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Theorem 1.4 of the article in question — its main result — is a prime orbit theorem for the geodesic flow of asymptotically hyperbolic manifolds with negative sectional curvatures. We correct a typo and supply a missing technical assumption to make the result and its proof correct. We also supply a remainder term required in the dynamical wave trace formula given in Theorem 1.1, relating the length spectrum to the regularized trace of the wave group. This correction does not affect the main terms in the trace formula nor its application by Borthwick and Perry (2011).

1. Corrections

Joint work with P. Suarez-Serrato and S. Tapie [Rowlett et al. 2011] has led to the discovery of a missing term in the dynamical wave trace formula given in Theorem 1.1 of [Rowlett 2009], due to a subtle gap in the proof. A correct version follows:

Theorem 1.1. *Suppose* (X, g) *is an asymptotically hyperbolic* $(n+1)$*-dimensional manifold with negative sectional curvatures. Let* $0\text{-tr}\cos(t\sqrt{\Delta - n^2/4})$ *denote the regularized trace of the wave group, and let* $t_0 > 0$*. Let* $\mathscr{L}_p$ *denote the set of primitive closed geodesics of* (X, g)*, and for* $\gamma \in \mathscr{L}_p$*, let* $l(\gamma)$ *denote the length of* γ*. Then*

$$0\text{-tr}\cos\big(t\sqrt{\Delta - n^2/4}\big) = \sum_{\gamma\in\mathscr{L}_p}\sum_{k\in\mathbb{N}} \frac{l(\gamma)\delta(|t| - kl(\gamma))}{2\sqrt{|\det(I - \mathscr{P}_\gamma^k)|}} + R(t), \tag{1-1}$$

as a distributional equality in $\mathscr{D}'([t_0, \infty))$[1]*, where* $\mathscr{P}_\gamma^k$ *is the k-times Poincaré map around* γ *in the cotangent bundle. The remainder* $R(t)$ *is continuous and can be written as the sum of two continuous terms*

$$R(t) = A(t) + B(t),$$

which satisfy the following.

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[1] $\mathscr{D}'(X)$ is the dual of $\mathscr{C}_0^\infty(X)$.

(1) *There exist constants* $\varepsilon, C > 0$ *such that*

$$\left| \int_0^\infty A(t)\cos(\lambda t)\rho(t)\,dt \right| \le C \tag{1-2}$$

for all $\lambda > 1$ *and* $\rho \in \mathscr{C}_0^\infty([t_0, \varepsilon \ln \lambda])$. *The constants* C *and* ε *depend only on* t_0 *and* $\|\rho\|_\infty$; *they are independent of* λ.

(2) $B(t)$ *is independent of the set of closed geodesics, and it is possible that* $B(t)$ *has exponential growth for large time.*

Theorem 1.1 of [Rowlett 2009] was restated in Theorem 5.1 of [Rowlett 2010], which is therefore subject to the same correction; it was also applied in [Borthwick and Perry 2011], but without use of the remainder estimate for large time, so the results there are unaffected.

Corollary 1.2 of [Rowlett 2009] (restated in [Rowlett 2010] as Corollary 5.2) should be corrected as follows.

Corollary 1.2. *Let* (X, g) *be a manifold with negative sectional curvatures that is hyperbolic near infinity. Then, for any* $t_0 > 0$, *as an element of* $\mathscr{D}'([t_0, \infty))$, *we have the distributional equality*

$$\sum_{s \in \mathscr{R}^{sc}} e^{(s-n/2)|t|} = \sum_{\gamma \in \mathscr{L}_p} \sum_{k \in \mathbb{N}} \frac{l(\gamma)\delta(|t| - kl(\gamma))}{\sqrt{|\det(I - \mathscr{P}_\gamma^k)|}} + A(t) + B(t).$$

Here the resonances $\mathscr{R}^{sc}$ *of the scattering operator* (see [Rowlett 2009]) *are summed* with multiplicity, *and the remainders* $A(t)$ *and* $B(t)$ *have the same properties as in Theorem 1.1.*

