Semistable K3 surfaces with icosahedral symmetry

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Abstract

In a Type III degeneration of $K3$ surfaces the dual graph of the central fibre is a triangulation of S^2. We realise the tetrahedral, octahedral and especially the icosahedral triangulation in families of $K3$ surfaces, preferably with the associated symmetry groups acting.

Introduction

A degeneration of surfaces is a 1-parameter family with general fibre a smooth complex surface. The case of $K3$ surfaces has attracted a lot of attention. A nice discussion is contained in the introductory first paper [F-M] of the bundle [SAGS]. One usually allows base change and modifications to obtain good models. After a ramified cover of the base and resolution of singularities we can assume that the degeneration $f\colon \mathcal{X} \to S \ni 0$ is *semistable*: the zero fibre $X = f^{-1}(0)$ is a reduced divisor with (simple) normal crossings in the smooth manifold $\mathcal{X}$. Further modifications of a $K3$ degeneration lead to a minimal model, which falls into one of three types.

In a Type III degeneration of $K3$ surfaces the dual graph of the central fibre is a triangulation of S^2. In this paper I construct an example with my favourite triangulation, the icosahedral one. A substantial part is taken up by the tetrahedral case, which is easier to handle and allows more explicit results.

It is not difficult to realise the tetrahedral and octahedral triangulations. In his monograph Ulf Persson invites the reader 'to find a degeneration into a dodekahedron of rational surfaces' [P, p. 126]. By deformation theory we show the existence of a degeneration of degree 60 with icosahedral symmetry, with central fibre consisting of twelve Del Pezzo surfaces of degree five, glued together along pentagons. Unfortunately the computations are too difficult to give explicit formulas. The same holds for a related problem in less variables, smoothing the

Stanley-Reisner ring of the icosahedron. A semistable model of such a degeneration has as central fibre a complexified football. The extra components come from singularities of the total space.

The last example suggests that one can get a dodecahedron out of a central fibre with less than 12 components. This requires a breaking of the symmetry. I obtain in (5.16) an explicit degeneration, whose general fibre is a smooth $K3$ surface of degree 12 in $\mathbb{P}^7$, with special fibre consisting of 6 planes with triangles as double curves and 3 quadric surfaces with rectangular double curve. Its total space has three singularities, which are isomorphic to cones over Del Pezzo surfaces of degree 5, and 18 A_1 points. The dual graph of the central fibre on a suitable smooth model is the icosahedron.

This paper is a shortened version of [S2], concentrating on the explicit computations.

1. Semistable degenerations of $K3$-surfaces

(1.1) The name $K3$ has been explained by André Weil: 'ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne $K2$ au Cachemire' [André Weil, Œuvres Scientifiques, p. 546]. He calls any surface a $K3$, if it has the differentiable structure of a smooth quartic surface in $\mathbb{P}^3(\mathbb{C})$. A Kummer surface is a quartic with 16 A_1-singularities. As these singularities admit simultaneous resolution, the minimal resolution of a Kummer surface deforms into a smooth quartic and is therefore a $K3$ surface. A quartic surface X is simply connected, so in particular $b_1(X) = 0$ and has trivial canonical sheaf by the adjunction formula: X is an anti-canonical divisor in $\mathbb{P}^3$.

(1.2) *Definition.* A smooth complex surface X is called a *$K3$ surface*, if $b_1(X) = 0$ and $\mathcal{K}_X$ is trivial.

The equivalence with Weil's definitions follows because all $K3$ surfaces form one connected family.

(1.3) Let $f\colon \mathcal{X} \to S \ni 0$ be a proper surjective holomorphic map of a 3-dimensional complex manifold $\mathcal{X}$ to a (germ of a) curve S such that the zero fibre $X = f^{-1}(0)$ is a reduced divisor with (simple) normal crossings; then the degeneration f is called *semistable*.

In the $K3$ case the following holds (see [F-M]) for exact references):

(1.4) Theorem (Kulikov). *Let $f\colon \mathcal{X} \to S$ be a semistable degeneration of $K3$ surfaces. If all components of $X = f^{-1}(0)$ are Kähler, then there exists a modification $\mathcal{X}'$ of $\mathcal{X}$ such that $K_{\mathcal{X}'} \equiv 0$.*

A degeneration as in the conclusion of the theorem ($K_{\mathcal{X}} \equiv 0$) is called a *Kulikov model*.

(1.5) Theorem (Persson, Kulikov)**.** *Let $f\colon \mathcal{X} \to S$ be a Kulikov model of a degeneration of $K3$ surfaces with all components of $X = f^{-1}(0)$ Kähler. Then either*

I *X is smooth, or*

II *X is a chain of elliptic ruled components with rational surfaces at the ends and all double curves are smooth elliptic curves, or*

III *X consists of rational surfaces meeting along rational curves which form cycles on each component. The dual graph is a triangulation of S^2.*

According to the case division in the theorem one speaks of degenerations of type I, II, or III. Without the Kähler assumption it is not always possible to arrange that $K_{\mathcal{X}} \equiv 0$ [K, N]. In particular it is possible that the central fibre contains surfaces of type VII_0. Even under the assumption $K_{\mathcal{X}} \equiv 0$ the list becomes longer (see [N, Thm. 2.1]).

A Kulikov model is not unique. The central fibre can be modified with flops. If C^- is a smooth rational curve in X with self intersection -1, lying in a component X_i and intersecting the double curve transversally in one point lying in X_j, then after the flop the curve C^+ lies in X_j. This operation is also called elementary modification of type I along C^-. An elementary modification of type II is a flop in a curve C^-, which is a component of the double curve and has self intersection -1 on both components X_i, X_j on which it lies. There are two triple points on C^- involving the components X_k and X_l. After the flop C^+ is a double curve lying in the components X_k and X_l. Note that we we might lose projectivity by using elementary transformations.

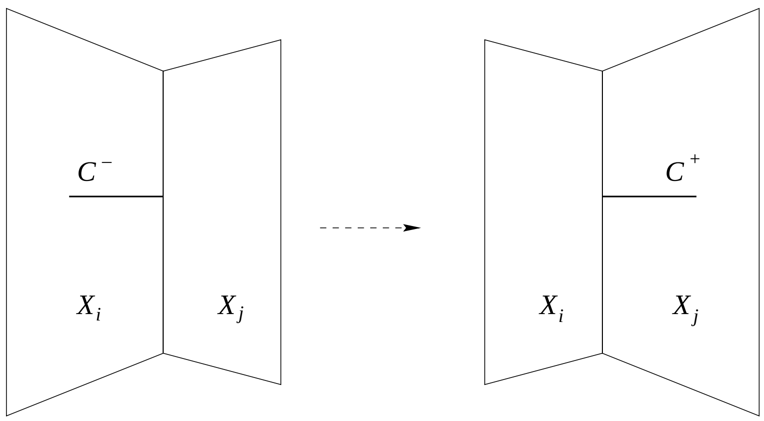


Figure 1: Elementary modification of type I.

