Part II: Singular integral operators (SIOs)

In part I of the course, we studied boundedness and invertibility of linear operators in B-spaces, using mainly abstract functional analysis methods.
In part II, we now turn to concrete function spaces, where we shall study boundedness of SIOs, which is the limit case of integral operators (failure of integrability of the kernel). We shall only consider the Lebesgue spaces $L_p(\mathbb{R}^n)$ on euclidean space $\mathbb{R}^n$, $n \geq 1$. However, much of the theory here can be generalized to more general metric measure spaces.

Defn 7.1 A SIO kernel on $\mathbb{R}^n$ is a function
$$k : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \quad \text{such that}$$
$$\forall x,y \in \mathbb{R}^n,\ x \neq y : \ |k(x,y)| \leq \frac{C}{|x-y|^n} \quad \text{for some } C<\infty.$$

Remark 7.2: In the literature it is standard to also impose some Hölder regularity for a SIO kernel. We shall not make this a standing assumption here since some of the theory does not use it, and moreover we shall study some SIOs (perfect dyadic ones) that do not have continuous kernel (for $x \neq y$).

Rem./Defn. 7.3: Let $X \geq 0, Y \geq 0$ be two quantities, depending on some variables. The analyst's $\lesssim$, denoted
$$X \lesssim Y$$
means that there exists a constant $C<\infty$ (and of course $C \geq 0$) such that $X \leq C \cdot Y$, uniformly for all values of the variables. For example, in defn. 7.1, the variables are x, y $(x \neq y)$ and

as an analyst we write

$$|k(x,y)| \lesssim \frac{1}{|x-y|^n}, \qquad (*)$$

Note that dimension n is not considered a varible here, and the implicit constant in $(*)$ is allowed to depend on n (but not on x, y). To ease notation, the analyst's inequality $\lesssim$ will be used frequently, and what are varibles (like x, y) and parameters (like n) is left to the readers common sense.

We also use use the analyst's equality $\approx$:

$$X \approx Y \iff X \lesssim Y \text{ and } Y \lesssim X$$

$$\iff X \leq C \cdot Y \text{ and } Y \leq C \cdot X \text{ for some } C < \infty.$$

Main problem for part II :

Let $k(x,y)$ be a SIO kernel on $\mathbb{R}^n$. Consider (formelly) the SIO

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy .$$

The first problem is to make precise the exact meaning of the SIO T. (In general the integral is divergent!)

The main problem is then to determine if T defines a bounded operator on $L_2(\mathbb{R}^n)$:

$$\text{is } \|Tf\|_{L_2(\mathbb{R}^n)} \lesssim \|f\|_{L_2(\mathbb{R}^n)} \ ?$$

We also ask this question for the B-spaces $L_p(\mathbb{R}^n)$.

Ex 7.4: It is easy to see that it is natural to restrict attention to $1 < p < \infty$. Indeed, by Prop 1.10 we have

$$\|T\|_{L_\infty(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |k(x,y)|\, dy$$

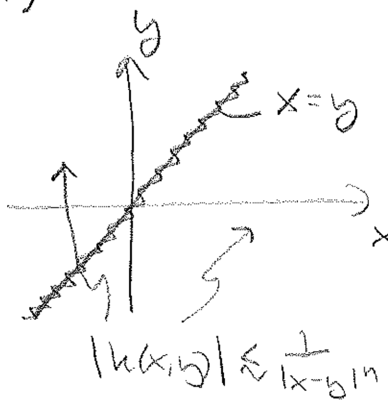
$$\|T\|_{L_1(\mathbb{R}^n) \to L_1(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |k(x,y)|\, dx.$$

For a SIO kernel, these norms are typically infinite.
It is important to note that the sign of the kernel $k(x,y)$ does not affect the norm of T in L_1 and L_∞. On the other hand, it is fundamental in the theory of SIOs that boundedness of SIOs on L_2 (and L_p, $1<p<\infty$) holds under certain cancellation (sign-changing) properties of the kernel $k(x,y)$.

We now make some basic observations about the problem of defining Tf. By the Schwartz kernel theorem, any continuous linear map $T: D(\mathbb{R}^n) \to D'(\mathbb{R}^n)$ is an integral operator, but in a generalized sense where k is a distribution on $\mathbb{R}^n \times \mathbb{R}^n$ in general. (A proof of this result can be built on Thm 2.13.)
For a SIO we know that the retriction of this distribution kernel to the complement

$\{(x,y);\ x \neq y\}$

of the diagonal is a function, with bounds given by defn. 7.1.



<u>Defn. 7.5</u>: A <u>SIO</u> T is a continuous linear operator $T: D(\mathbb{R}^n) \to D'(\mathbb{R}^n)$ such that
$$\langle g, Tf\rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} g(x)\, k(x,y)\, f(y)\, dx\,dy \quad \text{for all}$$
$g, f \in D(\mathbb{R}^n)$ such that $\operatorname{supp} g \cap \operatorname{supp} f = \emptyset$, with a SIO kernel k.

Note that T is not uniquely determined by its SIO kernel k. Indeed, the identity operator I

has its Schwartz kernel supported on the diagonal $x=y$, so $T+\alpha I$ has the same SIO kernel as T, for any $\alpha\in\mathbb{R}$. However, given $f\in\mathcal{D}(\mathbb{R}^n)$, $Tf(x)$ is well-defined and uniquely determined by k, for any $x\notin \operatorname{supp} f$, and we see that $Tf|_{(\operatorname{supp} f)^c}$ is a function. The following is an often useful estimate of this function.

<u>Prop. 7.6</u>: Let T be a SIO with kernel k. Let $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ denote the upper $(x_n>0)$ and lower $(x_n<0)$ half spaces. Then for any fixed $p\in(1,\infty)$ we have

$$\|Tf\|_{L_p(\mathbb{R}^n_+)} \lesssim \|f\|_{L_p(\mathbb{R}^n_-)} \quad \text{for all } f\in\mathcal{D}(\mathbb{R}^n) \text{ with } \operatorname{supp} f\subset\mathbb{R}^n_-.$$

<u>Proof</u>: We have for $x\in\mathbb{R}^n_+$

$$|Tf(x)|\lesssim\int_{\mathbb{R}^n_-}\frac{|f(y)|}{|x-y|^n}\,dy.$$

By a change of variables it suffices to estimate the norm of

$$f(x)\longmapsto\int_{\mathbb{R}^n}\frac{f(y)}{|x+y|^n}dy \quad\text{on } L_p(\mathbb{R}^n_+).$$

To this end, fix $\delta>0$ and apply the weighted Schur estimates 1.13 with

$$\omega(y)=y_n^{-\delta}, \quad \omega_1=\omega_2=\omega, \text{ and } \alpha=\tfrac1p,\ \beta=\tfrac1q.$$

We have

$$\int_{\mathbb{R}^n_+}\frac{1}{|x+y|^n}\frac{1}{y_n^{\delta q}}\,dy = \Big/\ \begin{matrix}x=(x',x_n)\\ y=(y',y_n)\end{matrix} \quad \begin{matrix}\text{Change of variables}\\ y'=-x'+(x_n+y_n)z\end{matrix}\ \Big/$$

$$=\int_{\mathbb{R}^n_+}\frac{1}{|((x_n+y_n)z,\,x_n+y_n)|^n}\frac{1}{y_n^{\delta q}}(x_n+y_n)^{n-1}\,dz\,dy_n$$

$$=\underbrace{\int_{\mathbb{R}^{n-1}}\frac{dz}{|(z,1)|^n}}_{<\infty}\int_0^\infty\frac{dy_n}{(y_n+x_n)\,y_n^{\delta q}} = \big/\ y_n=x_nt\ \big/$$

$$\lesssim\int_0^\infty\frac{x_n\,dt}{x_n(t+1)\,x_n^{\delta q}t^{\delta q}}=\frac{1}{x_n^{\delta q}}\int_0^\infty\frac{dt}{(t+1)\,t^{\delta q}}$$

$\lesssim \frac{1}{x_n^{\delta q}}$ if $0<\delta<\frac{1}{q}$

Similarly, if also $\delta < \frac{1}{p}$, then

$\int_{\mathbb{R}^n_+} \frac{\omega(x)^p}{|x+y|^n} dx \lesssim \omega(y)^p$, and the proof is complete. ∎

We next give a couple of examples of SIO which will serve as motivation for the theory developed here in part II.

Ex 7.7: The most basic SIO is the Cauchy integral on $\mathbb{R}$:

$$\mathcal{C}f(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

This is fundamental in complex analysis and elsewhere in analysis.

Ex 7.8: In $\mathbb{R}^n$, $n \geq 2$, the natural generalizations of the Cauchy integrals are the Riesz transforms

$$R_j f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad j = 1, \dots, n.$$

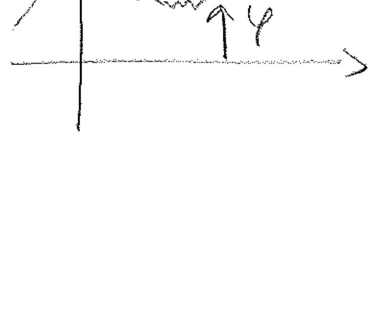
In vector notation, we have

$$Rf(x) = \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} f(y)\, dy,$$ where Rf is a vector field.

Ex 7.9: A much more non-trivial SIO, is the Cauchy integral on a Lipschitz graph γ in the complex plane. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, i.e. we assume

$$|\varphi(x) - \varphi(y)| \lesssim |x-y|, \quad x, y \in \mathbb{R},$$

such that γ is the graph of φ.



For functions $f(z)$ on γ, define the Cauchy integral (formally)

$$\int_\gamma \frac{f(w)}{z-w} dw, \quad z \in \gamma.$$

Changing variables $z = x + i\varphi(x)$, $w = y + i\varphi(y)$, this corresponds to the operator

$$\int_{\mathbb{R}} \frac{1 + i\varphi'(y)}{x - y + i(\varphi(x) - \varphi(y))} g(y)\,dy, \quad x \in \mathbb{R},$$

where $g(y) := f(y + i\varphi(y))$. We regard $1 + i\varphi'(y)$ as a bounded and invertible multiplication operator on $L_p(\mathbb{R})$. Ignoring this trivial operator (but useful as we shall see later!), we define

$$C^\varphi f(x) = \int_{\mathbb{R}} \frac{f(y)}{x - y + i(\varphi(x) - \varphi(y))} dy, \quad x \in \mathbb{R}.$$

<u>Ex 7.10</u>: The problem of proving L_2 boundedness of C^φ on any Lipschitz graph γ was called the Calderón problem. An early attempt to prove this famous conjecture was as follows.

Write

$$\frac{1}{x - y + i(\varphi(x) - \varphi(y))} = \sum_{k=0}^{\infty} \frac{1}{i^k} \frac{(\varphi(x) - \varphi(y))^k}{(x - y)^{k+1}},$$

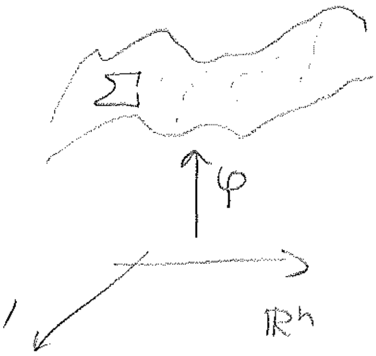
and consider the <u>Calderón commutators</u>

$$C_k^\varphi f(x) = \int_{\mathbb{R}} \frac{(\varphi(x) - \varphi(y))^k}{(x - y)^{k+1}} f(y)\,dy, \quad x \in \mathbb{R}, \quad k = 0, 1, 2, \ldots$$

Note that C_0^γ is the Cauchy integral on $\mathbb{R}$ from Ex 7.7. The hope was to obtain good enough bounds on C_k^φ, $k = 0, 1, 2, \ldots$, so that summing over k would yield boundedness of the Cauchy integral C^φ on γ.

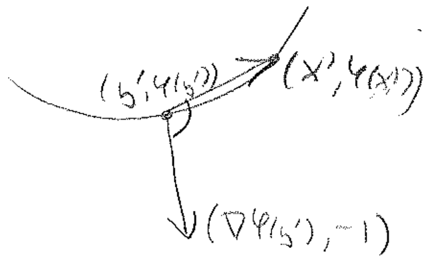
<u>Ex 7.11</u>: Our final, and main, example of SIOs, is the double layer potential operator on a Lipschitz hypersurface in $\mathbb{R}^{n+1}$. By a partition of unity argument, it suffices to consider a Lipschitz graph Σ. Let Σ be the graph

of a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R} : x' \mapsto x_n = \varphi(x')$.
Thus $|\varphi(x') - \varphi(y')| \lesssim |x' - y'|$.
In Defn 1.8, change variables from Σ to $\mathbb{R}^n$ (c.f. Ex. 7.9.) to obtain the operator $K^\varphi f(x) =$



$$= \int_{\mathbb{R}^n} \frac{(x', \varphi(x')) - (y', \varphi(y'))}{|(x', \varphi(x')) - (y', \varphi(y'))|^{n+1}} \cdot (\nabla\varphi(y'), -1) f(y)\, dy, \quad x' \in \mathbb{R}^n.$$

Note that when Σ / φ is $C^{1,\alpha}$-smooth ($\alpha > 0$), then as in Ex 1.12 this only weekly singular since $(x' - y', \varphi(x') - \varphi(y'))$ is almost orthogonal to $(\nabla\varphi(y'), -1)$ when $x' \approx y'$.



When Σ / φ is merely Lipschitz things are much more subtle. Even though the Lipschitz function φ is differentiable almost everywhere, we do not in general have better bounds for the kernel than that in Defn. 7.1.

Define Riesz transforms on Σ:

$$R_j^\varphi f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|(x-y, \varphi(x) - \varphi(y))|^{n+1}} f(y)\, dy, \quad j = 1, \dots, n$$

$$R_{n+1}^\varphi f(x) = \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|(x-y, \varphi(x) - \varphi(y))|^{n+1}} f(y)\, dy, \qquad x \in \mathbb{R}^n.$$

We aim to prove that these are bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, and hence that

$$K^\varphi = R_1^\varphi \cdot \partial_1\varphi + \dots + R_n^\varphi \partial_n \varphi - R_{n+1}^\varphi \quad \text{is bounded.}$$

<u>Note</u>: (1) All the operators $C, R_j', C^\varphi, C_h^\varphi$ and R_j^φ in examples 7.7–11 have <u>antisymmetric</u> kernels:

$$k(y, x) = -k(x, y).$$

(2) The operators C and R_j in Ex 7,7-8 are convolution SIOs:

$k(x,y) = k(x-y)$.

Antisymmetry of k here means convolution with an odd function.

Motivated by these examples, we shall mainly be interested in SIOs with antisymmetric kernels, and we consider first the more elementary case of convolution SIOs.

Classical convolution SIOs:

Consider a convolution SIO

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y) f(y)\, dy, \quad x \in \mathbb{R}^n,$$

where $|k(t)| \lesssim \frac{1}{|t|^n}$. In this case we can make use of the locally compact Abelian group $(\mathbb{R}^n, +)$ and the associated Fourier transform

$$F : L_2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{R}^n) : f(x) \longmapsto \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle} dx.$$

It is well known, Plancherel's theorem, that

$\boxed{\frac{1}{(2\pi)^n} \|\hat{f}\|_2^2 = \|f\|_2^2}$, i.e. $\frac{1}{(2\pi)^{n/2}} F$ is an isometry of $L_2(\mathbb{R}^n)$.

Equally well known is that F intertwines the convolution and pointwise products:

$\boxed{\widehat{f*g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)}$, where $(f*g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$.

The classical way to make sense of the SIO T is as a principal value integral. Define truncations

$$T_\varepsilon f(x) := \int_{|y-x|>\varepsilon} k(x-y) f(y)\, dy = \int_{\mathbb{R}} k_\varepsilon(x-y) f(y)\, dy,$$

where $k_\varepsilon(t) = \begin{cases} k(t), & |t| > \varepsilon \\ 0, & \text{else}. \end{cases}$

Note that $k_\varepsilon \in L_2(\mathbb{R}^n)$, so that

$T_\varepsilon f = k_\varepsilon * f$ is a well-defined continuous function.
We have $\widehat{T_\varepsilon f} = \hat{k}_\varepsilon \cdot \hat{f}(\xi) \in L_1(\mathbb{R}^n)$ if $f \in L_2(\mathbb{R}^n)$.

<u>Defn 7.12</u>: Under suitable convergence hypothesis on $T_\varepsilon f$, when $\varepsilon \to 0$, we define the <u>principal value SIO</u> with kernel k, to be the operator

$$(Tf)(x) = \text{p.v.}\int k(x-y) f(y)\,dy := \lim_{\varepsilon \to 0} \int_{|y-x|\geq\varepsilon} k(x-y) f(y)\,dy.$$

<u>Ex 7.12</u>: Consider the Cauchy integral

$$Cf(x) = \text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy$$

from Ex. 7.7, where $k_\varepsilon(t) = \begin{cases} 1/t, & |t|>\varepsilon \\ 0, & \text{else.}\end{cases}$

We calculate

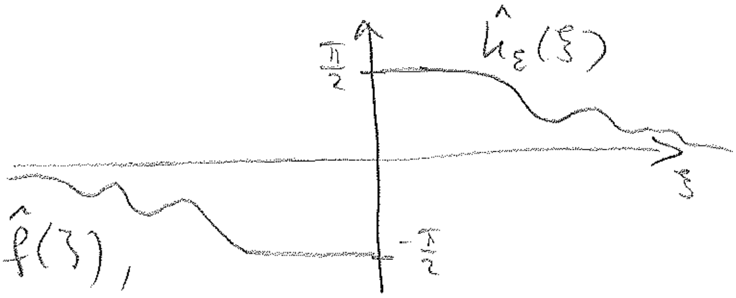
$$\hat{k}_\varepsilon(\xi) = \lim_{R\to\infty} \int_{\varepsilon<|x|<R} \frac{1}{x} e^{-ix\xi}\,dx = -2i \lim_{R\to\infty} \int_\varepsilon^R \frac{\sin(x\xi)}{x}\,dx.$$

Define the conditionally convergent integral

$$h(t) := \int_t^\infty \frac{\sin s}{s}\,ds$$

Then $\hat{k}_\varepsilon(\xi) = -2i\,\mathrm{sgn}(\xi)\cdot h(\varepsilon|\xi|)$. We have $\lim_{t\to 0^+} h(t) = \frac{\pi}{2}$ by residue calculus, and $\lim_{t\to\infty} h(t) = 0$.

From this it is clear that



$$\widehat{C_\varepsilon f}(\xi) \xrightarrow{L_2} -i\pi\,\mathrm{sgn}(\xi)\,\hat{f}(\xi),$$

that C_ε are uniformly bounded on $L_2(\mathbb{R})$ and converges strongly to an operator C such that FCF^{-1} is multiplication by $-i\pi\,\mathrm{sgn}(\xi)$.

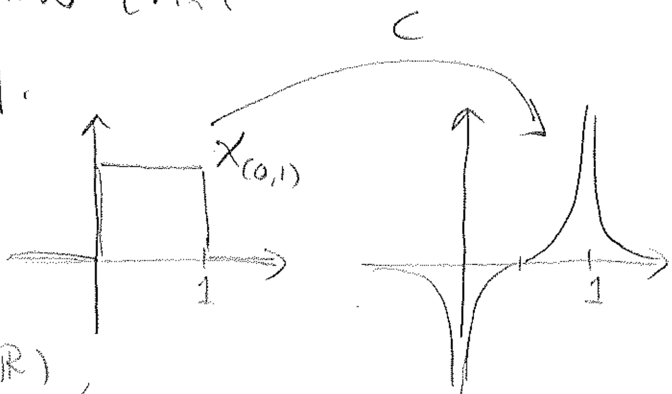
Exc. 7.13: Calculate the SIO

$$Tf(x) := \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus(x-\varepsilon,x+2\varepsilon)} \frac{f(y)}{x-y}\,dy$$

in terms of the Cauchy integral C from Ex. 7.12.

Exc. 7.14: Let C be the L_2 Cauchy integral as defined in Ex. 7.12. Show that

$$C(\chi_{(0,1)})(x) = \ln\left|\frac{x}{x-1}\right|.$$



Note that $C\chi_{(0,1)} \notin L_\infty(\mathbb{R})$, even though $\chi_{(0,1)} \in L_\infty(\mathbb{R})$. C.f. ex. 7.4.

Ex 7.15: Consider the convolution SIO

$$Tf(x) = \text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{|x-y|}\,dy,$$

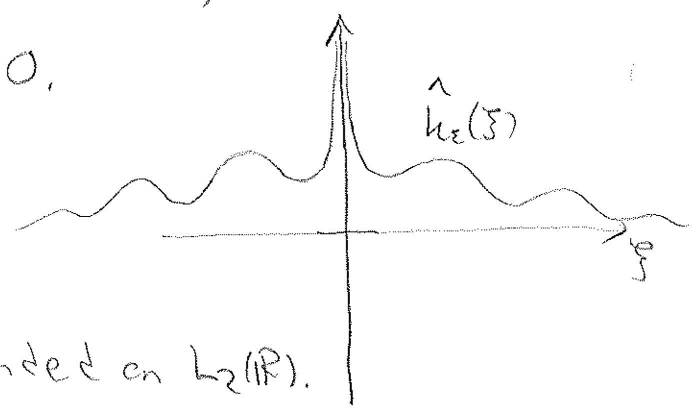
with positive kernel. Here $k_\varepsilon(t) = \begin{cases} 1/|t|, & |t|>\varepsilon \\ 0, & \text{else} \end{cases}$,

and we calculate

$$\hat{k}_\varepsilon(\xi) = \lim_{R\to\infty} 2\int_\varepsilon^R \frac{\cos(x\xi)}{x}\,dx.$$

Letting $h(t) := \int_t^\infty \frac{\cos(s)}{s}\,ds$ in this case, we have $\lim_{t\to 0^+} h(t) = +\infty$ and $\lim_{t\to\infty} h(t) = 0$.

Thus we have an unbounded multiplicator $\hat{k}_\varepsilon(\xi) = 2h(\varepsilon|\xi|)$ and we conclude that T_ε is not bounded on $L_2(\mathbb{R})$.



It is not possible to make sense of the formal expression $\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{|x-y|}\,dy$, since formally we have $\hat{k}(\xi) = \lim_{\varepsilon\to 0} 2h(\varepsilon|\xi|)$ being identically ∞.

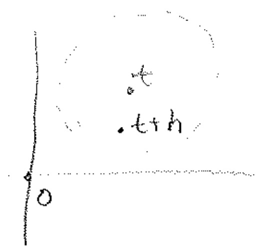
Note however that the unboundedness problem for the truncations T_ε is less serious, and is an effect of the unboundedness of $\mathbb{R}$.

Examples 7.12 and 7.15 illustrate the importance of the sign of kernel of an SIO. In these examples we "explicitly" calculated the symbol $\hat{k}_\varepsilon(\xi)$. If this is possible for a convolution SIO kernel $k(t)$, then we can check if $\sup_{\substack{\varepsilon>0 \\ \xi\in\mathbb{R}}} |\hat{k}_\varepsilon(\xi)| < \infty$, and in this case obtain an L_2-bounded SIO according to Plancherel's theorem.
The main result for classical convolution SIO that we shall prove, is the following result 7.18. It provides sufficient conditions directly on the kernel $k(t)$ (and not its Fourier transform $\hat{k}(\xi)$) for L_2-boundedness of the SIO.
As mentioned, boundedness results for SIO typically need some smoothness assumption on the kernel. A standard such is the following.

<u>Defn 7.16</u>: Let $k(x,y) = k(\underbrace{x-y}_{=t})$ be a convolution SIO kernel, and let $0<\gamma<1$. We say that k is <u>γ-Hölder regular convolution SIO kernel</u> if

$$\frac{|k(t+h)-k(t)|}{|h|^\gamma} \lesssim \frac{1}{|t|^{n+\gamma}} \quad \text{for } |h| \le \frac{|t|}{2}.$$

<u>Exc 7.17</u>:



(a) Show that in the above definition that if we replace $\frac{1}{2}$ in $|h|<\frac{|t|}{2}$ by any $0<c<1$ (possibly with a different implicit constant in $\lesssim$), then we obtain an equivalent definition.

(b) Show that k is a γ-Hölder SIO kernel if and only if k is γ-Hölder continuous in a ball $B(t, \frac{1}{2}|t|)$, with (homogeneous) Hölder norm $\lesssim \frac{1}{|t|^{n+\gamma}}$, for any $t \in \mathbb{R}^n$.

<u>Thm 7.18</u>: Let $k(x,y) = k(x-y)$ be a γ-Hölder regular convolution SIO kernel, for some $0<\gamma<1$.
Let
$$M(a,b) := \int_{a<|t|<b} k(t)\,dt \quad \text{and}$$
assume $\lim_{a \to 0^+} M(a,1)$ exists and $\sup_{b>1} |M(1,b)| < \infty$.
Then
$$T_\varepsilon f(x) = \int_{|y-x|>\varepsilon} k(x-y) f(y)\,dy$$
are uniformly bounded on $L_2(\mathbb{R}^n)$ for $\varepsilon>0$ and converges strongly on $L_2(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ to an operator $T \in L(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$.

Note that the above hypothesis is about cancellation. The integral
$$\int_{\mathbb{R}^n} \frac{dt}{|t|^n} \quad \text{is divergent, both at } t=0 \text{ and } t=\infty.$$
Thus, the bounds on $M(a,b)$ are only possible if k changes sign appropriately in annuli $a<|t|<b$. In this case the above theorem states that we have an L_2-bounded principal value SIO
$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x-y) f(y)\,dy.$$

<u>Proof</u>: (i) We first prove $\sup_{\varepsilon>0} \|T_\varepsilon\|_{L_2 \to L_2} < \infty$.

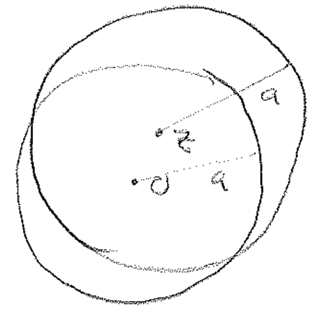
As T_ε is a convolution operator, it suffices to prove $\sup_{\varepsilon>0} \|\hat{k}_\varepsilon\|_{L_\infty} < \infty$. To this end, consider the integral
$$I := \int_{a<|x|<b} k(x) e^{-i\langle x, \xi\rangle}\,dx \quad \text{for some } 0<a<b<\infty \text{ and } \xi \in \mathbb{R}^n.$$

With $z := \pi\,\xi/|\xi|^2$ we have

$$I = \frac{1}{2}\int_{a<|x|<b} k(x)\, e^{-i\langle x,\xi\rangle}dx + \frac{1}{2}\int_{a<|x-z|<b} k(x-z)\, e^{-i\langle x,\xi\rangle}\underbrace{e^{i\langle x,\xi\rangle}}_{=-1}dx$$

$$= \frac{1}{2}\int_{a<|x|<b} (k(x)-k(x-z))\, e^{-i\langle x,\xi\rangle}dx + \text{error},$$

where the error term comes from the different domains of integration.



We get

$$|I| \le \frac{1}{2}\int_{a<|x|<b} |k(x)-k(x-z)|\,dx$$

$$+\frac{1}{2}\int_{a-|z|<|x|<a+|z|} |k(x)|\,dx + \frac{1}{2}\int_{b-|z|<|x|<b+|z|} |k(x)|\,dx.$$

To use the γ-Hölder regularity of k, we assume $|z| \le \frac{a}{2}$, i.e. $|\xi| \ge \frac{2\pi}{a}$, and get

$$|I| \lesssim \int_{a<|x|<b} \frac{|z|^\gamma}{|x|^{n+\gamma}}dx + \int_{a-|z|<|x|<a+|z|}\frac{dx}{|x|^n} + \int_{b-|z|<|x|<b+|z|}\frac{dx}{|x|^n}$$

$$\lesssim |z|^\gamma\left(\frac{1}{a^\gamma}-\frac{1}{b^\gamma}\right) + \ln\frac{a+|z|}{a-|z|} + \ln\frac{b+|z|}{b-|z|}$$

Since $\frac{|z|}{a} = \frac{\pi}{a|\xi|} \le \frac{1}{2}$, we have

$$|I| \le \left(\frac{1}{2}\right)^\gamma + \ln 3 + \ln\frac{b+|z|}{b-|z|}.$$

Letting $b\to\infty$, we get

$$\operatorname*{ess\,sup}_{a\ge 2\pi/|\xi|}\left|\int_{|x|>a} k(x)\, e^{-i\langle x,\xi\rangle}dx\right| < \infty.$$

- Now fix $\varepsilon>0$ and consider $\hat{k}_\varepsilon$.

If $|\xi| \ge \frac{2\pi}{\varepsilon}$, we take $a=\varepsilon$ and obtain $|\hat{k}_\varepsilon(\xi)| \le C$.