In Theorem 1.4 of [Rowlett 2009], a factor of h was omitted in the numerator; this was corrected in Theorem 4.4 of [Rowlett 2010]. An additional technical assumption, that the length-spectrum is nonarithmetic (see [Rowlett et al. 2011, §4]), was also missing. (Experts believe that this assumption always holds, but a proof is not known.) Thus the correct statement is this:

Theorem 1.4. *Suppose* (X, g) *is an asymptotically hyperbolic* $(n+1)$*-dimensional manifold with negative sectional curvatures and nonarithmetic length spectrum. Let* h *be the topological entropy of the geodesic flow, and assume* $h > 0$. *The dynamical zeta function*

$$Z(s) = \exp\left(\sum_{\gamma \in \mathscr{L}_p} \sum_{k \in \mathbb{N}} \frac{e^{-ksl_p(\gamma)}}{k} \right)$$

has a nowhere vanishing analytic extension to an open neighborhood of $\Re(s) \ge h$ *except for a simple pole at* $s = h$. *Let* $\mathscr{L}$ *denote the set of all closed geodesics, and* *for $\gamma \in \mathscr{L}$, let $l(\gamma)$ denote the length of γ. The length spectrum counting function*

$$N(T) := \#\{\gamma \in \mathscr{L} : l(\gamma) \leq T\} \tag{1-3}$$

satisfies $\lim_{T\to\infty} \frac{hTN(T)}{e^{hT}} = 1$.

The corrected statements of Theorem 1.1, Corollary 1.2 and Theorem 1.4 follow from the proofs in [Rowlett 2009], with the exception of statement (2) above, about the remainder term $B(t)$, which shall be proved in Section 2.

As we shall demonstrate below, the remainder term $B(t)$ may grow exponentially for large time. This was unexpected because in the compact case there is no such term, and in the noncompact model case of conformally compact hyperbolic manifolds, the remainder term has exponential decay as $t \to \infty$; see [Guillarmou and Naud 2006, Theorem 1.1].

Corollary 1.5 of [Rowlett 2009] (and the analogous Corollary 5.3 in [Rowlett 2010]) should both be replaced by the version below. Before stating it, we recall the standard big-O notation. For a function $f : \mathbb{R} \to \mathbb{C}$ and a function $F : \mathbb{R} \to \mathbb{R}$ we use the notation

$$f(t) = O(F(t)) \quad \text{as } t \to \infty$$

if there exist constants $T, C > 0$ such that $|f(t)| \leq CF(t)$ for all $t > T$.

Proposition 1.5. *Let (M^{n+1}, g) be a Riemannian manifold with negative sectional curvatures that is hyperbolic near infinity, and whose length spectrum is nonarithmetic. Assume the topological entropy of the geodesic flow is positive. Let W be the Sinai–Bowen–Ruelle potential, and let $\wp$ denote topological pressure.*

(1) *Assume the remainder $B(t)$ in Corollary 1.2 also satisfies a statement identical to (1) of Theorem 1.1, with the substitution of $B(t)$ for $A(t)$. If*

$$\wp(-W/2) > 0,$$

then the discrete spectrum of the Laplacian is nonempty and the infimum Λ_1 of the spectrum of the Laplacian satisfies the estimate

$$\Lambda_1 = \min \sigma_{\text{pp}}(\Delta) \leq \frac{n^2}{4} - \big(\wp(-W/2)\big)^2.$$

(2) *If both remainder terms are uniformly bounded as $t \to \infty$, then*

$$\wp(-W/2) > 0 \iff \sigma_{\text{pp}} \neq \varnothing,$$

and these equivalent conditions imply

$$\Lambda_1 = \frac{n^2}{4} - \big(\wp(-W/2)\big)^2.$$

(3) *The remainder satisfies*

$$|R(t)| = O\big(\sup\{e^{(s_1-n/2)t},\ e^{\wp(-W/2)t},\ t\}\big) \quad \text{as } t \to \infty.$$

(*If* $\sigma_{\mathrm{pp}} = \varnothing$, *the term* $e^{(s_1-n/2)t}$ *is omitted from the supremum.*)

(4) *The assumption about* $B(t)$ *in part* (1) *does not always hold.*

The proof of this proposition is based on Theorem 1.1, Corollary 1.2, and the following result, which will appear in [Rowlett et al. 2011].

