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One can hear the corners of a drum

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Abstract

We prove that the presence or absence of corners is spectrally determined in the following sense: any simply connected planar domain with piecewise smooth Lipschitz boundary and at least one corner cannot be isospectral to any connected planar domain, of any genus, that has smooth boundary. Moreover, we prove that amongst all planar domains with Lipschitz, piecewise smooth boundary and fixed genus, the presence or absence of corners is uniquely determined by the spectrum. This means that corners are an elementary geometric spectral invariant; one can hear corners.

1. *Introduction*

The isospectral problem is: if two compact Riemannian manifolds (M, g) and (M', g') have the same Laplace spectrum, then are they isometric? In this generality, the answer was first proved to be 'no' by Milnor [**10**]. He used a construction of lattices L_1 and L_2 by Witt [**15**] such that the corresponding flat tori $\mathbb{R}^{16}/L_i$ have the same Laplace spectrum for $i = 1, 2$, but are not isometric. Although the tori are not congruent, the fact that they are isospectral implies a certain amount of rigidity on their geometry. This rigidity is due to the existence of *geometric spectral invariants*, geometric features that are determined by the spectrum.

The first geometric spectral invariant was discovered by Hermann Weyl in 1912 [**14**]: the n-dimensional volume of a compact n-manifold is a spectral invariant. If the manifold has boundary, then the $n-1$-dimensional volume of the boundary is also a spectral invariant, as shown by Pleijel [**11**] in 1954. McKean and Singer [**9**] proved that certain curvature integrals are also spectral invariants. In the particular case of smooth surfaces and smoothly bounded planar domains, McKean–Singer and Kac [**7**] independently proved that the Euler characteristic is a spectral invariant.

Kac popularized the isospectral problem for planar domains by paraphrasing it as 'Can one hear the shape of a drum?' This makes physical sense because if a planar domain is the head of a drum, then the sound the drum makes is determined by the spectrum of the Laplacian with Dirichlet boundary condition. Therefore, if two such drums are isospectral, meaning they have the same spectrum for the Laplacian with Dirichlet boundary condition, then they sound the same. Do they necessarily have the same shape? The negative answer was demonstrated by Gordon, Webb, and Wolpert [**4**] who constructed two polygonal domains that are isospectral but not congruent; see Figure 1.

On the other hand, the isospectral problem has a positive answer in certain settings. It is easy to prove that if two rectangular domains are isospectral, then they are congruent. It requires more work to prove that if two triangular domains are isospectral, then they are congruent. That was demonstrated in Durso's technically demanding doctoral thesis at MIT which used both the wave and heat traces [**2**]. A simpler proof using only the heat trace was discovered

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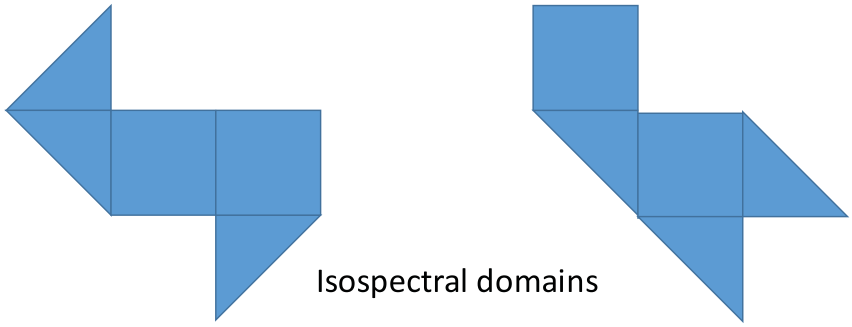

FIGURE 1 (colour online). *Identical sounding drums.*

by Grieser and Maronna [6]. Zelditch proved [16] that if two bounded domains with analytic boundary satisfying a certain symmetry condition are isospectral, then they are congruent.

We prove that the smoothness of the boundary, or equivalently the lack thereof, is a spectral invariant in the following sense: the spectrum determines whether or not a simply connected planar domain has corners.

THEOREM 1.1. *Let Ω be a simply connected planar domain with piecewise smooth Lipschitz boundary. If Ω has at least one corner, then Ω is not isospectral to any bounded planar domain with smooth boundary that has no corners.*

The proof of Theorem 1.1 can be adapted to demonstrate the following result.