(1.6) The Minus One Theorem [M-M]**.** *By modifications of type I and II one can achieve that every component of the double curve of the special fibre has self intersection -1 on both components on which it lies.*

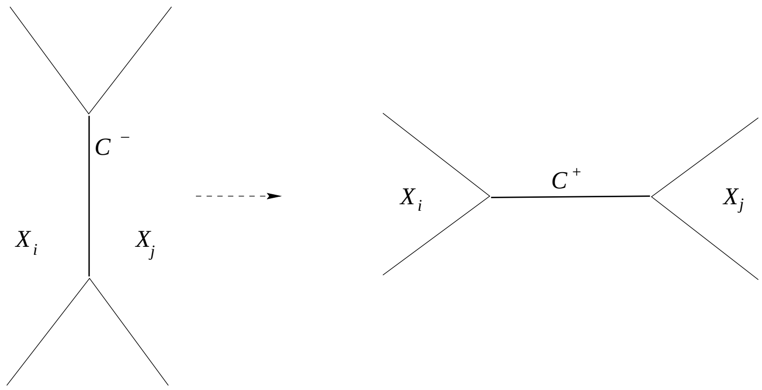


Figure 2: Elementary modification of type II.

(1.7) Let $X = \cup X_i$ be a normal crossings surface with double locus D. If X is a divisor in a smooth 3-fold M then one can define the infinitesimal normal bundle $\mathcal{O}_D(X)$ as $\mathcal{O}_D(X) = \mathcal{O}_M(X)|_D$. It can be defined independent of M. To this end, let I_{X_i} be the ideal sheaf of X_i in X. It is locally generated by one generator z_i, but not invertible as z_i is a zero divisor in $\mathcal{O}_X$. However $I_{X_i}|_D$ is locally free [F, (1.8)]. Following Friedman one makes the following definitions.

(1.8) *Definition.* The *infinitesimal normal bundle* $\mathcal{O}_D(X)$ is the line bundle dual to $\mathcal{O}_D(-X)$, where

$$\mathcal{O}_D(-X) = (I_{X_1}|_D) \otimes_{\mathcal{O}_D} \cdots \otimes_{\mathcal{O}_D} (I_{X_k}|_D) \ .$$

If X is a divisor in M the so defined bundle equals $\mathcal{O}_M(X)|_D$. In particular, if X is a central fibre in a semistable degeneration $\mathcal{X} \to S$, then $\mathcal{O}_{\mathcal{X}}(X) \equiv \mathcal{O}_{\mathcal{X}}$ so $\mathcal{O}_D(X) = \mathcal{O}_D$. This gives a necessary condition for being a central fibre.

(1.9) *Definition.* The normal crossings surface X is *d-semistable* if $\mathcal{O}_D(X) = \mathcal{O}_D$.

A consequence is the triple point formula: let $D_{ij} = X_i \cap X_j$ and denote by $(D_{ij})^2_{X_i}$ the self intersection of D_{ij} on X_i and by T_{ij} the number of triple points on D_{ij}. Then (cf. [P, Cor. 2.4.2])

$$(D_{ij})^2_{X_i} + (D_{ij})^2_{X_j} + T_{ij} = 0 \ .$$

(1.10) *Definition.* A compact normal crossings surface is a *d-semistable* $K3$ *surface of type* III if X is d-semistable, $\omega_X = \mathcal{O}_X$ and each X_i is rational, the double curves $D_i \subset X_i$ are cycles of rational curves and the dual graph triangulates S^2. If the conclusions of the Minus One Theorem hold, that every component of the double curve has self intersection -1 on either component of X on which it lies, the surface X is said to be in -1 *form*.

2. Tetrahedra

(2.1) To realise a tetrahedron we start out with four general planes in 3-space. They do not form a d-semistable $K3$ surface, but the dual graph is a tetrahedron. To write down a degeneration with this special fibre we just take the pencil spanned by $T = x_0x_1x_2x_3$ and a smooth quartic. The symmetry group of the tetrahedron (including reflections) acts if we only take S_4-invariant quartics:

$$Q = a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_2^2 + d\sigma_1\sigma_3 \ ,$$

where the σ_i are the elementary symmetric functions in the four variables x_i and a, b, c and d are constants.

To obtain a family $f\colon \mathcal{X} \to S$ one has to blow up the base locus of the pencil. This can be done in several ways. Blowing up $T = Q = 0$ gives a total space which is singular, with in general 24 ordinary double points coming from the 24 intersection points of Q with the double curve of the tetrahedron T. Arguably, this is the nicest model, and the best one can hope for in view of the theory of minimal models of 3-folds. A smooth model is obtained by a suitable small resolution of the 24 singularities. The quartic intersects each edge of the tetrahedron in four points. To get the (-1)-form two of them have to be blown up in one face and the other two in the other face. The central fibre then consists of four Del Pezzo surfaces of degree 3.

Alternatively one can blow up the irreducible components of $T = Q = 0$ one at a time. The advantage is that one has a projective model. However it is not in (-1)-form and furthermore the symmetry is not preserved. To achieve (-1)-form we have to apply modifications of type I; here we may loose projectivity.

(2.2) The tetrahedron of degree 12. We glue together four Del Pezzo surfaces of degree 3. Take coordinates $x_1, \ldots, x_4, y_1, \ldots, y_4$ on $\mathbb{P}^7$. Let $\{i,j,k,l\} = \{1,2,3,4\}$. The Del Pezzo surface X_i lies in $y_i = x_j = x_k = x_l = 0$ and has an equation of the form

$$y_jy_ky_l - x_iF_i(x_i, y_j, y_k, y_l) = 0 \ ,$$

where F_i is a quadratic form; more specifically,

$$F_i = \sum_{\alpha\neq i} f_i^{\alpha\alpha} y_\alpha^2 + \sum_{\alpha,\beta\neq i} f_i^{\alpha\beta} y_\alpha y_\beta + \sum_{\alpha\neq i} g_i^\alpha x_i y_\alpha + h_i x_i^2 \ .$$

The condition that X_i is nonsingular in the vertices of the triangle $x_i = y_jy_ky_l = 0$ is that the coefficients $f_i^{\alpha\alpha}$ in F_i do not vanish.