Theorem 1.6. *Let* (M, g) *be a conformally compact manifold whose sectional curvatures satisfy* $-b^2 \le K_g \le -a^2 < 0$, *with nonabelian fundamental group*[2] *and nonarithmetic length spectrum. The weighted zeta function*

$$\tilde{Z}(s) = \exp\biggl(\sum_{\gamma\in\mathcal{L}_p}\sum_{k\in\mathbb{N}} \frac{e^{-ksl_p(\gamma)}}{k\sqrt{|\det(I-\mathcal{P}_\gamma^k)|}}\biggr)$$

is an analytic nonzero function on the half-plane $\Re(s) > \wp(-W/2)$. *It admits a meromorphic extension to the half plane*

$$\Re(s) > \wp(-W/2) - \inf\Bigl\{\frac{\lambda a}{b}, \frac{\lambda}{2}\Bigr\},$$

where λ *is the expansion factor of the geodesic flow on the nonwandering set. Moreover, with the exception of a simple pole at* $\wp(-W/2)$, *this extension is analytic and nonvanishing in an open neighborhood of* $\{\Re(s) \ge \wp(-W/2)\}$. *If* $\wp(-W/2) > 0$, *then we have the counting estimate*

$$\sum_{\gamma\in\mathcal{L}_T} |\det(I-\mathcal{P}_\gamma)|^{-1/2} \sim \frac{\exp\big(\wp(-W/2)T\big)}{\wp(-W/2)T} \quad \text{as } T \to \infty,$$

where $\mathcal{L}_T$ *denotes the set of closed geodesics of length at most* T.

2. Proofs

The relationship between the support of the test function and its oscillation in (1-2) is known in the context of semiclassical analysis as *Ehrenfest time.* Therefore, we will say that $(\lambda, T) \in (1,\infty)^2$ is an ε-*Ehrenfest pair* if it satisfies $T \le \varepsilon \ln \lambda$, where $\varepsilon > 0$ is a constant. Due to the oscillation of test functions needed to control the remainder, we shall use the "Dirichlet box principle" technique of Jakobson, Polterovich, and Toth:

[2]Note that this is equivalent to the positivity of the topological entropy of the geodesic flow (see [Rowlett et al. 2011, Proposition 3.12]).

Proposition 2.1 (Dirichlet box principle [Jakobson et al. 2008]). *There are infinitely many ε-Ehrenfest pairs $(\lambda, T(\lambda))$ such that*

$$\text{(2-1)} \quad \textit{for each } \gamma \in \mathscr{L} \textit{ with } l(\gamma) \le T, \quad \textit{there exists } k(\gamma) \in \mathbb{Z} \textit{ such that } |\lambda l(\gamma) - 2\pi k(\gamma)| \le \tfrac{1}{2}.$$

Remark. For such an ε-Ehrenfest pair $(\lambda, T(\lambda))$, for each $\gamma \in \mathscr{L}$ with $l(\gamma) \le T$,

$$\cos(\lambda l(\gamma)) \ge \tfrac{1}{2}.$$

Proof of Proposition 1.5. From the assumption of negative sectional curvatures and hyperbolicity near infinity, it follows that the manifold is asymptotically hyperbolic and has pinched negative sectional curvatures; see [Rowlett et al. 2011, §2]. We may therefore apply Corollary 1.2 and Theorem 1.6.

We first prove part (1). Let the assumptions in that statement hold and set

$$\eta := \wp(-W/2) > 0.$$

The spectral side of the trace formula can be reformulated in terms of the resonances of the resolvent as follows:

$$\sum_{s \in \mathscr{R}} m(s) e^{(s-n/2)|t|} + \sum_{k \in \mathbb{N}} d_k e^{-k|t|}.$$

There are at most finitely many terms in the first sum with $\Re(s) > n/2$, and these are in bijection with the pure point (discrete) spectrum (see [Mazzeo and Melrose 1987]) via

$$s(n-s) = \Lambda \in \sigma_{\text{pp}} \subset \left(0, \frac{n^2}{4}\right).$$

Write

$$\mathscr{R} = \mathscr{R}_{\text{pp}} \cup \mathscr{R}_0,$$

where

$$\mathscr{R}_{\text{pp}} = \{s \in \mathscr{R} \mid \Re(s) > n/2\}, \quad \mathscr{R}_0 = \{s \in \mathscr{R} \mid \Re(s) \le n/2\}.$$