If $|\xi| < \frac{2\pi}{\varepsilon}$, we have from above

$$\left|\int_{|x|>2\pi/|\xi|} k(x)\, e^{-i\langle x,\xi\rangle}dx\right| \le C.$$

It remains to estimate

$$\int_{\varepsilon<|x|<\frac{2\pi}{|\xi|}} k(x) e^{-i\langle x,\xi\rangle} dx =: \mathrm{II}$$

Above (for $|x|>\frac{2\pi}{|\xi|}$) we use smoothness of k.
Here (for $|x|<\frac{2\pi}{|\xi|}$) we use smoothness of $e^{-i\langle x,\xi\rangle}$.
We write

$$\mathrm{II} = \int_{\varepsilon<|x|<\frac{2\pi}{|\xi|}} k(x)\left(e^{-i\langle x,\xi\rangle}-1\right)dx + \int_{\varepsilon<|x|<\frac{2\pi}{|\xi|}} k(x)\,dx,$$

and obtain

$$|\mathrm{II}| \lesssim \int_{\varepsilon<|x|<\frac{2\pi}{|\xi|}} \frac{1}{|x|^n}(|x||\xi|)\,dx + \left|M\left(\varepsilon,\frac{2\pi}{|\xi|}\right)\right|$$

$$\approx |\xi|\left(\frac{2\pi}{|\xi|}-\varepsilon\right) + \left|M\left(\varepsilon,\frac{2\pi}{|\xi|}\right)\right| \le C.$$

This proves $\sup_{\varepsilon>0}\|\hat{k}_\varepsilon\|_\infty<\infty$.

(ii) Next we prove that T_ε converges strongly on $L_2(\mathbb{R}^n)$.
By (i) above and exercise 7.19 below, it suffices to prove that $\{T_\varepsilon f\}_{\varepsilon\to0}$ is a Cauchy sequence in $L_2(\mathbb{R}^n)$, for any fixed $f\in C_c^\alpha(\mathbb{R}^n)$ = compactly supported α-Hölder continuous.
For $\varepsilon_1<\varepsilon_2$, we have

$$T_{\varepsilon_1}f(x)-T_{\varepsilon_2}f(x) = \int_{\varepsilon_1<|y-x|<\varepsilon_2} k(x-y)f(y)\,dy$$

$$= \int_{\varepsilon_1<|y-x|<\varepsilon_2} k(x-y)(f(y)-f(x))\,dy + M(\varepsilon_1,\varepsilon_2)f(x)$$

Regarding $f\in C_c^\alpha(\mathbb{R}^n)$ as fixed, we get

$$|T_{\varepsilon_1}f(x)-T_{\varepsilon_2}f(x)| \lesssim \int_{\varepsilon_1<|y-x|<\varepsilon_2} \frac{1}{|x-y|^n}|x-y|^\alpha\,dy + |M(\varepsilon_1,\varepsilon_2)|$$

$$\approx \varepsilon_2^\alpha-\varepsilon_1^\alpha + |M(\varepsilon_1,\varepsilon_2)| \longrightarrow 0,\quad \varepsilon_1,\varepsilon_2\to0$$

using the hypothesis on M.
Since $\operatorname{supp}(T_{\varepsilon_1}f - T_{\varepsilon_2}f)\subset \operatorname{supp} f + B(0;\varepsilon_2)$, it follows that $\{T_\varepsilon f\}$ is a Cauchy sequence in $L_2(\mathbb{R}^n)$. ▤

Exc. 7.19: Let X, Y be B-spaces and let $T_n \in L(X,Y)$. Assume that $\{T_n f\}_{n=1}^{\infty}$ converges in Y, for each f in a dense subspace $X_0 \subset X$, and that $\sup_n \|T_n\| < \infty$. Show that T_n converges strongly, i.e.

$\forall f \in X$: $T_n f$ converges in Y.

We note that in Thm 7.18, the hypothesis on M is satisfied for any odd function $k(t)$, since in this case $M(a,b) = 0$ for all a, b.

Ex. 7.20:

Applying Thm 7.18, we recover the result from Ex. 7.12 that the Cauchy integral

$$Cf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy$$

is L_2-bounded. We also obtain the result that the Riesz transforms from Ex. 7.8 are bounded on $L_2(\mathbb{R}^n)$, where these SIOs are defined as in Defn. 7.12.

Ex 7.21:

Consider the SIO kernel $k(x,y) = \frac{1}{|x-y|}$ in $\mathbb{R}$ from Ex. 7.15. This is seen to be a γ-Hölder regular SIO kernel. But the cancellation hypothesis on k in Thm 7.18 does not hold. Indeed,

$$M(a,b) = \int_{a<|t|<b} \frac{dt}{|t|} = 2 \ln \frac{b}{a}, \text{ which is unbounded!}$$

Non-convolution SIOs

In lecture 7 we studied convolution SIOs, where $k(x,y) = k(x-y)$ and we could make use of the Fourier transform. We proved sufficient conditions on $k(x,y)$ for the SIO to be L_2-bounded.
We now consider more general SIOs

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\,dy,$$

of non-convolution type, motivated by Ex. 7.9-11.
Our goal is conditions on k which are both sufficient and necessary for T to be L_2-bounded. For these problems, the Fourier transform turns out to be a not flexible enough tool. It is mainly useful for SIOs of convolution type.

Instead of trigonometric bases, as used for the Fourier transform, the natural type of bases for SIOs are wavelet type bases. Working with $L_2(\mathbb{R}^n)$, where only size and not regularity is controlled, non-smooth wavelets (Haar wavelets) suffices for us. For technical reason, we shall therefore restrict our definition 7.5 of a SIO as follows.

Defn. 8.1: Let $0<s<1$ and $0<t<\infty$. For $f:\mathbb{R}^n \to \mathbb{R}$ define the norm

$$\|f\|^2_{H^s_t} := \iint_{|x-y|<1} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} |f(x)|^2 (1+|x|)^t\,dx.$$

Let $H^s_t(\mathbb{R}^n)$ denote the H-space of functions f such that $\|f\|_{H^s_t} < \infty$.

- By a SIO on $\mathbb{R}^n$, we mean a continuous linear operator $T: H^s_t(\mathbb{R}^n) \to (H^s_t(\mathbb{R}^n))^*$, with a SIO kernel k in the sense of Defn. 7.5 and $0<s<1/2$, $0<t<\infty$.

Note that $\mathcal{D}(\mathbb{R}^n) \subset H^s_t(\mathbb{R}^n) \subset L_2(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n) \supset H^s_t(\mathbb{R}^n)^* \supset L_2(\mathbb{R}^n)$. Therefore any SIO $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ in the sense of Defn 8.1 restricts to $T: \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, i.e a SIO in the sense of Defn 7.5. Motivation is the following.

(1) Any SIO with antisymmetric kernel (which all examples 7.7–11 have) is defined in a natural way $T: H^s_t \to (H^s_t)^*$, for any $s>0$, $t>0$.

(2) The space H^s_t contains the Haar basis functions when $s<\frac{1}{2}$.

Note also that the spaces H^s_t generalizes well to more general metric measure spaces, as compared to $\mathcal{D}(\mathbb{R}^n)$.

Prop. 8.2: Let $k(x,y) = -k(y,x)$ be an antisymmetric SIO kernel on $\mathbb{R}^n$, and let $0<s<1$ and $0<t<\infty$. Then

$$T_\varepsilon f(x) := \int_{|y-x|>\varepsilon} k(x,y) f(y)\, dy$$

defines a function $T_\varepsilon f \in (H^s_t(\mathbb{R}^n))^* \cap L_\infty(\mathbb{R}^n)$, for any $f \in H^s_t(\mathbb{R}^n)$. Moreover, $T_\varepsilon : H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ are uniformly bounded, and T_ε converges in $\mathcal{L}(H^s_t(\mathbb{R}^n), H^s_t(\mathbb{R}^n)^*)$ when $\varepsilon \to 0$.

Proof: (i) Let $f, g \in H^s_t(\mathbb{R}^n)$. Clearly

$$|T_\varepsilon f(x)| \lesssim \left(\int_{|y-x|>\varepsilon} \frac{dy}{|y-x|^{2n}} \right)^{1/2} \|f\|_2 \le C_\varepsilon \|f\|_{H^s_t}, \ \forall x.$$

To estimate $\langle g, T_\varepsilon f\rangle = \iint_{|y-x|>\varepsilon} g(x) k(x,y) f(y)\, dy$, write

$$\langle g, T_\varepsilon f\rangle = \iint_{\varepsilon<|y-x|<1} \cdots + \iint_{|y-x|>1} \cdots =: I + II$$

For II we have

$$\Big(\int_{|y-x|>1} \frac{|f(y)|}{|x-y|^n}dy\Big)^2 \le \underbrace{\Big(\int_{|y-x|>1} \frac{dy}{|x-y|^n\cdot|y|^\alpha}\Big)}_{=:\,\mathrm{III}}\Big(\int_{|y-x|>1}\frac{1}{|x-y|^n}|f(y)|^2|y|^\alpha dy\Big),$$

for any $\alpha>0$, where

$$\mathrm{III} = \Big/\begin{smallmatrix} y=|x|z\\ \hat{x}:=x/|x|\end{smallmatrix}\Big/ = \frac{1}{|x|^\alpha}\int_{|z-\hat{x}|>\frac{1}{|x|}} \frac{dz}{|z-\hat{x}|^n|z|^\alpha} = \frac{1}{|x|^\alpha}\int_{|z|>\frac{1}{|x|}}\frac{dz}{|z|^n|z-e_1|^\alpha},$$

where $e_1=(1,0,\dots,0)$.

Assuming $0<\alpha<n$, the function $f(z):=\frac{1}{|z|^n|z-e_1|^\alpha}$ is in $L_1(B(0;\frac12)^c)$ with $\int_{|x|>R} f(z)\approx\frac{1}{R^\alpha}$ as $R\to\infty$, and $\int_{\varepsilon<|z|<\frac12} f(z) \approx \ln\frac{1}{\varepsilon}$ as $\varepsilon\to0$.

Together this shows $\sup_{x\in\mathbb{R}^n}\mathrm{III}<\infty$.

Now fix $\alpha:=\min(t,\frac{n}{2})$ to obtain

$$\Big(\iint_{|y-x|>1}|g(x)||k(x,y)|\cdot|f(y)|dxdy\Big)^2\lesssim\Big(\int|g(x)|^2|x|^\alpha dx\Big)\cdot\Big(\int\Big(\int_{|y-x|>1}\frac{|f(y)|}{|x-y|^n}dy\Big)^2\frac{1}{|x|^\alpha}dx\Big)$$

$$\lesssim\|g\|^2_{H^s_t}\int\underbrace{\Big(\int_{|x-y|>1}\frac{1}{|x-y|^n}\frac{1}{|x|^\alpha}dx\Big)}_{\le C\text{ as above}}|f(y)|^2|y|^\alpha dx$$

$$\lesssim\|g\|^2_{H^s_t}\|f\|^2_{H^s_t}. \qquad \text{(c.f. Prop. 1,10 = Schur estimate)}.$$

For I we make use of antisymmetry and write

$$\Big|\iint_{\varepsilon<|y-x|<1} g(x)k(x,y)f(y)dxdy\Big| = \Big|\iint_{\varepsilon<|y-x|<1}\tfrac12 k(x,y)(g(x)f(y)\underbrace{-}_{-g(x)f(x)+g(x)f(x)}g(y)f(x))dxdy\Big|$$

$$\lesssim\iint_{|y-x|<1}\frac{1}{|x-y|^n}|g(x)|\cdot|f(x)-f(y)|dxdy + A$$

$$\lesssim\Big(\iint_{|y-x|<1}\frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}dxdy\Big)^{1/2}\Big(\iint_{|y-x|<1}\frac{|g(x)|}{|y-x|^{n-2s}}dxdy\Big)^{1/2}+A,$$

where A denotes the same term but with f & g interchanged.

Since $\int |g(x)|^2 \Big(\int\limits_{|y-x|<1} \frac{dy}{|y-x|^{n-2s}} \Big) dx \lesssim \|g\|_{L_2}^2 \lesssim \|g\|_{H_t^s}^2$,

we have proved that

$$\|T_\varepsilon f\|_{(H_t^s)^*} = \sup_{\|g\|_{H_t^s}=1} \langle g, T_\varepsilon f \rangle \lesssim \|f\|_{H_t^s}, \text{ uniformly for } \varepsilon>0.$$

(ii) To prove convergence of T_ε in $L(H_t^s, (H_t^s)^*)$, consider $T_{\varepsilon_1} - T_{\varepsilon_2}$ for $0<\varepsilon_1<\varepsilon_2<<1$.

For $f, g \in H_t^s(\mathbb{R}^n)$, we have

$$|\langle g, (T_{\varepsilon_1}-T_{\varepsilon_2}) f \rangle| \lesssim \iint\limits_{\varepsilon_1<|y-x|<\varepsilon_2} \frac{1}{|y-x|^n} |g(x)f(y)-g(y)f(x)|\, dx\, dy$$

$$\lesssim \Big(\int\limits_{\varepsilon_1<|z|<\varepsilon_2} \frac{dz}{|z|^{n-2s}} \Big) \|f\|_{H_t^s} \|g\|_{H_t^s} \quad \text{as above.}$$

Thus $\|T_{\varepsilon_1}-T_{\varepsilon_2}\|_{H_t^s\to(H_t^s)^*} \lesssim \int\limits_{\varepsilon_1<r<\varepsilon_2} \frac{dr}{r^{1-2s}} \to 0, \quad \varepsilon_1, \varepsilon_2 \to 0.$ ∎

<u>Exc 8.3</u>: Let Q be a cube in $\mathbb{R}^n$, and let χ_Q denote the characteristic function of Q. Show that $\chi_Q \in H_t^s(\mathbb{R}^n)$ if $0<s<\frac{1}{2}$ and $\chi_Q \notin H_t^s(\mathbb{R}^n)$ if $s\geq \frac{1}{2}$.

Let us compare $H_t^s(\mathbb{R}^n)$ with $L_2(\mathbb{R}^n)$. A larger $t>0$ forces $f\in H_t^s$ to decay faster at ∞ than an L_2-function. On the hand, $0<s<1$ denotes the fractional Sobolev differentiability of f.
Indeed, for $f\in H^s(\mathbb{R}^n) = H_0^s(\mathbb{R}^n)$ (i.e. $t=0$), it is well known that

$$\iint\limits_{|x-y|<1} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}} dx\,dy + \int\limits_{\mathbb{R}^n} |f(x)|^2 dx \approx \int\limits_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|)^{2s} d\xi.$$

More common is to define the $H^s(\mathbb{R}^n)$ norm as this right hand side, using the Fourier transform.

Defn 8.4: Let $k(x,y)$ be a SIO kernel on $\mathbb{R}^n$, and let $0<\gamma<1$. We say that k is a γ-Hölder regular SIO kernel if

$t \mapsto k(y+t, y)$ and

$t \mapsto k(x, x+t)$

are so in the sense of Defn. 7.16, uniformly for $x, y \in \mathbb{R}^n$.

We note that if $\gamma_1 < \gamma_2$, then k is γ_1-Hölder regular if it is γ_2-Hölder regular, and that estimates

$$|\nabla_x k(x,y)| \lesssim \frac{1}{|x-y|^{n+1}} \text{ and } |\nabla_y k(x,y)| \lesssim \frac{1}{|x-y|^{n+1}}$$

implies γ-Hölder regularity of k for any $0<\gamma<1$.

Ex 8.5: Recall from Thm 7.18 that $k(x-y)$ being an odd γ-regular convolution SIO kernel implies that the principal value SIO is L_2-bounded. We now show that antisymmetry of a non-convolution γ-regular SIO kernel does not imply L_2-boundedness.

Let $n=1$ and define

$$k(x,y) := \sum_0^\infty 2^k \theta(2^k(x-y)) \sin(2^k(x+y)),$$

where $\theta \in C_0^\infty(\mathbb{R})$ is an odd function such that

$\operatorname{supp} \theta \subset [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ and

$\int \theta(t) \sin(t) = 1$.

Claim: $|k(x,y)| \lesssim \frac{1}{|x-y|}$.

If $|x-y| > 1$, then $k(x,y) = 0$.

If $2^{-k_0-1} < |x-y| < 2^{-k_0}$, then

$$|k(x,y)| = 2^{k_0} |\theta(2^{k_0}|x-y|) \sin(2^{k_0}(x+y))| \lesssim 2^{k_0} < \frac{1}{|x-y|},$$

Termwise derivation (the sum is locally finite!) similarly give

$$|\partial_x k(x,y)|, |\partial_y k(x,y)| \lesssim 2^{2k_0} < \frac{1}{|x-y|^2}.$$

Thus we see that k is an antisymmetric γ-regular SIO kernel, so by Prop. 8.2 we have a well defined SIO

$$T = \lim_{\varepsilon\to 0} T_\varepsilon : H_t^s(\mathbb{R}) \longrightarrow H_t^s(\mathbb{R})^*.$$

Now consider $T_\varepsilon f(x)$ for $|x|\leq 1$, $f\in H_t^s(\mathbb{R})$ such $f=1$ on $B(0;2)$, and with $\varepsilon = 2^{-m}$:

$$T_\varepsilon f(x) = \int_{|y-x|>2^{-m}} \sum_{k=0}^{\infty} 2^k \theta(2^k(x-y)) \sin(2^k(x+y)) f(y)\,dy$$

$$= \sum_{k=0}^{m-1} 2^k \int_{|y-x|>2^{-m}} \theta(2^k(x-y)) \sin(2^k(x+y)) f(y)\,dy$$

$$= \Big/ |y| \leq |x| + |x-y| \leq 1+1 = 2 \text{ when } \theta(2^k(x-y)) \neq 0 \Big/$$

$$= \sum_{k=0}^{m-1} 2^k \int_{|t|>2^{k-m}} \theta(t) \sin(2^{k+1}x - t)\, 2^{-k}\,dt$$

$$= \sum_{k=0}^{m-1} \operatorname{Im}\Big\{-ie^{i2^{k+1}x} \underbrace{\int \theta(t)\sin t\,dt}_{1}\Big\}$$

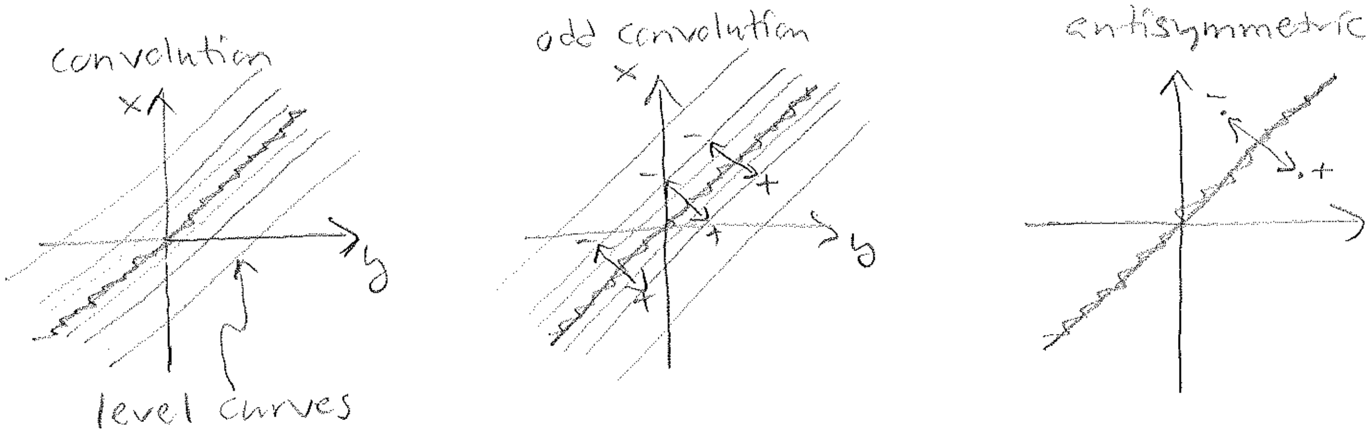
$$= -\sum_{k=0}^{m-1} \cos(2^{k+1}x) = -\sum_{k=1}^{m} \cos(2^k x)$$

As $m\to\infty$ ($\varepsilon\to 0$), this series does not converge in L_2. In fact, by orthogonality we have

$$\|T_\varepsilon f\|_2 \gtrsim \sqrt{m} \to \infty, \quad m\to\infty.$$

However, by Prop. 8.2 $T_\varepsilon f$ converges to a distribution Tf in $H_t^s(\mathbb{R})^*$.

Note the following for kernels $k(x,y)$:



Before continuing, we formulate our problem precisely.

Main problem for part II:

Let $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ be a SIO, with SIO kernel $k(x,y)$. Find conditions on k, and T, such that

$\forall f \in H^s_t(\mathbb{R}^n): Tf \in L_2(\mathbb{R}^n)$, with

uniform bounds

$\|Tf\|_{L_2} \lesssim \|f\|_{L_2}$, $\forall f \in H^s_t(\mathbb{R}^n)$.

Since $H^s_t(\mathbb{R}^n)$ is dense in $L_2(\mathbb{R}^n)$, T can in this case be extended by continuity to $T \in L(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$ in a unique way.

The Haar basis for $L_2(\mathbb{R})$:

Since the solution of the problem is somewhat technical, we first study in this section a "toy model" to familiarize the reader with notation and ideas. The goal in this section is to reprove the L_2-boundedness of the Cauchy integral on $\mathbb{R}$ from Ex. 7.12, using the Haar basis instead of trigonometric bases. The proof of the Tb theorem later will use these same ideas, but with some additional technicalities.

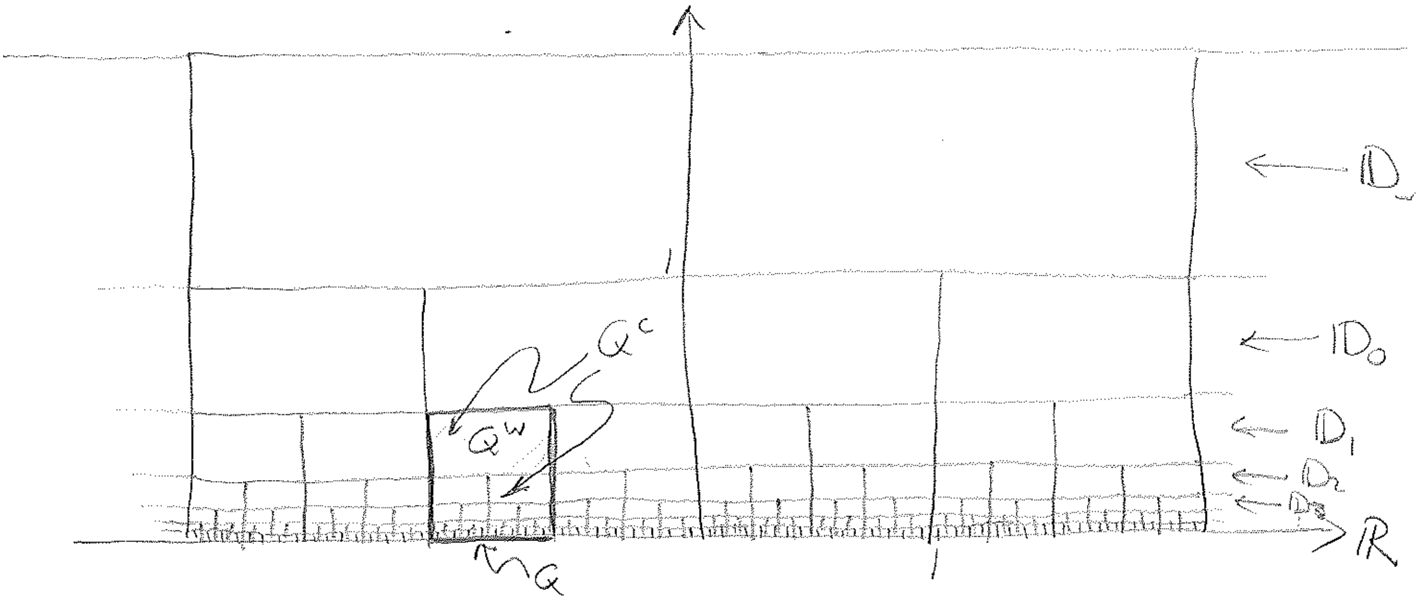
Defn. 8.6: We define the set of <u>dyadic intervals</u> in $\mathbb{R}$ as follows.

$$\mathbb{D}_j = \{(k\cdot 2^{-j}, (k+1)2^{-j}) \subset \mathbb{R} \,;\, k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}$$

$$\mathbb{D} := \bigcup_{j\in\mathbb{Z}} \mathbb{D}_j$$

For $Q \in \mathbb{D}_j$, its length is $\ell(Q) = 2^{-j}$ and its measure/1-volume is $|Q| = 2^{-j}$.

We here use notation adapted to $\mathbb{R}^n$: Q will in general be an n-dimensional cube, and $|Q| = \ell(Q)^n$.
Since intervals Q in $\mathbb{D}$ overlap, it is often useful to visualize $\mathbb{D}$ as a <u>Whitney partition</u> of the upper half plane $\mathbb{R}^2_+$.



Defn. 8.7: Associated to $Q = (k2^{-j}, (k+1)2^{-j}) \in \mathbb{D}$, we define the <u>Whitney region</u>

$$Q^w := (k2^{-j}, (k+1)2^{-j}) \times (\tfrac{1}{2}2^{-j}, 2^{-j})$$

and the <u>Carleson box</u>

$$Q^{ca} := (k2^{-j}, (k+1)2^{-j}) \times (0, 2^{-j}).$$

We note the following:

(1) The Whitney regions form a partition

(modulo zero sets) of $\mathbb{R}^2_+$: $\mathbb{R}^2_+ = \bigcup_{Q\in\mathbb{D}} Q^w$.

(2) Dyadic intervals $Q \subset \mathbb{R}$ and Whitney regions $Q^w \subset \mathbb{R}^2_+$ are in natural one-to-one correspondence.

(3) The dyadic intervals come with a natural <u>tree ordering</u>: for $Q, R \in \mathbb{D}$ there are three and only three possibilities:

$$Q \cap R = \emptyset \quad , \quad Q \subset R \quad \text{or} \quad R \subset Q.$$

(Here $\subset$ does not mean strict inclusion.)
Inclusion $\subset$ defines a partial order on $\mathbb{D}$. For an interval $Q = (k2^{-j}, (k+1)2^{-j}) \in \mathbb{D}_j \subset \mathbb{D}$, we speak of its

- <u>left and right children</u> $Q^l, Q^r \in \mathbb{D}_{j+1}$, where $Q = Q^l \cup Q^r$.
- <u>parent</u> $Q^p \in \mathbb{D}_{j-1}$, $Q \subset Q^p$.
- <u>sibling</u> $Q^s \in \mathbb{D}_j$, $Q^p = Q \cup Q^s$.

Note that under the correspondence $Q \leftrightarrow Q^w$, the <u>subtree</u>

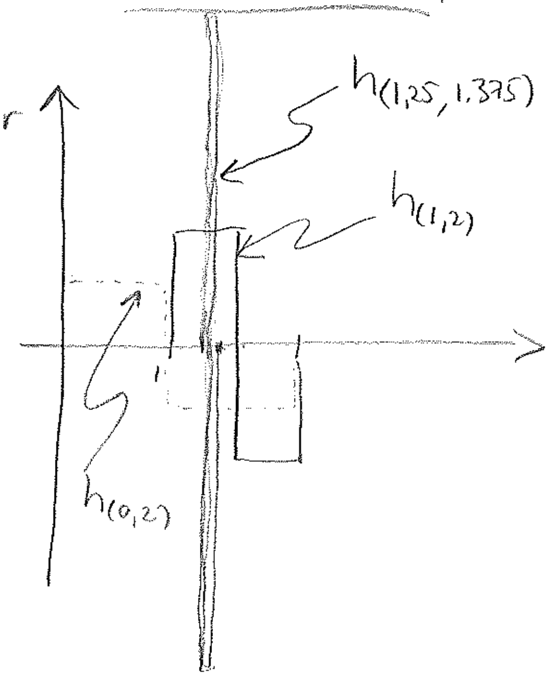
$$\{R \in \mathbb{D} \,;\, R \subset Q\}$$

corresponds to the Carleson box Q^{ca}.

We shall use the dyadic intervals as index set for the Haar basis.

<u>Defn 8.8</u>: For $Q \in \mathbb{D}$, define the <u>Haar function</u>

$$h_Q(x) := \begin{cases} \ell(Q)^{-1/2} & ; \ x \in Q^l \\ -\ell(Q)^{-1/2} & ; \ x \in Q^r \\ 0 & ; \ \text{else} \end{cases}$$



<u>Lem. 8.9</u>:
$\{h_Q \,;\, Q \in \mathbb{D}\}$ is an ON-set in $L_2(\mathbb{R})$.