COROLLARY 1.2. *Amongst all planar domains of fixed genus with piecewise smooth Lipschitz boundary, those that have at least one corner are spectrally distinguished.*

The corollary means, in the spirit of Kac, that 'one can hear corners', as depicted in Figure 2. Whereas many spectral invariants are complicated abstract objects, there are relatively few known elementary geometric spectral invariants. Although experts recognize that polygonal domains have an extra, purely local corner contribution to the heat trace, we are not aware of this having been exploited previously to show that corners are spectrally determined. It therefore appears that we have found the next elementary geometric spectral invariant for simply connected planar domains, since [7] and [9] showed, half a century ago, that the number of holes is a spectral invariant for smoothly bounded planar domains.

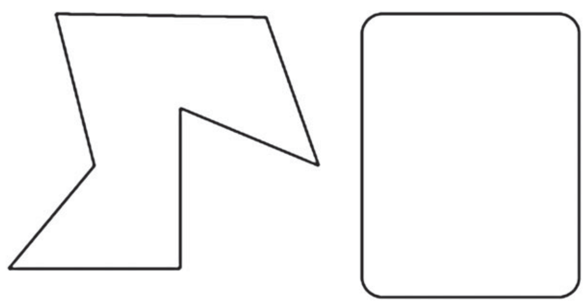

FIGURE 2. *No polygonal domain is isospectral to a smoothly bounded domain. One can hear that the left domain has corners, and one can also hear that the right domain has no corners.*

2. *Proofs*

2.1. *Preliminaries*

Let Ω be a bounded domain in the plane. The Laplace equation with Dirichlet boundary condition is

$$u_{xx} + u_{yy} = -\lambda u; \quad u|_{\partial\Omega} = 0.$$

The numbers λ such that there exists a non-trivial solution u are the eigenvalues of the Laplace operator, and the set of all these eigenvalues is the spectrum, which satisfies

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \uparrow \infty.$$

The rate at which $\lambda_k \to \infty$ as $k \to \infty$ was determined by Weyl in 1912, and is known as Weyl's law [14],

$$\lim_{k\longrightarrow\infty} \frac{|\Omega|\lambda_k}{4\pi k} = 1.$$

Above, $|\Omega|$ denotes the area of Ω.

Positive isospectral results, such as [2] and [6], often employ the heat trace,

$$h(t) := \sum_{k\geqslant 1} e^{-\lambda_k t}.$$

Since this quantity is determined by the spectrum, it is a spectral invariant. It can be shown using Weyl's law that the heat trace is asymptotic to

$$h(t) \sim \frac{|\Omega|}{4\pi t} \quad t \downarrow 0. \tag{2.1}$$

It is also possible to compute the next two terms in the short-time asymptotic expansion of the heat trace. In 1954, Pleijel proved [11] that

$$h(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}}, \quad t \downarrow 0. \tag{2.2}$$

Above $|\partial\Omega|$ denotes the perimeter of Ω. The next term in the expansion, which we denote by a_0, is the key to the proof of our results, and so we briefly explain how to compute it.

As the name suggests, the heat trace is equivalently defined by taking the trace of the heat kernel, that is, the fundamental solution of the heat equation. For an $\mathcal{L}^2$ orthonormal basis of eigenfunctions $\{\varphi_k\}_{k\geqslant 1}$ of the Laplacian with corresponding eigenvalues $\{\lambda_k\}_{k\geqslant 1}$, the heat kernel is

$$H(x, y, x', y', t) = \sum_{k\geqslant 1} e^{-\lambda_k t}\varphi_k(x, y)\varphi_k(x', y').$$

Another way to construct the heat kernel is to begin with an approximate solution to the heat equation, or parametrix, and then solve away the error term using the heat semi-group property and Duhamel's principle; for a manifold without boundary, see, for example, [12, Chapter 3]. A fundamental property of the heat kernel is that it is asymptotically local for short times. Kac referred to this locality principle as 'not feeling the boundary' [7]. He exploited this principle to compute the first three terms in the heat trace expansion as $t \downarrow 0$.