We shall require the counting estimate demonstrated in Theorem 1.1 of [Borthwick 2008], which implies

$$\text{(2-2)} \qquad \mathcal{N}(R) := \sum_{\substack{s \in \mathscr{R} \\ |s| \le R}} m(s) = O(R^{n+1}), \quad \sum_{k=1}^{R} d_k = O(R^{n+1}).$$

Let $(\lambda, T(\lambda))$ be an ε-Ehrenfest pair that satisfies (2-1). Let

$$l_0 := \inf\{l(\gamma) \mid \gamma \in \mathscr{L}_p\}.$$

For $T > 0$, let $\rho_T \in \mathscr{C}^\infty((0, \infty))$ be a smooth, nonnegative function such that

$$\rho_T(t) = \begin{cases} 0 & \text{if } t \le l_0/2, \\ 1 & \text{if } l_0 \le t \le T, \\ 0 & \text{if } T+1 \le t. \end{cases} \tag{2-3}$$

Let

$$f(t) := \cos(\lambda t)\rho_{T(\lambda)}(t), \quad \rho(t) := \rho_{T(\lambda)}(t). \tag{2-4}$$

By Corollary 1.2 and the assumption of statement (1), we have $I = II$, where

$$I := \sum_{s\in\mathcal{R}} m(s) \int_0^\infty e^{(s-n/2)t} f(t)\,dt + \sum_{k=1}^\infty d_k \int_0^\infty e^{-kt} f(t)\,dt$$

and

$$II := \sum_{\gamma\in\mathcal{L}_p} \sum_{k=1}^\infty \frac{l(\gamma) f(kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}_\gamma^k)|}} + O(1).$$

By the Dirichlet box principle, for all $\gamma \in \mathcal{L}_p$ and for any $k \in \mathbb{N}$ with $kl(\gamma) = \ell(k\gamma) \le T(\lambda)$, there exists $j(\gamma) \in \mathbb{Z}$ such that

$$|\lambda l(k\gamma) - 2\pi j(\gamma)| \le \tfrac{1}{2}.$$

It follows that

$$f(kl(\gamma)) = \cos(\lambda k l(\gamma))\rho(kl(\gamma)) \ge \tfrac{1}{2}.$$

Therefore,

$$II \ge \sum_{\substack{\gamma\in\mathcal{L}_p,\, k\in\mathbb{N} \\ kl(\gamma)\le T}} \frac{l(\gamma)}{2\sqrt{|\det(I-\mathcal{P}_\gamma^k)|}} + O(1).$$

By Theorem 1.6, there exists a constant $C > 0$, independent of T, such that

$$II \ge C\frac{e^{\eta T}}{T} + O(1).$$

On the other hand, we can estimate I from above. Define

$$F(s) := \int_0^\infty e^{(s-n/2)t} f(t)\,dt.$$

Since the pure point spectrum is finite or empty, and $0 < \Lambda_1 \le \Lambda_2 \le \cdots < n^2/4$, with $\Lambda_j = s_j(n-s_j)$, it follows that

$$s_1 \ge s_2 \ge \cdots > \frac{n}{2}.$$

Therefore, for each $s \in \mathcal{R}_{\text{pp}}$,

$$|F(s)| \le \int_0^\infty e^{(s_1-n/2)t} \rho(t)\,dt = O(e^{(s_1-n/2)T}).$$

Next we estimate for $s \in \mathcal{R}_0$. Since

$$F(s) = \tfrac{1}{2}\int_0^\infty e^{st} e^{-(n/2)t}(e^{i\lambda t} + e^{-i\lambda t})\rho(t)\,dt,$$

for $s = 0$ and $s = n/2 \pm i\lambda$, we have

$$F(s) = O(T).$$

For $s \neq 0$, $s \neq n/2 \pm i\lambda$, integrating by parts k times (compare [Guillarmou and Naud 2006, §3]), we have

$$\begin{aligned} F(s) = \quad & \frac{(-1)^k}{2(s-n/2+i\lambda)^k}\int_0^\infty e^{(s-n/2+i\lambda)t}\rho^{(k)}(t)\,dt \\ & + \frac{(-1)^k}{2(s-n/2-i\lambda)^k}\int_0^\infty e^{(s-n/2-i\lambda)t}\rho^{(k)}(t)\,dt. \end{aligned}$$