Proof: Clearly

$$\|h_Q\|_2^2 = \int_Q (\ell(Q)^{-1/2})^2 dx = 1.$$

If $R \neq Q$, $R, Q \in \mathbb{D}$, then we have 3 cases:

- $R \cap Q = \emptyset$. Clearly $\langle h_Q, h_R \rangle = 0$.
- $R \subsetneq Q$. Then $R \subset Q^l$ or $R \subset Q^r$. In the first case $\langle h_Q, h_R \rangle = \underbrace{\int \ell(Q)^{-1/2} h_R(x)\, dx}_{R} = 0$.

  Case $R \subset Q^r$ is similar.
- $Q \subset R$. Similar. ∎

We wrote out the proof above, since it uses the most fundamental principle in harmonic analysis:

"if f is localized (small outside a compact set) with $\int f dx = 0$ and if g is close to constant on the set where f is localized to, then

$$\langle f, g \rangle = \int f(x) g(x)\, dx \approx 0."$$

Sometimes this is called "the principle of almost orthogonality". Often one applies it as follows.

(1) If g is $C^1(\mathbb{R})$ and $\operatorname{supp} f \subset K$, then we do integration by parts:

$$\left|\int f g\, dx\right| = \left|-\int F g' dx\right| \le \left(\int_K |F|\, dx\right) \sup_K |g'| \approx 0$$

since g' is assumed small on K. Note that $\int f = 0$ ensures existence of F such that $F' = f$ and $\operatorname{supp} F \subset K$.

(2) If g is in $C^\alpha(\mathbb{R})$, $0 < \alpha < 1$, then we pick $x_0 \in K$ if $\operatorname{supp} f \subset K$ and do a calculation

$$\left|\int f(x) g(x)\, dx\right| = \left|\int f(x)(g(x) - g(x_0))\, dx\right|$$
$$\le \left(\int_K |f(x)| \cdot |x - x_0|^\alpha dx\right) \cdot \sup_K \frac{|g(x) - g(x_0)|}{|x - x_0|^\alpha} \approx 0$$

since the second factor is assumed small.

The observant reader has noted that we used the almost orthogonality principle in the proof of thm 7.18. Lemma 8.9 used a more precise version of this principle.

Next we introduce some useful projection operators associated with the Haar basis.

<u>Defn 8.10</u>: Let $f \in L_1^{loc.}(\mathbb{R})$.

- The <u>average of f on $Q \in \mathbb{D}$</u> is

$$E_Q f := \frac{1}{|Q|}\int_Q f(x)\,dx = ⨍_Q f(x)\,dx.$$

- Define the piecewise constant function

$$E_j f(x) = \sum_{Q \in \mathbb{D}_j} (E_Q f)\,\chi_Q(x), \quad j \in \mathbb{Z}, x \in \mathbb{R}.$$

- Let $\Delta_j f(x) := E_{j+1} f(x) - E_j f(x)$, $j \in \mathbb{Z}$,

and $\Delta_Q f(x) := \Delta_j f(x) \cdot \chi_Q(x) = \begin{cases} E_{j+1} f(x) - E_Q f, & x \in Q \\ 0, & x \notin Q \end{cases}$, $Q \in \mathbb{D}$.

The notation E_j comes from <u>ex</u>pectation and probability theory, where $\{E_j\}$ is an example of a martingale.

<u>Exc. 8.11</u>: Let P_1, P_2 be two projections in a B-space X (defn. 3.3.). Prove that

$$P_1 P_2 = P_2 \iff R(P_1) \supset R(P_2)$$
$$P_2 P_1 = P_2 \iff N(P_1) \subset N(P_2).$$

<u>Lemma 8.12</u>: (1) As a bounded operator on $L_2(\mathbb{R})$, E_j is the orthogonal projection onto

$$R(E_j) = \{ f \in L_2(\mathbb{R}^n);\ f|_Q = \text{constant},\ \forall Q \in \mathbb{D}_j \}$$

(2) $E_i E_j = E_j E_i = E_{\min(i,j)}$, $i, j \in \mathbb{Z}$.

(3) Δ_Q is orthogonal projection onto $\operatorname{span}\{h_Q\}$, and Δ_j is orthogonal projection onto $\operatorname{span}_{Q \in \mathbb{D}_j}\{h_Q\}$.

Proof: It is straightforward to verify (1), and it is also clear that $R(E_i) \supset R(E_j)$ if $i \geq j$.
Since $E_k^* = E_k$, this implies $N(E_i) \subset N(E_j)$ if $i \geq j$.
(2) follows from Exc. 8.11.
We see that Δ_j is orthogonal projection onto $R(E_{j+1}) \cap R(E_j)^{\perp}$. By (1), it follows that this range is $\operatorname{span}_{Q \in \mathbb{D}_j} \{h_Q\}$. (How?)
The conclusion for Δ_Q follows. ∎

Prop. 8.13: The Haar functions $\{h_Q\}_{Q \in \mathbb{D}}$ form an ON-basis for $L_2(\mathbb{R})$.
For a fixed $Q_0 \in \mathbb{D}$, the functions $\{h_Q\}_{Q \in \mathbb{D},\, Q \subset Q_0}$ together with $\ell(Q_0)^{-1/2} \chi_{Q_0}$ form an ON-basis for $L_2(Q_0)$.

Proof: For $L_2(\mathbb{R})$ it remains to show that $\{h_Q\}_{Q \in \mathbb{D}}$ span $L_2(\mathbb{R})$. Given $f \in C_c^\infty(\mathbb{R})$, write

$$\sum_{j=-N}^{N} \Delta_j f = /\text{telescope}/ = E_{N+1} f - E_{-N} f.$$

In $L_2(\mathbb{R})$, this converges to $f - 0 = f$ when $N \to \infty$. (Why?). This shows that

$$f(x) = \sum_{Q \in \mathbb{D}} \Delta_Q f(x) = \sum_{Q \in \mathbb{D}} c_Q' h_Q(x) \quad \text{in } L_2(\mathbb{R}),$$

and the conclusion follows since $C_c^\infty(\mathbb{R})$ is dense in $L_2(\mathbb{R})$.

For $L_2(Q_0)$, $\ell(Q_0) = 2^{-j_0}$, the stated set is clearly ON. That is spans is derived from the identity

$$f(x) = E_{j_0} f(x) + \sum_{j=j_0}^{\infty} \Delta_j f(x), \quad x \in Q_0. \ ∎$$

Exc 8.14: Resolve the following "paradox":

$$f(x) := e^{-|x|} = \sum_{Q \in \mathbb{D}} c_Q h_Q$$

$$\forall Q \in \mathbb{D} : \int h_Q = 0 \Rightarrow \int_{-\infty}^{\infty} e^{-|x|} dx = 0 \ldots$$

Having constructed and understood the Haar basis, we now use it to prove L_2-boundedness of the Cauchy integral on $\mathbb{R}$. We note the "technical problem" that the index set $\mathbb{D}$ is not totally/linearly ordered as for Fourier bases. Let us first make this comparison more clear, by considering the periodic analogue of Ex. 7.12.

<u>Ex. 8.15</u>: The trigonometric functions

$$\{\ldots, e^{-i3x}, e^{-i2x}, e^{-ix}, 1, e^{ix}, e^{i2x}, e^{i3x}, \ldots\}$$

form an ON-basis for $L_2(-\pi,\pi)$, with inner product $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\bar{g}\,dx$.

Consider the Cauchy integral on the circle:

$$Cf(z) := \text{p.v.}\int_{\mathbb{T}} \frac{f(w)}{z-w}\,dw,$$

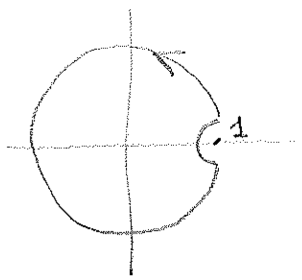
where $\mathbb{T}\subset\mathbb{C}$ is the unit circle. In polar coordinates

$$Cf(t) = \text{p.v.}\int_{-\pi}^{\pi} \frac{f(s)}{e^{i(t-s)}-1}\, i\,ds.$$

The kernel of this SIO is $k(t) = \dfrac{i}{e^{it}-1}$, with Fourier coefficients

$$\hat{k}_n = \langle k, e^{int}\rangle = \frac{i}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} \frac{e^{-int}}{e^{it}-1}\,dt$$

$$= \frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\substack{\mathbb{T}\\ |z-1|>\varepsilon}} \frac{1}{(z-1)z^{1+n}}\,dz.$$



Residue calculus gives

$$\hat{k}_n = \begin{cases} -i/2, & n\geq 0\\ +i/2, & n<0.\end{cases}$$

This means that the trigonometric basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ is perfectly adapted to the SIO C: the matrix of C in this basis is diagonal!

This is the case for any convolution operator.

The deepest fact about SIOs is that any such (even non-convolution ones) is "almost diagonal" in Haar/wavelet bases. Almost here means that the off-diagonal elements in the matrix of the SIO are "small enough" so that a Schur estimate suffices to prove boundedness on $\ell_2(\mathbb{D}) \approx L_2(\mathbb{R})$. Smallness of the off-diagonal elements follows from the principle of almost orthogonality.

<u>Ex 8.16</u>:

We end this lecture by proving, once again, the L_2-boundedness of the Cauchy integral on $\mathbb{R}$:

$$Cf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy, \quad x \in \mathbb{R},$$

this time using the Haar basis. Let $\{c_{R,Q}\}_{R,Q\in\mathbb{D}}$ denote the matrix of C in $\{h_Q\}_{Q\in\mathbb{D}}$, i.e.

$c_{R,Q} = \langle h_R, Ch_Q\rangle$.

(Note that $c_{R,Q}$ is well-defined by Prop. 8.2 since $h_Q \in H_t^S(\mathbb{R})$ according to Exc. 8.3.)

Since $C^* = -C$, it suffices to consider $|R| \le |Q|$. Furthermore, we may assume that $Q = (0,1)$. Indeed, let $A_Q f(x) := |Q|^{1/2} f(\ell(Q)x + x_0)$, where x_0 is the left endpoint of Q. Then

- $A_Q : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is an isometry,
- $CA_Q = A_Q C$
- $A_Q h_Q = h_{(0,1)}$ and $A_Q h_R = h_{R'}$ for some $R' \subset \mathbb{D}$, $|R'| \le 1$.

So $\langle h_R, Ch_Q\rangle = \langle A_Q h_R, A_Q C h_Q\rangle = \langle h_{R'}, Ch_{(0,1)}\rangle$.

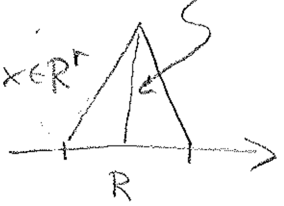
- From Exc. 7.14 we know that

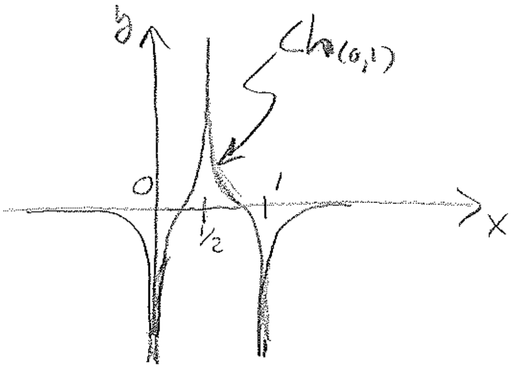
$$Ch_{(0,1)}(x) = C(\chi_{(0,\frac{1}{2})} - \chi_{(\frac{1}{2},1)})(x) = \ln\left|\frac{x}{x-1/2}\right| - \ln\left|\frac{x-1/2}{x-1}\right|$$
$$= \ln\left|\frac{x(x-1)}{(x-1/2)^2}\right|$$

We let

$$H_R(x) := \begin{cases} 2^{j/2}(x - k2^{-j}), & x \in R^l \\ -2^{j/2}(x-(k+1)2^{-j}), & x \in R^r \end{cases}$$

if $R = (k2^{-j}, (k+1)2^{-j})$, $j \geq 0$.





Using the principle of almost orthogonality, we write

$$c_{R,Q} = \langle h_R, Ch_{(0,1)} \rangle = -\langle H_R, (Ch_{(0,1)})' \rangle$$

$$= -\int_R H_R(x)\left(\frac{1}{x} + \frac{1}{x-1} - \frac{2}{x-1/2}\right) dx = -\int_R \frac{H_R(x)\, dx}{x(x-1)(2x-1)}.$$

Since $C^* = -C$, it is clear that $c_{Q,Q} = 0$, $\forall Q \in \mathbb{D}$, so we may assume $R \neq (0,1)$, $|R| \leq |Q|$.

We consider four cases:

(A) $R \subset (-\infty, -1) \cup (2, \infty)$

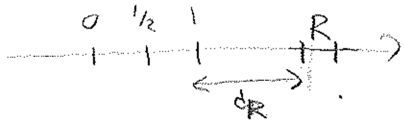
(B) $R \subset (-1, -\ell(R)) \cup (1+\ell(R), 2)$

(C) $R \subset (\ell(R), 1/2 - \ell(R)) \cup (1/2 + \ell(R), 1 - \ell(R))$

(D) $\overline{R} \cap \{0, 1/2, 1\} \neq \emptyset$

Let $d_R := \operatorname{dist}(R, \{0, \frac{1}{2}, 1\}) = \inf_{x \in R} \{|x|, |x - \frac{1}{2}|, |x-1|\}$.

(A) Here



$$|c_{RQ}| \lesssim \int_R \frac{|H_R(x)|\, dx}{d_R^3} = 2^{-3j/2} \cdot d_R^{-3}.$$

(B) Wlog we may assume $R \subset (1,2)$. Here

$$|c_{RQ}| \lesssim \int_R \frac{|H_R(x)|\, dx}{d_R} = 2^{-3j/2} d_R^{-1}.$$

(C) Same as in (B).

(D) Here

$$|c_{RQ}| \lesssim \int_0^{2^{-j}} \frac{2^{j/2} x}{x}\, dx = 2^{-j/2}.$$

We now apply the Schur estimate Prop. 1.10, with $\Omega_1 = \Omega_2 = \mathbb{D}$. It suffices to show

$$\sup_{R\in\mathbb{D}} \sum_{Q\in\mathbb{D}} |C_{RQ}| < \infty \quad \text{and} \quad \sup_{Q\in\mathbb{D}} \sum_{R\in\mathbb{D}} |C_{RQ}| < \infty.$$

Since $C_{RQ} = -C_{QR}$, it suffices to consider the second estimate. As above, it suffices to show

$$\sum_{R\in\mathbb{D}} |C_{R,(0,1)}| < \infty.$$

We write $\displaystyle \sum_{j\geq 0} \underbrace{\sum_{R\in\mathbb{D}_j} |C_{R(0,1)}|}_{=: c_j} + \sum_{j<0} \sum_{R\in\mathbb{D}_j} |C_{R(0,1)}|.$

For $j \geq 0$, we have

$$\begin{aligned} c_j &= \sum_{A} + \sum_{B} + \sum_{C} + \sum_{D} \\ &\lesssim 2^{-3j/2} \sum_{h=0}^{\infty} \frac{1}{(1+h2^{-j})^3} + \left(2^{-3j/2} \sum_{h=1}^{2^j} \frac{1}{(h2^{-j})^1}\right)\cdot 2 + 2^{-j/2} \\ &\approx 2^{-3j/2}\cdot 2^j + 2^{-j/2} \ln 2^j + 2^{-j/2} \\ &\approx 2^{-j/2}(1+j) \end{aligned}$$

Thus $\sum_{j\geq 0} c_j < \infty$. For $j<0$, we again use the rescaling operator A_R to obtain

$$\sum_{R\in\mathbb{D}_j} |C_{R(0,1)}| = \sum_{R'\in\mathbb{D}_{-j}} |C_{(0,1),R'}| = \sum_{R'\in\mathbb{D}_{-j}} |C_{R',(0,1)}| \lesssim 2^{j/2}(1-j),$$

giving in total

$\displaystyle\sum_{R\in\mathbb{D}} |C_{R,(0,1)}| < \infty$. This proves that $C: L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is bounded. ∎

<u>Exc. 8.17</u>: The proof in Ex. 8.16 used (but not in a crucial way) that the Cauchy integral on $\mathbb{R}$ is both translation and dilation invariant. Show the following:

If T is a SIO, in the sense of defn 8.1, which commutes with all translation/dilation operators $A_{t,p}f(x) := t^{1/2} f(tx+p)$, $t>0$, $p\in\mathbb{R}$, then

$$T = \alpha I + \beta C, \quad \text{for some } \alpha,\beta\in\mathbb{R}.$$

identity   Cauchy

We next aim to develop a theory of so-called perfect dyadic SIOs in $\mathbb{R}^n$, where the γ-Hölder regularity of the SIO kernel is replaced by a perfect dyadic regularity condition. This requires a definition of dyadic cubes in $\mathbb{R}^n$, $n \geq 1$. In Defn. 8.6, the <u>main branch</u> in the tree of dyadic intervals is

$$\ldots \subset (0,\tfrac{1}{16}) \subset (0,\tfrac{1}{8}) \subset (0,\tfrac{1}{4}) \subset (0,\tfrac{1}{2}) \subset (0,1) \subset (0,2) \subset (0,4) \subset (0,8) \subset (0,16) \subset \ldots$$

For technical reason we prefer to, from <u>now on</u>, use a modified version of this main branch as follows.

<u>Defn 9.1</u>: Let the <u>main branch</u> of dyadic intervals be the sequence of (translated) dyadic intervals

$$\ldots \subset Q_4^0 \subset Q_3^0 \subset Q_2^0 \subset Q_1^0 \subset Q_0^0 \subset Q_{-1}^0 \subset Q_{-2}^0 \subset Q_{-3}^0 \subset Q_{-4}^0 \subset \ldots$$

where Q_{j+1}^0 is the left (right) child of Q_j^0 if j is even (odd), and $\bigcap_{j\in\mathbb{Z}} Q_j^0 = \{0\}$.

Explicitly

$$Q_j^0 = \left(-\frac{1}{3}\frac{1}{4^{\lfloor j/2 \rfloor}}, \frac{1}{2^j} - \frac{1}{3}\frac{1}{4^{\lfloor j/2\rfloor}}\right), \text{ where}$$

$$\lfloor x \rfloor := \max\{k \in \mathbb{Z} ; k \leq x\}.$$

Define dyadic cubes in $\mathbb{R}^n$:

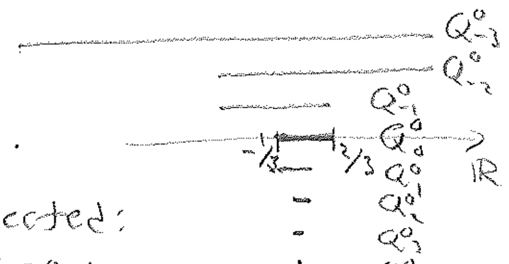
$$\mathbb{D}_j := \{(Q_j^0)^n + 2^{-j}k ;\ k \in \mathbb{Z}^n\}, \quad j \in \mathbb{Z}.$$

$$\mathbb{D} := \bigcup_{j\in\mathbb{Z}} \mathbb{D}_j$$

<u>Ex 9.2</u>: The advantage of the modified main branch is that

$$\bigcup_{j\in\mathbb{Z}} Q_j^0 = \mathbb{R},$$

as compared to $\bigcup_{j\in\mathbb{Z}} (0,2^{-j}) = \mathbb{R}_+$.



This makes tree $(\mathbb{D}, \subset)$ connected: If $Q_1, Q_2 \in \mathbb{D}$, then $Q_1, Q_2 \subset (Q_j^0)^n$ for some $j \in \mathbb{Z}$ clearly.

Hence $\mathrm{lub}(Q_1, Q_2) := \bigcap\{Q \in \mathcal{D};\, Q_1, Q_2 \subset Q\}$ is well defined. (For the mean branch $\{(0, 2^{-j})\}$, $\mathrm{lub}(Q_1, Q_2)$ would not exist if $Q_1 \subset \mathbb{R}_-$ and $Q_2 \subset \mathbb{R}_+$.)

Exc. 9.3: For $x, y \in \mathbb{R}^n$, $x \neq y$, let
$$\mathrm{lub}(x,y) := \bigcap\{Q \in \mathcal{D};\, x, y \in Q\}.$$
Show that for any $0 < C < \infty$, there exists $x, y \in \mathbb{R}^n$ such that $x \neq y$ and
$$\mathrm{diam}(\mathrm{lub}(x,y)) \geq C|x-y|.$$

This shows that, with the mean branch $\{Q_j^a\}$, $\mathrm{lub}\{Q_1, Q_2\}$ always exists, but may be a much larger cube than Q_1, Q_2.

Similar to dyadic intervals, we make the following definitions for the dyadic cubes from Defn 9.1.

Defn. 9.4: For $Q \in \mathcal{D}_j$, its side length is $\ell(Q) := 2^{-j}$ and its n-volume is $|Q| = 2^{-nj}$, and we define associated subsets of $\mathbb{R}^{n+1}_+ := \mathbb{R}_+ \times \mathbb{R}^n = \{(t,x);\, t > 0\}$:

Whitney region: $Q^W := (\frac{1}{2}2^{-j}, 2^{-j}) \times Q$

Carleson box: $Q^{ca} := (0, 2^{-j}) \times Q$.

A dyadic cube $Q' \in \mathcal{D}$ is
- a child of Q if $Q' \in \mathcal{D}_{j+1}$ and $Q' \subset Q$
- Q is parent if Q is a child of Q'
- a sibling to Q if Q and Q' has same parent.

We recall that:

(1) there is a one-to-one correspondence between dyadic cubes $Q \subset \mathbb{R}^n$ and Whitney regions $Q^W \subset \mathbb{R}^{n+1}_+$.

(2) the Whitney regions Q^W, $Q \in \mathcal{D}$, form a partition (modulo zero sets) of $\mathbb{R}^{n+1}_+$.

(3) the is a one-to-one correspondence between Carleson boxes $Q^{ca} \subset \mathbb{R}^{n+1}_+$ and subtrees $\{R \in \mathcal{D};\, R \subset Q\} \subset \mathcal{D}$.

Carleson's theorem and BMO

Preliminary to our study of non-convolution SIOs, we here first study some fundamental estimates from harmonic analysis. A very well known estimate basic estimate is the following.

Thm 9.4: For $f \in L_1^{loc}(\mathbb{R}^n)$, define the Hardy-Littlewood maximal function

$$(Mf)(x) := \sup_{t>0} \fint_{|y-x|<t} |f(y)|\,dy, \quad x \in \mathbb{R}^n.$$

(Here $\fint_X = \frac{1}{|X|} \cdot \int_X$, where $|X|$ denotes Lebesgue $\mathbb{R}^n$-measure.)

Then

$$\sup_{\lambda>0} \lambda \cdot |\{x ; Mf(x) > \lambda\}| \le \|f\|_{L_1(\mathbb{R}^n)}$$

and $\|Mf\|_{L_p(\mathbb{R}^n)} \lesssim \|f\|_{L_p(\mathbb{R}^n)}$ for $1<p\le\infty$.

The proof of this is classical (see eg. E.M. Stein: "Singular integrals and differentiability properties of functions". Ingredients of the proof are the Vitali covering theorem and the Marcinkiewicz interpolation theorem. Note that M is not a linear operator, only sub-additive: $M(f+g) \le Mf + Mg$.

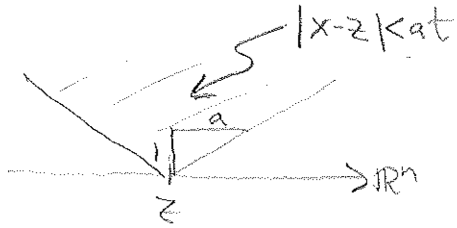
Defn 9.5: Fix $0<a<\infty$. For functions $u \in L_\infty^{loc}(\mathbb{R}^{1+n}_+)$, define the non-tangential (NT) maximal function

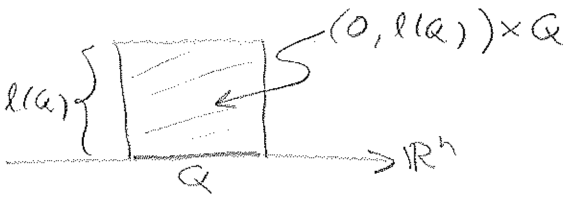
$$(N_a u)(z) := \sup\{|u(t,x)| ; |x-z| < at\}, \quad z \in \mathbb{R}^n.$$

- For (positive) measures μ on $\mathbb{R}^{1+n}_+$, define it Carleson norm

$$\|\mu\|_C := \sup_Q \frac{1}{|Q|} \mu(Q^{ca}), \quad Q^{ca} := (0, \ell(Q)) \times Q, \text{ where}$$

sup is over cubes $Q \subset \mathbb{R}^n$ with sidelength $\ell(Q)$ and volume $|Q| = \ell(Q)^n$.





Exc 9.6: Define the dyadic Carleson norm

$$\|\mu\|_{C_d} := \sup_{Q\in\mathbb{D}} \frac{1}{|Q|}\mu(Q^{CQ}).$$

Show that $\|\mu\|_{C_d} \approx \|\mu\|_C$ in the sense of Defn. 7.3.
Show also that

$$\sup_{\substack{x\in\mathbb{R}^n\\ t>0}} \frac{1}{|B(x;t)|}\mu(T_{B(x;t)}) \approx \|\mu\|_C$$

where $T_{B(x;t)} := \{(s,y)\in\mathbb{R}^{1+n}_+;\ |y-x|<t-s\}$
is the tent over the ball
$B(x,t) := \{y\in\mathbb{R}^n;\ |y-x|<t\}$.

Similarly for the NT max. function, there is a dyadic version:

$$N_d u(z) := \sup_{\substack{Q\in\mathbb{D}\\ z\in Q}} \|u\|_{L_\infty(Q^W)}$$

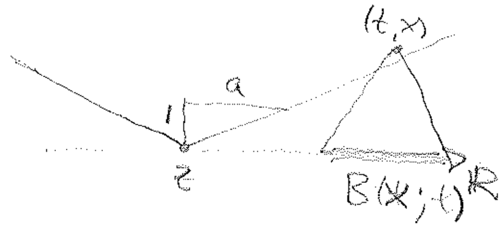
There is a simple pointwise estimate $N_d u(z) \le N_a u(z)$ for appropriate $a>0$: if $z\in Q$ and $(t,x)\in Q^W$, then $|x-z|\le\sqrt{n}\,\ell(Q)\le 2\sqrt{n}\,t$. Thus $N_d u(z)\le N_{2\sqrt{n}}u(z)$.

We have reverse estimates, but not pointwise, only in L_p sense.

Lemma 9.7: Let $1<a<\infty$ and $1\le p\le\infty$. Then

$$\|N_a u\|_{L_p(\mathbb{R}^n)} \lesssim \|N_1 u\|_{L_p(\mathbb{R}^n)}.$$

Proof: Assume $N_a u(z)>\lambda$. Then $u(t,x)>\lambda$ for some $|x-z|<at$.
Thus $N_1 u>\lambda$ on $B(x;t)$.



Let $E_\lambda := \{y; N_1 u(y) > \lambda\}$. We have

$$M(\chi_{E_\lambda})(z) \geq \frac{1}{|B(z,(a+1)t)|} \int_{B(z,(a+1)t)} |\chi_{E_\lambda}(y)|\,dy$$

$$\geq \frac{1}{|B(z,(a+1)t)|} |B(x,t)| = \frac{1}{(a+1)^n}.$$

$$\Rightarrow |\{N_a u > \lambda\}| \leq |\{M(\chi_{E_\lambda}) \geq \frac{1}{(a+1)^n}\}| \overset{\text{Thm 10.4}}{\lesssim} (a+1)^n \|\chi_{E_\lambda}\|_{L_1(\mathbb{R}^n)}$$

$$\approx |E_\lambda| = |\{N_1 u > \lambda\}|.$$

Thus

$$\int_{\mathbb{R}^n} |N_a u|^p dz = \int_{\mathbb{R}^n} \left(\int_0^{|N_a u(z)|} p\lambda^{p-1} d\lambda \right) dz = \int_0^\infty \left(\int_{z : N_a u(z) > \lambda} dz \right) p\lambda^{p-1} d\lambda$$

$$= \int_0^\infty p\lambda^{p-1} |\{N_a u > \lambda\}| d\lambda \lesssim \int_0^\infty p\lambda^{p-1} |\{N_1 u > \lambda\}| d\lambda$$

$$= \dots = \int_{\mathbb{R}^n} |N_1 u|^p dz. \quad \blacksquare$$

<u>Exc. 9.8</u>: Show for any $0 < a < \infty$ and $1 \leq p \leq \infty$ that $\|N_d u\|_{L_p(\mathbb{R}^n)} \approx \|N_a u\|_{L_p(\mathbb{R}^n)}$.

We now prove the key harmonic analysis estimate for §10s.