The ideal of the tetrahedron $X = \bigcup_i X_i$ has 14 generators, the 4 cubic Del Pezzo equations $y_jy_ky_l - x_iF_i$ and 10 quadratic monomials: the six products x_ix_j

and the four products x_iy_i. The relations among them are:

$$\begin{aligned}(x_ix_j)x_k-(x_ix_k)x_j\\ (x_iy_i)x_j-(x_ix_j)y_i\\ (y_jy_ky_l-x_iF_i)x_l-(x_ly_l)y_jy_k+(x_ix_l)F_i\\ (y_iy_jy_k-x_lF_l)y_l-(y_jy_ky_l-x_iF_i)y_i-(x_ly_l)F_l+(x_iy_i)F_i\ .\end{aligned}\tag{1}$$

(2.3) Proposition. *The tetrahedron X is d-semistable if and only if the four equations*

$$f_k^{jj}f_l^{kk}f_j^{ll}-f_j^{kk}f_k^{ll}f_l^{jj}=0$$

are satisfied.

Proof. We look at the chart $y_4=1$. Then $x_4=0$ and we have the equations x_ix_j, x_iy_i, $y_1y_2y_3$ and $y_iy_j-x_kF_k$. In all points near the origin $y_1y_2y_3\mapsto 1$ is generator of the infinitesimal normal bundle $\mathcal{O}_D(X)$. We now look on the y_2-axis. The equation $y_2y_1-x_3F_3$ shows that the section $y_1y_2y_3\mapsto 1$ has a pole in the zeroes of F_3 restricted to the y_2-axis, and likewise in the zeroes of F_1 (using the equation $y_2y_3-x_1F_1$).

This shows that the expression, given in homogeneous coordinates by

$$\frac{(f_1^{22}y_2^2+f_1^{24}y_2y_4+f_1^{44}y_4^2)(f_3^{22}y_2^2+f_3^{24}y_2y_4+f_3^{44}y_4^2)}{y_1y_2y_3y_4}\ ,$$

represents a nonvanishing holomorphic section of $\mathcal{O}_D(X)$ on the whole line $y_1=y_3$. Similar expression can be found for the other edges of the tetrahedron. We get a global section if and only if we can find a quaternary form f of degree 4 which restricts to a multiple of the above denominator for each line.

For each face we find from the following Lemma the condition that the 12 points in the corresponding hyperplane are cut out by a quartic. We obtain four equations, which in fact are not independent: under our assumption that all f_{jj}^i are different from 0 one can derive one equation from the remaining three. They give necessary and sufficient conditions for the existence of the quaternary quartic. □

(2.4) Lemma. *Consider $3n$ points $P_{i,\alpha}$ with $P_{i,1},\dots,P_{i,n}$ smooth points of the triangle $x_1x_2x_3=0$, lying on the side $x_i=0$ and given by the binary form $B_i(x_j,x_k)=\sum_{m=0}^n b_{im}x_j^mx_k^{n-m}$, where (i,j,k) is a cyclic permutation of $(1,2,3)$. These points are cut out by a ternary form of degree n if and only if*

$$b_{10}b_{20}b_{30}=b_{1n}b_{2n}b_{3n}\ .$$

Proof. Suppose $A(x_1,x_2,x_3)=\sum_{l+m+p=n}a_{lmp}x_1^lx_2^mx_3^p$ cuts out the points. Then $A(0,x_2,x_3)$ is proportional to $B_1(x_2,x_3)$, so $(a_{0n0}:a_{00n})=(b_{10}:b_{1n})$. Likewise we have that $(a_{00n} : a_{n00}) = (b_{20} : b_{2n})$ and $(a_{n00} : a_{0n0}) = (b_{30} : b_{3n})$. Multiplying these ratios gives the condition.

Conversely, to find A we may suppose that $b_{10} = b_{3n} = 1 = a_{0n0}$ (as no point lies at one of the vertices). We put $A(0, x_2, x_3) = B_1(x_2, x_3)$, $A(x_1, x_2, 0) = B_3(x_1, x_2)$. We also can take $b_{20} = a_{00n}$. As $b_{1n} = a_{00n}$ the condition gives now $b_{2n} = b_{30} = a_{n00}$ and we can set $A(x_1, 0, x_3) = B_2(x_1, x_3)$. The remaining monomials in A are divisible by $x_1x_2x_3$ and do not matter. □

(2.5) *Remark.* It is not surprising that only the extremal coefficients b_{i0}, b_{in} are involved, as they depend only on the product of the coordinates of the points. Ignoring the other coefficients we rename: $b_{i0} =: b_{jk}$, $b_{in} =: b_{kj}$. The condition becomes $b_{jk}b_{ki}b_{ij} = b_{ji}b_{ik}b_{kj}$, which is the form used in the proposition above.

(2.6) Infinitesimal deformations. We compute embedded deformations modulo coordinate transformations. To this end we look at the equations as defining the affine cone $C(X)$ over X. We follow the standard procedure (see e.g. [S1]): given equations f_i, satisfying relations $\sum f_i r_{ij} = 0$, we have to lift the equations to $F_i = f_i + \varepsilon f_i'$ and the relations to $R_{ij} = r_{ij} + r_{ij}'$, satisfying $\sum F_i R_{ij} \equiv 0 \pmod{\varepsilon^2}$. This means that we have to find f_i' such that $\sum f_i' r_{ij}$ lies in the ideal generated by the f_i. Using undetermined coefficients this is a finite dimensional problem for each degree. The deformations of $C(X)$ in degree 0 give embedded deformations of X in $\mathbb{P}^7$, while those in degree < 0 have an interpretation in terms of extensions of X: they tells us of which varieties X is a hyperplane section. Our main interest lies in the degree 0 deformations, but as preparation we first compute those of negative degrees.

(2.7) Proposition. *The dimension of* $T^1_{C(X)}(-2)$ *equals 4 and* $\dim T^1_{C(X)}(-1) = 16$. *In case X is d-semistable* $\dim T^1_{C(X)}(0) = 22$, *otherwise it is* 21.

Proof. Degree -2: we perturb the quadratic equations with constants and the cubic equations with linear terms. Write $x_ix_j + a_{ij}$. The first type of the relations (1) then gives $a_{ij}x_k - a_{ik}x_j = 0 \in \mathcal{O}_{C(X)}$, so $a_{ik} = 0$. Also the equation x_iy_i are not perturbed. Consider $y_jy_ky_l - x_iF_i + \sum a_i^\alpha x_\alpha + \sum b_i^\alpha y_\alpha$. The third relation gives $a_i^jx_j^2 + \sum_{\alpha\neq j} b_i^\alpha x_jy_\alpha = 0$ so we conclude that all coefficients vanish, except the a_i^i, which we may choose arbitrary. The last type of relation is then also satisfied.