We therefore have the estimate

$$|F(s)| = O\left(\frac{T}{|s|^k + |\lambda|^k}\right) = O\left(\frac{T}{|s|^k}\right).$$

For $s \neq 0$, $s \neq n/2 \pm i\lambda$, using the counting estimate (2-2) and the above with $k = n+2$, we estimate (compare [Guillarmou and Naud 2006, §3])

$$\sum_{s\in\mathcal{R}_0} |F(s)| = O(T) + \int_1^\infty \frac{d\mathcal{N}(R)}{R^{n+2}} = O(T).$$

By the counting estimate (2-2), it follows that there is a constant $c > 0$ (independent of k) such that

$$d_k \le ck^{n+1} \quad \text{for all } k \in \mathbb{N}.$$

Therefore the sum

$$\sum_{k=1}^\infty d_k e^{-kt}$$

converges uniformly on $[l_0/2, \infty)$ and is uniformly bounded. Consequently

$$\left|\int_0^\infty f(t)\sum_{k=1}^\infty d_k e^{-kt}\,dt\right| = O(T).$$

Putting this together with the estimate for $\mathcal{R}_{\mathrm{pp}}$, we have

$$I = O(e^{(s_1-n/2)T}) + O(T).$$

If the pure point spectrum of the Laplacian is empty, then there is no contribution to I from $\mathcal{R}_{\text{pp}} = \varnothing$, and we have

$$|I| = O(T) = |II| \geq C\frac{e^{\eta T}}{T}.$$

This implies that we cannot have $\eta > 0$. We have thus shown that $\sigma_{\text{pp}} = \varnothing \Longrightarrow \eta \leq 0$.

By the contrapositive, if $\eta > 0$ then $\sigma_{\text{pp}}(\Delta) \neq \varnothing$. In this case, denote the infimum of the spectrum σ_{pp} by Λ_1, with

$$s_1(n - s_1) = \Lambda_1.$$

By our above estimates,

$$O(e^{(s_1 - n/2)T}) + O(T) = |I| = |II| \geq C\frac{e^{\eta T}}{T}. \tag{2-5}$$

By the Dirichlet box principle, there exist infinitely many Ehrenfest pairs, so we may let $T \to \infty$, from which it follows that

$$s_1 - \frac{n}{2} \geq \eta, \quad \text{and hence} \quad s_1 \geq \eta + \frac{n}{2}.$$

This in turn implies the upper bound

$$\Lambda_1 \leq \left(\eta + \frac{n}{2}\right)\left(\frac{n}{2} - \eta\right) = \frac{n^2}{4} - \eta^2.$$

We now turn to part (2), and thus assume that $A(t)$ and $B(t)$ are both uniformly bounded as $t \to \infty$. In this case we do not need the Dirichlet box principle and may simply use the test function ρ. If $\eta > 0$, then by part (1) the pure point spectrum is nonempty and

$$s_1 \geq \eta + \frac{n}{2}.$$

We have the estimate

$$\int_0^\infty e^{(s_1 - n/2)t} \rho(t)\, dt \geq C e^{(s_1 - n/2)T}, \quad C > 0.$$

Combining this with our estimates above, we have

$$I \geq C e^{(s_1 - n/2)T} - C'T,$$

for some constant C'. Since $\eta > 0$, Theorem 1.6 implies that

$$II \leq C''\frac{e^{\eta T}}{T}$$

for some constant $C'' > 0$. Putting this together, we have

$$C e^{(s_1 - n/2)T} - C'T \leq I = II \leq C''\frac{e^{\eta T}}{T}.$$

It follows that

$$s_1 - \frac{n}{2} \le \eta,$$

which, combined with the inequality from (1) (note that the condition in (2) implies the one in (1)), gives

$$\eta > 0 \implies \sigma_{\mathrm{pp}} \neq \varnothing \text{ and } s_1 - \frac{n}{2} = \eta.$$

The last equality implies $\Lambda_1 = n^2/4 - \eta^2$.