<u>Thm 9.9.</u> (Carleson's theorem)

Let $u \in L_\infty^{loc}(\mathbb{R}^{1+n}_+)$ and let μ be a <u>Carleson measure</u>, i.e. $\|\mu\|_C < \infty$. Then

$$\left| \int_{\mathbb{R}^{1+n}_+} u(t,x)\, d\mu(t,x) \right| \lesssim \|\mu\|_C \cdot \|N_* u\|_{L_1(\mathbb{R}^n)},$$

where N_* denotes any of the equivalent N_d, N_a.

<u>Proof</u>: By Exc. 9.6, 9.8 it suffices to prove the dyadic version of Carleson's estimate:

$$\int_{\mathbb{R}^{1+n}_+} u\, d\mu \lesssim \|\mu\|_{C_d} \cdot \|N_d u\|_{L_1(\mathbb{R}^n)},$$ where we may assume $u \geq 0$. To this end, let

$A_\lambda := \bigcup \{ Q^w ;\ \|u\|_{L^\infty(Q^w)} > \lambda,\ Q \in \mathbb{D} \}$

$B_\lambda := \{ z \in \mathbb{R}^n ;\ N_d u(z) > \lambda \}$, for $\lambda > 0$.

- By definition of B_λ:

$$Q \in \mathbb{D},\ Q \cap B_\lambda \neq \emptyset \Rightarrow Q \subset B_\lambda .$$

Thus $B_\lambda = \bigcup_{Q \in \mathbb{D}^\lambda} Q$, where $\mathbb{D}^\lambda := \{ Q \in \mathbb{D} ;\ Q \subset B_\lambda \}$.

Let $\mathbb{D}^\lambda_{max}$ be all maximal cubes in $\mathbb{D}^\lambda$:

$$\mathbb{D}^\lambda_{max} := \{ Q \in \mathbb{D}^\lambda ;\ R \in \mathbb{D}^\lambda,\ R \cap Q \neq \emptyset \Rightarrow R \subset Q \}.$$

Then $B_\lambda = \bigcup_{Q \in \mathbb{D}^\lambda_{max}} Q$ and $|B_\lambda| = \sum_{Q \in \mathbb{D}^\lambda_{max}} |Q|$.

- We claim: $A_\lambda \subset \bigcup_{Q \in \mathbb{D}^\lambda_{max}} Q^{ca}$.

Indeed, if $(t,x) \in Q_1^w$ and $u(t,x) > \lambda$, $Q_1 \in \mathbb{D}$, then $Q_1 \subset B_\lambda$, so $Q_1 \subset Q$ for some $Q \in \mathbb{D}^\lambda_{max}$.

This shows the claim since $Q_1^w \subset Q^{ca}$.

We get

$$\mu(A_\lambda) \le \sum_{Q \in \mathbb{D}^\lambda_{max}} \mu(Q^{ca}) \overset{\|\mu\|_{Cd} < \infty}{\lesssim} \sum_{Q \in \mathbb{D}^\lambda_{max}} |Q| = |B_\lambda| .$$

Integration gives

$$\int_{\mathbb{R}^{1+n}_+} u\, d\mu = \int_{\mathbb{R}^{1+n}_+} \left(\int_0^{u(t,x)} d\lambda \right) d\mu(t,x) = \int_0^\infty \mu\{(t,x) ; u(t,x) > \lambda\}\, d\lambda$$

$$\le \int_0^\infty \mu(A_\lambda)\, d\lambda \lesssim \int_0^\infty |B_\lambda|\, d\lambda = \dots = \int_{\mathbb{R}^n} N_d u(z)\, dz .$$

∎

<u>Ex. 9.10</u>: The classical way the function $u(t,x)$ appears in Carleson's lemma is as a Poisson extension of a function $f : \mathbb{R}^n \to \mathbb{R}$:

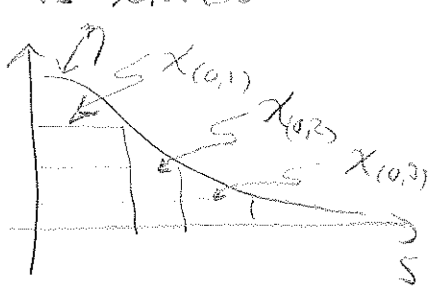
$$u(t,x) = (P_t f)(x) = \int_{\mathbb{R}^n} P_t(x,y) f(y)\, dy ,$$ where

the kernel is

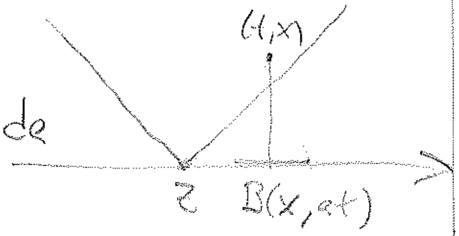
$$P_t(x,y) = \frac{2}{\sigma_n} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}}$$ / $\sigma_n :=$ area of unit $\mathbb{R}^{n+1}$-sphere /

This u is the solution to the Dirichlet problem
$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{1+n}_+ \\ u = f & \text{on } \mathbb{R}^n \end{cases}$$
which decays at ∞.

Let us estimate $N_*(P_t f)$. We shall not use the exact formula for p_t, but only the estimate
$$|p_t(x,y)| \lesssim \frac{1}{t^n}\,\eta\left(\frac{|x-y|}{t}\right),$$
where $\eta : \mathbb{R}_+ \to \mathbb{R}$ is differentiable with
$$\int_0^\infty |\eta'(s)|(1+s)^n\,ds < \infty \quad \text{and} \quad \lim_{s\to\infty} \eta(s) = 0. \qquad (*)$$
We want to write η as a continuous linear combination



$$\eta = \int_0^\infty \varphi(a)\,\chi_{(0,a)}\,da,$$
with coefficients $\varphi(a)$. This means $\eta(s) = \int_s^\infty \varphi(a)\,da$,
i.e. $\varphi(s) = -\eta'(s)$. We get
$$|P_t f(x)| \lesssim \int_{\mathbb{R}^n} \frac{1}{t^n}\,\underbrace{\eta\left(\frac{|x-y|}{t}\right)}_{= -\int_0^\infty \eta'(a)\,\chi_{B(x;at)}(y)\,da} |f(y)|\,dy$$
$$= -\int_0^\infty \eta'(a)\left(\frac{1}{t^n}\int_{B(x;at)} |f(y)|\,dy\right) da$$
Now consider $N_1(P_t f)(z)$ and $|x-z| < t$.
If $|y-x| < at$, then $|y-z| < (a+1)t$,
so
$$N_1(P_t f)(z) \lesssim \int_0^\infty |\eta'(a)|\,(a+1)^n\,Mf(z)\,da$$



$$\lesssim Mf(z)$$
$\therefore$ For any Poisson-like extension of f satisfying $(*)$, we have
$$N_1(P_t f)(z) \lesssim Mf(z) \text{ pointwise, hence}$$
$$\|N_*(P_t f)\|_{L_p} \lesssim \|f\|_{L_p} \text{ for } 1 < p \leq \infty \text{ by Thm 9.4.}$$

Next we consider some examples of Carleson measures.

Ex. 9.11: Consider a measure $d\mu(t,x) = a(t,x)\,dt\,dx$, for some function $a : \mathbb{R}^{1+n}_+ \to \mathbb{R}_+$.

- If $a(t,x) = \frac{1}{t}$, then μ is not a Carleson measure, since $\int_0^\infty \frac{dt}{t}$ is divergent.
- If $a(t,x) = \frac{1}{|(t,x)|} = \frac{1}{\sqrt{t^2+|x|^2}}$, then μ is a Carleson measure

An example of a Carleson measure to have in mind is therefore $d\mu = b(t,x)\frac{dt\,dx}{t}$, where $b = 0$ at $t=0$ and ∞ "almost everywhere" in a way quantified by $\|\mu\|_C < \infty$.

We next describe the usual way Carleson measures arise in harmonic analysis.

Defn. 9.12: Let $f \in L_1^{loc}(\mathbb{R}^n)$. We say that $f \in BMO(\mathbb{R}^n)$ (bounded mean oscillation) if

$$\|f\|_{BMO} := \sup_{\substack{Q \subset \mathbb{R}^n \\ \text{cube}}} \Big(\frac{1}{|Q|} \int_Q |f(y) - \underbrace{E_Q f}_{\text{mean value}}|^2 dy \Big)^{1/2} < \infty.$$

We remark that

- $L_\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$: $\|f\|_{BMO} \lesssim \|f\|_{L_\infty(\mathbb{R}^n)}$.
- $\|f\|_{BMO} = 0$ if $f = $ constant, so BMO is really a B-space of functions modulo constants. (Thus we should speak of a map $L_\infty \to BMO : f \mapsto f + \{\text{const.}\}$.)

Ex. 9.13: Similar to Ex 9.10, we consider, for functions $f : \mathbb{R}^n \to \mathbb{R}$, the function

$$(Q_t f)(x) = \int_{\mathbb{R}^n} q_t(x,y) f(y)\,dy \quad , \quad (t,x) \in \mathbb{R}^{1+n}_+.$$

We here assume

$$q_t(x,y) = \frac{1}{t^n} \varphi\Big(\frac{x-y}{t}\Big)$$

for some $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\boxed{\int_{\mathbb{R}^n} \varphi(t)\,dt = 0}.$$

- We first prove "square function estimates":

$$\iint_{\mathbb{R}^{1+n}_+} |Q_t f(x)|^2 \frac{dt\,dx}{t} \lesssim \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

Using the Fourier transform, this amounts to

$$\int_0^\infty \left(\int_{\mathbb{R}^n} |\hat\varphi(t\xi)\hat f(\xi)|^2 d\xi \right) \frac{dt}{t} \lesssim \int_{\mathbb{R}^n} |\hat f(\xi)|^2 d\xi$$

$$\Longleftrightarrow \sup_{\xi \in \mathbb{R}^n} \int_0^\infty |\hat\varphi(t\xi)|^2 \frac{dt}{t} < \infty.$$

By change of variable $t = s/|\xi|$, we have

$\int_0^\infty |\hat\varphi(\frac{\xi}{|\xi|} s)|^2 \frac{ds}{s}$, so we may assume $|\xi| = 1$.

The hypothesis ensures that $\hat\varphi \in \mathcal{S}(\mathbb{R}^n)$ ← Schwartz class

and $\boxed{\hat\varphi(0) = 0}$.

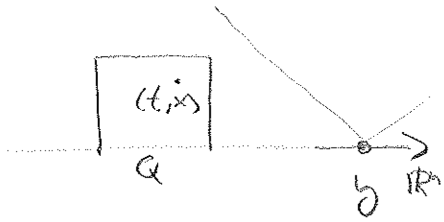
From this the square function estimate follows.

- Now let $f \in BMO(\mathbb{R}^n)$ and consider the measure $d\mu(t,x) := |Q_t f(x)|^2 \frac{dt\,dx}{t}$.

We claim $\|\mu\|_C \lesssim \|f\|_{BMO}$.

To see this, fix a cube Q and consider

$$\iint_{Q^{ca}} |Q_t f(x)|^2 \frac{dt\,dx}{t}.$$



Assume $\varphi = 0$ for $|x| > a$.

Then $q_t(x,y) = 0$ if $(t,x) \in Q^{ca}$ and $\operatorname{dist}(y, Q) > \ell(Q)\cdot a$.

Let $\tilde Q$ be the cube with same center as Q and $\ell(\tilde Q) = (2a+1)\cdot \ell(Q)$.

Then

$$\iint_{Q^{ca}} |Q_t f(x)|^2 \frac{dt\,dx}{t} = \iint_{Q^{ca}} |\overbrace{Q_t(f - \underbrace{E_{\tilde Q} f}_{\text{const.}})}^{\int = 0}|^2 \frac{dt\,dx}{t}$$

$$= \iint_{Q^{ca}} |Q_t((f - E_{\tilde Q} f)\chi_{\tilde Q})|^2 \frac{dt\,dx}{t} \lesssim \Big/ \text{square function estimate} \Big/$$

$$\lesssim \int_{\mathbb{R}^n} |(f-E_{\tilde{Q}}f)\chi_{\tilde{Q}}|^2 dx \le |\tilde{Q}|\cdot\|f\|_{BMO}^2 \lesssim |Q|\cdot\|f\|_{BMO}^2$$

This shows that

$$BMO(\mathbb{R}^n) \longrightarrow \{\text{Carleson meas. on } \mathbb{R}^{1+n}_+\} : f \longmapsto |Q_t f(x)|^2 \frac{dt\,dx}{t}$$

is a bounded map.

We end this section by noting the following corollary to Carleson's theorem 9.9.

<u>Cor 9.14</u>: Let $\|\mu\|_C < \infty$, let P_t be as in Ex. 10.10 and let $1 < p \le \infty$. Then

$$\int_{\mathbb{R}^n} |P_t f(x)|^p d\mu(t,x) \lesssim \|\mu\|_C \int_{\mathbb{R}^n} |f(x)|^p dx.$$

In particular, if Q_t is as in Ex. 9.13, then

$$\int_{\mathbb{R}^n} |Q_t b(x)|^2 \cdot |P_t f(x)|^2 \frac{dt\,dx}{t} \lesssim \|b\|^2_{BMO(\mathbb{R}^n)} \|f\|^2_{L_2(\mathbb{R}^n)}$$

<u>Proof</u>: What possibly remains to be seen is that

$$\int_{\mathbb{R}^n} |N_*(|P_t f|^p)|\,dx = \int_{\mathbb{R}^n} |N_*(P_t f)|^p dx$$
$$\lesssim \int_{\mathbb{R}^n} |M(f)|^p dx \lesssim \int_{\mathbb{R}^n} |f|^p dx. \quad \boxtimes$$

<u>Perfect dyadic SIOs</u>:

<u>Defn. 9.15</u>: A SIO kernel $k(x,y)$ on $\mathbb{R}^n$ is called <u>perfect dyadic</u> if

$k|_{Q\times R} = \text{constant}$ whenever $Q, R \in \mathbb{D}$ are disjoint.

There is an equivalent characterization:

$k(x,y) = \text{constant}$ for $x \in Q$, $y \in R$ whenever $Q, R \in \mathbb{D}$ are siblings.

Indeed, consider the lowest upper bound $\mathrm{lub}(Q,R)$, and let $\tilde{Q} \supset Q$ and $\tilde{R} \supset R$

be children of $\mathrm{lib}(Q,R)$.
Then $k_{\tilde{Q}\times\tilde{R}} = \text{const.} \Rightarrow k_{Q\times R} = \text{const.}$

Exc. 9.16: Take an A4 paper (with vertical + horizontal lines 5×5 mm) and put the x,y origin in the middle. Draw the diagonal $x=y$ and the set of all points where a perfect dyadic SIO kernel may be discontinuous (as accurately as possible). Use units $1 = 1.5$ cm, and the dyadic intervals from Defn. 9.1.

- Then do the same exercise for the dyadic intervals from Defn. 8.6. (Use here units $1 = 1$ cm, and Defn. 9.15). What is the advantage of Defn. 9.1?

It should be clear from Exc. 9.16 that Defn. 9.15 is a dyadic analogue of Defn. 8.4. The importance of these additional regularity assumptions on k is that they imply certain improved local behaviour of T.

Lemma 9.17: Let $T: H^s_c(\mathbb{R}^n) \to H^s_c(\mathbb{R}^n)^*$ be a perfect dyadic SIO. If $R, Q \in \mathbb{D}$ are disjoint, then $Tf|_R = 0$ if $\operatorname{supp} f \subset Q$ and $\int_Q f = 0$.

Proof: If $\operatorname{supp} g \subset R$, then

$$\langle g, Tf\rangle = \iint_{R\times Q} g(x)\underbrace{k(x,y)}_{= c = \text{const.}} f(y)\,dx\,dy = c \cdot \int_R g\,dx \underbrace{\int_Q f\,dy}_{=0}$$

$$= 0. \ \blacksquare$$

This local behaviour allow us to extend the domain of definition of T from $H^s_c(\mathbb{R}^n)$ to $H^s_{loc}(\mathbb{R}^n) = \{f\,;\ \varphi f \in H^s(\mathbb{R}^n),\ \forall \varphi \in C_0^\infty(\mathbb{R}^n)\}$.
Let $b \in H^s_{loc}(\mathbb{R}^n)$ be given. Consider some bounded open set $\Omega \subset \mathbb{R}^n$. We want to define $\langle \varphi, Tb\rangle$ for any test function $\varphi \in H^s(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \Omega$.
This will show $Tb \in H^{-s}_{loc}(\mathbb{R}^n)$.

Let $Q \in \mathbb{D}$ be such that $Q \supset \Omega$.
Assume that $b_i \in H^s(\mathbb{R}^n)$, supp b_i is compact, and $b_i = b$ on Q, for $i=1,2$.
If $R \in \mathbb{D}$, $R \cap Q = \emptyset$, then $\chi_R b_i \in H^s(\mathbb{R}^n)$ by Exc. 8.3 (assuming $s<\frac{1}{2}$). Lemma 9.17 gives

$$\langle \varphi, Tb_1 - Tb_2 \rangle = \langle T^*\varphi, b_1 - b_2 \rangle$$
$$= \sum_{\substack{R : R\cap Q = \emptyset \\ |R| = |Q|}} \langle T^*\varphi, \chi_R(b_1 - b_2)\rangle = 0 \quad \underline{\text{if}} \quad \int_Q \varphi = 0,$$

finite!

since T^* is also a perfect dyadic SIO.

Thus $Tb_1|_\Omega = Tb_2|_\Omega + c$ for some constant c.

<u>Defn 9.18</u>: Let $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ be a perfect dyadic SIO. Extend T to an operator
$$T : H^s_{loc}(\mathbb{R}^n) \longrightarrow H^{-s}_{loc}(\mathbb{R}^n)/\{\text{constants}\}, \text{ setting}$$
$$Tb := \lim_{R\to\infty} T(b\chi_{|x|<R}).$$
The above argument shows that the exact choice of approximating sequence $\{b\chi_{|x|<R}\}_{R>0}$ to b is not important. In fact, it is clear that
$$(Tb)|_Q = (T(b\chi_Q))|_Q \quad (\text{mod consts}) \quad \text{for } Q \in \mathbb{D}.$$

Recall our main problem:
Under what hypothesis does
$$T : H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$$
extend to
$$T : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \ ?$$
We here show a necessary condition, namely $T(1) \in BMO(\mathbb{R}^n)$, which we shall see in the next lecture is essentially sufficient also.

We need the following dyadic analogue of Defn 9.12.

<u>Defn. 9.19</u>: If $f \in L_1^{loc}(\mathbb{R}^n)$, we say that $f \in BMO_d(\mathbb{R}^n)$ (dyadic BMO) if

$$\|f\|_{BMO_d(\mathbb{R}^n)} := \sup_{Q \in \mathbb{D}} \left(\frac{1}{|Q|} \int_Q |f(y) - E_Q f|^2 dy \right)^{1/2} < \infty.$$

<u>Prop. 9.20</u>: Let $T : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ be a perfect dyadic SIO (that is, T is assumed to extend to a bounded L_2-operator). Then

$$Tb := \lim_{R \to \infty} T(b \chi_{|x|<R})$$

defines a bounded operator

$$T : L_\infty(\mathbb{R}^n) \to BMO_d(\mathbb{R}^n).$$

If furthermore $T(1) = 0$ (mod constants), then we have a bounded operator

$$T : BMO_d(\mathbb{R}^n) \to BMO_d(\mathbb{R}^n).$$

<u>Proof</u>: Fix $Q \in \mathbb{D}$ and note that

$$Tb = T(b\chi_Q) \text{ mod consts on } Q.$$

Thus

$$\int_Q |Tb - E_Q(Tb)|^2 dx = \int_Q |T(b\chi_Q) - E_Q(T(b\chi_Q))|^2 dx$$

$$\lesssim \int_Q |T(b\chi_Q)|^2 dx \leq \int_{\mathbb{R}^n} |T(b\chi_Q)|^2 dx$$

$$\underset{\substack{\uparrow \\ L_2\text{-bounded}}}{\lesssim} \int_{\mathbb{R}^n} |b\chi_Q|^2 dx = \int_Q |b|^2 dx \leq |Q| \cdot \|b\|_\infty^2.$$

This proves $T : L_\infty \to BMO_d$.

Assume now $T(1) = 0$. Then we can replace $T(b\chi_Q)$ in the above calculation by $T(b\chi_Q - (E_Q b)\chi_Q)$ since $T(\chi_Q) = T(1) = 0$ mod constants on Q. This gives

$$\int_Q |Tb - E_Q(Tb)|^2 dx \lesssim \int_Q |b - E_Q b|^2 dx \leq |Q| \cdot \|b\|_{BMO}^2. \quad \boxtimes$$

Recall from Exc. 7.14 that a SIO is not in general bounded on $L_\infty(\mathbb{R}^n)$.

Recall from Ex. 9.13 that there is a close link between BMO and Carleson measures. By Exc. 9.6, $\|\mu\|_{C_d} \approx \|\mu\|_C$. However, the corresponding result for BMO is not true. We clearly have

$$\|b\|_{BMO_d} \le \|b\|_{BMO} \quad \text{and} \quad BMO(\mathbb{R}^n) \subset BMO_d(\mathbb{R}^n),$$

but the converse is not true.

Exc. 9.21:
- Consider $BMO(\mathbb{R}^n)$ from Defn. 9.12.
  The standard example of $f \in BMO \setminus L_\infty$ is
  $$f_1(x) = \ln|x|, \quad x \in \mathbb{R}.$$
  Show that $f_1 \in BMO(\mathbb{R})$
- Unlike L_∞, BMO-functions cannot in general be "cut". Show that
  $$f_2(x) = \begin{cases} \ln x, & x > 0 \\ 0, & x < 0 \end{cases}$$ does not belong to $BMO(\mathbb{R})$.
- Consider $BMO_d^0(\mathbb{R})$ defined using standard dyadic intervals from Defn. 8.6. Show that $f_2 \in BMO_d^0(\mathbb{R})$.
- Consider finally $BMO_d(\mathbb{R})$ from Defn. 9.19 (using dyadic intervals from Defn. 9.1. Show that $f_2 \notin BMO_d(\mathbb{R})$. Modify f_2 appropriately and construct $f_3 \in BMO_d(\mathbb{R}) \setminus BMO(\mathbb{R})$.

A classical result for BMO is the John-Nirenberg inequality:

Thm 9.22: There are constants $0 < a, c < \infty$ such that

$$\sup_{\text{cubes } Q} \frac{1}{|Q|} \int_Q |f - E_Q f|\,dx \le K \Rightarrow$$

$$\sup_{\substack{\text{cubes } Q \\ \lambda > 0}} \frac{1}{|Q|} e^{\frac{a}{K}\lambda} |\{x \in Q; |f(x) - E_Q f| > \lambda\}| \le c.$$

Note that the strong version of the RHS, i.e.

$$\int_Q e^{\frac{a}{K}|f(x) - E_Q f|}\,dx \le c|Q|,$$

fails for the function $f(x) = \ln|x| \in BMO(\mathbb{R})$.

We start this lecture by extending the theory 8.8–13 of Haar functions to $\mathbb{R}^n$. It is here more convenient to focus on the projection operators E_j, Δ_j from Defn. 8.10, rather than the Haar functions h_Q from Defn 8.8. (We shall not define Haar functions in $\mathbb{R}^n$, $n \geq 2$.)

<u>Defn 10.1</u>: Let $\mathbb{D}$ be the dyadic cubes in $\mathbb{R}^n$ from Defn 9.1. Let $f \in L_1^{loc}(\mathbb{R}^n)$.

- The <u>average of f</u> on $Q \in \mathbb{D}$ is
$$E_Q f := \frac{1}{|Q|} \int_Q f(x)\,dx$$
- For $j \in \mathbb{Z}$, define the expectation operator
$$(E_j f)(x) := \sum_{Q \in \mathbb{D}_j} (E_Q f)\, \chi_Q(x), \quad x \in \mathbb{R}^n.$$
- For $j \in \mathbb{Z}$, define the martingale difference operator
$$(\Delta_j f)(x) := (E_{j+1} f)(x) - (E_j f)(x), \quad x \in \mathbb{R}^n.$$
- For $Q \in \mathbb{D}_j$, let
$$\Delta_Q f(x) := \Delta_j f(x) \cdot \chi_Q(x) = \begin{cases} (E_{j+1} f)(x) - E_Q f, & x \in Q \\ 0, & x \notin Q. \end{cases}$$

Define $L_2(\mathbb{R}^n)$-ranges:
$$V_j := R(E_j), \quad W_j := R(\Delta_j), \quad W_Q := R(\Delta_Q).$$

As before we have an increasing sequence of subspaces
$$\ldots \subset V_{-3} \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset \ldots \subset L_2(\mathbb{R}^n),$$
where $W_j = V_{j+1} \ominus V_j \; (= V_{j+1} \cap V_j^{\perp})$

and $W_j = \bigoplus_{Q \in \mathbb{D}_j} W_Q$

Since $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R}^n)$, we have
$$\boxed{L_2(\mathbb{R}^n) = \bigoplus_{Q \in \mathbb{D}} W_Q}$$

It is seen that
$$W_Q = \{ f \in L_2(\mathbb{R}^n) ;\ \operatorname{supp} f \subset Q,\ \int_Q f = 0 \text{ and } f|_{Q'} = \text{constant},\ \forall \text{children } Q' \text{ to } Q \}.$$

Thus for all $Q \in \mathbb{D}$:

$\dim W_Q = 2^n - 1.$

If $n=1$, we before had $\dim W_Q = 1$ and $W_Q = \mathrm{span}(h_Q)$.

When $n \geq 2$, we can choose a basis $\{h_Q^1, \dots, h_Q^{2^n-1}\}$ for W_Q, but there is no canonical way to do this.

Instead we adopt the following point of view: given $f \in L_2(\mathbb{R}^n)$, write

$$f = \sum_{Q \in \mathbb{D}} \Delta_Q f \quad , \quad \text{where } \Delta_Q f \in W_Q.$$

For two cubes $Q, R \in \mathbb{D}$, there is a canonical isometry

$$W_Q \to W_R : f(x) \longmapsto \sqrt{\frac{|Q|}{|R|}}\, f\left(x_Q + \frac{\ell(Q)}{\ell(R)}(x - x_R)\right),$$

where x_Q, x_R denotes the centers of the cubes.

In this way, we obtain an isometry

$$L_2(\mathbb{R}^n) = \ell_2(\mathbb{D}; W_{Q_0}), \qquad Q_0 \in \mathbb{D} \text{ fixed.}$$

<u>Paraproducts</u>:

We shall now show that there is a close relationship between perfect dyadic SIOs T and multiplication operators

$$L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) : f(x) \longmapsto b(x) f(x).$$

Let us expand both the multiplier b and the variable function f in the above "vector-valued Haar basis", and multiply termwise:

$$b \cdot f = \Big(\sum_{Q \in \mathbb{D}} \Delta_Q b\Big)\Big(\sum_{R \in \mathbb{D}} \Delta_R f\Big) = \sum_Q (\Delta_Q b) \cdot (\Delta_Q f)$$

$$+ \sum\sum_{Q \subsetneq R} (\Delta_Q b) \cdot (\Delta_R f) + \sum\sum_{Q \supsetneq R} (\Delta_Q b)(\Delta_R f) + \cancel{\sum\sum_{Q \cap R = \emptyset} (\Delta_Q b)(\Delta_R f)} \to 0$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

III: We can rewrite this term

$$\mathrm{III} = \sum\sum_{j<k} (\Delta_j b)(\Delta_k f) = \sum_k \Big(\sum_{j=-\infty}^{k-1} \Delta_j b\Big) \Delta_k f = \sum_k (E_k b)(\Delta_k f)$$

$$= \sum_{Q \in \mathbb{D}} (E_Q b)(\Delta_Q f).$$

Note that $E_Q b$ is constant on $Q \supset \operatorname{supp} \Delta_Q f$.
In the $\ell_2(\mathbb{D}; W_{Q_0})$ picture $\underline{III}$ multiplies the vector $\Delta_Q f \in W_Q \approx W_{Q_0}$ by the constant $E_Q b$.

<u>Defn. 10.2</u>: Given $m : \mathbb{D} \to L(W_{Q_0})$, i.e. to each $Q \in \mathbb{D}$ there is a linear map $W_Q \to W_Q$, we define the Haar multiplier

$$\Pi_m^0(f)(x) := \sum_{Q \in \mathbb{D}} m_Q \Delta_Q f(x).$$

operator norm

It is clear that $\| \Pi_m^0 \|_{L_2 \to L_2} = \sup_{Q \in \mathbb{D}} |m_Q|$.
Note that the term $\underline{III}$ above is the Haar multiplier with the symbol $m : Q \mapsto E_Q b$. (Here each matrix m_Q is a scalar multiple of the identity.)

<u>Prop. 10.3</u>: If $\sup_{Q \in \mathbb{D}} |m_Q| < \infty$, then Π_m^0 defines an L_2-bounded perfect dyadic SIO, with $\| \Pi_m^0 \|_{L_2 \to L_2} = \| m \|_\infty$.