Degree -1: consider the perturbations

$$x_ix_j + \sum a_{ij}^\alpha x_\alpha + \sum b_{ij}^\alpha y_\alpha \, .$$

In the local ring we obtain the equation

$$a_{ij}^k x_k^2 + \sum_{\alpha\neq k} b_{ij}^\alpha x_ky_\alpha - a_{ik}^k x_j^2 + \sum_{\alpha\neq j} b_{ik}^\alpha x_jy_\alpha = 0 \, ,$$

from which we get $b_{ij}^{\alpha}=0$, $a_{ij}^{k}=a_{ij}^{l}=0$. We now put

$$x_iy_i+\sum a_{ii}^{\alpha}x_{\alpha}+\sum b_{ii}^{\alpha}y_{\alpha}\ .$$

We find

$$a_{ii}^{j}x_j^2+\sum_{\alpha\neq j} b_{ii}^{\alpha}x_jy_{\alpha}-a_{ij}^{j}x_jy_i=0\ .$$

We conclude $a_{ii}^{j}=0$, $b_{ii}^{j}=0$ for all $j\neq i$ and finally $a_{ij}^{j}=b_{ii}^{i}$. In particular a_{ij}^{j} is independent of j. We can use the coordinate transformation $x_i\mapsto x_i-b_{ii}^{i}$ to get rid of the a_{ij}^{j}-term. So the equations x_ix_j are not perturbed at all. This means that x_iy_j is only perturbed with the term $a_{ii}^{i}x_i$, which can be made to vanish by coordinate transformations in the y_i-variables. As above we find that the only allowable pertubations of the cubic equation $y_jy_ky_l-x_iF_i$ are those divisible by x_i. As we have used all coordinate transformations, all monomials x_i^2, x_iy_j can occur. This makes the dimension of $T^1(-1)$ into 4×4.

Degree 0: we proceed in the same way by first considering the perturbations

$$x_ix_j+\sum a_{ij}^{\alpha}x_{\alpha}^2+\sum b_{ij}^{\alpha\beta}x_{\alpha}y_{\beta}+\sum c_{ij}^{\alpha}y_{\alpha}^2+\sum d_{ij}^{\alpha\beta}y_{\alpha}y_{\beta}\ .$$

Multiplied with x_k this gives the following terms in the local ring

$$a_{ij}^{k}x_k^3+\sum_{\beta\neq k} b_{ij}^{k\beta}x_k^2y_{\beta}+\sum_{\alpha\neq k} c_{ij}^{\alpha}x_ky_{\alpha}^2+\sum_{\alpha,\beta\neq k} d_{ij}^{\alpha\beta}x_ky_{\alpha}y_{\beta}\ .$$

We conclude that all coefficients occurring here vanish. In particular $a_{ij}^{k}=0$. Using the coordinate transformations $x_j\mapsto x_j-a_{ij}^{i}x_i$ we may suppose that all a_{ij}^{α} vanish. We are left with

$$x_ix_j+\sum b_{ij}^{i\beta}x_iy_{\beta}+\sum b_{ij}^{j\beta}x_jy_{\beta}+d_{ij}^{kl}y_ky_l\ .$$

With the perturbations

$$x_iy_i+\sum_{\alpha\neq i} a_{ii}^{\alpha}x_{\alpha}^2+\sum_{\alpha\neq i} b_{ii}^{\alpha\beta}x_{\alpha}y_{\beta}+\sum_{\alpha\neq i} c_{ii}^{\alpha}y_{\alpha}^2+\sum_{\alpha,\beta\neq i} d_{ii}^{\alpha\beta}y_{\alpha}y_{\beta}\ ,$$

where we used coordinate transformations $x_i\mapsto x_i-c_{ii}^{i}y_i$, $x_i\mapsto x_i-d_{ii}^{ij}y_j$, $y_i\mapsto y_i-a_{ii}^{i}x_i$ and $y_i\mapsto y_i-b_{ii}^{ij}y_j$ to remove some coefficients, we now get (using the jth Del Pezzo equation)

$$a_{ii}^{j}x_j^3+\sum b_{ii}^{j\beta}x_j^2y_{\beta}+\sum_{\alpha\neq i} c_{ii}^{\alpha}x_jy_{\alpha}^2+\sum_{\alpha,\beta\neq i} d_{ii}^{\alpha\beta}x_jy_{\alpha}y_{\beta}-\sum b_{ij}^{j\beta}x_jy_iy_{\beta}-d_{ij}^{kl}x_jF_j=0\ .$$

Using the explicit expression for F_j we obtain the equations

$$\begin{array}{rclrclrcl} a_{ii}^{j} &=& d_{ij}^{kl}h_j, & b_{ii}^{j\beta} &=& d_{ij}^{kl}g_j^{\beta}, & c_{ii}^{\alpha} &=& d_{ij}^{kl}f_j^{\alpha\alpha},\\ -b_{ij}^{ji} &=& d_{ij}^{kl}f_j^{ii}, & d_{ii}^{kl} &=& d_{ij}^{kl}f_j^{kl}, & -b_{ij}^{j\beta} &=& d_{ij}^{kl}f_j^{i\beta}. \end{array}$$

We can determine all coefficients, but because c^α_{ii} does not depend on j, we get two equations for it

$$d^{kl}_{ij} f^{ll}_j = c^l_{ii} = d^{jl}_{ik} f^{ll}_k \;.$$

We view these as one linear equation for the unknowns d^{kl}_{ij}. The coefficient matrix of the resulting linear system is

$$\begin{pmatrix}
0 & f^{jj}_k & -f^{jj}_l & 0 & 0 & 0 \\
-f^{kk}_j & 0 & f^{kk}_l & 0 & 0 & 0 \\
f^{ll}_j & -f^{ll}_k & 0 & 0 & 0 & 0 \\
f^{kk}_i & 0 & 0 & 0 & -f^{kk}_l & 0 \\
-f^{ll}_i & 0 & 0 & f^{ll}_k & 0 & 0 \\
0 & 0 & 0 & -f^{ii}_k & f^{ii}_l & 0 \\
0 & f^{ll}_i & 0 & -f^{ll}_j & 0 & 0 \\
0 & -f^{jj}_i & 0 & 0 & 0 & f^{jj}_l \\
0 & 0 & 0 & f^{ii}_j & 0 & -f^{ii}_l \\
0 & 0 & f^{jj}_i & 0 & 0 & -f^{jj}_k \\
0 & 0 & -f^{kk}_i & 0 & f^{kk}_j & 0 \\
0 & 0 & 0 & 0 & -f^{ii}_j & f^{ii}_k
\end{pmatrix}$$