To show how the third term, a_0, in the asymptotic expansion of the heat trace for $t \downarrow 0$ arises, we must also outline the calculation of the first two terms. The first term comes from considering the most elementary approximation to the heat kernel for a planar domain: the Euclidean heat kernel on $\mathbb{R}^2$,

$$\frac{e^{|(x,y)-(x',y')|^2/4t}}{4\pi t}.$$

By the locality principle (see also [8, Lemma 2.2]), the trace of the actual heat kernel for the domain Ω is asymptotically equal to the corresponding trace of the Euclidean heat kernel. Along the diagonal, $(x,y)=(x',y')$, so the Euclidean heat kernel is simply

$$\frac{1}{4\pi t}.$$

The integral over the domain Ω is therefore $|\Omega|/4\pi t$. Consequently, the leading order asymptotic of the heat trace is given by (2.1).

Whereas the Euclidean heat kernel is a good parametrix on the interior, it does not 'feel the boundary'. Since we assume the boundary is piecewise smooth, in a small neighborhood of each boundary point away from the corner points, the heat kernel for a half plane with Dirichlet boundary condition is a good parametrix, meaning it is a good approximation to the actual heat kernel for the domain. For the $x \geqslant 0$ half plane, this heat kernel is

$$\frac{1}{4\pi t}\left(e^{-(x-x')^2/4t} - e^{-(x+x')^2/4t}\right) e^{-(y-y')^2/4t}. \tag{2.3}$$

For curved boundary neighborhoods away from any corners, since the boundary is smooth, it is a one-dimensional manifold. Therefore, the smaller the boundary neighborhood is, the closer the boundary is to being straight in this neighborhood. Consequently, by the locality principle, the trace of the heat kernel for the half plane over this neighborhood is asymptotically equal to the trace of the heat kernel for Ω along the same neighborhood. We calculate that integrating (2.3) with $x=x'$ and $y=y'$ yields

$$\frac{|\mathcal{N}|}{4\pi t} - \frac{|\partial\mathcal{N}|}{8\sqrt{\pi t}},$$

where $|\mathcal{N}|$ denotes the area of the neighborhood and $|\partial\mathcal{N}|$ denotes the length of the boundary along this neighborhood. Since the first term gives the same local contribution as that of the Euclidean heat kernel, cumulatively we arrive at the first two terms in the asymptotic expansion of the heat trace (2.2).

In the case of smooth boundary, the next term in the heat trace expansion is best understood by the local parametrix method as in [12, Chapter 3]. McKean and Singer [9] and Kac [7] independently proved that the next term is

$$\frac{1}{12\pi}\int_{\partial\Omega} k\, ds,$$

where k is the geodesic curvature of the boundary (see also [13, Proposition 1]). By the Gauss–Bonnet Theorem, this integral is equal to $2\pi\chi(\Omega)$, where $\chi(\Omega)$ is the Euler characteristic of Ω. So, if there are no corners, then the heat trace

$$h(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{\chi(\Omega)}{6}, \quad t\downarrow 0, \quad a_0 = \frac{\chi(\Omega)}{6}. \tag{2.4}$$

REMARK 1. The difference between $h(t)$ and the above expansion is $O(\sqrt{t})$ as $t\downarrow 0$. This bound depends only on the $\mathcal{L}^2$-norm of the curvature of the boundary; see D_3 on p. 449 of [13].

For smoothly bounded planar domains, the number of holes is equal to $1-\chi(\Omega)$, so this shows that the number of holes in the domain is an elementary geometric spectral invariant *within the class of smoothly bounded domains*. Interestingly, it is not known whether or not this is a spectral invariant for domains that may have corners. The reason is that if there are corners, then these each contribute an extra, purely local term, to the coefficient a_0. This has been called a 'corner defect' or an 'anomaly' due to the fact that the coefficient a_0 is not continuous under Lipschitz convergence of domains; see [8].

FIGURE 3. *A domain with a generalized corner at the point p.*

This heat trace coefficient a_0 for a domain with polygonal boundary can be computed using the trace of a suitable parametrix in a neighborhood of the corner. This is precisely what Kac did [7] using a formula for the heat kernel of a wedge. His method unfortunately requires that the corner, and the wedge that opens with the same angle as the corner, are convex. So, the opening angle, that is the interior angle at the corner, may only be between 0 and π. Since we do not require convexity, that method will not suffice.