<u>Proof</u>: Fix $Q_0 \in \mathbb{D}$ and consider the kernel $k_Q(x,y)$ of $m_Q \Delta_Q$. If $\{h_{Q_0}^k\}_{k=1}^{2^n-1}$ is an ON-basis for W_{Q_0}, then $h_Q^k(x) := \sqrt{\frac{|Q_0|}{|Q|}}\, h_{Q_0}^k\left(x_{Q_0} + \frac{\ell(Q_0)}{\ell(Q)}(x - x_Q)\right)$ are ON-basis for W_Q. We see that

$$|k_Q(x,y)| = \left| \sum_{i,j=1}^{2^n-1} m_Q^{ij} h_Q^i(x) h_Q^j(y) \right| \lesssim \frac{1}{\sqrt{|Q|}} \cdot \frac{1}{\sqrt{|Q|}} = \frac{1}{|Q|}$$

and $k_Q(x,y) = 0$ unless $x, y \in Q$.
For the kernel $k(x,y)$ of $\Pi_m^0 = \sum_Q m_Q \Delta_Q$, we thus have

$$|k(x,y)| \lesssim \sum_{Q : x,y \in Q} \frac{1}{|Q|} \lesssim \sum_{k=0}^{\infty} \frac{1}{(2^k |x-y|)^n} \approx \frac{1}{|x-y|^n}.$$

Furthermore, if Q_1, Q_2 are siblings, with parent Q_0, then if

$$k(x,y) = \sum_{\substack{R \in \mathbb{D} \\ R \supset Q_0}} k_R(x,y) = \text{constant for } x \in Q_1,\ y \in Q_2$$

since h_R^k are constant on Q:s children. ∎

(II): we can rewrite this term

$$\mathrm{II} = \sum\sum_{j>k} (\Delta_j b)\cdot(\Delta_k f) = \sum_j \Delta_j b \Big(\sum_{k=-\infty}^{j-1} \Delta_k f\Big) = \sum_j (\Delta_j b)(E_j f)$$

$$= \sum_Q (\Delta_Q b)(E_Q f).$$

<u>Defn 10.4</u>: Given $b \in BMO_d(\mathbb{R}^n)$, the (dyadic) <u>paraproduct</u> with b is the operator

$$\pi_b^+(f)(x) = \sum_Q \Delta_Q b(x) \cdot (E_Q f).$$

To estimate π_b^+, we use the following.

<u>Lemma 10.5</u>: Let $Q \in \mathbb{D}$. Then

$$\int_Q |f(x) - E_Q f|^2 dx = \sum_{\substack{R \in \mathbb{D} \\ R \subset Q}} |\Delta_R f|_2^2.$$

L_2-norm on the 2^n-1-dimensional space $W_R \subset L_2(\mathbb{R}^n)$.

Equivalently $L_2(Q) = \bigoplus_{\substack{R \in \mathbb{D} \\ R \subset Q}} W_R \oplus \operatorname{span}\{\chi_Q\}$.

<u>Proof</u>: If $Q \in \mathbb{D}_j$, write $\sum_{k=j}^{\infty} \Delta_k f = f - E_j f$.

Restricting to Q gives $\sum_{\substack{R \in \mathbb{D} \\ R \subset Q}} \Delta_R f = f|_Q - E_Q f$. ∎

<u>Prop. 10.6</u>: If $b \in BMO_d(\mathbb{R}^n)$, then π_b^+ is an L_2-bounded perfect dyadic SIO, with

$$\|\pi_b^+\|_{L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)} \lesssim \|b\|_{BMO_d(\mathbb{R}^n)}.$$

<u>Proof</u>: • We first prove the L_2-estimate.

Partition $\mathbb{R}^{1+n}_+$ into Whitney regions $\mathbb{R}^{1+n}_+ = \bigcup_{Q \in \mathbb{D}} Q^w$, and define the measure

$d\mu = a(t,x)\, dt\, dx$, where

$$a(t,x) := \frac{1}{|Q^w|} \cdot |\Delta_Q b|_2^2 \text{ on } Q^w.$$

Using Exc. 9.6. we have

$$\|\mu\|_C \approx \|\mu\|_{C_d} = \sup_{Q \in \mathbb{D}} \frac{1}{|Q|} \sum_{R \subset Q} \int_{R^w} \frac{1}{|R^w|} |\Delta_R b|_2^2 \, dt\, dx$$

$$= \sup_{Q\in\mathcal{D}} \frac{1}{|Q|} \sum_{R\subset Q} |\Delta_R b|_2^2 = /\text{lem. } 10.5/ = \sup_{Q\in\mathcal{D}} \frac{1}{|Q|} \int_Q |b - E_Q b|^2 dx$$

$$= \|b\|_{BMO_d}.$$

Now note that $E_Q f$ is constant on $\operatorname{supp} \Delta_Q b$, so

$$\|\pi_b^+ f\|_2^2 = \sum_Q \underbrace{|\Delta_Q b|_2^2}_{\text{norm in } W_Q} \cdot \underbrace{|E_Q f|^2}_{\text{absolute value}} \quad \text{since } \Delta_Q b \cdot E_Q f \in W_Q.$$

In order to apply Cor. 9.14, let P_t from Ex. 9.10 be the dyadic averaging operator. More precisely, let

$$(P_t f)(x) := E_Q f \quad \text{when } (t,x) \in Q^W$$

Then P_t satisfies the estimate in Ex. 9.10 and we get

$$\sum_Q |\Delta_Q b|_2^2 \cdot |E_Q f|^2 = \sum_{Q\in\mathcal{D}} \int_{Q^W} \underbrace{|E_Q f|^2}_{= |P_t f|^2} d\mu \lesssim \|b\|_{BMO}^2 \cdot \|f\|_{L_2(\mathbb{R}^n)}^2$$

This proves $\|\pi_b^+\|_{L_2\to L_2} \lesssim \|b\|_{BMO_d}$

- Next consider the kernel of $\Delta_Q b\, E_Q$.

With notation as in the proof of Prop. 10.3, we have

$$k_Q(x,y) = \sum_{k=1}^{2^n-1} \langle b, h_Q^k\rangle h_Q^k(x) \frac{1}{|Q|} \chi_Q(y),$$

so that $|k_Q(x,y)| \lesssim \underbrace{|\Delta_Q b|_2}_{\le \sqrt{|Q|}\cdot \|b\|_{BMO_d}} \frac{1}{\sqrt{|Q|}} \cdot \frac{1}{|Q|} \lesssim \frac{1}{|Q|}$

and $k_Q(x,y) = 0$ unless $x, y \in Q$.

$$\Rightarrow |k(x,y)| \le \sum_{\substack{Q\in\mathcal{D}:\\ x,y\in Q}} \frac{1}{|Q|} \lesssim \frac{1}{|x-y|^n}.$$

Since $k_R(x,y)$ is constant for $x\in Q_1$, $y\in Q_2$, where $R \supset Q$ = parent to the siblings Q_1, Q_2, it follows that π_b^+ is a perfect dyadic SIO. ∎

I: This term is

$$I = \sum_{Q\in\mathcal{D}} (\Delta_Q b)\cdot(\Delta_Q f) = \sum_{j\in\mathbb{Z}} (\Delta_j b)\cdot(\Delta_j f).$$

We shall relate I to the following adjoint paraproduct:

<u>Defn. 10.7</u>: For $b \in BMO_d(\mathbb{R}^n)$, let

$$\pi_b := (\pi_b^+)^*.$$

Fix $Q \in \mathbb{D}$ and $f, g \in L_2(\mathbb{R}^n)$, and calculate

$$\langle (\Delta_Q b)(\Delta_Q f), g \rangle = \langle (\Delta_Q b)(\Delta_Q f), E_Q g + \sum_{\substack{R \in \mathbb{D} \\ R \subset Q}} \Delta_R g \rangle$$

$$= \langle \Delta_Q f, \underbrace{(\Delta_Q b)(E_Q g)}_{\in W_Q} \rangle + \langle (\Delta_Q b)(\Delta_Q f), \Delta_Q g \rangle + \sum_{\substack{R \in \mathbb{D} \\ R \subsetneq Q}} \underbrace{\langle \underbrace{\Delta_Q b \Delta_Q f}_{= \text{const. on supp } \Delta_R g}, \Delta_R g \rangle}_{0}$$

$$= \langle f, (\Delta_Q b)(E_Q g) \rangle + \langle f, (\Delta_Q \underbrace{M_{\Delta_Q b}}_{\text{multiplication operator on } L_2(\mathbb{R}^n)} \Delta_Q) g \rangle$$

We obtain the following

<u>Prop. 10.8</u>: For $Q \in \mathbb{D}$, define linear maps $m_Q^1, m_Q^0 \in \mathcal{L}(W_Q)$ by

$$(m_Q^1 f)(x) := (E_Q b)(x) \cdot f(x)$$
$$(m_Q^0 f) := \Delta_Q(\Delta_Q b \cdot f).$$

Then $m_Q := m_Q^1 + m_Q^0 : f \longmapsto \Delta_Q(E_{j+1} b \cdot f)$ if $Q \in \mathbb{D}_j$, and

$$\sum_j \Delta_j b \cdot \Delta_j f = \Pi_b^- f + \Pi_m^0 f.$$

Thus

$$b \cdot f = \Pi_m^0(f) + \Pi_b^+(f) + \Pi_b^-(f),$$

with $m = (m_Q)_{Q \in \mathbb{D}}$ as above.

<u>Proof</u>: This is clear from the above discussion. Only note that

$$E_Q b \cdot f + \Delta_Q(\underbrace{\Delta_Q b}_{\substack{\text{constant on } Q\text{'s children} \\ \& \int_Q = 0}} f) = \Delta_Q(\underbrace{E_{j+1} b}_{\text{constant on } Q\text{'s children}} \cdot f), \quad f \in W_Q.$$

<u>Exc. 10.9</u>: Show that if $n=1$ (one-dimensional case), then $m_Q^0 = 0$ in Prop. 10.8 for any b. (So that Π_m^0 is the scalar Haar multiplicator from term $\overline{\underline{III}}$ only). For $n=2$, give example which shows that m_Q^0 may be non-zero.

To summerize, the pointwise multiplication operator $f \mapsto b \cdot f$ can be written as the sum of three operators:

$$b \cdot f = \pi_m^0 f + \pi_b^+ f + \pi_b^- f$$

= Haar multiplier + paraproduct + adjoint paraproduct.

Here $\|\pi_m^0\|_{L_2 \to L_2} \lesssim \|b\|_\infty$

$\|\pi_b^\pm\|_{L_2 \to L_2} \lesssim \|b\|_{BMO} \lesssim \|b\|_\infty$.

Coming to the relation between these operators and perfect dyadic SIOs, we note the following mapping properties.

$\underline{\pi_m^0}$: For $f_Q \in W_Q$, $g_R \in W_R$, $Q, R \in \mathbb{D}$, we have

$$\langle \pi_m^0 f_Q, g_R \rangle = 0 \text{ unless } R = Q.$$

$\underline{\pi_b^+}$: For the paraproduct, we have

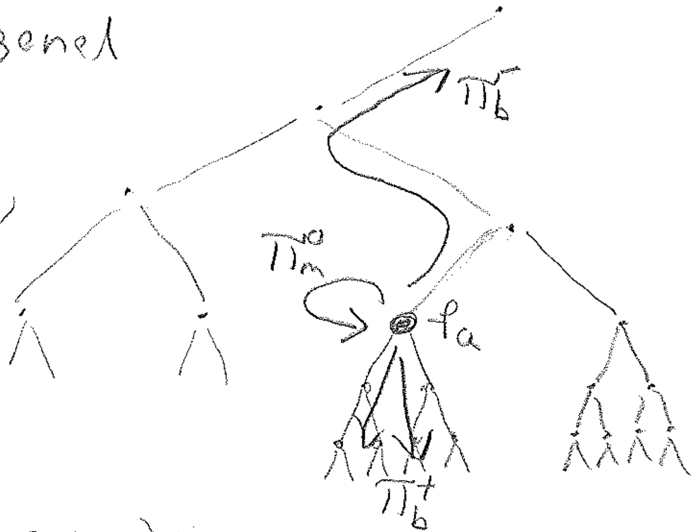
$$\langle \pi_b^+ f_Q, g_R \rangle = \langle \underbrace{\Delta_R b \, E_R(f_Q)}_{\in W_R}, g_R \rangle$$

$$= \langle b, g_R \rangle E_R(f_Q) = 0 \text{ unless } R \subsetneqq Q.$$

$\underline{\pi_b^-}$: For the adjoint paraproduct, we have

$$\langle \pi_b^- f_Q, g_R \rangle = \langle f_Q, \pi_b^+ g_R \rangle = 0 \text{ unless } R \supsetneqq Q.$$

Thus, using the orthogonal decomposition $L_2(\mathbb{R}^n) \approx \ell_2(\mathbb{D}, W_{Q_0})$, we have the following picture.



That is, π_m^0 is "diagonal", π_b^+ maps down into the subtree under Q and π_b^- maps up along the branch above Q. Note that none of these operators maps f_Q to a cube R not related to Q, i.e. $R \cap Q = \emptyset$.

Recall from Lemma 9.17 that all perfect dyadic SIOs have this mapping property. More precisely, we have the following main result for perfect dyadic SIOs: the "T(1) theorem".

Thm 10.10: Let $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ be a perfect dyadic SIO, for some $0 < s < \frac{1}{2}$, $t > 0$. Define $T(1), T^*(1) \in H^{-s}_{loc}(\mathbb{R}^n)$ as in Defn. 9.18, and let $m = (m_Q)_{Q \in \mathcal{D}}$, $m_Q \in L(W_Q)$, be such that
$$\langle m_Q f, g \rangle = \langle Tf, g \rangle \quad , \quad f, g \in W_Q.$$
Assume that

- $T(1), T^*(1) \in BMO_d(\mathbb{R}^n)$
- $\sup_{Q \in \mathcal{D}} |m_Q| < \infty$, $\quad (L_2(W_Q) \to L_2(W_Q)$ norm$)$

Then T extends to a bounded operator $T: L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ and
$$T = \Pi^0_m + \Pi^+_{T(1)} + \Pi^-_{T^*(1)}.$$

Proof: By the discussion above, it suffices to show for $f_Q \in W_Q$, $g_R \in W_R$, $Q, R \in \mathcal{D}$ that

(1) $\langle T f_Q, g_R \rangle = \langle \Pi^0_m f_Q, g_R \rangle$ if $R = Q$

(2) $\langle T f_Q, g_R \rangle = \langle \Pi^+_{T(1)} f_Q, g_R \rangle$ if $R \subsetneq Q$

(3) $\langle T f_Q, g_R \rangle = \langle \Pi^-_{T^*(1)} f_Q, g_R \rangle$ if $R \supsetneq Q$.

(1) is obvious from the definition of m.

To see (2):
$$\text{RHS} = \langle \underbrace{\Delta_R(T(1)) \cdot E_R f_Q}_{\in W_R}, g_R \rangle = \langle T(1), g_R \rangle E_R f_Q$$
$= \langle T(\chi_R), g_R \rangle E_R f_Q$ by the remark after Defn. 9.18, since $\int g_R = 0$.

$\text{LHS} = \langle f_Q, T^* g_R \rangle = / R \subsetneq Q /$
$$= \langle f_Q \cdot \chi_R, T^* g_R \rangle = \langle T(\chi_R), g_R \rangle E_R f_Q = \text{RHS}.$$

Note that $f_Q(x) = E_R f_Q$ = constant on R.

To see (3):

$\langle T f_Q, g_R \rangle = \langle f_Q, T^* g_R \rangle = /(2)/ = \langle f_Q, \Pi^*_{T^*(1)} g_R \rangle$

$= \langle \Pi_{T^*(1)} f_Q, g_R \rangle$ if $R \supsetneq Q$. ∎

Note that Thm 10.10 is very sharp:
<u>if</u> T extends to an L_2-bounded operator, then clearly $\sup_{Q\in\mathbb{D}} |m_Q| < \infty$, and

$T(1), T^*(1) \in BMO_d(\mathbb{R}^n)$ by Prop. 9.20.

- Now consider the condition $\sup_{Q\in\mathbb{D}} |m_Q| < \infty$.
  Fixing a cube $Q_0 \in \mathbb{D}$, we have
  $|m_{Q_0}| < \infty$ since $W_{Q_0} \subset H^s_t(\mathbb{R}^n)$.
  Introducing the scaling / translation operator

  $$(A_Q f)(x) := \sqrt{\frac{|Q|}{|Q_0|}}\, f\left(x_Q + \frac{l(Q)}{l(Q_0)}(x - x_{Q_0})\right),$$

  we have an L_2-isometry $W_Q \to W_{Q_0}$.

<u>Defn 10.11</u>: Let $T : H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ be a SIO. We say that T is (dyadically) <u>weakly bounded</u>

$$\sup_{Q\in\mathbb{D}} \|A_Q T A_Q^{-1}\|_{H^s_t(\mathbb{R}^n)\to H^s_t(\mathbb{R}^n)^*} < \infty.$$

Note that <u>if</u> T is L_2-bounded, then

$$\|A_Q T A_Q^{-1}\|_{H^s_t \to H^s_t{}^*} \lesssim \|A_Q T A_Q^{-1}\|_{L_2\to L_2}$$
$$\le \|A_Q\|_{L_2\to L_2} \cdot \|T\|_{L_2\to L_2} \cdot \|A_Q^{-1}\|_{L_2\to L_2} = \|T\|_{L_2\to L_2}.$$

<u>Exc. 10.12</u>: Let k be an antisymmetric SIO kernel, and let T be the associated SIO from Prop. 8.2. Show that such T are weakly bounded.

<u>Prop. 10.13</u>: With notation as in Thm 10.13, if T is weakly bounded, then

$$\sup_{Q\in\mathbb{D}} |m_Q| < \infty.$$

<u>Proof</u>: Let $f, g \in W_{Q_0}$. We have

$$|\langle m_Q A_Q^{-1} f, A_Q^{-1} g \rangle| = |\langle T A_Q^{-1} f, A_Q^{-1} g\rangle| = |\langle A_Q T A_Q^{-1} f, g\rangle|$$

$\in W_Q$

$$\lesssim \|f\|_{H_t^s} \|g\|_{H_t^s} \approx \text{/uniformly in } f, g \in W_{Q_0} \text{ since } \dim W_{Q_0} < \infty /$$

$$\approx \|f\|_{L_2} \|g\|_{L_2} = \|A_Q^{-1} f\|_{L_2} \|A_Q^{-1} g\|_{L_2}$$

$\therefore \sup_{Q \in \mathbb{D}} |m_Q| < \infty$. ∎

In particular, for antisymmetric perfect dyadic SIOs, we have

$$T(1) \in BMO(\mathbb{R}^n) \implies T: L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \text{ bounded.}$$

Thus the action of T on the single constant function 1, determines whether T is L_2-bounded. Compare this with the situation for convolution SIOs, where boundedness of $\hat{k}(\xi)$ give L_2-boundedness.

For general perfect dyadic SIOs, it follows from the discussion above that

$$\|T\|_{L_2 \to L_2} \approx \sup_{Q \in \mathbb{D}} \|A_Q T A_Q^{-1}\|_{H_t^s \to H_t^{s*}} + \|T(1)\|_{BMO_d} + \|T^*(1)\|_{BMO_d}$$

The T(1) theorem for Hölder regular SIOs

After having studied the ideal situation of perfect dyadic SIOs in lecture 10, we now consider the "real" SIOs. We assume

- $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ for some $0<s<\frac{1}{2}$, $t>0$.
- T:s Schwartz kernel restricts to a SIO kernel $k(x,y)$ for $x \neq y$ as in Defn 7.1.
- k is a γ-Hölder regular SIO kernel as in Defn. 8.4, for some $0<\gamma<1$.

We start by adapting Defn 9.18 and Prop. 9.20 to Hölder regular SIOs.

To define $T(b)$, where $b \in H^s_{loc}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, consider a large ball $B(0;r)$ and assume $b = b_1 = b_2$ on $B(0,R)$, $R>2r$. If $\varphi \in L_2(B(0;r))$, $\operatorname{supp}\varphi \subset B(0,r)$, $\int \varphi = 0$, then

$$|\langle \varphi, Tb_1 - Tb_2\rangle| = |\langle \varphi, T(b_1-b_2)\rangle|$$

$= /$principle of almost orthogonality$/$

$$= \left| \int_{B(0,r)} \int_{B(0,R)^c} \varphi(x) \left(k(x,y) - \overset{\downarrow}{k(0,y)}\right) \left(b_1(y) - b_2(y)\right) dy\, dx \right|$$

$$\lesssim \iint_{B(0,r)\ B(0,R)^c} |\varphi(x)| \cdot \frac{|x|^\gamma}{|y|^{n+\gamma}} \|b_1 - b_2\|_{L_\infty(B(0,R)^c)}\, dy\, dx$$

$$\lesssim \|b_1 - b_2\|_{L_\infty(B(0,R)^c)} \frac{r^{\gamma+\frac{n}{2}}}{R^\gamma} \|\varphi\|_{L_2(B(0,r))}.$$

Thus

$$\|Tb_1 - Tb_2\|_{L_2(B(0,r))/\{\text{constants}\}} \lesssim \frac{r^{\gamma+\frac{n}{2}}}{R^\gamma} \|b_1 - b_2\|_{L_\infty(B(0,R)^c)}.$$

<u>Defn 11.1</u>: Let T be a SIO as above. Extend T to an operator $T: H^s_{loc} \cap L_\infty(\mathbb{R}^n) \to H^{-s}_{loc}(\mathbb{R}^n)/\{\text{constants}\}$, setting

$$T(b) := \lim_{R\to\infty} T(b \cdot \chi_{|x|<R}).$$

We note that the exact choice of approximating sequence $\{b \cdot \chi_{|x|<R}\}$ to b is not important.
As before, we have the following necessary condition for L_2-boundedness of T.

<u>Prop. 11.2</u>: Let T be a SIO as above. Assume that T extends to a bounded L_2-operator. Then
$$Tb := \lim_{R\to\infty} T(b\chi_{|x|<R})$$
defines a bounded operator
$$T: L_\infty(\mathbb{R}^n) \to BMO(\mathbb{R}^n).$$

We shall not prove the implication $T1=0 \Rightarrow T: BMO \to BMO$ since the proof is more elaborate than in Prop. 9.20, and since we shall not need this result.

<u>Proof</u>: Fix a cube $Q \subset \mathbb{R}^n$, with center x_Q.
Let $2Q$ be the cube with center x_Q, and $l(2Q) = 2l(Q)$.
Write $b = b\cdot\chi_{2Q} + b\chi_{(2Q)^c}$. For the first term we obtain as in Prop. 9.20 the estimate
$$\int_Q |T(b\chi_{2Q}) - E_Q(T(b\chi_{2Q}))|^2 dx \lesssim \int_{2Q} |b|^2 dx \lesssim |Q| \cdot \|b\|_\infty^2.$$
Replacing $b_1 - b_2$ by $b \cdot \chi_{(2Q)^c}$ in the calculation before Defn. 11.1, and adapting it, gives
$$\int_Q |T(b\chi_{(2Q)^c}) - E_Q(T(b\chi_{(2Q)^c}))|^2 dx = \|T(b\chi_{(2Q)^c})\|^2_{L_2(Q)/\text{const.}}$$
$$\lesssim |Q| \cdot \|b\|_\infty^2.$$
This proves $T: L_\infty \to BMO$. ∎

We also need the non-dyadic version of weak boundedness Defn. 10.11.

<u>Defn. 11.3</u>: Define L_2-isometries
$$(A_{t,x}f)(y) := t^{n/2} f(ty+x), \quad t>0,\ x\in\mathbb{R}^n.$$
We say that a SIO $T: H^s_c(\mathbb{R}^n) \to H^s_c(\mathbb{R}^n)^*$ is

weakly bounded if

$$\sup_{t>0, x\in\mathbb{R}^n} \| A_{t,x} T A_{t,x}^{-1} \|_{H^s_t(\mathbb{R}^n)\to H^s_t(\mathbb{R}^n)^*} < \infty.$$

Note:

(1) If T has SIO kernel $k(x,y)$, then $A_{t_0,x_0} T A_{t_0,x_0}^{-1}$ has SIO kernel $t_0^n\, k(t_0 x + x_0, t_0 y + x_0) =: k_{t_0,x_0}(x,y)$.
Weak boundedness means that we make a uniform hypothesis on all these SIOs.

(2) Exc. 10.12 extends to show that all antisymmetric SIOs are (non-dyadically) weakly bounded.

We now formulate our main result.

Thm 11.4 (The T1 theorem)
Let $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ be a SIO $(0<s<\frac{1}{2}, t>1)$.
Assume that
- the SIO kernel $k(x,y)$ of T is γ-Hölder regular as in Defn. 8.4 $(0<\gamma<1)$,
- T is weakly bounded as in Defn. 11.3,
- $T(1)$, $T^*(1) \in BMO(\mathbb{R}^n)$ as in Defn. 11.2.

Then T extends to a bounded operator
$$T: L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n).$$

Given our previous work, the proof of thm 11.4 is quite straightforward (but technical): we approximate T by a perfect dyadic SIO as in Thm 10.10, and estimate the error using γ-Hölder regularity and Schur estimates.
Before the proof, we give some examples.

Ex. 11.5: Let T be a SIO with an odd γ-Hölder regular convolution kernel $k(x,y) = k(x-y)$. To show L_2-boundedness it suffices to show $T1 = 0$.
Consider $T1|_{B(0,r)}$, for some ball $B(0,r)$.
For $|x| < r < R$, we have

$$|T(\chi_{|y|<R})(x)| = \left|\int_{|y|<R} k(x-y)\,dy\right| = \left|\int_{|t-x|<R} k(t)\,dt\right|$$

$$\leq \text{/k is odd/} \lesssim \int_{R-|x|<|t|<R+|x|} \frac{dt}{|t|^n} \approx \ln\frac{R+|x|}{R-|x|} \leq \ln\frac{R+r}{R-r}$$

Letting $R \to \infty$, we get $T1 = 0$ since r is arbitrary.
More generally, note the similarity between the condition on $T1$ in Thm 11.4 and the condition on $\int k(t)\,dt$ in Thm 7.18.

Ex 11.6: Consider the (non-convolution!) Calderón commutators C_k^φ from Ex. 7.10. For $k=0$, C_0^φ is the Cauchy integral on $\mathbb{R}$ from Ex. 7.7.
We now use this to prove C_1^φ is L_2-bounded.
To this end, we need to show $C_1^\varphi(1) \in BMO$.
For $|x| < r < R$, we have

$$C_1^\varphi(\chi_{|y|<R})(x) = \lim_{\varepsilon\to 0} \int_{\substack{|y|<R \\ |y-x|>\varepsilon}} \frac{\varphi(x)-\varphi(y)}{(x-y)^2}\,dy \quad \text{in } H_t^s(\mathbb{R}^n)^* \text{ by Prop. 8.2}$$

$$\int_{\substack{|y|<R \\ |y-x|>\varepsilon}} \frac{\varphi(x)-\varphi(y)}{(x-y)^2}\,dy = \text{/int. by parts/} = \left[\frac{\varphi(x)-\varphi(y)}{x-y}\right]_{x+\varepsilon}^{R} + \left[\frac{\varphi(x)-\varphi(y)}{x-y}\right]_{-R}^{x-\varepsilon} + \int_{\substack{|y|<R \\ |y-x|>\varepsilon}} \frac{\varphi'(y)}{x-y}\,dy$$

$$= \underbrace{\frac{\varphi(R)+\varphi(-R)}{R}}_{=:\,C_R \,=\, \text{constant in } x} + \underbrace{\left[\frac{\varphi(x)-\varphi(R)}{x-R} - \frac{\varphi(x)-\varphi(-R)}{x+R} - \frac{\varphi(R)+\varphi(-R)}{R}\right]}_{=\,\mathcal{O}(\frac{1}{R}) \text{ uniformly for } |x|<r.} +$$

$$\underbrace{\left[-\frac{\varphi(x)-\varphi(x+\varepsilon)}{-\varepsilon}+\frac{\varphi(x)-\varphi(x-\varepsilon)}{\varepsilon}\right]}_{\substack{\to -\varphi'(x)+\varphi'(x)=0 \\ \text{in } L_2(B(0,r)) \text{ as } \varepsilon\to 0.}} + \int_{\substack{|y|<R\\ |y-x|>\varepsilon}} \frac{\varphi'(y)}{x-y}\,dy$$

We get

$$C_1^\varphi(\chi_{|y|<R}) = c_R + O(\tfrac{1}{R}) + C_0^\varphi(\varphi'\cdot\chi_{|y|<R})$$

Letting $R\to\infty$, we get

$$C_1^\varphi(1) = C_0^\varphi(\varphi') \quad \text{mod constants.}$$

Since C_0^φ is known to be L_2-bounded, and $\varphi'\in L_\infty(\mathbb{R})$, Prop. 11.2 give $C_1^\varphi(1)\in BMO(\mathbb{R})$.
Since C_1^φ is antisymmetric, the hypothesis in Thm 11.4 is satisfied, so $C_1^\varphi: L_2(\mathbb{R})\to L_2(\mathbb{R})$ is bounded.