It has a nontrivial solution if all 6×6 minors vanish. Among those are

$$f^{kk}_j f^{ll}_k f^{jj}_l (f^{jj}_k f^{kk}_l f^{ll}_j - f^{kk}_j f^{ll}_k f^{jj}_l)$$

from which we obtain that the square of

$$f^{jj}_k f^{kk}_l f^{ll}_j - f^{kk}_j f^{ll}_k f^{jj}_l$$

lies in the ideal of the minors. This is one of the four conditions for d-semistability. There are three more equations

$$f^{jj}_k f^{ll}_j f^{ii}_l f^{kk}_i - f^{kk}_j f^{jj}_l f^{ll}_i f^{ii}_k$$

in the reduction of the ideal of minors, which do not give new conditions if the $f^{jj}_i \neq 0$, as

$$\begin{aligned}
&f^{kk}_l (f^{jj}_k f^{ll}_j f^{ii}_l f^{kk}_i - f^{kk}_j f^{jj}_l f^{ll}_i f^{ii}_k) \\
&\qquad = f^{ii}_l f^{kk}_i (f^{jj}_k f^{kk}_l f^{ll}_j - f^{kk}_j f^{ll}_k f^{jj}_l) + f^{kk}_j f^{jj}_l (f^{ii}_l f^{kk}_i f^{ll}_k - f^{ll}_i f^{ii}_k f^{kk}_l) \;.
\end{aligned}$$

Under the d-semistability conditions the rank of the matrix is 5, and we obtain one infinitesimal deformation, where the quadratic equations are perturbed. Furthermore one has the perturbations of the cubic equations alone, which as before have to be divisible by x_i. We already used 44 coordinate transformations. The coefficient of $x_i y_\alpha y_\beta$ can be made to vanish with a transformation of the type $y_\gamma \mapsto y_\gamma - \varepsilon x_i$. So we have 28 coefficients left and the diagonal coordinate transformations, giving dimension 21. □

The computations in negative degree show that the tetrahedron X is only a hyperplane section of threefolds with two-dimensional singular locus, obtained by glueing together four cubic threefolds.

(2.8) We want to describe an explicit deformation in the d-semistable case. We use the coordinate transformation $x_i \mapsto (f_4^{ii}/f_i^{44})x_i$, $i = 1, 2, 3$, which gives $f_i^{jj} f_4^{ii}/f_i^{44}$ as coefficient of y_j^2 in the new F_i. The d-semistability conditions yield that the new coefficients satisfy $f_i^{jj} = f_j^{ii}$. We will denote them by f_{ij}. A solution to the linear equations above is then $d_{ij} = df_{kl}$, with d a new deformation variable. Furthermore we use coordinate transformations to remove the $y_\alpha y_\beta$ terms from the F_i.

We set $H_i = g_i^j y_j + g_i^k y_k + g_i^l y_l + h_i x_i$. With this notation we get the following infinitesimal deformation:

$$x_i x_j + df_{kl} y_k y_l - df_{ij} f_{kl}(x_i y_j + x_j y_i) \; ,$$

$$x_i y_i + d(f_{kl} x_j H_j + f_{jl} x_k H_k + f_{jk} x_l H_l + f_{jk} f_{jl} y_j^2 + f_{kl} f_{kj} y_k^2 + df_{lj} f_{lk} y_l^2) \; ,$$

$$\begin{aligned} &y_j y_k y_l - x_i(f_{ij} y_j^2 + f_{ik} y_k^2 + f_{il} y_l^2) - x_i^2 H_i \\ &\quad + dy_i((f_{ik} f_{jl} + f_{il} f_{jk}) f_{ij} y_j^2 + (f_{ij} f_{kl} + f_{il} f_{kj}) f_{ik} y_k^2 + (f_{ij} f_{lk} + f_{ik} f_{lj}) f_{il} y_l^2) \\ &\qquad + df_{ij} f_{ik} f_{il} y_i^3 + dy_i(f_{ik} f_{il} x_j H_j + f_{ij} f_{il} x_k H_k + f_{ij} f_{ik} x_l H_l) \; . \end{aligned}$$

If we try to lift to higher order complicated formulas arise, and it is not clear whether the computation is finite. It does stop if we restrict ourselves to the case of tetrahedral symmetry. Then f_{ij} does not depend on (i,j), and we call the common value f; likewise g is the value of all g_i^j and h of the h_i. We retain the notation $H_i = g(y_j + y_k + y_l) + hx_i$. By a coordinate transformation $x_i \mapsto x_i + df^2 y_i$ we simplify the expression for the first 6 equations. We write t for the deformation parameter.

(2.9) Proposition. *The following set of equations defines a degeneration of $K3$ surfaces with special fibre a tetrahedron of degree 12:*

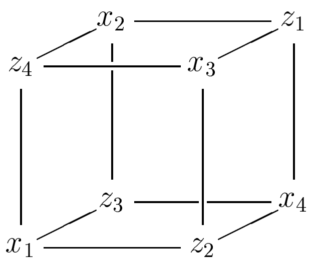
$$\begin{aligned} &x_i x_j + tf(y_k - tfH_k)(y_l - tfH_l) \; , \\ &x_i y_i + tf(x_j H_j + x_k H_k + x_l H_l) + tf^2(y_i^2 + y_j^2 + y_k^2 + y_l^2) \; , \\ &y_j y_k y_l - fx_i(y_i^2 + y_j^2 + y_k^2 + y_l^2) - x_i^2 H_i \\ &\qquad - t^2 f^2 (y_j H_k H_l + y_k H_j H_l + y_l H_j H_k) + 2t^3 f^3 H_j H_k H_l \; . \end{aligned}$$

The general fibre is a smooth $K3$ surface lying on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. We suppress the computation. For $t \neq 0$ the cubic equations lie in the ideal of the quadrics. We can write three independent equations

$$x_i(y_i - tfH_i) - x_j(y_j - tfH_j)$$

which together with the first six define $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$: each square in the following picture gives an equation, where we put $z_i := \sqrt{-tf}(y_i - tfH_i)$.



We obtain the $K3$ by taking the complete intersection with the quadric $\sum_i (x_i y_i + 3tf x_i H_i + 4tf^2 y_i^2)$. $\square$

(2.10) The relation with quartics. We can construct a degree 12 tetrahedron from four planes in $\mathbb{P}^3$ by first blowing up each plane in 6 points and then glueing them back together.

Therefore we first describe the blow up in a way adapted to our situation. The problem of actually writing down a cubic equation is not often treated in the literature (see however [Ma]). Here we want to fix a particular hyperplane section, which limits the choice of coordinates.