Conversely, if $\sigma_{\mathrm{pp}} \neq \varnothing$, we may similarly estimate I from below, which shows that the dynamical side must also grow exponentially as $T \to \infty$. Since the remainder terms are bounded, by Theorem 1.6 we must have $\eta > 0$. This concludes the proof that $\sigma_{\mathrm{pp}} \neq \varnothing \iff \eta > 0$, and as we have seen we have $\Lambda_1 = n^2/4 - \eta^2$ in this case.

We shall prove (3) by contradiction. Assume that for any $C > 0$, for each $N \in \mathbb{N}$ there is $T_N > N$ with

$$|R(T_N)| > C \sup\{e^{(s_1 - n/2)T_N}, e^{\wp(-W/2)T_N}, T_N\}.$$

By the continuity of R, for each $N \in \mathbb{N}$ there exists a nonnegative test function $\rho_N \in \mathscr{C}_0^\infty(0, T_N + 1)$ such that

$$\int_0^\infty \rho_N(t)\, dt = 1,$$

and

$$\left| \int_0^\infty R(t) \rho_N(t)\, dt \right| > C \sup\{e^{(s_1 - n/2)T_N}, e^{\wp(-W/2)T_N}, T_N\}.$$

By Corollary 1.2,

$$\sum_{s \in \mathscr{R}} m(s) \int_0^\infty e^{(s-n/2)t} \rho_N(t)\, dt + \sum_{k=1}^\infty d_k \int_0^\infty e^{-kt} \rho_N(t)\, dt - \sum_{\gamma \in \mathscr{L}_p} \sum_{k=1}^\infty \frac{l(\gamma) \rho_N(kl(\gamma))}{\sqrt{|\det(I - \mathscr{P}_\gamma^k)|}} = \int_0^\infty R(t) \rho_N(t)\, dt.$$

By Theorem 1.6 and our above estimates, the norm of the left side is bounded above by

$$C' \sup\{e^{(s_1 - n/2)T_N}, e^{\wp(-W/2)T_N}, T_N\} \tag{2-6}$$

for a fixed constant $C' > 0$ that is independent of N and ρ_N. This in turn implies the same upper bound for the right side. This is a contradiction. Therefore, there exist constants $C > 0$ and $T > 0$ such that

$$|R(t)| \le C \sup\{e^{(s_1 - n/2)t}, e^{\wp(-W/2)t}, t\} \quad \text{for all } t > T,$$

which is the conclusion of (3).

To prove (4), we will describe a counterexample suggested by Gilles Carron and Samuel Tapie. Let $S = \mathbb{H}^2/\Gamma$ be a convex cocompact hyperbolic surface whose topological entropy satisfies $h = h(g_H) > \frac{1}{2}$, where g_H is the hyperbolic metric on S, and assume that the length spectrum is nonarithmetic. Then by [Sullivan 1979] its Laplacian admits an isolated first eigenvalue $\Lambda_H = h(1-h) \in \left(0, \frac{1}{4}\right)$. We shall assume for the sake of contradiction that for any manifold that has negative sectional curvatures and is hyperbolic near infinity, the remainder term $B(t)$ in Corollary 1.2 also satisfies the estimate (1-2).

Let Ω be a compact, convex subset of S that properly contains the nonwandering set,[3] and let

$$\Omega \subset B_R \subset B_{R'} \subset B_{R''},$$

with B_R, $B_{R'}$, $B_{R''}$ balls of radii $0 < R < R' < R''$.

Let $\alpha : S \to [1-\varepsilon, 1]$ be a smooth function satisfying

$$\alpha(x) = \begin{cases} 1 & \text{for } x \in \Omega, \\ 1-\varepsilon & \text{for } x \in B_{R'} \setminus B_R, \\ 1 & \text{for } x \in S \setminus B_{R''}. \end{cases} \tag{2-7}$$