Repeating this calculation, we have

$$C_{k+1}^\varphi(1) = C_k^\varphi(\varphi')$$

$$C_k^\varphi(\varphi')\in BMO \Rightarrow C_{k+1}^\varphi: L_2\to L_2 \text{ bounded.}$$

Thus L_2-boundedness of all Calderón commutators C_k^φ, $k=0,1,2,\dots$, follows from the $T(1)$ theorem.

Proof of the T(1) theorem 11.4

Given the SIO T, we compute $T(1), T^*(1)\in BMO(\mathbb{R}^n)$ as in Prop. 11.2, and the Haar multipliers

$$m_Q: W_Q\to W_Q : f\longmapsto \Delta_Q(Tf) \quad \text{for } Q\in\mathcal{D},$$

as in Thm 10.10. We look at the operator

$$\tilde{T} := T - (\pi_m^0 + \pi_{T(1)}^+ + \pi_{T^*(1)}^-).$$

Extending Prop. 10.13 to the non-dyadic setting, $\sup_{Q\in\mathcal{D}}|m_Q|<\infty$ follows from the weak boundedness of T.
Thus T is L_2-bounded if and only if $\tilde{T}$ is L_2-bounded.
We shall estimate the tail term $\tilde{T}$ using the γ-Hölder regularity of $k(x,y)$, following quite closely Ex. 8.16.

With the isometry $L_2(\mathbb{R}^n) \approx l_2(\mathbb{D}, W_{Q_0})$, we have a matrix of $\tilde{T}$ with matrix elements $\Delta_R \tilde{T} \Delta_Q : W_Q \to W_R$ being $(2^n-1)\times(2^n-1)$ matrix, i.e we have an infinite block matrix. We need to estimate

$$\|\Delta_R \tilde{T} \Delta_Q\| = \sup_{\substack{f\in W_Q, g\in W_R \\ \|f\|_2 = \|g\|_2 = 1}} |\langle g, \tilde{T} f\rangle|$$

Below we assume $Q, R \in \mathbb{D}$, $f \in W_Q$, $g \in W_R$, $\|f\|_2 = \|g\|_2 = 1$, so that $\sup_{x\in\mathbb{R}^n} |f(x)| \lesssim \frac{1}{\sqrt{|Q|}}$, $\sup_{x\in\mathbb{R}^n} |g(x)| \lesssim \frac{1}{\sqrt{|R|}}$.

(A) Case $R = Q$. Here

$$\langle g, \tilde{T} f\rangle = \langle g, Tf\rangle - \langle g, \Pi_m^0 f\rangle - 0 - 0 = 0.$$

(B) Case $R \cap Q = \emptyset$, $|R| \le |Q|$. Here

$$\langle g, \tilde{T} f\rangle = \langle g, Tf\rangle - 0 - 0 - 0 = \iint_{R\,Q} g(x)\, k(x,y) f(y)\, dy\, dx$$

We use the principle of almost orthogonality to get

$$|\langle g, \tilde{T} f\rangle| = \Big|\iint_{R\,Q} g(x)\big(k(x,y) - k(x_R, y)\big) f(y)\, dy\, dx\Big|$$

center of R.

$$\lesssim \int_R \int_Q |g(x)| \frac{|x - x_R|^\gamma}{|y - x_R|^{n+\gamma}} |f(y)|\, dy\, dx.$$

Here we assume that $\bar{R} \cap \bar{Q} = \emptyset$, so that $d(R,Q) \ge l(R)$, where $d(R,Q) := \inf_{y\in R, x\in Q} |y - x|$,

Thus
$$|\langle g, \tilde{T} f\rangle| \lesssim \frac{1}{\sqrt{|R|}} \frac{1}{\sqrt{|Q|}} \underbrace{\int_R |x - x_R|^\gamma dx}_{\approx l(R)^{n+\gamma}} \cdot \int_Q |y - x_R|^{-n-\gamma} dy$$

Case 1: $d(R,Q) \ge l(Q)$.

Then $|y - x_R| \approx d(R,Q)$ on Q, so

$$|\langle g, \tilde{T} f\rangle| \lesssim \frac{1}{\sqrt{|R|}} \frac{1}{\sqrt{|Q|}} |R|\, l(R)^\gamma \frac{|Q|}{d(R,Q)^{n+\gamma}} = \sqrt{\frac{|R|}{|Q|}}\, l(R)^\gamma \frac{|Q|}{d(R,Q)^{n+\gamma}}.$$

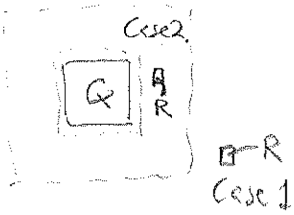
Case 2: $l(R) \le d(R,Q) \le l(Q)$.

In this case

$$\int_Q |y - x_R|^{-n-\gamma} dy \le \int_{|y| > d(R,Q)} \frac{dy}{|y|^{n+\gamma}} \approx \frac{1}{d(R,Q)^\gamma}, \text{ so}$$

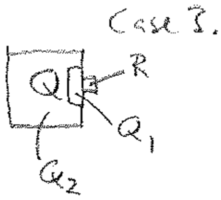
$$|\langle g, \tilde{T} f\rangle| \lesssim \frac{1}{\sqrt{|R|}} \frac{1}{\sqrt{|Q|}} |R|\, l(R)^\gamma \frac{1}{d(R,Q)^\gamma} = \sqrt{\frac{|R|}{|Q|}}\, l(R)^\gamma \frac{1}{d(R,Q)^\gamma}.$$

Case 3: $d(R,Q)=0$



Here we write $Q = Q_1 \cup Q_2$, where
$Q_1 := \{y \in Q;\ d(y,R) \geq \ell(R)\}$
$Q_2 := \{y \in Q;\ d(y,R) > \ell(R)\}$.

As in case 2, we get



$$|\langle g, \vec{T}(f \cdot \chi_{Q_2})\rangle| \lesssim \sqrt{\frac{|R|}{|Q|}}\, \ell(R)^{\gamma} \frac{1}{\ell(R)^{\gamma}} = \sqrt{\frac{|R|}{|Q|}}.$$

For Q_1, we cannot/should not use the γ-Hölder regularity, but instead the simple estimate

$$|\langle g, \vec{T}(f \cdot \chi_{Q_1})\rangle| \lesssim \int_R \int_{Q_1} |g(x)| \frac{1}{|x-y|^n} |f(y)|\, dy\, dx$$

$$\lesssim \frac{1}{\sqrt{|R|}} \frac{1}{\sqrt{|Q|}} \int_R \int_{Q_1} \frac{dy\, dx}{|x-y|^n} \lesssim \frac{1}{\sqrt{|R|}} \frac{1}{\sqrt{|Q|}} \ell(R)^{n+n-n} = \sqrt{\frac{|R|}{|Q|}}.$$

The last estimate is by rescaling and an argument similar to Prop. 7.6.

We summarize:

$$\begin{cases} \|\Delta_R \vec{T} \Delta_Q\| \lesssim \sqrt{\frac{|R|}{|Q|}}\, \ell(R)^{\gamma} \frac{|Q|}{d(R,Q)^{n+\gamma}} & \text{if } d(R,Q) \geq \ell(Q) \\ \|\Delta_R \vec{T} \Delta_Q\| \lesssim \sqrt{\frac{|R|}{|Q|}}\, \ell(R)^{\gamma} \frac{1}{d(R,Q)^{\gamma}} & \text{if } \ell(Q) \geq d(R,Q) \geq \ell(R) \\ \|\Delta_R \vec{T} \Delta_Q\| \lesssim \sqrt{\frac{|R|}{|Q|}} & \text{if } d(R,Q) = 0 \end{cases}$$

(c) Case $R \subsetneq Q$. Here

$$\langle g, \vec{T} f\rangle = \langle g, Tf\rangle - 0 - \langle g, \pi^{*}_{T(1)} f\rangle - 0$$

Let $Q_1, \dots, Q_{2^n}$ be the children of Q, and let NQ be the cube with same center as Q but $\ell(NQ) = N \cdot \ell(Q)$.

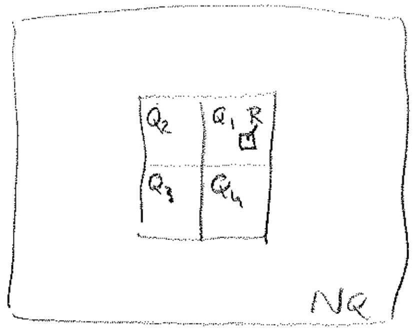
Then

$$\langle g, \pi^{*}_{T(1)} f\rangle = \langle g, \Delta_R(T(1)) E_R f\rangle$$

$$= \lim_{N \to \infty} \langle g, T\chi_{NQ}\rangle E_R f, \text{ and}$$

$$\langle g, Tf\rangle = \sum_{j=1}^{2^n} \langle g, T\chi_{Q_j}\rangle E_{Q_j} f$$

Note that f is constant on each Q_j. We assume the enumeration is such that $R \subset Q_1$.



Since $E_R f = E_{Q_1} f$, we get

$$\langle g, Tf\rangle - \langle g, \Pi^{+}_{T(1)} f\rangle = \lim_{N\to\infty} \langle g, T(\chi_{NQ\setminus Q_1})\rangle E_R f - \sum_{j=2}^{2^n} \langle g, T\chi_{Q_j}\rangle E_{Q_j} f$$

Since $|E_R f|, |E_{Q_j} f| \le \frac{1}{\sqrt{|Q|}}$,

we get

$$|\langle g, \tilde{T} f\rangle| \lesssim \int_R \int_{NQ\setminus Q_1} |g(x)| \frac{|x-x_R|^\gamma}{|y-x_R|^{n+\gamma}} \, dy \, dx \cdot \frac{1}{\sqrt{|Q|}}$$

and similar to (B), case 2, we have

$$|\langle g, \tilde{T} f\rangle| \lesssim \sqrt{\frac{|R|}{|Q|}} \, \ell(R)^\gamma \frac{1}{d(R,Q)^\gamma} \quad \text{if} \quad d(R, Q_1^c) \ge \ell(R).$$

With the modification from (B), case 3, when $d(R, Q_1^c) = 0$, we get

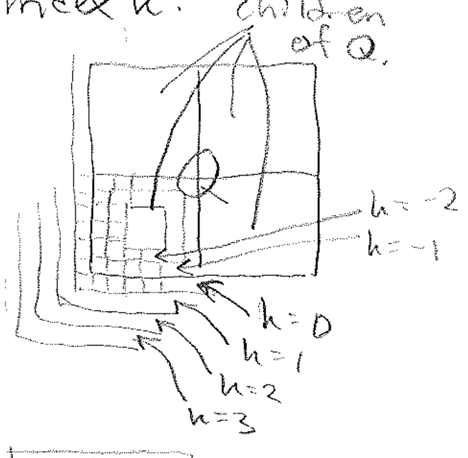
$$\begin{cases} \|\Delta_R \tilde{T} \Delta_Q\| \lesssim \sqrt{\frac{|R|}{|Q|}} \frac{\ell(R)^\gamma}{d(R,Q)^\gamma} & \text{if} \quad d(R, Q_1^c) \ge \ell(R) \\ \|\Delta_R \tilde{T} \Delta_Q\| \lesssim \sqrt{\frac{|R|}{|Q|}} & \text{if} \quad d(R, Q_1^c) = 0 \end{cases}$$

We now aim for a Schur estimate of $\tilde{T}$, and consider $c_j := \sum_{\substack{R\in\mathcal{D} \\ |R| = 2^{-j}|Q|}} \|\Delta_R \tilde{T} \Delta_Q\|$ for a fixed $Q \in \mathcal{D}$, similar to Ex. 8.16.

<u>Estimate for $j \ge 0$:</u>

For fixed j, we consider all $R \in \mathcal{D}$ with $\ell(R) = 2^{-j}\ell(Q) \le \ell(Q_1)$

Partition these cubes R in layer around Q ($k \ge 0$) an inside Q's children ($k < 0$)

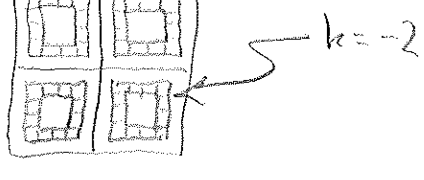


$k = 0$: R touching Q on the outside

$k = 1, 2, 3, \ldots$ consecutive layers where $d(R,Q) \approx 2^{-j} \cdot k$

$k = -1$: R touching the boundary ∂Q_1 of a child Q_1 of Q from the inside.

$k = -2, -3, \ldots, -2^{j-2}$ consecutive layers inside the children Q_1, where $d(R, Q_1^c) \approx 2^{-j} \cdot |k|$



We count the number of cubes R in these layers

$k \geq 0$: $\# \approx (2^j + 2k)^{n-1}$

$k < 0$: $\# \approx (2^{j-1} + 2k)^{n-1}$

This and estimate (B) & (C) gives

$$S_j \lesssim \sum_{k=2^j}^{\infty} + \sum_{k=1}^{2^j} + (k=0) + \sum_{k=-2^{j-2}}^{-1} + (k=-1)$$

$$\lesssim \left[\sum_{k=2^j}^{\infty} (2^j+2k)^{n-1} \sqrt{\frac{2^{-nj}|Q|}{|Q|}} \left(2^{-j}\ell(Q)\right)^{\gamma} \frac{|Q|}{(k2^{-j}\ell(Q))^{n+\gamma}}\right] +$$

$$+ \left[\sum_{k=1}^{2^j} (2^j+2k)^{n-1} \sqrt{2^{-nj}} \left(\frac{2^{-j}\ell(Q)}{k2^{-j}\ell(Q)}\right)^{\gamma}\right] + \left[(2^j)^{n-1}\sqrt{2^{-nj}}\right]$$

$$+ \left[\sum_{k=-2^{j-2}}^{-1} (2^{j-1}+2k)^{n-1} \sqrt{2^{-nj}} \left(\frac{2^{-j}\ell(Q)}{|k|2^{-j}\ell(Q)}\right)^{\gamma}\right] + \left[(2^{j-1})^{n-1}\sqrt{2^{-nj}}\right]$$

$$\lesssim 2^{\frac{n}{2}j} \sum_{k=2^j}^{\infty} k^{-\gamma-1} + 2^{(\frac{n}{2}-1)j} \sum_{k=1}^{2^j} k^{-\gamma} + 2^{(\frac{n}{2}-1)j}$$

$$+ 2^{(\frac{n}{2}-1)j} \sum_{k=-2^{j-2}}^{-1} \frac{1}{|k|^{\gamma}} + 2^{(\frac{n}{2}-1)j}$$

$$\lesssim 2^{\frac{n}{2}j}\, 2^{-\gamma j} + 2^{(\frac{n}{2}-1)j}\, 2^{j(1-\gamma)} + 2^{(\frac{n}{2}-1)j} \lesssim 2^{(\frac{n}{2}-\gamma)j}.$$

<u>Estimate for $j<0$:</u>

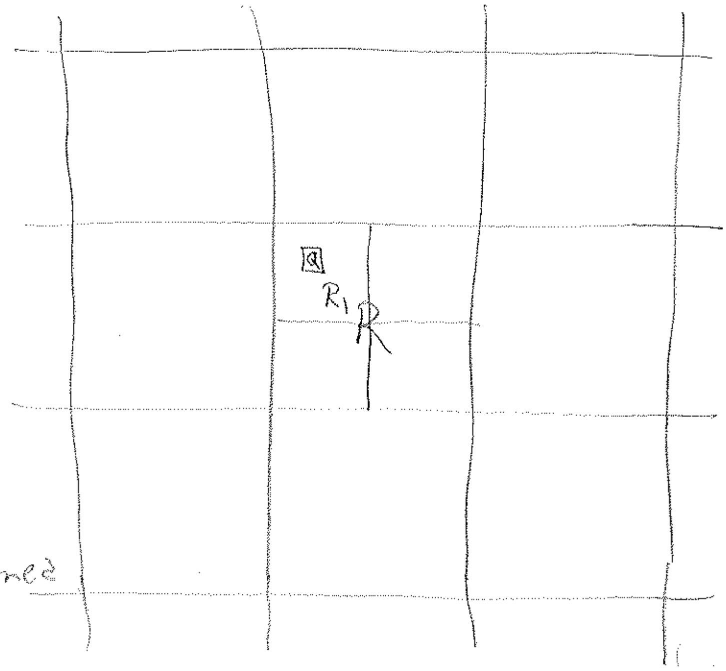
When $\ell(R) = 2^{-j}\ell(Q) > \ell(Q)$, we argue by duality:

$$\|\Delta_R \tilde{T} \Delta_Q\|_{L_2\to L_2} = \sup |\langle g, \tilde{T} f\rangle| = \sup |\langle \tilde{T}^* g, f\rangle| = \|\Delta_Q \tilde{T}^* \Delta_R\|_{L_2\to L_2};$$

Since our hypothesis is invariant upon replacing T by T^*, we have by (B) & (C):

$$\|\Delta_R \tilde{T} \Delta_Q\| \lesssim \begin{cases} \sqrt{\frac{|Q|}{|R|}}\, \ell(Q)^{\gamma} \frac{|R|}{d(R,Q)^{n+\gamma}}, & d(R,Q) \geq \ell(R) \\ \sqrt{\frac{|Q|}{|R|}} \left(\frac{\ell(Q)}{d(R,Q)}\right)^{\gamma}, & \ell(Q) \leq d(R,Q) \leq \ell(R) \\ & \text{or } d(Q,R_1^c) \geq \ell(Q) \\ \sqrt{\frac{|Q|}{|R|}}, & d(R,Q)=0 \text{ and } R\cap Q = \emptyset, \text{ or} \\ & d(Q,R_1^c)=0,\ Q\subset R_1 \subset R. \end{cases}$$

Here, analogous to (C), R_1 denotes the child of R containing Q, We see in the picture that there are at most



- 3^n cubes R where $\ell(Q) \le d(R,Q) \le \ell(R)$ or $d(Q, R_1^c) \ge \ell(Q)$
- 2^n cubes R where $d(R,Q) = 0$ and $R \cap Q = \emptyset$, or $d(Q, R_1^c) = 0$ and $Q \subset R_1^c \subset R$.

The remaining cubes R have $d(R,Q) \ge \ell(R)$, and are partitioned into shells $k = 1, 2, 3, \dots$ containing $\lesssim (3+2k)^{n-1}$ cubes.

We get

$$c_j \lesssim 3^n \sqrt{\frac{|Q|}{2^{-nj}|Q|}} \left(\frac{\ell(Q)}{\ell(Q)}\right)^{\gamma} + 2^n \sqrt{\frac{|Q|}{2^{-nj}|Q|}}$$
$$+ \sum_{k=1}^{\infty} (3+2k)^{n-1} \sqrt{\frac{|Q|}{2^{-nj}|Q|}}\, \ell(Q)^{\gamma} \frac{2^{-nj}|Q|}{(k2^{-j}\ell(Q))^{n+\gamma}}$$
$$\lesssim 2^{\frac{n}{2}j} + 2^{(\frac{n}{2}+\gamma)j} \sum_{k=1}^{\infty} k^{-1-\gamma} \lesssim 2^{n/2\, j}.$$

In total we have

$$c_j \lesssim \begin{cases} 2^{(n/2-\gamma)j}, & j \ge 0 \\ 2^{n/2\, j}, & j < 0. \end{cases}$$

We now apply a weighted Schur estimate as in Prop. 1.13. Replace there the kernel $k(x,y)$ by the discrete block matrix $(\Delta_R \tilde{T} \Delta_Q)_{R,Q \in \mathbb{D}}$, and use the weight function

$$\omega(R) := \ell(R)^{(n-\gamma)/2}, \quad R \in \mathbb{D}.$$

We need to show $\sup_{Q \in \mathbb{D}} \left(\frac{1}{\omega(Q)} \sum_{R \in \mathbb{D}} \|\Delta_R \tilde{T} \Delta_Q\| \omega(R) \right) < \infty$.

(The dual estimate

$\sup_{R \in \mathbb{D}} \left(\frac{1}{\omega(R)} \sum_{Q \in \mathbb{D}} \|\Delta_R \tilde{T} \Delta_Q\| \omega(Q) \right) < \infty$ follows from replacing T by T^* in our entire proof.)

For $Q \in \mathbb{D}$, we have

$$\sum_{R \in \mathbb{D}} \|\Delta_R \tilde{T} \Delta_Q\| \omega(R) = \sum_{j \in \mathbb{Z}} c_j (2^{-j})^{\frac{n-\gamma}{2}} \omega(Q) = \sum_{-\infty}^{\infty} 2^{-\frac{\gamma}{2}|j|} \omega(Q) \lesssim \omega(Q).$$

This completes the proof. ∎

The T(b) theorem:

We note that, under the assumption $T: H^s_t(\mathbb{R}^n) \to H^s_t(\mathbb{R}^n)^*$ has a γ-regular SIO kernel, the T(1)-theorem 11.4 is sharp, i.e. the hypothesis

weak boundedness & $T(1), T^*(1) \in BMO$

is both necessary and sufficient.

A disadvantage of the T(1)-theorem though, is that the conditions $T(1), T^*(1) \in BMO$ often are difficult to check. For the Calderón commutators (Ex. 7.10, 11.6) we were lucky. For the Cauchy integral (Ex. 7.9) and the double layer potential (Ex. 7.11) on Lipschitz curves/surfaces, the T(1) condition is hard to verify. To show L_2-boundedness of these operators, we shall now prove the more general T(b)-theorem, where the constant function 1 is replaced by a more arbitrary auxiliary function $b(x)$. Moreover, for applications to Ex 7.9 + 7.11 we need to generalize to vector-valued functions $f: \mathbb{R}^n \to \mathbb{R}^N$. For example, the Cauchy integral uses $\mathbb{R}^N = \mathbb{R}^2 \simeq \mathbb{C}$.

Defn 12.1 A vector-valued SIO kernel on $\mathbb{R}^n$ is a function $k: \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^N)$ such that

$$|k(x,y)| \lesssim \frac{1}{|x-y|^n}.$$

(Here $|k(x,y)|$ denotes the operator norm $|k(x,y)|_{\mathbb{R}^N \to \mathbb{R}^N}$.)
It is γ-Hölder regular if each scalar component SIO kernel $k^{ij}(x,y)$, $i,j = 1, \ldots, N$, is so in the sense of Defn. 8.4.
A vector-valued SIO is an bounded operator $T: H^s_t(\mathbb{R}^n;\mathbb{R}^N) \to H^s_t(\mathbb{R}^n;\mathbb{R}^N)^*$ with a vector-valued SIO kernel for $x \neq y$. As before, we always assume $0 < s < \frac{1}{2}$, $t > 0$.

The setup for the T(b)-theorem will be as follows.

Consider a vector-valued SIO

$$\underbrace{T_0 f(x)}_{\mathbb{R}^N} = \int_{\mathbb{R}^n} \underbrace{k_0(x,y)}_{L(\mathbb{R}^N)} \underbrace{f(y)}_{\mathbb{R}^N} dy, \text{ with a } \gamma\text{-Hölder}$$

regular SIO kernel. As we shall ~~soon~~ see in examples, it is sometimes the case that there is a (non-smooth!) function $b \in L_\infty(\mathbb{R}^n, L(\mathbb{R}^N))$, which is "adapted" to T_0. We want to define $T_0(b)(x) = \int k_0(x,y) b(y) dy \in L(\mathbb{R}^N)$, but unfortunately we typically have $b \notin H^s_{loc}$. We are lead to consider the closely related SIO kernel

$$k(x,y) := b(x)^t k_0(x,y) b(y).$$

Here b^t denotes the transpose of the matrix b. For simplicity, assume here that $k_0(x,y)^t = k_0(x,y)$. Thus $k(x,y)$ is antisymmetric if and only if $k_0(x,y)$ is so. (This is the reason for considering $b^t k_0 b$ rather than $k_0 b$.)
Applying Prop. 8.2 (componentwise) gives a SIO

$$T: H^s_t(\mathbb{R}^n; \mathbb{R}^N) \longrightarrow H^s_t(\mathbb{R}^n; \mathbb{R}^N)^*$$

with kernel $k(x,y) = b(x)^t k_0(x,y) b(y)$.
In terms of T_0, this means that we have an operator

$$T_0: b \cdot H^s_t(\mathbb{R}^n; \mathbb{R}^N) \longrightarrow (b \cdot H^s_t(\mathbb{R}^n; \mathbb{R}^N))^*$$

i.e. T_0 acts on "non-smooth test functions"

$$b \cdot H^s_t := \{ b \cdot \varphi \,;\, \varphi \in H^s_t \}.$$

- We shall assume that $k_0(x,y)$ is a γ-Hölder regular SIO kernel and $b^{-1} \in L_\infty(\mathbb{R}^n; L(\mathbb{R}^N))$. Since

$$\text{"} b^t(x) \cdot (T_0 b)(x) = T(1)(x) \text{"},$$

we have two point of views:

(A) A "Tb" theorem for a γ-Hölder regular SIO, or equivalently

(B) A "T1" theorem for a SIO with a non-smooth kernel.

We need to define $T_0 b$, or equivalently $T(1)$, analogous to Defn. 11.1. Denote the $\mathbb{R}^N$ inner product by

$$\langle u, v \rangle := u_1 v_1 + \dots + u_N v_N.$$

Defn 12.2: Let $T = b^t T_0 b : H^s_t(\mathbb{R}^n;\mathbb{R}^N) \to H^s_t(\mathbb{R}^n;\mathbb{R}^N)^*$ be a SIO as above, and assume that the SIO kernel $k_0(x,y)$ for T_0 is γ-Hölder regular. We define

$$T_0 b \in H^{-s}_{loc}(\mathbb{R}^n; L(\mathbb{R}^N)) / \{\text{constant}\}$$

(a matrix valued function, defined modulo constant matrices), through

$$\langle u, T_0 b(x) v\rangle := \lim_{R\to\infty} \langle u, (b(x)^t)^{-1} T(\chi_{|y|<R} \cdot v)\rangle, \quad u,v \in \mathbb{R}^N.$$

Let us verify that $T_0 b$ is well-defined. Consider $\varphi \in L_2(\mathbb{R}^n;\mathbb{R}^N)$, $\operatorname{supp}\varphi \subset B(0;r)$, $\int \varphi\, dx = 0$. For $2r < R_1 < R_2 < \infty$, we have

$$\left| \int_{\mathbb{R}^n} \langle \varphi(x), (b(x)^t)^{-1} (T(\chi_{|y|<R_1} \cdot v) - T(\chi_{|y|<R_2} \cdot v))\rangle dx \right|$$

$/\int\varphi = 0/$
$$= \left| \int_{\mathbb{R}^n} \int_{R_1<|y|<R_2} \langle \varphi(x), (k_0(x,y) - k_0(0,y))\, b(y)\cdot v\rangle\, dy\, dx \right|$$

$$\lesssim \int_{\mathbb{R}^n} |\varphi(x)|\cdot|x|^\gamma dx \cdot \int_{R_1<|y|<R_2} \frac{dy}{|y|^{n+\gamma}} \cdot \|b\|_\infty$$

$$\lesssim \|\varphi\|_{L_2} \cdot r^{\frac{n}{2}+\gamma} \left(\frac{1}{R_1^\gamma} - \frac{1}{R_2^\gamma}\right) \cdot \|b\|_\infty \to 0, \quad R_1, R_2 \to \infty$$

This shows that $T_0 b$ is a well-defined distribution in H^{-s}_{loc} mod. constants. We note:

- equivalently $T(1) = b^t(T_0 b)$ belongs to $b^t \cdot (H^{-s}_{loc}/\{\text{constants}\})$, so $T(1)$ is well-defined modulo multiples of b^t.
- if T extends to a bounded L_2-operator (or equivalently $T_0 : L_2 \to L_2$ is bounded), then $T_0 b \in BMO(\mathbb{R}^n; L(\mathbb{R}^N))$, i.e. each component of $T_0 b$ is in scalar $BMO(\mathbb{R}^n)$. The proof is similar to Prop. 11.2.