A more general method follows an idea of Ray. Apparently, this calculation was previously performed by Fedosov and published in Russian [3]. The idea was to express the Green's function for a wedge, with any opening angle between 0 and 2π, explicitly as a Kontorovich–Lebedev transform using special functions. The heat kernel is the inverse Laplace transform of the Green's function, so one can compute its trace using these special functions and integral transformations. Although Ray's work was never published, and Fedesov's work poses a difficulty to those who cannot read Russian, van den Berg and Srisatkunarajah independently worked through this calculation and published it in [1, § 2]. We have verified their calculation using the reference [5] by Gradshteyn and Ryzhik on special functions and integral transformations.

Although van den Berg's and Srisatkunarajah's calculation of the corner contribution to the heat trace was for polygonal boundary, we will show below how to compute for a *generalized corner*.

DEFINITION 1. A *generalized corner* is a point p on the boundary at which the following are satisfied.

(i) The boundary in a neighborhood of p is defined by a continuous curve $\gamma(t) : (-a, a) \to \mathbb{R}^2$ for $a > 0$ with $\gamma(0) = p$, such that γ is smooth on $(-a, 0]$ and $[0, a)$ and such that

$$\lim_{t \uparrow 0} \dot{\gamma}(t) = v_1, \quad \lim_{t \downarrow 0} \dot{\gamma}(t) = v_2.$$

Above v_1 and v_2 are tangent vectors based at p.

(ii) The *opening angle* at the point p is the *interior angle* at that corner, which is the angle between the vectors $-v_1$ and v_2. In particular, this angle lies in $(0, \pi) \cup (\pi, 2\pi)$.

We use the term generalized corner, because often a corner is assumed to have straight, non-curved edges in some neighborhood of the corner point. However, we only require that the boundary curve is asymptotically straight on each side of the corner point. It is interesting to note that a generalized corner, which may have curved boundary, nevertheless contributes the same purely local term in the heat trace like a classical straight corner.

THEOREM 2.1. *A generalized corner as defined above with opening angle θ gives a purely local contribution to the heat trace*

$$\frac{\pi^2 - \theta^2}{24\pi\theta}.$$

For a bounded planar domain Ω with piecewise smooth boundary and a finite set of generalized corners at the points $\mathfrak{P} = \{p_j\}_{j=1}^n$ with corresponding interior angles $\{\theta_j\}_{j=1}^n$, the heat trace

$$h(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{12\pi}\int_{\partial\Omega\setminus\mathfrak{P}} k\, ds + \sum_{j=1}^{n} \frac{\pi^2 - \theta_j^2}{24\pi\theta_j}, \quad t \downarrow 0. \tag{2.5}$$

Proof. We shall first prove the theorem for the case in which there is only one generalized corner, as illustrated in Figure 3. Let θ denote the opening angle at the corner point p. Consider two domains, $\Omega_+(\epsilon)$ and $\Omega_-(\epsilon)$, which we shall simply denote by $\Omega_\pm$, slightly abusing notation because these domains depend on a parameter ϵ. These domains are chosen such that away from a neighborhood of radius ϵ about p, denoted by $\mathcal{N}_\epsilon$, the boundaries of these domains coincide with that of Ω. In a neighborhood of radius $\epsilon/2$ about p, Ω_+ and Ω_- each have a corner with straight edges and respective opening angles $\theta_- < \theta < \theta_+$. We choose these domains such that

$$\Omega_- \subset \Omega \subset \Omega_+,$$

and all three domains have smooth boundary away from the point p. As the parameter $\epsilon \downarrow 0$, $\Omega_\pm \to \Omega$ in the sense of Lipschitz convergence of domains. Consequently, the areas and perimeters

$$|\Omega_\pm| \longrightarrow |\Omega|, \quad \text{and} \quad |\partial\Omega_\pm| \longrightarrow |\partial\Omega|, \quad \epsilon \downarrow 0. \tag{2.6}$$

Moreover, since the curvature is uniformly bounded, the integrals

$$\int_{\partial\Omega_\pm\setminus\mathcal{N}_\epsilon} k\, ds \longrightarrow \int_{\partial\Omega\setminus\{p\}} k\, ds, \quad \epsilon \downarrow 0. \tag{2.7}$$

