} |\underset{g}{\overset{\shortparallel}{u}}|^2 dx$$

Localization as on p. 11 gives $(*)$.

- For general $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we do the change of variables in Ex 1, p. 4. The corresponding coeffs. A_1 in the Lipschitz domain Ω_1 satisfies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -\nabla h \\ 0 & I \end{bmatrix} A_1 \begin{bmatrix} 1 & 0 \\ -\nabla h^T & I \end{bmatrix}$$

$$\Rightarrow A_1 = \begin{bmatrix} 1 & \nabla h \\ 0 & I \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \nabla h^T & I \end{bmatrix}$$

$$= \begin{bmatrix} a + \nabla h \cdot c + c \cdot \nabla h^T & b + \nabla h \cdot d \\ c + \nabla h^T \cdot d & d \end{bmatrix}$$

We get triangular coeff $A_1 = \begin{bmatrix} a' & 0 \\ c' & d' \end{bmatrix}$ if we can choose $h(x)$ s.t. $b + \nabla h \cdot d = 0$.

This is possible when $n = 1$, see [KKPT:00, thm 3.1], where they generalize the triangular estimate from $\mathbb{R}^{1+n}_+$ to Ω_1.

- In higher dimension $n \geq 2$, there is in general no scalar function $h : \mathbb{R}^n \to \mathbb{R}$ s.t. $\nabla h = -b \cdot d^{-1}$

We can however make a Helmholtz / Hodge splitting of the vector field $v = -b \cdot d^{-1}$:

$v = \nabla h + \tilde{v}$, where $\tilde{v}$ is a

divergence free vector field on $\mathbb{R}^n$.
Choosing instead $\vec{v}$ in an A-adapted complement to $\{\text{gradients } \nabla h\}$, similar to 2nd prop. on p. 15, the above reduction to triangular matrices is extended to dim. $n \geq 2$. See [HKMP: 15, sec. 2–3]. $\square$

We now sketch the proof of $\omega \in A_\infty(dy)$.

<u>Need</u>: $\exists\, \delta, \varepsilon \in (0,1)$:
$\omega(E) \leq \delta\, \omega(Q) \Rightarrow |E| \leq \varepsilon |Q|$, $\forall$ cubes Q, $E \subset Q$

The A_∞ condition is "symmetric", so this gives
$|E| \leq (1-\varepsilon)|Q| \Rightarrow |Q \setminus E| > \varepsilon |Q| \Rightarrow$
$\omega(Q\setminus E) > \delta\, \omega(Q) \Rightarrow \omega(E) \leq (1-\delta)\,\omega(Q)$, so $\omega \in A_\infty(dy)$

A toy model for the needed construction is as follows.

<u>Ex</u>: $Q \subset \mathbb{R}^n$ dyadic cube, $E \subset Q$ small subset, $\gamma \in (0,1)$ given.
Select the maximal dyadic subcubes $\{Q_k\}$ s.t.
$|Q_k \cap E| > \gamma |Q_k|$
Disjointness $\Rightarrow$
$$\sum_k |Q_k| \leq \frac{1}{\gamma} \sum_k |Q_k \cap E| \leq \frac{1}{\gamma} |E| \quad (\ll |Q|)$$
maximality $\Rightarrow$ $|Q_k \cap E| \leq |\underset{\substack{\uparrow \\ \text{parent of } Q_k}}{\tilde{Q}_k} \cap E| \leq \gamma |\tilde{Q}_k| \overset{\text{doubling property of } |\cdot|}{\leq} \gamma \cdot 2^n |Q_k|$

Set $E_1 = \cup Q_k$. Replace E by E_1 and repeat construction $\Rightarrow$
$E \subset E_1 \subset E_2 \subset \ldots \subset E_N \subset Q$
- $|E_{j+1}| \leq \frac{1}{\gamma} |E_j|$
- E_j is "small in E_{j+1}": $|E_j \cap Q_k^{j+1}| \leq 2^n \gamma \cdot |Q_k^{j+1}|$, $\forall k$

where $E_{j+1} = \bigcup_k Q_k^{j+1}$

Need $\left(\frac{1}{\gamma}\right)^N \cdot |E| \lesssim |Q|$, so $N \lesssim \log \frac{|Q|}{|E|}$.

$\therefore$ E can be enlarged $\sim \log \frac{|Q|}{|E|}$ times.