Let two points lie on each side of the coordinate triangle in $\mathbb{P}^2$ with coordinates $(z_1 : z_2 : z_3)$. We describe them by $z_k = z_i^2 + a_{ij} z_i z_j + b_{ij} z_j^2 = 0$, where (ijk) is a cyclic permutation of (123) (this means that we choose an orientation on the triangle). As cubics through the six points we take the coordinate triangle and three cubics, each consisting of a side and a quadric passing through the remaining four points. More precisely, we take

$$\begin{aligned} x_0 &= z_1 z_2 z_3 \\ y_i &= z_i (z_k^2 - a_{ki} z_k z_i + b_{ki}(z_i^2 - a_{ij} z_i z_j + b_{ij} z_j^2)) \ . \end{aligned}$$

One computes the relations

$$z_k\, y_j - b_{ij} z_j\, y_k - ((1 - b_{ij} b_{jk} b_{ki}) z_i - a_{ij} z_j + b_{ij} b_{jk} a_{ki} z_k)\, x_0 = 0 \ .$$

By the Hilbert-Burch theorem the maximal minors of the relation matrix

$$\begin{pmatrix} 0 & z_3 & -b_{12} z_2 & -(1 - b_{12} b_{23} b_{31}) z_1 + a_{12} z_2 - b_{12} b_{23} a_{31} z_3 \\ -b_{23} z_3 & 0 & z_3 & -(1 - b_{12} b_{23} b_{31}) z_2 + a_{23} z_3 - b_{23} b_{31} a_{12} z_1 \\ z_2 & -b_{31} z_1 & 0 & -(1 - b_{12} b_{23} b_{31}) z_3 + a_{31} z_1 - b_{31} b_{12} a_{23} z_2 \end{pmatrix}$$

give the cubics, up to a common factor $1 - b_{12} b_{23} b_{31}$. By Lemma (2.4) this factor vanishes exactly when the 6 points lie on a conic.

Viewing the relations as holding between the z_i gives the coefficient matrix

$$\begin{pmatrix} (1-b_{12}b_{23}b_{31})x_0 & b_{12}y_3 - a_{12}x_0 & b_{12}b_{23}a_{31}x_0 - y_2 \\ b_{23}b_{31}a_{12}x_0 - y_3 & (1-b_{12}b_{23}b_{31})x_0 & y_1 b_{23} - a_{23}x_0 \\ b_{31}y_2 - a_{31}x_0 & b_{12}b_{31}a_{23}x_0 - y_1 & (1-b_{12}b_{23}b_{31})x_0 \end{pmatrix} .$$

Its determinant is the equation of the surface. After dividing by $b_{12}b_{23}b_{31}-1$ it equals

$$\begin{aligned} &y_1y_2y_3 - x_0(b_{12}y_3^2 + b_{23}y_1^2 + b_{31}y_2^2) \\ &-x_0^2(a_{12}a_{23}b_{31}y_2 + a_{23}a_{31}b_{12}y_3 + a_{31}a_{12}b_{23}y_1 - (a_{12}y_3 + a_{23}y_1 + a_{31}y_2)(1+b_{12}b_{23}b_{31})) \\ &-x_0^3((1-b_{12}b_{23}b_{31})^2 + (1+b_{12}b_{23}b_{31})a_{12}a_{23}a_{31} + b_{12}b_{23}a_{31}^2 + b_{23}b_{31}a_{12}^2 + b_{12}b_{31}a_{23}^2) . \end{aligned}$$

This last formula also works if the 6 points lie on a conic, but then it is easier to take the y_i as product of a side and the conic through the 6 points; this means adding a multiple of x_0 to each y_i. The equation then becomes $y_1y_2y_3 - x_0Q(y)$ with $Q(z)$ the conic.

Now we apply this to our tetrahedron. We choose an orientation and orient the faces with the induced orientation. We get variables x_i and y_i. For the face i we take $x_i = z_jz_kz_l$ as before, but we multiply y_j by a factor λ_{ij} to be determined later. So we set $y_j = \lambda_{ij}z_j(z_l^2 + \ldots)$. Now we look at the line $z_3 = z_0 = 0$, with coordinates $(z_1 : z_2)$. Via the coordinates of face 0 we get the embedding $(y_1 : y_2) = (\lambda_{01}b_{31}z_1 : \lambda_{02}z_2)$ whereas face 3 gives $(y_1 : y_2) = (\lambda_{31}z_1 : \lambda_{32}b_{02}z_2)$. The condition that the Del Pezzo surfaces are glued in the same way as the planes yields the equations $\lambda_{01}\lambda_{32}b_{31}b_{02} = \lambda_{02}\lambda_{31}$. By even permutations of (0123) we get in total six equations. They are solvable if and only if

$$b_{01}b_{10}b_{02}b_{20}b_{03}b_{30}b_{12}b_{21}b_{13}b_{31}b_{23}b_{32} = 1 ,$$

a condition obtained by multiplying the six equations.

The d-semistability conditions $f_k^{jj}f_l^{kk}f_j^{ll} - f_j^{kk}f_k^{ll}f_l^{jj} = 0$ give for the cubic above, using $(jkl) = (123)$:

$$\frac{b_{30}}{\lambda_{21}^2}\frac{b_{10}}{\lambda_{32}^2}\frac{b_{20}}{\lambda_{13}^2} = \frac{b_{03}}{\lambda_{12}^2}\frac{b_{01}}{\lambda_{23}^2}\frac{b_{02}}{\lambda_{31}^2} .$$

Using that the λ_{ij} satisfy the equations $\lambda_{32}b_{02}/\lambda_{31} = \lambda_{02}/(\lambda_{01}b_{31})$ we see that this condition is equivalent to $b_{21}^2b_{32}^2b_{13}^2 = b_{12}^2b_{23}^2b_{31}^2$, which is one of the conditions that the 24 points are cut out by a quartic.

We can ask which choices of 24 points give our symmetric tetrahedron. The condition $\prod b_{ij} = 1$ limits the possibilities. In particular, if all $b_{ij} = 1$, the six points in each face lie on a conic, giving a singular tetrahedron. If we take the quartic $Q = (a\sigma_1^2 + b\sigma_2)^2$ then each element of the pencil has 12 singular points. We can blow up them and blow down the six conics in the faces by embedding the pencil in $\mathbb{P}^7 \times \mathbb{P}^1$ with the linear system of cubics in $\mathbb{P}^3$ with as base points the 12 singular points. We set

$$\begin{aligned} x_i &= z_j z_k z_l \\ y_i &= z_i(a\sigma_1^2 + b\sigma_2) \,. \end{aligned}$$

We obtain a symmetric tetrahedron with $g = h = 0$.

We get nonsingular Del Pezzo surfaces by taking all $b_{ij} = -1$, and $a_{ij} = a$. Then $f = -1$, $g = -a^2$ and $h = a^2 + 4$. The points on the side of the tetrahedron are given by

$$(z_i^2 + az_iz_j - z_j^2)(-z_i^2 + az_iz_j + z_j^2) = (-z_i^4 + (2 + a^2)z_i^2z_j^2 - z_j^4) \,.$$

In particular, we obtain different smoothings of the same tetrahedron, those embedded in $\mathbb{P}^7$ and other ones, where the general fibre is embeddable in $\mathbb{P}^3$. They belong to different 19-dimensional hypersurfaces in the 20-dimensional subspace of the versal deformation whose general fibre is a smooth $K3$ surface.