By the definition of generalized corner, the angles $\theta_\pm$ may be chosen to converge to θ as $\epsilon \downarrow 0$ which shows that

$$\frac{\pi^2 - \theta_\pm^2}{24\pi\theta_i} \longrightarrow \frac{\pi^2 - \theta^2}{24\pi\theta}, \quad \epsilon \downarrow 0. \tag{2.8}$$

To relate the heat traces of $\Omega_\pm$ to the heat trace of Ω, we use domain monotonicity. Since $\Omega_- \subset \Omega \subset \Omega_+$, the eigenvalues,

$$\lambda_k^- = \lambda_k(\Omega_+) \leqslant \lambda_k = \lambda_k(\Omega) \leqslant \lambda_k(\Omega_-) = \lambda_k^+ \quad \forall k.$$

Therefore, the heat traces satisfy

$$h_-(t) = \sum_{k\geqslant 1} e^{-\lambda_k^- t} \leqslant \sum_{k\geqslant 1} e^{-\lambda_k t} = h(t) \leqslant \sum_{k\geqslant 1} e^{-\lambda_k^+ t} = h_+(t). \tag{2.9}$$

We use the locality principle of the heat kernel to compute the traces. Let $H_\pm$ denote the heat kernels for $\Omega_\pm$, respectively. Let χ be a discontinuous or sharp cutoff function such that in the neighborhood of radius $\epsilon/2$ about the corner point p, denoted by $\mathcal{N}_\epsilon$, $\chi \equiv 1$, and $\chi \equiv 0$ off this neighborhood. Let $H_{W_\pm}$ be the heat kernels for infinite wedges of opening angles $\theta_\pm$, respectively. By Lemma 2.2 of [8], the heat traces satisfy

$$|h_\pm(t) - \mathrm{Tr}(\chi H_{W_\pm} + (1-\chi)H_\pm)(t)| = O(t^\infty), \quad t \downarrow 0.$$

In § 2 of [1], the trace of the heat kernel for an infinite wedge was computed over a finite wedge with the same opening angle as the infinite one. This trace is

$$\frac{|W|}{4\pi t} - \frac{|\partial W|}{8\sqrt{\pi t}} + \frac{\pi^2 - \theta^2}{24\pi\theta} + O(e^{-c/t}), \tag{2.10}$$

where $|W|$ and $|\partial W|$ denote the area and the length of the sides of the (finite) wedge, and θ is the opening angle. Theorem 1 of [1] gives an estimate for the positive constant c.

Combining (2.10) with the trace of $H_\pm$ away from the corner, as computed in [9] (see also [13]), we have

$$\mathrm{Tr}(\chi H_{W_\pm} + (1-\chi)H_\pm)(t) = \frac{|\Omega_\pm|}{4\pi t} - \frac{|\partial\Omega_\pm|}{8\sqrt{\pi t}} + \frac{\pi^2 - \theta_\pm^2}{24\pi\theta_\pm} + \frac{1}{12\pi}\int_{\partial\Omega_\pm \setminus \mathcal{N}_\epsilon} k\, ds + O(\sqrt{t}).$$

Therefore,

$$h_\pm(t) = \frac{|\Omega_\pm|}{4\pi t} - \frac{|\partial\Omega_\pm|}{8\sqrt{\pi t}} + \frac{\pi^2 - \theta_\pm^2}{24\pi\theta_\pm} + \frac{1}{12\pi}\int_{\partial\Omega_\pm \setminus \mathcal{N}_\epsilon} k\, ds + O(\sqrt{t}). \tag{2.11}$$

By Remark 1, the remainders in (2.11) are uniformly $O(\sqrt{t})$ as $t \downarrow 0$, due to the boundedness of the curvature of the boundary. Letting $\epsilon \downarrow 0$, by (2.6), (2.7), (2.8), (2.11), and (2.9),

$$h(t) = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{\pi^2 - \theta^2}{24\pi\theta} + \frac{1}{12\pi}\int_{\partial\Omega \setminus \{p\}} k\, ds + O(\sqrt{t}), \quad t \downarrow 0.$$