$E \subset E_1 \subset \ldots \subset E_N$ is said to be a good $2^n \cdot \gamma$-cover of E of length N.

The proof of [KKPT:16, thm 3.1.] goes as follows.

Given $E \subset Q$ with $\omega(E) \leq \delta\,\omega(Q)$.

Replace Lebesgue meas. $|\cdot|$ by harm. meas. ω^X is construction of good cover. This works since

- ω is known to be a doubling measure: $\omega(2Q) \leq C \cdot \omega(Q)$
- "dyadic cubes" Q_k adapted to ω can be constructed. See [C:90].

We get a good cover $E \subset E_1 \subset \ldots \subset E_N \subset Q$ where E_j is "ω-small" in E_{j+1}: $\omega(E_j \cap Q_k^{j+1}) \ll \omega(Q_k^{j+1})$.

$N \approx \log \frac{\omega(Q)}{\omega(E)}$

From this good cover, one constructs a subset $H \subset Q$ and considers the solution u to

$$\begin{cases} \operatorname{div} A \nabla u = 0 & \text{in } \mathbb{R}^{1+n}_+ \\ u = 1_H & \text{on } \mathbb{R}^n \end{cases}$$

The idea is that u is very oscillatory on E, and it is proved in [KKPT:16, prop. 3.1 and thm 3.1] that

$$N \cdot |E| \lesssim \iint_{\hat{Q}} |\nabla u|^2 t \, dt\, dx$$

With (*), p. 29, we get $N \cdot |E| \lesssim \|1_H\|_\infty \cdot |Q|$

So $|E| \lesssim \dfrac{1}{\log \frac{\omega(Q)}{\omega(E)}} \cdot |Q|$

Since $\dfrac{1}{\log 1/t} \to 0$ as $t \to 0$, this proves
$\exists\, \delta, \varepsilon \in (0,1) : \omega(E) \leq \delta\, \omega(Q) \Rightarrow |E| \leq \varepsilon |Q|$.

[M:61] J. Moser : On Harnack's theorem for elliptic differential equations.

[HK:07] S. Hofmann, S. Kim : The Green function estimates for strongly elliptic systems of 2nd order.

[N:67] J. Nečas : Direct methods in the theory of elliptic equations
(original = 1967, eng. transl. = 2012)

[T:96] M. E. Taylor : Partial differential equations basic theory.

[CFK:81] L. Caffarelli, E. Fabes, C. Kenig : Completely singular elliptic-harmonic measures.

[KKPT:00] C. Kenig, H. Koch, J. Pipher, T. Toro : A new approach to absolute continuity of elliptic measure, with appl. to non-symmetric equations.

[D:77] B. Dahlberg : On estimates for harmonic measure.

[DKP:10] M. Dindos, C. Kenig, J. Pipher : "BMO solvability and the A_∞ condition for elliptic operators."

[GT:83]: D. Gilbarg, N. Trudinger: Elliptic partial differential equations of second order.

[LSW:63]: W. Littman, G. Stampacchia, H. Weinberger Regular points for elliptic equations with discontinuous coefficients.

[K:91] C. Kenig : Harmonic analysis techniques for second order elliptic boundary value problems.

[AAM^c:10a]: P. Auscher, A. Axelsson, A. McIntosh: Solvability of elliptic systems with square integrable boundary data.

[AAM^c:10b]: P. Auscher, A. Axelsson, A. McIntosh: On a quadratic estimate related to the Kato conjecture and boundary value problems.

[S:74]: I. Sneiberg: Spectral properties of linear operators in interpolation families of Banach spaces.

[AMM:12] P. Auscher, A. McIntosh, M. Mourgoglou: On L_2 solvability of BVPs for elliptic systems.

[A:09] A. Axelsson: Non-unique solutions to boundary value problems for non-symmetric divergence form equations.

[HKMP:15] S. Hofmann, C. Kenig, S. Mayboroda, J. Pipher: Square function / non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators.

[G:81] J. Garnett: Bounded analytic functions

[D:80] B. Dahlberg: Approximation of harmonic functions.

[DJK:84] B. Dahlberg, D. Jerison, C. Kenig: Area integral estimates for elliptic differential operators with non-smooth coefficients.

[KKPT:16] C. Kenig, B. Kirchheim, J. Pipher, T. Toro: Square functions and the A_∞ property of elliptic measures.

[C:90] M. Christ: A T(b) theorem with remarks on analytic capacity and the Cauchy integral.