Ex 12.3:

We will now demonstrate how Ex 7.9 and 7.11 of SIOs, fits into the framework $T = b^t T_0 b$.
Suppose $n \geq 1$, Σ is the graph of a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$, and $R_1, \dots, R_n, R_{n+1}$ are the Riesz transforms on Σ as in Ex. 7.11. Define

$$T_0 f(x) = \text{p.v.} \int_{\mathbb{R}^n} \underbrace{\frac{x - y + e_{n+1}(\varphi(x) - \varphi(y))}{\left(|x-y|^2 + |\varphi(x) - \varphi(y)|^2\right)^{\frac{n+1}{2}}}}_{=: k_0(x,y)} f(y)\, dy.$$

The question is now how to see $k_0(x,y)$ as a linear operator on some $\mathbb{R}^N$. (Here $x = \sum_{i=1}^n x_i e_i$, $y = \sum_{i=1}^n y_i e_i$, and $e_1, \dots, e_n, e_{n+1}$ is the standard basis for $\mathbb{R}^{n+1}$)
For this we make use of higher dimensional complex algebra: Clifford algebra. Consider all $2^{n+1} =: N$ products

$$e_{s_1} \cdot e_{s_2} \cdot \ldots \cdot e_{s_k}, \quad 1 \leq s_1 < s_2 < \dots < s_k \leq n+1, \quad 0 \leq k \leq n+1,$$

(where $k=0$ means the empty product 1),
and view these as a basis for $\mathbb{R}^N$, $N := 2^{n+1}$
The defining relations for the Clifford product are

$$e_i \cdot e_j = -e_j \cdot e_i, \quad 1 \leq i, j \leq n+1, \quad i \neq j.$$
$$e_i^2 = +1, \quad 1 \leq i \leq n+1.$$

It is well known that this defines an associative (but not commutative) product on

$$\mathbb{R}^N = \text{span}\{\underset{\mathbb{R}}{1}, \underbrace{e_1, e_2, \dots, e_{n+1}}_{=\mathbb{R}^{n+1}},$$
$$e_1e_2, e_1e_3, \dots, e_ne_{n+1},$$
$$e_1e_2e_3, e_1e_2e_4, \dots, e_{n-1}e_ne_{n+1},$$
$$\vdots$$
$$e_1e_2e_3 \cdots e_{n+1}\}.$$

In a natural way $\mathbb{R}^N$ contains all scalars $\mathbb{R}$ and all vectors $\mathbb{R}^{n+1}$.

Importent is the formula

$$u\cdot v = \Big(\sum_{i=1}^{n+1} u_i e_i\Big)\cdot\Big(\sum_{j=1}^{n+1} v_j e_j\Big) = \sum_{i=1}^{n+1} u_i v_i + \sum\sum_{1\le i<j\le n+1} (u_i v_j - u_j v_i) e_i e_j$$

$$= \underbrace{\langle u,v\rangle}_{\text{inner product}} + \underbrace{u\wedge v}_{\text{exterior product.}}$$

Now return to the SIO T_0.

We let $f: \mathbb{R}^n \to \mathbb{R}^N$ take values in the Clifford algebra, and $k_0(x,y)$ means the operator of left Clifford multiplication $f(y) \mapsto k_0(x,y)\cdot f(y)$ by the vector $k_0(x,y)$. Thus $k_0(x,y) \in L(\mathbb{R}^N)$.

As we saw in Ex. 7.11, the vector field $b(x) := \nabla\varphi(x) - e_{n+1}$ is important.

As for $k_0(x,y)$, we identify the vector $b(x)$ with the operator $(f \mapsto b(x)\cdot f) \in L(\mathbb{R}^N)$.

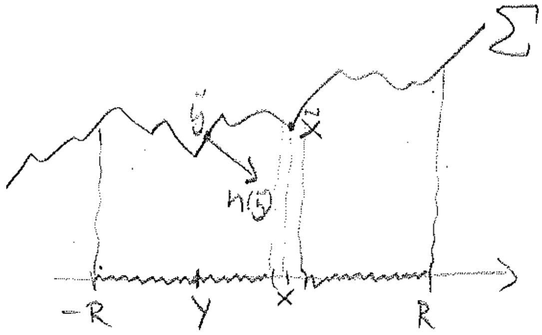
- We now show $T_0 b \in BMO(\mathbb{R}^n; L(\mathbb{R}^N))$.

Similar to Ex. 11.6, consider

$$\int_{\substack{|y-x|>\varepsilon \\ |y|<R}} k_0(x,y)\, b(y)\, dy \qquad \text{for} \quad |x| < r \ll R.$$

Making a change of varibles, this integral is

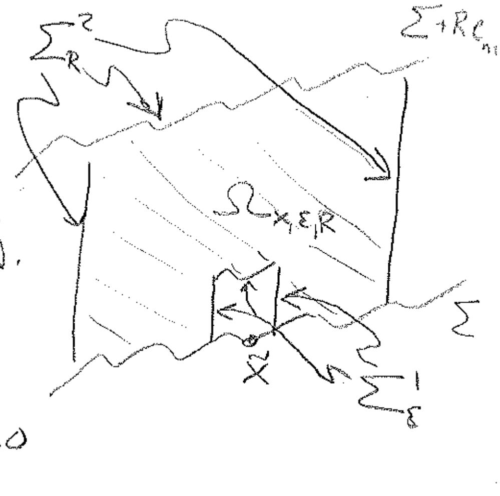
$$\int_{\Sigma_{x,\varepsilon,R}} \frac{\tilde{x}-\tilde{y}}{|\tilde{x}-\tilde{y}|^{n+1}}\cdot n(\tilde{y})\, d\sigma(\tilde{y})$$



where $\Sigma_{x,\varepsilon,R}$ is the part of the Lipschitz graf above $B(0,R)\setminus B(x,\varepsilon)$, $n(\tilde{y})$ is the unit normal vector on Σ and $d\sigma$ is surfacemeasure on Σ.

Now consider

$$\Omega_{x,\varepsilon,R} := \{y + s e_{n+1};\ y\in\mathbb{R}^n,\ |y|<R,$$
$$\varphi(y) < s < \varphi(y)+R\} \setminus$$
$$\setminus \{y+se_{n+1};\ y\in\mathbb{R}^n,\ |y-x|<\varepsilon,\ \varphi(y)<s<\varphi(y)+\varepsilon\}.$$



The vector field $E_x(\tilde{y}) := \dfrac{\tilde{x}-\tilde{y}}{|\tilde{x}-\tilde{y}|^{n+1}}$ is divergence and curl free in $\Omega_{x,\varepsilon,R}$, so

$$\int_{\partial\Omega_{x,\varepsilon,R}} E_x(\tilde y)\, n(\tilde y)\, d\sigma(\tilde y) = \int_{\partial\Omega_{x,\varepsilon,R}} \langle n(\tilde y), E_x(\tilde y)\rangle\, d\sigma(\tilde y) - \int_{\partial\Omega_{x,\varepsilon,R}} n(\tilde y)\wedge E_x(\tilde y)\, d\sigma(\tilde y)$$

$$= \int_{\Omega_{x,\varepsilon,R}} \operatorname{div} E_x(\tilde y)\, d\tilde y + \int_{\Omega_{x,\varepsilon,R}} \operatorname{curl} E_x(\tilde y)\, d\tilde y = 0,$$

by Gauss' and Stokes' theorems.

Besides $\Sigma_{x,\varepsilon,R}$, $\partial\Omega_{x,\varepsilon,R}$ consists of the surfaces Σ^1_ε and Σ^2_R as in the picture. Thus

$$\int_{\Sigma_{x,\varepsilon,R}} E_x(\tilde y)\cdot n(\tilde y)\, d\sigma(\tilde y) = -\int_{\Sigma^1_\varepsilon} E_x(\tilde y)\, n(\tilde y)\, d\sigma(\tilde y) - \int_{\Sigma^2_R} E_x(\tilde y)\, n(\tilde y)\, d\sigma(\tilde y)$$

Fix $R \gg 1$ and let $\varepsilon \to 0$. Using Rademacher's theorem (φ is differentiable almost everywhere) and rescaling the integral, shows that

$$\int_{\Sigma^1_\varepsilon} E_x(\tilde y)\, n(\tilde y)\, d\sigma(\tilde y) \longrightarrow \tfrac{1}{2}\sigma_n \quad \text{(area of unit } \mathbb{R}^{n+1} \text{ sphere)}, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

(see Exc. 12.10)

Therefore, for $|x|<r$, by dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0} \int_{\Sigma_{x,\varepsilon,R}} E_x(\tilde y)\, n(\tilde y)\, d\sigma(\tilde y) = -\int_{\Sigma^2_R} E_x(\tilde y)\, n(\tilde y)\, d\sigma(\tilde y)$$

$$= \int_{\Sigma^2_R} \big(E_0(\tilde y) - E_x(\tilde y)\big)\, n(\tilde y)\, d\sigma(\tilde y)$$

modulo constants.

Thus

$$T_0(b\cdot \chi_{|y|<R})(x) = \int_{\Sigma^2_R} \underbrace{\big(E_0(\tilde y) - E_x(\tilde y)\big)}_{=\,\mathcal{O}\left(\frac{1}{R^{n+1}}\right)}\, n(\tilde y)\, d\sigma(\tilde y) \longrightarrow 0, \quad R\to\infty$$

so $T_0 b = 0$ modulo constants.

In order to derive L_2-boundedness of T_0 from $T_0 b \in BMO$, we need the following non-degeneracy condition for b.

Defn. 12.4: We say that $b \in L_\infty(\mathbb{R}^n; L(\mathbb{R}^N))$ is (dyadic) *paraaccretive* if for all (dyadic) cubes $Q \subset \mathbb{R}^n$, we have

$$\left\| \left(\frac{1}{|Q|}\int_Q b(x)\, dx \right)^{-1} \right\|_{L(\mathbb{R}^N)} \le C < \infty.$$

The condition is satisfied in Ex. 12,3:

$$\left|\left(\frac{1}{|Q|}\int_Q (\nabla\varphi(x) - e_{n+1})dx\right)^{-1}\right| = \left|\left(\frac{1}{|Q|}\int_Q \nabla\varphi dx - e_{n+1}\right)^{-1}\right|$$

$$= \left|\frac{\frac{1}{|Q|}\int_Q \nabla\varphi dx - e_{n+1}}{\left|\frac{1}{|Q|}\int_Q \nabla\varphi dx\right|^2 + 1}\right| \lesssim 1 + \|\nabla\varphi\|_\infty.$$

Thm 12.5 (The $T(b)$-theorem)

Let $T: H^s_t(\mathbb{R}^n;\mathbb{R}^N) \to H^s_t(\mathbb{R}^n;\mathbb{R}^N)^*$ be vector-valued SIO ($0<s<\frac{1}{2}$, $t>0$), with SIO kernel

$$k(x,y) = b_1(x)^t k_0(x,y) b_2(y).$$

Assume that

- $b_1, b_2 \in L_\infty(\mathbb{R}^n; L(\mathbb{R}^N))$ are paraaccretive functions
- $k_0(x,y)$ is a γ-Hölder regular SIO kernel ($0<\gamma<1$)
- T is weakly bounded as in Defn. 11,3
- $T_0 b_2 \in BMO(\mathbb{R}^n; L(\mathbb{R}^N))$ and
  $T_0^* b_1 \in BMO(\mathbb{R}^n; L(\mathbb{R}^N))$ as in Defn. 12,2.

Then T (and T_0) extends to a bounded operator

$$T: L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n).$$

Let us first demonstrate that without loss of generality we can and will assume $b_1 = b_2$ and $T^* = -T$ in the proof:

Given $k(x,y) = b_1(x)^t k_0(x,y) b_2(y)$, consider the identity

$$\begin{bmatrix} b_1^t(x) & 0 \\ 0 & b_2^t(x)\end{bmatrix}\begin{bmatrix} 0 & -k_0(x,y) \\ k_0(y,x)^t & 0\end{bmatrix}\begin{bmatrix} b_1(y) & 0 \\ 0 & b_2(y)\end{bmatrix} = \begin{bmatrix} 0 & -b_1(x)^t k_0(x,y) b_2(y) \\ (b_1(y)^t k_0(y,x) b_2(x))^t & 0\end{bmatrix}$$

The RHS contains the SIO kernel of $-T$ and T^*, hence the LHS is a $\mathbb{R}^{2N}$-valued SIO $\tilde{T}$, where $\tilde{T}^* = -\tilde{T}$.

$\tilde{b} := \begin{bmatrix} b_1 & 0 \\ 0 & b_2\end{bmatrix} \in L_\infty(\mathbb{R}^n; L(\mathbb{R}^{2N}))$ is paraaccretive and

$$\tilde{T}_0\tilde{b} = \begin{bmatrix} 0 & -T_0 b_2 \\ T_0^* b_1 & 0\end{bmatrix} \in BMO(\mathbb{R}^n; L(\mathbb{R}^{2N})).$$

Having proved thm 12.5 for $\tilde{T} = \begin{bmatrix} 0 & -T \\ T^* & 0\end{bmatrix}$, this implies L_2-boundedness of T.

Ex. 12.6: We apply the Tb-theorem to $T := b^t T_0 b$ in Ex. 12.3. We saw that $T_0 b = 0 \in BMO$. Note that T is antisymmetric (componentwise), so it is weakly bounded, and $T_0^* b = 0$. We conclude that

$$T,\ T_0 : L_2(\mathbb{R}^n;\mathbb{R}^N) \longrightarrow L_2(\mathbb{R}^n;\mathbb{R}^N)$$

are bounded. Checking component-operators shows that the Riesz transforms

$$R_1, R_2, \ldots, R_n, R_{n+1}$$

are L_2-bounded. As corollaries we obtain L_2-boundedness of

- the double layer potential in Ex. 7.11,
- the Cauchy integral in one dimension from Ex. 7.9,

on Lipschitz surfaces / curves.

Proof of the Tb-theorem 12.5

The proof of 12.5 is an adaption of the proof of the T(1)-theorem 11.4. A key observation is the following. In case (B) in 11.4, we used the principle of almost orthogonality to estimate

$$\int \langle g, \tilde{T} f\rangle dx = \int_R \int_Q \langle g(x), k(x,y) f(y)\rangle dy\,dx$$

for $g \in W_R$, $f \in W_Q$, $R \cap Q$, $|R| \le |Q|$.

Here in thm 12.5, this is not possible since our $k(x,y)$ is not γ-Hölder regular. However $k_0(x,y)$ is, so we write

$$\int_R \int_Q \langle g(x), k(x,y) f(y)\rangle dy\,dx = \int_R \int_Q \langle b(x) g(x), k_0(x,y)(b(y) f(y))\rangle dy\,dx$$

$$= \int_R \int_Q \langle b(x) g(x), (k_0(x,y) - k(x_R,y))(b(y) f(y))\rangle dy\,dx.$$

To be able to do this, we must have the cancellation condition $\int_R b(x) g(x)\,dx = 0$ rather than $\int_R g(x)\,dx = 0$.

Thus we are lead to the question whether we have a decomposition

$$L_2(\mathbb{R}^n;\mathbb{R}^N) = \bigoplus_{Q\in\mathbb{D}} W_Q^b, \quad \text{where}$$
(not orthogonal, just topological)

$W_Q^b := \{ f:\mathbb{R}^n \to \mathbb{R}^N;\ f = \text{const. on each child of } Q,$
$\operatorname{supp} f \subset Q,\ \int_Q b(x) f(x)\,dx = 0 \}.$

Associated to this decomposition, we want (oblique) projections Δ_Q^b such that

$$\|f\|_{L_2}^2 \approx \sum_{Q\in\mathbb{D}} \|\Delta_Q^b f\|_2^2 .$$

We have the following analogue of Defn. 10.1.

<u>Defn. 12.7</u>: Let $\mathbb{D}$ be the dyadic cubes in $\mathbb{R}^n$ from Defn. 9.1, and let $f \in L_1^{loc}(\mathbb{R}^n;\mathbb{R}^N)$.

(1) $E_Q^b f := \left(\int_Q b(x)\,dx\right)^{-1}\left(\int_Q b(x) f(x)\,dx\right), \quad Q\in\mathbb{D}.$

(2) $(E_j^b f)(x) := \sum_{Q\in\mathbb{D}_j} (E_Q f)\cdot \chi_Q(x), \quad j\in\mathbb{Z},\ x\in\mathbb{R}^n.$

(3) $\Delta_j^b f := E_{j+1} f - E_j f, \quad j\in\mathbb{Z}.$

(4) $\Delta_Q^b f(x) := \Delta_j^b f(x)\cdot\chi_Q(x) = \begin{cases} E_{j+1}f(x) - E_Q f, & x\in Q \\ 0, & x\notin Q \end{cases}, \quad Q\in\mathbb{D}.$

We note that since $b\in L_\infty$ is paraaccretive, we have
$|E_Q f| \lesssim \int_Q |f(x)|\,dx \le \|f\|_{L_2(Q)}, \quad \|E_j^b f\|_{L_2} \lesssim \|f\|_{L_2}.$
Clearly E_j^b are uniformly bounded (but not in general orthogonal!) projections onto the subspaces $V_j = R(E_j^b) = R(E_j^1)$ from Defn. 10.1.

<u>Lemma 12.8</u>: $(E_j^b)^* = b^t E_j^{b^t} (b^t)^{-1}$ holds for $j\in\mathbb{Z}$, and therefore $(\Delta_j^b)^* = b^t \Delta_j^{b^t}(b^t)^{-1}$ and
$$(\Delta_Q^b)^* = b^t \Delta_Q^{b^t}(b^t)^{-1}.$$

<u>Proof</u>: $\int_{\mathbb{R}^n} \langle g, E_j^b f\rangle dx = \sum_{Q\in\mathbb{D}_j} \int_Q \langle g(x), (\int_Q b)^{-1}(\int_Q b(y) f(y)\,dy)\rangle dx$
$= \sum \int_Q \langle b^t(y)(\int_Q b^t)^{-1}(\int_Q g(x)\,dx), f(y)\rangle dy = \int_{\mathbb{R}^n} \langle b^t E_j^{b^t}(b^t)^{-1} g, f\rangle dy.$

From Lemma 12.8, we get

$$N(E^b_{j+1}) = R((E^b_{j+1})^*)^\perp \subset R((E^b_j)^*)^\perp = N(E^b_j).$$

Therefore Δ^b_j is the projection onto

$$W^b_j := R(E^b_{j+1}) \cap N(E^b_j) = V^b_{j+1} \cap (b^t V^{b^t}_j)^\perp$$
$$= \bigoplus_{Q\in\mathbb{D}} W^b_Q ,$$

and Δ^b_Q is the projection onto W^b_Q.

The key result for the projections Δ^b_j is the following.

<u>Prop. 12.9:</u> We have estimates

$$\|f\|^2_{L_2} \approx \sum_{Q\in\mathbb{D}} \|\Delta^b_Q f\|^2_{L_2} = \sum_{j=-\infty}^{\infty} \|\Delta^b_j f\|^2_{L_2}.$$

<u>Proof</u>:

(1) We first prove $\sum_Q \|\Delta^b_Q f\|^2_{L_2} \lesssim \|f\|^2_{L_2}$.

If $x\in R$, R being a child of Q, then

$$(\Delta^b_Q f)(x) = (⨍_R b)^{-1}(⨍_R bf) - (⨍_Q b)^{-1}(⨍_Q bf)$$
$$= (⨍_R b)^{-1}\left[⨍_R bf - ⨍_Q bf\right] + \left[(⨍_R b)^{-1} - (⨍_Q b)^{-1}\right]⨍_Q bf$$
$$= (E_R b)^{-1}\Delta_Q(bf)(x) \cdot (E_R b)^{-1}(\Delta_Q b)(x)(E_Q b)^{-1} E_Q(bf)(x)$$

where E_Q, Δ_Q from Defn. 10.1 act componentwise and $⨍_X := \frac{1}{|X|}\int_X$. We get

$$|\Delta^b_Q f(x)| \lesssim |\Delta_Q(bf)(x)| + |\Delta_Q b(x)|\cdot|E_Q(bf)|$$

For the first term, we have $\sum_{Q\in\mathbb{D}} \|\Delta_Q(bf)\|^2_{L_2} \leq \|bf\|^2_{L_2} \lesssim \|f\|^2_{L_2}$ by orthogonality. For the second term, we need to prove $\sum_{Q\in\mathbb{D}} \int_Q |\Delta_Q b(x)|^2 dx \, |E_Q(bf)|^2 \lesssim \|f\|^2_2$.

As in the proof of Prop. 10.6, we have

$\text{LHS} \lesssim \|b\|^2_{BMO} \cdot \|bf\|^2_2 \lesssim \|f\|^2_2$ using Carleson's theorem.

(2) To prove the reverse estimate $\sum_Q \|\Delta_Q^b f\|_2^2 \gtrsim \|f\|_2^2$, we use duality. For $g \in L_2(\mathbb{R}^n; \mathbb{R}^m)$, we have

$$\int_{\mathbb{R}^n} \langle f, g \rangle dx = \lim_{N\to\infty} \int_{\mathbb{R}^n} \langle E_{N+1}^b f - E_{-N}^b f, g \rangle dx$$

$$\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \langle \underbrace{\Delta_j^b}_{=(\Delta_j^b)^2} f, g \rangle dx = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \langle \Delta_j^b f, b^t \Delta_j^{b^t}(b^t g) \rangle dx$$

$$\lesssim \left(\sum_{j=-\infty}^{\infty} \|\Delta_j^b f\|_2^2 \right)^{1/2} \underbrace{\left(\sum_{j=-\infty}^{\infty} \|\Delta_j^{b^t}(b^t g)\|_2^2 \right)^{1/2}}_{\lesssim \|b^t g\|_2 \lesssim \|g\|_2 \text{ by (1) with } b \rightsquigarrow b^t.}$$

This proves the prop. since g is arbitrary. ∎

We now return to the proof of the Tb-theorem.

Let $C_{RQ} := \sup_{\substack{g \in W_R^b \\ f \in W_Q^b \\ \|g\|_2 = \|f\|_2 = 1}} |\langle g, Tf \rangle|$.

A variant of the Schur estimate technique 1.13 is

$$|\langle g, Tf \rangle| \leq \sum_R \sum_Q |\langle \Delta_R^b g, T \Delta_Q^b f \rangle|$$

$$\leq \sum_R \sum_Q C_{RQ} \|\Delta_R^b g\|_2 \cdot \|\Delta_Q^b f\|_2$$

$$= \sum_R \sum_Q \left(\|\Delta_R^b g\|_2 \sqrt{C_{RQ}} \sqrt{\frac{\omega(Q)}{\omega(R)}} \right) \left(\|\Delta_Q^b f\|_2 \sqrt{C_{RQ}} \sqrt{\frac{\omega(R)}{\omega(Q)}} \right) = \Big/ \text{Cauchy-Schwarz} \Big/$$

$$\leq \left(\sum_R \|\Delta_R^b g\|_2^2 \frac{1}{\omega(R)} \sum_Q C_{RQ}\, \omega(Q) \right)^{1/2} \left(\sum_Q \|\Delta_Q^b f\|_2^2 \frac{1}{\omega(Q)} \sum_R C_{RQ}\, \omega(R) \right)^{1/2}$$

Therefore, as in the proof of thm 11.4, it suffices to show $\forall Q: \sum_R C_{RQ}\, \omega(R) \lesssim \omega(Q)$.

(The estimate $\sum_Q C_{RQ}\,\omega(Q) \lesssim \omega(R)$ follows upon replacing T by T^*.)

However, this Schur estimate cannot be proved for the SIO T in general, but only for the remainder $\tilde{T}$ after having subtracted a suitable perfect dyadic SIO as in thm 11.4.

Since we have been using notation b too much, let us denote the Haar multiplier and paraproducts simply by Π^0, Π^+, Π^-. The remainder is then

$$\tilde{T} := T - \Pi^0 - \Pi^+ - \Pi^-.$$

$\underline{\Pi^0}$: Set $\Pi^0 f := \sum_{Q\in\mathcal{D}} (\Delta_Q^b)^* T \Delta_Q^b f$. Then

$$\langle g, \Pi^0 f\rangle = \sum_{Q\in\mathcal{D}} \langle \Delta_Q^b g, T(\Delta_Q^b f)\rangle.$$

A difference to the case $b=1$ before, is that now there is not a natural L_2-isometry between the $N\cdot(2^n-1)$ dimensional subspaces W_Q^b. However $W_Q^b \subset V_Q$, where

$$V_Q := \{f:\mathbb{R}^n\to\mathbb{R}^N;\ \operatorname{supp} f\subset Q,\ f=\text{const. on each child of } Q\}$$

Using weak boundedness of T and the natural L_2-isometries between spaces V_Q, gives

$$|\langle \Delta_Q^b g, T(\Delta_Q^b f)\rangle| \lesssim \|\Delta_Q^b g\|_2 \cdot \|\Delta_Q^b f\|_2 \quad \text{uniformly in } Q.$$

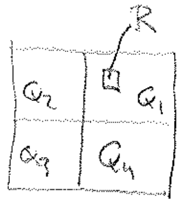
Thus $\|\Pi^0\|_{L_2\to L_2} \le C$.

$\underline{\Pi^+}$: To see what the paraproduct Π^+ should be, we look into case (C) in the proof of thm 11.4.

We here have $g\in W_R^b$, $f\in W_Q^b$, $R\subsetneq Q$, and

$$\int \langle g, Tf\rangle\,dx = \sum_{j=1}^{2^n} \int_R \langle g, T(\underbrace{f\chi_{Q_j}}_{\text{const. on } Q_j})\rangle\,dx,$$

where $Q_1,\dots,Q_{2^n}$ are the children of Q, with $R\subset Q_1$.



The term $j=1$ is the problematic one, since this uses T on the diagonal $x=y$. We define the paraproduct type operator

$$\Pi^+ f(x) := \sum_{Q\in\mathcal{D}} \left[(\Delta_Q^b)^*(b^t T_0 b)(x)\right]\cdot E_Q^b f$$

Assume for the moment that $\Pi^+: L_2\to L_2$ is bounded, and consider

$$\int \langle g, \Pi^+ f\rangle\,dx = \int_R \langle bg, T_0(b E_R^b f)\rangle\,dx$$

Thus

$$\int \langle g, \tilde{T} f\rangle dx = \sum_{j=2}^{2^n} \int_{\mathbb{R}} \langle bg, T_0(b \cdot f \cdot \chi_{Q_j})\rangle dx$$

$$- \int_{\mathbb{R}} \langle bg, T_0(b \cdot (E_{Q_j} f) \cdot \chi_{Q_j^c})\rangle dx.$$

Inspection of the proof of thm 11.4 now shows that this goes through for

$$\tilde{T} := T - \Pi^a - \sum_{Q \in \mathcal{D}} (\Delta_Q^b)^*(T(1)) \cdot E_Q^b - \Big(\sum_{Q \in \mathcal{D}} (\Delta_Q^b)^*(T^*(1)) E_Q^b\Big)^*,$$

using the principle of almost orthogonality, the cancellation $\int_Q bf = 0$, $f \in W_Q^b$, and γ-Hölder regularity of $k_0(x,y)$.

It remains to show L_2-boundedness of Π^+. However, this is almost immediate: by lemma 12.8,

$$\Pi^+ f(x) = b(x)^t \sum_{Q \in \mathcal{D}} \Delta_Q^{b^t}(T_0(b))(x) \, (⨍_Q b)^{-1} E_Q(bf).$$

Thus

$$\|\Pi^+ f\|_2^2 \approx \sum_{Q \in \mathcal{D}} \|\Delta_Q^{b^t}(T_0(b))\|_2^2 \cdot |E_Q(bf)|^2$$

$$\lesssim \|T_0 b\|_{BMO}^2 \|bf\|_2^2 \lesssim \|f\|_2^2 \quad \text{by Carleson's}$$

theorem as in Prop. 10.6.

Together with the calculations analogous to those for the $T(1)$-theorem, this completes the proof of the $T(b)$-theorem. ∎

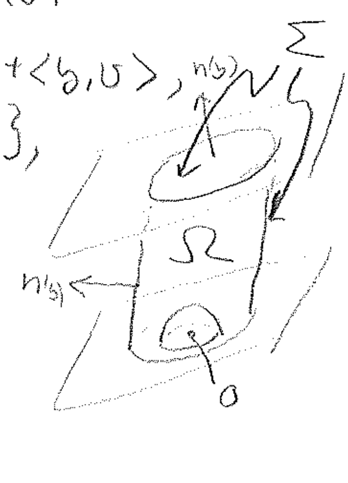
Exc. 12.10: (See end of Ex. 12.3.)

Let $v \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and let e_{n+1} be the basis vector for $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$. For $0 < \delta < \varepsilon$, let

$$\Omega := \{y + s e_{n+1} \,;\, y \in \mathbb{R}^n, |y| < \varepsilon, \langle y, v\rangle < s < \varepsilon + \langle y, v\rangle, |y|^2 + s^2 > \delta^2\},$$

and

$$\Sigma := \{y + s e_{n+1} \,;\, |y| = \varepsilon, \langle y, v\rangle < s < \varepsilon + \langle y, v\rangle\}$$
$$\cup \{y + s e_{n+1} \,;\, |y| < \varepsilon, s = \varepsilon + \langle y, v\rangle\}.$$



Apply Gauss' and Stokes' theorems to Ω, to show

$$\int_{\Sigma} \frac{y}{|y|^{n+1}} \cdot n(y)\, d\sigma(y) = \frac{1}{2}\sigma_n$$

Clifford product,

L_p-boundedness of SIOs

In previous lectures, we have proved $T(1)$ and $T(b)$-theorem, which give sufficient and necessary conditions for L_2-boundedness of a SIO T. We now take the L_2-boundedness for granted, and deduce L_p-boundedness for $p \neq 2$. Recall from Ex. 7.4 that $1<p<\infty$ is the interval of interest.