In case the domain has a finite number of corners, this argument applied at each corner together with the locality principle of the heat kernel completes the proof. □

If $\partial\Omega$ has corners, then the curvature integral

$$\int_{\partial\Omega \setminus \mathfrak{P}} k\, ds = \sum_{j=1}^{n} \theta_j + \pi(2\chi(\Omega) - n), \tag{2.12}$$

and therefore

$$a_0 = \frac{1}{12}(2\chi(\Omega) - n) + \sum_{j=1}^{n} \frac{\pi^2 + \theta_j^2}{24\pi\theta_j}, \tag{2.13}$$

If Ω has no corners, then a_0 is given by (2.4). We shall now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. For a planar domain with at least one corner, the third heat trace invariant a_0 is given by (2.13). For a smoothly bounded planar domain, this heat trace invariant is given in (2.4). Any two domains that are isospectral have the same heat trace and therefore must have the same heat trace invariant a_0. We shall prove the theorem by showing this is impossible if one domain has at least one corner and the other domain is smoothly bounded.

Let Ω be a simply connected planar domain with piecewise smooth Lipschitz boundary and n generalized corners with corresponding interior angles θ_k for $k = 1, \ldots, n$. Then,

$$a_0 = \frac{1}{24}\sum_{k=1}^{n}\left(\frac{1}{x_k} + x_k\right) - \frac{n}{12} + \frac{1}{6}, \quad x_k = \frac{\theta_k}{\pi}.$$

Consider the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad f(x) = \sum_{k=1}^{n}\left(\frac{1}{x_k} + x_k\right).$$

If any $x_k \downarrow 0$, then $f \uparrow \infty$, so f does not have a maximum. The gradient of f vanishes precisely when $x_k = 1$ for all k. This is therefore the unique global minimum of the function f restricted to $x \in \mathbb{R}^n$ such that $x = (x_1, \ldots, x_n)$ with all $x_k > 0$.

Note that $x_k = 1$ corresponds to an interior angle equal to π at the kth corner, which is impossible if there is indeed a corner there. Consequently, we cannot have $x_k = 1$ for all k, and so

$$f(x) > f(1, \ldots, 1) = 2n.$$

We therefore have

$$a_0 = \frac{1}{24} f(x) - \frac{n}{12} + \frac{1}{6} > \frac{2n}{24} - \frac{n}{12} + \frac{1}{6} = \frac{1}{6} \implies a_0 > \frac{1}{6}. \tag{2.14}$$

For any smoothly bounded domain, the Euler characteristic is at most 1, and therefore the heat trace invariant (2.4) is at most $\frac{1}{6}$. □

An immediate consequence of our proof is the following.

Corollary 2.2. *One can hear corners. Specifically, a bounded simply connected planar domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary has at least one corner if and only if*

$$a_0 > \tfrac{1}{6}.$$

Equivalently, a bounded connected planar domain $\Omega \subset \mathbb{R}^2$ has smooth boundary if and only if

$$a_0 \leqslant \tfrac{1}{6}.$$

Proof. For any smoothly bounded domain, the heat trace invariant a_0 is one-sixth of the Euler characteristic (2.4) and consequently is at most 1/6. On the other hand, in (2.14), we have that for any simply connected planar domain with piecewise smooth boundary and at least one corner, $a_0 > 1/6$. These are mutually exclusive. □

We conclude with the proof of Corollary 1.2.

Proof of Corollary 1.2. For a connected bounded domain Ω with piecewise smooth boundary and at least one corner, then as above we have a_0 given by (2.13), which we can equivalently express as

$$a_0 = \frac{1}{24} \sum_{k=1}^{n} \left(\frac{1}{x_k} + x_k \right) - \frac{n}{12} + \frac{\chi(\Omega)}{6}, \quad \theta_k = \pi x_k.$$

It therefore follows that if the Euler characteristic is fixed, by the above arguments,

$$a_0 > \frac{\chi(\Omega)}{6}.$$

Since for any smoothly bounded domain Ω' with Euler characteristic identical to that of Ω, we have

$$a_0(\Omega') = \frac{\chi(\Omega')}{6} = \frac{\chi(\Omega)}{6} < a_0(\Omega),$$

and the result follows. □

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