Let g be the Riemannian metric defined for all $x \in S$ by $g(x) = \alpha(x) g_H(x)$. For $\varepsilon > 0$ sufficiently small, the sectional curvatures of g remain negative. Thus (S, g) is again hyperbolic near infinity, and a function is in $\mathscr{L}^2(S, g)$ if and only if it is in $\mathscr{L}^2(S, g_H)$. Since $\alpha \equiv 1$ on the nonwandering set, $g = g_H$ on Ω. It follows that the length spectrum of (S, g) is identical to the length spectrum of (S, g_H), and hence is also nonarithmetic. Moreover, $h_g = h_{g_H}$, and therefore the topological entropy of (S, g) is also positive; equivalently, the fundamental group of (S, g) is nonabelian. Since $W(g) = W(g_H)$ along all closed geodesics,

$$\eta := \wp\left(\frac{-W(g)}{2}\right) = \wp\left(\frac{-W(g_H)}{2}\right) = h - \tfrac{1}{2} > 0.$$

Since $B(t)$ satisfies the estimate (1-2) and the hypotheses of Proposition 1.5, it follows from (1) that the discrete spectrum of (S, g) is nonempty and

$$\Lambda_1 \le \tfrac{1}{4} - \eta^2 = \Lambda_H.$$

It follows that there exists $\phi : S \to [0, \infty)$ (not identically zero) such that

$$\frac{\int_S \|\nabla\phi\|_g^2 \, dv_g}{\int_S \|\phi\|_g^2 \, dv_g} = \Lambda_1 \le \Lambda_H.$$

[3]Note that since S is convex cocompact and hyperbolic, it is asymptotically hyperbolic as well as convex cocompact with pinched negative curvatures and therefore the nonwandering set is a compact subset; see [Joshi and Sá Barreto 2001; Rowlett et al. 2011].

For any positive smooth function ϕ,

$$\int_S \|\nabla\phi\|_g^2\,dv_g = \int_S g^{-1}(d\phi, d\phi)\alpha^2\,dv_{g_H} = \int_S \alpha^{-2} g_H^{-1}(d\phi, d\phi)\alpha^2\,dv_{g_H} = \int_S g_H^{-1}(d\phi, d\phi)\,dv_{g_H} = \int_S \|\nabla\phi\|_{g_H}^2\,dv_{g_H}.$$

By the maximum principle, ϕ cannot vanish identically on $B_{R'} \setminus B_R$, so by definition of α we have

$$\int_S \|\phi\|_{g_H}^2 \alpha^2\,dv_{g_H} < \int_S \|\phi\|_{g_H}^2\,dv_{g_H}.$$

Since $0 < \Lambda_1 \le \Lambda_H$, we have

$$\int_S \|\nabla\phi\|_{g_H}^2\,dv_{g_H} = \int_S \|\nabla\phi\|_g^2\,dv_g = \Lambda_1 \int_S \|\phi\|_g^2\,dv_g = \Lambda_1 \int_S \|\phi\|_{g_H}^2 \alpha^2\,dv_{g_H} < \Lambda_1 \int_S \|\phi\|_{g_H}^2\,dv_{g_H} \le \Lambda_H \int_S \|\phi\|_{g_H}^2\,dv_{g_H},$$

which leads to the estimate

$$\frac{\int_S \|\nabla\phi\|_{g_H}^2\,dv_{g_H}}{\int_S \|\phi\|_{g_H}^2\,dv_{g_H}} < \Lambda_H.$$

This contradicts the definition of Λ_H as the infimum of the spectrum of the Laplacian on S with respect to the hyperbolic metric. □

For manifolds of higher dimension, it is also possible to build counterexamples to the long-time estimate (1-2) for $B(t)$ using conformal deformations.

Proof of Theorem 1.1(2). The remainder term $A(t)$ is *defined* to depend only on the set of closed geodesics. We shall use the preceding example to prove statement (2) of Theorem 1.1. Since the set of closed geodesics is contained in the nonwandering set, which is a compact, convex subset of the manifold (see [Rowlett et al. 2011]), the estimate (1-2) follows from the Ehrenfest estimate for compact manifolds with pinched negative curvature as in [Jakobson et al. 2008] and in the original proof of this estimate in [Rowlett 2009]. The unexpected news is that the difference

$$B(t) := R(t) - A(t),$$

may have exponential growth for large time. To prove this, we shall use the example used to prove part (4) of Proposition 1.5. Let (S, g_H), (S, g), Λ_H, and Λ_1 be defined as above. Since the assumption that