Defn. 13.1: A Calderón-Zygmund (CZ) operator is a bounded linear operator $T : L_2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{R}^n)$, with a γ-Hölder regular SIO kernel $k(x,y)$.

Note that we now again only consider scalar SIOs, as compared to lecture 12.

We aim to prove the following classical result.

Thm 13.2: If $T: L_2 \to L_2$ is a CZ-operator, then for each $1<p<\infty$, the operator

$$T : L_2 \cap L_p \longrightarrow L_2$$

extends to a bounded operator

$$T : L_p \longrightarrow L_p.$$

The main step in the proof is an estimate of T on $f \in L_1 \cap L_2$. This makes use of the following CZ decomposition of functions $f \in L_1$.

Thm 13.3: Let $f \in L_1(\mathbb{R}^n)$ and $\lambda > 0$ be given. Then there are disjoint cubes $Q_k \in \mathbb{D}$, $k = 1,2,\ldots$ such that

(1) $|f(x)| \leq \lambda$, a.e. $x \notin \bigcup_{k=1}^{\infty} Q_k$

(2) $\frac{1}{|Q_k|}\int_{Q_k} |f(x)| \lesssim \lambda$, $k=1,2,\ldots$

(3) $\sum_{k=1}^{\infty} |Q_k| \lesssim \frac{1}{\lambda} \|f\|_{L_1}$.

Proof: Consider the dyadic Hardy-Littlewood maximal function $M_d f(x) := \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \frac{1}{|Q|} \int_Q |f(x)| dx$.

Clearly we have a pointwise estimate $M_d f(x) \lesssim M f(x)$, so Thm 9.4 yields

$$\sup_{\lambda > 0} \lambda |\underbrace{\{x ; M_d f(x) > \lambda\}}_{=: \Omega}| \lesssim \|f\|_1 .$$

Let Q_k be the maximal dyadic cubes contained in Ω. As in Thm 9.9, we have

$$\Omega = \dot{\bigcup_{k=1}^{\infty}} Q_k .$$

At each Lebesgue point $x \notin \Omega$ to f, we have $|f(x)| \le M_d f(x) \le \lambda$, so (1) holds.

Moreover (3) holds, since $\sum |Q_k| = |\Omega| \lesssim \frac{1}{\lambda} \|f\|_1$.

To prove (2), let R be the parent of Q_k. Since Q_k is maximal, $R \not\subset \Omega$, so

$$\int_R |f(x)| dx \le |R| .$$

We get

$$\int_{Q_k} |f(x)| dx \le \int_R |f(x)| dx \le |R| \lesssim |Q_k| . \blacksquare$$

In Thm 13.3, the cubes Q_k are often referred to as "bad cubes". These are the places where "f is big", and (3) is saying that the bad cubes are not too big.

Proof of thm 13.2:

We claim that it suffices to show the weak L_1 estimate

$$\sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n ; |Tf(x)| > \lambda\}| \lesssim \|f\|_{L_1(\mathbb{R}^n)} , \quad \text{for all } f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n).$$

Indeed, if this holds, then interpolation with $T : L_2 \to L_2$ and the Marcinkiewicz interpolation theorem shows boundedness

$$T : L_p \to L_p , \quad 1 < p < 2 .$$

Duality then shows boundedness for $2 < p < \infty$.

To show the weak L_1 estimate for $f \in L_1 \cap L_2$, let Q_k be the cubes from Thm 13.3. Write

$$f = g + \sum_{k=1}^{\infty} b_k,$$

where $g(x) := \begin{cases} f(x) \,;\, x \notin \bigcup Q_k \\ \frac{1}{|Q_k|}\int_{Q_k} f\,dx \,;\, x \in Q_k \end{cases}$ and $b_k(x) := \begin{cases} 0 \,;\, x \notin \bigcup Q_k \\ f(x) - \frac{1}{|Q_k|}\int_{Q_k} f\,dx \,;\, x \in Q_k. \end{cases}$

Then $\|g\|_{L_\infty} \lesssim \lambda$, $\|g\|_{L_1} \le \|f\|_{L_1}$,

$\int_{Q_k} b_k\,dx = 0$, and $\frac{1}{|Q_k|}\int_{Q_k} |b_k|\,dx < \lambda$.

Note that

$$\{x; |Tf| > \lambda\} \subset \{x; |Tg| > \lambda/2\} \cup \{x; |\Sigma T b_k| > \lambda/2\}$$

We have

$$|\underbrace{\{x; |Tg| > \lambda/2\}}_{=:A}| = \int_A 1\,dx \le \left(\tfrac{2}{\lambda}\right)^2 \int_A |Tg|^2\,dx$$

$$\lesssim \lambda^{-2}\int_{\mathbb{R}^n} |Tg|^2\,dx \lesssim \lambda^{-2}\int_{\mathbb{R}^n} \underbrace{|g|^2}_{\lesssim \lambda \cdot |g|}\,dx$$

$$\lesssim \lambda^{-1}\|f\|_{L_1}.$$

Next let Q_k' be the cube with same center as Q_k but $\ell(Q_k') = 2\sqrt{n}\,\ell(Q_k)$. We have

$$|\underbrace{\{x; |\Sigma T b_k| > \lambda/2\}}_{=:B}| \le \sum_{k=1}^{\infty} |Q_k'| + \frac{2}{\lambda}\int_{B \setminus \bigcup Q_k'} |\Sigma T b_k|\,dx$$

$$\le (2\sqrt{n})^n \sum_{k=1}^{\infty} |Q_k| + \frac{2}{\lambda}\sum_{k=1}^{\infty}\int_{\mathbb{R}^n \setminus Q_k'} |T b_k|\,dx$$

$$\lesssim \frac{1}{\lambda}\|f\|_{L_1} + \frac{1}{\lambda}\sum_{k=1}^{\infty}\int_{\mathbb{R}^n \setminus Q_k'} \left| \int_{Q_k} (k(x,y) - k(x, \overset{\text{center of } Q_k}{y_k}))\, b_k(y)\,dy \right| dx$$

principle of almost orth.

$$\lesssim \frac{1}{\lambda}\|f\|_{L_1} + \frac{1}{\lambda}\sum_{k=1}^{\infty} \underbrace{\int_{\mathbb{R}^n \setminus Q_k'} \frac{dx}{|x-y_k|^{n+\gamma}}}_{\approx \frac{1}{\ell(Q_k)^\gamma}} \int_{Q_k} \underbrace{|y-y_k|^\gamma}_{\lesssim \ell(Q_k)^\gamma} |b_k(y)|\,dy$$

$$\lesssim \frac{1}{\lambda}\|f\|_{L_1} + \frac{1}{\lambda}\sum_{k=1}^{\infty} \underbrace{\int_{Q_k} |b_k(y)|\,dy}_{\lesssim |Q_k|\cdot\lambda} \lesssim \frac{1}{\lambda}\cdot\|f\|_{L_1}.$$

This proves the theorem. ∎

Invertibility of SIOs

This course has focused on two problems:

(I) Invertibility of linear equations
$Tx = y$, where typically T was a weakly singular integral operator.

(II) Boundedness of SIOs
$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy.$

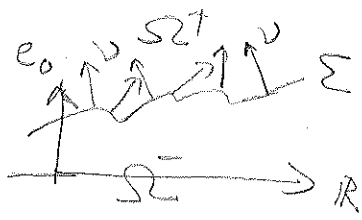
We now invertibility of SIOs. In general this is a very hard question and much remains to be understood here. For convolution SIO though it is fairly straightforward: T is bounded if $\|\hat{k}\|_\infty < \infty$ and T is invertible if furthermore $\|\hat{k}^{-1}\|_\infty < \infty$. Coming back to non-convolution SIOs, we shall here study our main example: the double layer potential on a Lipschitz graph from Ex. 7.11.

Ex. 13.4: Let $\Sigma \subset \mathbb{R}^{n+1}$ be the graph of a Lipschitz function $\varphi: \mathbb{R}^n \to \mathbb{R}$. Let K be the double layer potential operator on Σ:

$$Kh(x) = 2\,\text{p.v.} \int_\Sigma \underbrace{\frac{1}{\sigma_n} \frac{y-x}{|y-x|^n}}_{= E(y,x)} \cdot \nu(y)\, h(y)\, d\sigma(y)$$

See Defn 1.8.

By Ex. 12.6, the SIO $K: L_2(\Sigma) \to L_2(\Sigma)$ is bounded. Recall from lecture 6 that solvebility of the Dirichlet and Neumann problem in the regions Ω^+/Ω^- above / below Σ follows from invertibility of $I \pm K$ and $I \pm K^*$. We shall now show that this is indeed the case in the function space $L_2(\Sigma)$.



A key fact: Σ being Lipschitz means that for the unit normal $\nu(x)$ pointing into Ω^+

$$\exists c > 0\ \forall x \in \Sigma: \nu_0(x) := \langle \nu(x), e_0 \rangle \geq c.$$

The proof relies on a simple, but somewhat tricky, application of Stokes' theorem, referred to as a "Rellich estimate". We consider the bilinear and symmetric form

$$B(f,g) = \int_\Sigma \langle f(x),\nu(x)\rangle\langle g(x),e_0\rangle + \langle f(x),e_0\rangle\langle g(x),\nu(x)\rangle - \langle \nu(x),e_0\rangle\langle f(x),g(x)\rangle \, d\sigma(x)$$

for vector fields $f,g \in L_2(\Sigma;\mathbb{R}^{1+n})$ on Σ.

Here $e_0, e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^{1+n}$ and e_0 is pointing "upwards". Note that in coordinates

$$B(f,g) = \sum_0^n \int_\Sigma (f_i\nu_i)g_0 + (f_0\nu_i - \nu_0 f_i)g_i$$

$$= \sum_0^n \int_\Sigma (g_i\nu_i)f_0 + (g_0\nu_i - \nu_0 g_i)f_i .$$

(We remark that in Clifford algebra notation as in Ex. 12.3, we simply have $B(f,g) = \int \langle \nu f, g e_0\rangle d\sigma(x)$.)

The form B has the following remarkable properties.

(1) If $f = F|_\Sigma$, $g = G|_\Sigma$, for vector fields F and G in Ω^+, which are divergence and curl-free, and "decays appropriately" at ∞, then

$$B(f,g) = 0.$$

The same holds for Ω^- in place of Ω^+.

(2) If f and g are normal vector fields (pointwise parallel to ν), then

$$B(f,g) = + \int_\Sigma \langle f(x), g(x)\rangle \nu_0(x)\, d\sigma(x).$$

(3) If f and g are tangential vector fields (pointwise orthogonal to ν), then

$$B(f,g) = - \int_\Sigma \langle f(x), g(x)\rangle \nu_0(x)\, d\sigma(x).$$

(4) If f is tangential and g is normal, then

$$B(f,g) = \int_\Sigma \langle f(x), e_0\rangle\langle g(x), \nu(x)\rangle\, d\sigma(x).$$

Properties (2)-(4) are straightforward to check, and (1) follows directly from Gauss'/Stokes' theorems.

Note that invertibility of $I \pm K$ follows from that of $I \pm K^*$. Recall the following from lecture 6.
Given a scalar function $h \in L_2(\Sigma)$, define the single layer potential

$$v^\pm(x) := \int_\Sigma \Gamma(y,x)\, h(y)\, d\sigma(y), \quad x \in \Omega^\pm.$$

Since $v^\pm$ are harmonic in $\Omega^\pm$, their gradients $F^\pm := \nabla v^\pm$ are divergence and curl free vector fields.
For the traces $f^\pm := F^\pm|_\Sigma$, we have

$$\begin{cases} 2\langle f^+, \nu\rangle = (-I + K^*) h \\ 2\langle f^-, \nu\rangle = (I + K^*) h. \end{cases}$$

Write $f_t^\pm := f^\pm - \underbrace{\langle f^\pm, \nu\rangle \nu}_{=: f_n^\pm}$ for the tangential parts.
From lecture 6 it follows that $f_t^+ = f_t^- =: f_t$

By (1)–(4) we have

$$(i)\ 0 = B(f^+, f^+) = B(f_n^+, f_n^+) + 2B(f_t, f_n^+) + B(f_t, f_t)$$
$$= \frac{1}{4}\int_\Sigma |-h + K^* h|^2 \nu_0 + \int_\Sigma \langle f_t, e_0\rangle(-h + K^* h) - \int_\Sigma |f_t|^2 \nu_0,$$

$$(ii)\ 0 = B(f^-, f^-) = B(f_n^-, f_n^-) + 2B(f_t, f_n^-) + B(f_t, f_t)$$
$$= \frac{1}{4}\int_\Sigma |h + K^* h|^2 \nu_0 + \int_\Sigma \langle f_t, e_0\rangle(h + K^* h) - \int_\Sigma |f_t|^2 \nu_0.$$

We now use these "Rellich formulae" to prove lower bounds on $I \pm K^*$ as in Prop. 3.10 (even without compact term). From (i), we get estimates

$$\|-h + K^* h\| \le C\left(\|f_t\| \cdot \|-h + K^* h\| + \|f_t\|^2\right)$$
$$/\text{use } 0 \le \left(\sqrt{c/2}\, x - \tfrac{1}{\sqrt{2c}}\, y\right)^2 = \tfrac{c}{2}x^2 + \tfrac{1}{2c}y^2 - xy\ /$$
$$\le \frac{C^2}{2}\|f_t\|^2 + \frac{1}{2}\|-h + K^* h\|^2 + C\|f_t\|^2$$

$$\Rightarrow \|-h + K^* h\|^2 \le (C^2 + 2C)\|f_t\|^2 \lesssim \|f_t\|^2$$

Similarly we get from (i)

$$\|f_t\|^2 \le C\left(\|f_t\| \cdot \|-h + K^* h\| + \|-h + K^* h\|^2\right)$$

and a similar "absorption argument" gives

$$\|f_t\|^2 \lesssim \|-h + K^* h\|^2.$$

This, and a similar calculation from (ii), yields
$\|-h+K^*h\| \approx \|f_t\| \approx \|h+K^*h\|$.

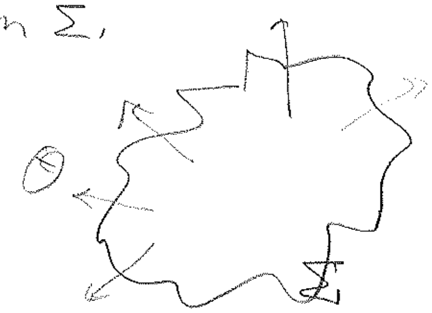
$$\Rightarrow \|h\| = \left\| \frac{h+K^*h}{2} - \frac{-h+K^*h}{2} \right\| \lesssim \|h+K^*h\| + \|-h+K^*h\|$$
$$\approx \|h+K^*h\| \approx \|-h+K^*h\|.$$

This proves $I \pm K^* \in SF_+(L_2(\Sigma), L_2(\Sigma))$.
We next want to show that $I \pm K^*$ is invertible.
For this it suffices to show $i(I \pm K^*) = 0$. We apply the method of continuity 5.19 to
$\lambda \mapsto \lambda I - K^*$.
Since $\|K^*\|_{L_2 \to L_2} < \infty$, $\lambda I - K^*$ is invertible for $|\lambda|$ large enough. Thus it suffices to show

<u>Claim</u>: $\|f\|_2 \lesssim \|(\lambda I - K^*)f\|_2$ for each $\lambda \in \mathbb{R}, |\lambda| > 1$.

In (i) and (ii), write
$$|-h+K^*h|^2 = |(\lambda h - K^*h) - (\lambda-1)h|^2 = |\lambda h - K^*h|^2 + (\lambda-1)^2|h|^2$$
$$- 2(\lambda-1)\, h \cdot (\lambda h - K^*h),$$
$$|h+K^*h|^2 = |(\lambda h - K^*h) - (\lambda+1)h|^2 = |\lambda h - K^*h|^2 + (\lambda+1)^2|h|^2$$
$$- 2(\lambda+1)\, h\, (\lambda h - K^*h)$$
To eliminate the mixed term, we form
$(\lambda+1)\cdot(i) - (\lambda-1)\cdot(ii)$, and get
$$0 = \frac{1}{4}\left\{ 2\int_\Sigma |\lambda h - K^*h|^2 \nu_0 - 2(\lambda^2-1) \int_\Sigma |h|^2 \nu_0 \right\}$$
$$- 2\int_\Sigma \langle f_t, e_0\rangle (\lambda h - K^*h) + 2\int_\Sigma |f_t|^2 \nu_0.$$
$$\Leftrightarrow (\lambda^2-1)\int_\Sigma |h|^2\nu_0 + 4\int_\Sigma |f_t|^2 \nu_0 = \int_\Sigma |\lambda h - K^*h|^2 \nu_0 + 4\int_\Sigma \langle f_t, e_0\rangle (\lambda h - K^*h)$$
Since $\nu_0 \geq c > 0$ and $\|f_t\|_2 \lesssim \|h\|_2$, we get
$$\|h\|_2^2 \lesssim \|\lambda h - K^*h\|_2^2 + \|h\|_2 \cdot \|\lambda h - K^*h\|_2.$$
As before an absorption argument gives
$$\|h\|_2 \lesssim \|\lambda h - K^*h\|_2.$$
This shows that $\lambda I - K^*$ is invertible for $\lambda \in \mathbb{R}, |\lambda| \geq 1$.

- For bounded Lipschitz domains, Rellich estimates can also be done. In this case we replace e_0 by a smooth vector field $\theta(x)$ which is transversal on Σ.
In this case we only get lower bounds modulo compact terms as in Prop. 3.10. These arise in Gauss'/Stokes' theorem application in (1).

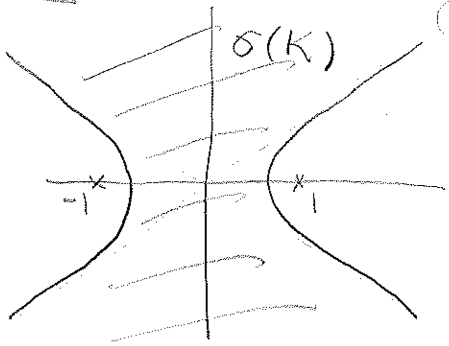


- Coming back to the Lipschitz graph in Ex. 13.4, one can prove the following estimate of the complex spectrum of K:

$$\sigma(K; L_2(\Sigma)) \subset \{x+iy \in \mathbb{C} ; \ x^2 \geq L^2(y^2 + \tfrac{1}{L^2+1})\},$$

where L is the Lipschitz constant for Σ:

$$L := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|}.$$



For general Lipschitz surfaces, this is what is know about $\sigma(K)$ today.

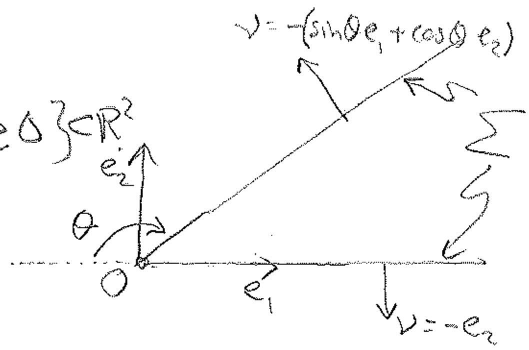
We next show how one in simple special cases can compute exactly the spectrum.

<u>Ex. 13.5</u>: Fix $0<\theta<\pi$ and consider the Lipschitz curve

$$\Sigma := \{(t,0) ; t\geq 0\} \cup \{t(-\cos\theta, \sin\theta) ; t\geq 0\} \subset \mathbb{R}^2$$

We want to compute

$$\sigma(K; L_2(\Sigma)) \subset \mathbb{C}.$$



- Consider the isometry

$$V_1 : L_2(\Sigma) \to L_2(\mathbb{R}_+; \mathbb{C}^2), \text{ where } (V_1 f)(t) := \begin{bmatrix} f(t) \\ f(-t\cos\theta, t\sin\theta)\end{bmatrix}.$$

We have

$$\sigma(K; L_2(\Sigma)) = \sigma(\underbrace{V_1 K V_1^{-1}}_{=:K_1}; L_2(\mathbb{R}_+; \mathbb{C}^2)).$$

From Defn. 1.8, we get

$$K_1 \begin{bmatrix} f \\ 0 \end{bmatrix}(t) = \begin{bmatrix} 0 \\ \frac{1}{\pi}\int_0^\infty \frac{se_1 - t(-\cos\theta e_1 + \sin\theta e_2)}{s^2+t^2+2st\cos\theta} \cdot (-e_2) f(s)\, ds \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ \frac{1}{\pi}\int_0^\infty \frac{t\sin\theta}{t^2+s^2+2\cos\theta\, st} f(s)\,ds \end{bmatrix}$$

$$K_1\begin{bmatrix} 0 \\ f \end{bmatrix}(t) = \begin{bmatrix} \frac{1}{\pi}\int_0^\infty \frac{s.(-\cos\theta e_1+\sin\theta e_2)-te_1}{s^2+t^2+2st\cos\theta}\cdot(-\sin\theta e_1-\cos\theta e_2) f(s)\,ds \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\pi}\int_0^\infty \frac{\sin\theta\, t}{t^2+s^2+2\cos\theta\, st} f(s)\,ds \\ 0 \end{bmatrix}$$

Define the operator $T: L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})$ by

$(Tf)(t) = \frac{1}{\pi}\int_0^\infty \frac{t\sin\theta}{t^2+s^2+2\cos\theta\, st} f(s)\,ds$, so that

$$K_1 = \begin{bmatrix} 0 & T \\ T & 0 \end{bmatrix}.$$

- To analyse T, we make use of the isometry $V_2: L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})$, where $(V_2 f)(x) = e^{x/2} f(e^x)$.

(Note that $\int_{-\infty}^\infty |e^{x/2} f(e^x)|\,dx = \int_0^\infty |f(t)|^2\,dt$.)

We compute the similar operator $V_2 T V_2^{-1}$.

$$(V_2 T V_2^{-1} g)(x) = e^{x/2}(TV_2^{-1}g)(e^x)$$

$$= e^{x/2}\frac{1}{\pi}\int_0^\infty \frac{e^x\sin\theta}{e^{2x}+s^2+2\cos\theta\, s e^x}(V_2^{-1}g)(s)\,ds$$

$$= /s=e^y/ = e^{x/2}\frac{1}{\pi}\int_{-\infty}^\infty \frac{e^x\sin\theta}{e^{2x}+e^{2y}+2\cos\theta\, e^{x+y}}(e^{-y/2}g(y))(e^y\,dy)$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{\frac{x-y}{2}}\sin\theta}{e^{x-y}+e^{y-x}+2\cos\theta}\, g(y)\,dy.$$

Let $k(t) := \frac{1}{\pi}\,\frac{e^{t/2}\sin\theta}{e^t+e^{-t}+2\cos\theta}$.

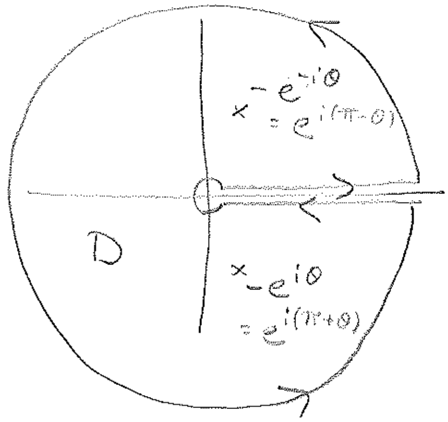
Then $V_2 T V_2^{-1} g = k * g$.

We want to compute the Fourier transform

$$\hat{k}(\xi) = \frac{\sin\theta}{\pi}\int_{-\infty}^\infty \frac{e^{x/2}\cdot e^{-ix\xi}}{e^x+e^{-x}+2\cos\theta}\,dx = /e^x=t/ = \frac{\sin\theta}{\pi}\int_0^\infty \frac{t^{1/2}\cdot t^{-i\xi}\,dt}{(t+\cos\theta)^2+\sin^2\theta}$$

where $t^{-i\xi} = e^{-i\xi\log(t)}$, using residue calculus

Set $f(z) := \dfrac{z^{1/2 - i\xi}}{(z+\cos\theta)^2 + \sin^2\theta}$



Then $f(z)$ is meromorphic in D as in picture, with poles $e^{\pm i\theta}$. We choose branch $0 < \arg z < 2\pi$ and note that $z^{-i\xi}$ is bounded in D. Residue calculus as indicated in the picture gives

$$\int_0^\infty \frac{t^{1/2 - i\xi}}{(t+\cos\theta)^2 + \sin^2\theta}\,dt - \int_0^\infty \frac{t^{1/2-i\xi} e^{(\frac{1}{2}-i\xi)2\pi i}}{(t+\cos\theta)^2+\sin^2\theta}\,dt$$

$$= 2\pi i \left(\frac{e^{(\frac{1}{2}-i\xi) i(\pi-\theta)}}{-e^{i\theta}+e^{-i\theta}} + \frac{e^{(\frac{1}{2}-i\xi)(\pi+\theta)i}}{-e^{-i\theta}+e^{i\theta}} \right)$$

$$= \frac{\pi}{\sin\theta}\left(-e^{(\pi-\theta)(\xi+\frac{i}{2})} + e^{(\pi+\theta)(\xi+\frac{i}{2})} \right)$$

$$\Rightarrow \hat{k}(\xi) = \frac{-e^{(\pi-\theta)(\xi+\frac{i}{2})} + e^{(\pi+\theta)(\xi+\frac{i}{2})}}{1+e^{2\pi\xi}} = \frac{i e^{\pi\xi}}{1+e^{2\pi\xi}}\left(-e^{-\theta(\xi+\frac{i}{2})} + e^{\theta(\xi+\frac{i}{2})}\right)$$

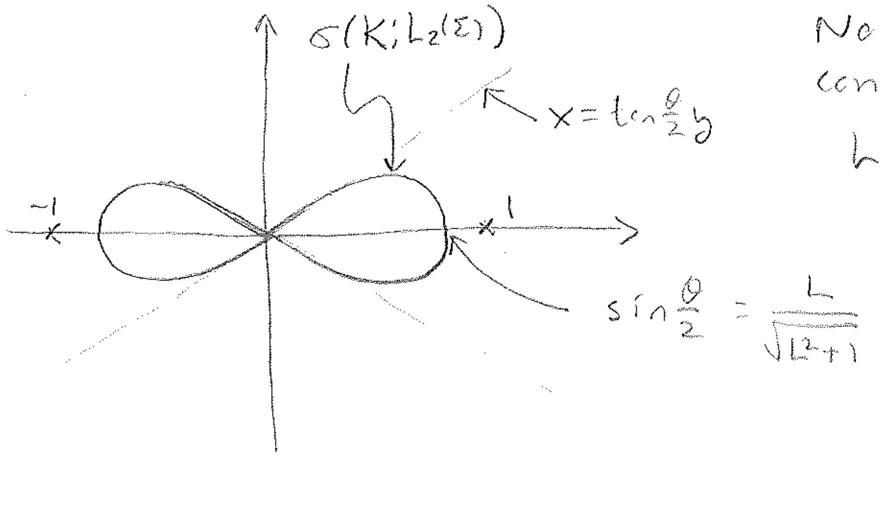
$$= \frac{i}{2}\frac{1}{\cosh(\pi\xi)}\, 2\underbrace{\sinh\left(\theta\xi + \frac{\theta i}{2}\right)}_{=\sinh\theta\xi \underbrace{\cosh\frac{\theta i}{2}}_{=\cos\frac{\theta}{2}} + \cosh\theta\xi \underbrace{\sinh\frac{\theta i}{2}}_{= i\sin\frac{\theta}{2}}}$$

$$= \frac{-\sin\frac{\theta}{2}\cosh\theta\xi + i\cos\frac{\theta}{2}\sinh\theta\xi}{\cosh(\pi\xi)}$$

Going back in calculations, we have $\lambda I - K$ invertible $\Leftrightarrow$ $\lambda I - K_1$ inv. $\Leftrightarrow$ $\begin{bmatrix} \lambda I & -T \\ -T & \lambda I \end{bmatrix}$ inv. $\Leftrightarrow$ $\sup_{\xi\in\mathbb{R}} \left\| \begin{bmatrix} \lambda & -\hat{k}(\xi) \\ -\hat{k}(\xi) & \lambda \end{bmatrix}^{-1} \right\| < \infty$

$\Leftrightarrow \inf_{\xi\in\mathbb{R}} |\lambda^2 - \hat{k}(\xi)^2| > 0$. Thus

$$\sigma(K; L_2(\Sigma)) = \left\{ \pm \frac{\sin\frac{\theta}{2}\cosh(\theta t) + i\cos\frac{\theta}{2}\sinh(\theta t)}{\cosh(\pi t)} \,;\, t\in\mathbb{R} \right\} \cup \{0\}$$



Note that the Lipschitz constant for Σ is $L = \tan\frac{\theta}{2}$