3. Deformation theory

(3.1) Friedman developed a general theory for deformations of normal crossings varieties [F]. Let $X = \cup X_i$ be a normal crossings surface with normalisation $\widetilde{X} = \coprod X_i$. The components of the double locus D are $D_{ij} = X_i \cap X_j$. The divisor $D_i := \cup_j D_{ij}$ is a normal crossings divisor in X_i. We set $\overline{D} = \coprod D_i$.

As X is locally a hypersurface in a 3-fold M, its cotangent cohomology sheaves $\mathcal{T}_X^i$ vanish for $i \geq 2$ and

$$0 \longrightarrow \mathcal{T}_X^0 \longrightarrow \Theta_M|_X \longrightarrow N_{X/M} \longrightarrow \mathcal{T}_X^1 \longrightarrow 0 \,.$$

There is a canonical isomorphism $\mathcal{T}_X^1 \cong \mathcal{O}_D(X)$ and in particular, if X is d-semistable, then $\mathcal{T}_X^1 \cong \mathcal{O}_D$ [F, Prop. 2.3].

The $K3$-case was the main motivation for Friedman. There one can compute everything explicitly (for details see [S2]). The homological computations give:

(3.2) Theorem. *Let $X = \cup_{i=1}^k X_i$ be a d-semistable $K3$ surface of type III in (-1)-form, with k components. Then*

$$\begin{aligned} \dim H^1(X, \mathcal{T}_X^0) &= k + 18 \\ \dim H^0(X, \mathcal{T}_X^1) &= 1 \\ \dim H^1(X, \mathcal{T}_X^1) &= k - 1 \end{aligned}$$

So $\dim T_X^1 = k + 19$, $\dim T_X^2 = k - 1$.

(3.3) Theorem [F, (5.10)]. *A d-semistable $K3$ surface X of type III is smoothable. Its versal base space is the union $V_1 \cup V_2$, where V_1 is a smooth hypersurface corresponding to locally trivial deformations of X, which meets transversally a 20-dimensional smooth subspace V_2, with $V_2 \setminus V_1$ parametrising smooth $K3$ surfaces and $V_2 \cap V_1$ locally trivial deformations of X for which $\mathcal{O}_D(X)$ remains trivial.*

Locally trivial deformations of a d-semistable $K3$ surface X are unobstructed and fill up a codimension one smooth subspace of the base of the versal deformation with tangent space $H^1(X, \mathcal{T}_X^0)$. This means that every equation of the base is divisible by the equation of this hypersurface. As one obtains the base space as fibre of a map $T^1 \to T^2$, we look at the map

$$\mathrm{Ob}\colon H^1(\mathcal{T}_X^0) \times H^0(\mathcal{T}_X^1) \to H^1(\mathcal{T}_X^1)\ .$$

Let ξ be a global generator of $\mathcal{T}_X^1$. The existence of a second smooth component (of dimension 20) follows, if one can show that $\mathrm{Ob}(\ .\ ,\xi)\colon H^1(\mathcal{T}_X^0) \to H^1(\mathcal{T}_X^1)$ is a surjective linear map. So the deformation computation, which is in general a highly non-linear problem, is here reduced to a linear one, which can be solved by computing suitable cohomology groups.

(3.4) Embedded deformations. Comparing the above results to direct computations with generators and relations for the tetrahedron, we see that the dimension of the space of embedded deformations is one less. This fits in with the fact that algebraic $K3$ surfaces have non-algebraic deformations: the versal deformation of a smooth $K3$ has dimension 20, whereas algebraic families have maximal dimension 19. It is one condition to preserve the polarisation (which by definition is the embedding line bundle). The same holds true for d-semistable $K3$ surfaces.

(3.5) Theorem [F-S, (5.5)]. *A d-semistable $K3$ surface X of type III in $\mathbb{P}^n$ is smoothable by embedded deformations. They form a 19-dimensional smooth component.*

4. Hodge algebras

(4.1) Stanley–Reisner rings.

Let Δ be a simplicial complex with set of vertices $V = \{v_1, \dots, v_n\}$. A monomial on V is an element of $\mathbb{N}^V$. Each subset of V determines a monomial on V by its characteristic function. The support of a monomial $M\colon V \to \mathbb{N}$ is the set $\operatorname{supp} M = \{v \in V \mid M(v) \neq 0\}$. The set Σ_Δ of monomials whose support is not a face is an ideal, generated by the monomials corresponding to minimal non-simplices.

Given a ring R and an injection $\phi\colon V \to R$ we can associate to each monomial M on V the element $\phi(M) = \prod_{v\in V} \phi(v)^{M(v)} \in R$. We will usually identify V and

$\phi(V)$ and write $M \in R$ for $\phi(M)$. This applies in particular to the polynomial ring $K[V]$ over a field K. The ideal Σ_Δ gives rise to the Stanley-Reisner ideal $I_\Delta \subset K[V]$. The *Stanley-Reisner ring* is $A_\Delta = K[V]/I_\Delta$.

Deformations of Stanley-Reisner rings are studied in [A-C].

(4.2) *Example.* Let Δ be an octahedron. We map the set of vertices to $\mathbb{C}[x_1, \ldots, x_6]$ such that opposite vertices correspond to variables with index sum 7. The Stanley-Reisner ring is minimally generated by the three monomials

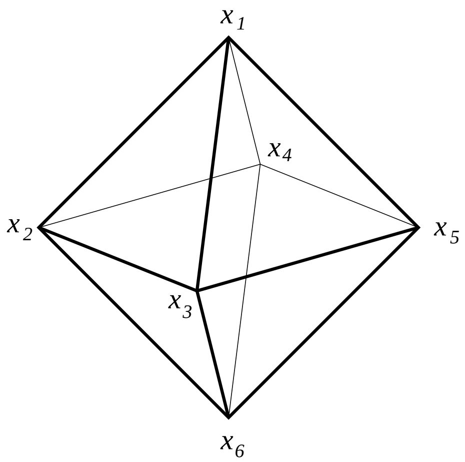


Figure 3: Octahedron.

$x_i x_{7-i}$. The spaces smoothes to a $K3$ surface, the complete intersection of three general quadrics. A general 1-parameter deformation is not semi-stable, because the total space has singularities at the six quadruple points of the special fibre.

(4.3) *Remark.* To get an octahedron as dual graph we need the incidence relations of a cube. The toric variety associated to a cube is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The general anticanonical divisor is a smooth $K3$, whereas the complement of the torus is the unions of six quadrics (each of the form $\mathbb{P}^1 \times \mathbb{P}^1$) with dual graph the octahedron. A small resolution of a general pencil yields a semistable degeneration.