$$\Lambda_1 \le \Lambda_H$$

leads to a contradiction, we have $\Lambda_1 > \Lambda_H$, and hence

$$s_1 < s_H = h = \eta + \tfrac{1}{2} = \wp\big(-W(g)/2\big) + \tfrac{1}{2},$$

where

$$s_1 = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \Lambda_1}.$$

For the constant curvature metric (S, g_H), the Guillarmou–Naud trace formula [2006, Theorem 1.1, Theorem 1.2] implies

$$\sum_{s \in \mathcal{R}} m(s) e^{(s-n/2)|t|} + \sum_{k \in \mathbb{N}} d_k e^{-k|t|} = \sum_{\gamma \in \mathcal{L}_p} \sum_{k=1}^{\infty} \frac{l(\gamma)\delta(|t| - kl(\gamma))}{\sqrt{|\det(I - \mathcal{P}_\gamma^k)|}} + R_H(t),$$

where

$$R_H(t) = O(e^{-t/2}) \quad \text{as } t \to \infty.$$

Let us write

$$R_H(t) = A_H(t) + B_H(t),$$

where A_H is defined to be the contribution from closed geodesics, and

$$B_H := R_H - A_H.$$

The perturbation (S, g) is hyperbolic near infinity and has negative sectional curvatures. Let $R(t)$ denote the remainder in the trace formula Theorem 1.1 for (S, g), and similarly write

$$R(t) = A(t) + B(t).$$

Since the perturbation did not change the set of closed geodesics,

$$A = A_H.$$

Estimating as in the proof of Proposition 1.5, by (2-5),

$$O(e^{(s_1 - 1/2)T}) + O(T) = |I| = |II| \geq C \frac{e^{\eta T}}{T} - \left| \int_0^\infty R(t) f(t)\, dt \right|.$$

Rearranging, we have

$$\left| \int_0^\infty R(t) f(t)\, dt \right| \geq C \frac{e^{\eta T}}{T} - O(e^{(s_1 - 1/2)T}). \tag{2-8}$$

Since

$$s_1 - \tfrac{1}{2} < s_H - \tfrac{1}{2} = h - \tfrac{1}{2} = \eta,$$

both sides of (2-8) have exponential growth as $t \to \infty$. For the test function f as defined in (2-3), (2-4),

$$\left| \int_0^\infty R_H(t) f(t)\, dt \right| = \left| \int_0^\infty A_H(t) f(t)\, dt + \int_0^\infty B_H(t) f(t)\, dt \right| = O(e^{-t/2}).$$

This shows that the perturbation of the metric $g_H \mapsto g$, which did not affect $A_H = A$, had a rather drastic effect on the second part of the remainder term, B_H.

In particular, for the original metric, since A_H satisfies the estimate (1-2), and the total remainder R_H decays exponentially, the remainder term B_H must also satisfy (1-2). However, this clearly cannot be the case for B. Hence, the remainder in the trace formula Theorem 1.1 may include a contribution that is independent of the set of closed geodesics and that grows exponentially for large time. □

3. Concluding remarks

The estimate in (1-2) [Rowlett 2009] arises solely from the nonwandering set, which is a compact, convex subset of the manifold (see [Rowlett et al. 2011]), and therefore reduces to the Ehrenfest estimate for compact manifolds with pinched negative curvature as in [Jakobson et al. 2008]. This estimate corresponds to the remainder term $A(t)$. The proof of Proposition 1.5 and Theorem 1.1(2) shows that one may perturb the manifold away from the nonwandering set, which does not change the leading term in the dynamical side of the trace formula nor the remainder $A(t)$, but which does change the bottom of the spectrum. It follows that this perturbation affects the long-time asymptotics of the dynamical side of the trace formula and therefore must change the long-time estimate of the remainder term. Since the perturbation does not affect the nonwandering set, which contains all closed geodesics, the remainder term must have an additional contribution arising from nonclosed geodesics; this is the term $B(t)$ in Theorem 1.1 and Corollary 1.2 above. This unexpected contribution is not seen in either the compact case or the model case of convex cocompact hyperbolic manifolds. It would be interesting to identify this contribution more precisely.

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