(4.4) *Definition* [D-E-P]. Let H be a finite partially ordered set and Σ an ideal of monomials on H. Let K be a commutative ring and R a commutative K-algebra. Suppose an injection $\phi\colon H \to R$ is given. Then R is a *Hodge algebra* (or algebra with straightening law) *governed by* Σ if

H-1 R is a free K-module admitting the set of monomials not in Σ as basis

H-2 for each generator M of Σ in the unique expression

$$M = \sum_{N \notin \Sigma} k_{M,N} N, \qquad k_{M,N} \in K \,, \tag{$*$}$$

guaranteed by H-1, for each $x \in H$ dividing M and each $N \notin \Sigma$ with $k_{M,N} \neq 0$ there is a $y_{M,N}$ dividing N and satisfying $y_{M,N} < x$.

The relations $(*)$ are called *straightening relations* for R.

(4.5) If R is graded and the elements of $\phi(H)$ are homogeneous the straightening relations give a presentation for R [D-E-P, p. 15].

We note that R is a deformation of the discrete Hodge algebra governed by Σ, whose ideal is generated by the monomials M.

(4.6) *Example.* The equations of the tetrahedron of degree 12 of (2.2) are straightening relations. We take Σ as Stanley-Reisner ideal Σ_Δ, where Δ is the stellation of the tetrahedron: in each top-dimensional face we take an additional vertex, which is joined to all vertices on the face. The partial order on the set of vertices is obtained by declaring the new vertices to be smaller. The discrete Hoge algebra has then equations x_ix_j, x_iy_j and $y_jy_ky_l$.

5. The dodecahedron

(5.1) To get an icosahedron as dual graph we need the incidence relations of a dodecahedron. Each side should be a rational surface and the intersection with the other surfaces should have a pentagon as dual graph. A pentagon occurs as hyperplane section of a Del Pezzo surface of degree 5. So we can realise our dodecahedron by glueing together 12 Del Pezzo surfaces.

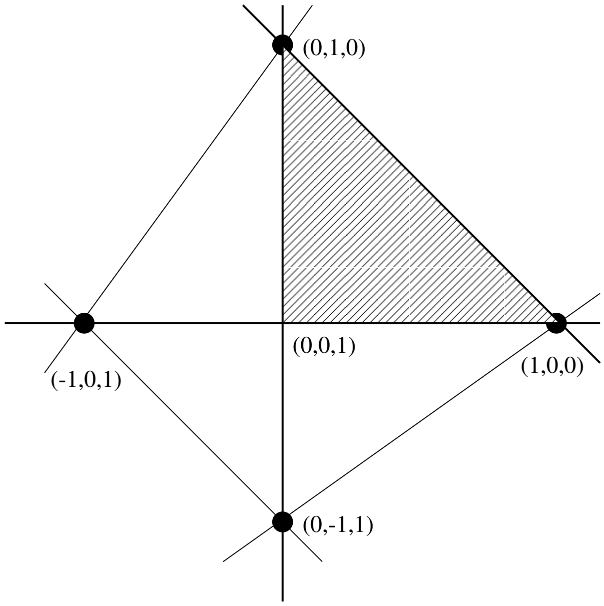
We first describe the Del Pezzo surfaces. Each of those is an extension of its pentagonal hyperplane section. Its coordinate ring can be obtained as Stanley-Reisner ring of a pentagon as 1-dimensional simplicial complex. Introducing variables y_i, we get the equations $y_{i-1}y_{i+1}$. With an extra variable x the Del Pezzo surface has equations

$$y_{i-1}y_{i+1} - xy_i - x^2 \ .$$

These are the pfaffians of the matrix

$$\begin{pmatrix} 0 & y_1 & x & -x & -y_5 \\ -y_1 & 0 & y_2 & x & -x \\ -x & -y_2 & 0 & y_3 & x \\ x & -x & -y_3 & 0 & y_4 \\ y_5 & x & -x & -y_4 & 0 \end{pmatrix} .$$

We can check that this is indeed a smooth Del Pezzo of degree 5 by giving an explicit birational map from $\mathbb{P}^2$, which blows up four points, see Fig. 4. To the variable x corresponds a new vertex at the centre of the pentagon. By joining it to all other vertices we obtain a 2-dimensional simplicial complex, and the homogeneous coordinate ring of the Del Pezzo surface is a graded Hodge algebra governed by the Stanley-Reisner ideal of the complex: to satisfy H-2 we take x to be less than all y_i.



$$
\begin{aligned}
y_1 &= v^2 u \\
y_2 &= vw(v+w) \\
y_3 &= w^2(u+v+w) \\
y_4 &= uw(u+w) \\
y_5 &= u^2 v \\
x &= uvw
\end{aligned}
$$

Figure 4: Blowing up $\mathbb{P}^2$ in 4 points.

(5.2) To construct a normal crossings dodecahedron of degree 60 we glue twelve Del Pezzo surfaces. We get a simplicial complex Δ by stellating a dodecahedron: we take in each face the center of the pentagon as extra vertex. A non-convex realisation of this complex is the great stellated dodecahedron.

(5.3) Proposition. *The coordinate ring of the dodecahedron of degree 60 is a graded Hodge algebra governed by* Σ_Δ.

We describe the equations for a dodecahedron X with icosahedral symmetry in more detail. We have 20 variables y_α, one for each dodecahedral vertex and 12 variables x_i from the extra vertices in the faces. We will denote the vertices by α and i. As two face vertices are not connected by an edge we have 66 equations $x_i x_j = 0$, $i \neq j$. If $\overline{i\alpha}$ is not an edge, we have $x_i y_\alpha = 0$; there are 180 such equations. The non-edges $\overline{\alpha\beta}$ come in two types: in 100 cases the line $\overline{\alpha\beta}$ does not lie in a face of the original dodecahedron, leading to $y_\alpha y_\beta = 0$; if it lies in such a face we get a Del Pezzo equation (in $5 \times 12 = 60$ cases).

We summarise:

type	equation	conditions	#
(1)	$x_i x_j$	$i \neq j$	66
(2)	$x_i y_\alpha$	$\overline{i\alpha}$ not an edge	180
(3)	$y_\alpha y_\beta$	$\overline{\alpha\beta}$ not in face	100
(4)	$y_\alpha y_\gamma - x_i y_\beta - x_i^2$	Del Pezzo	60

The relations follow from the relations between the generators of the Stanley-Reisner ideal of the great stellated dodecahedron, which have a particularly simple form: we get a relation for each pair of equations which have one variable in common.

This gives the following list of relations, where we suppress the conditions on the indices; they can be deduced from the list of equations.

type	relation	#
$(1-1)$	$(x_ix_j)x_k-(x_ix_k)x_j$	440
$(2-1)$	$(x_iy_\alpha)x_j-(x_ix_j)y_\alpha$	1980
$(2-2)$	$(x_iy_\alpha)y_\beta-(x_iy_\beta)y_\alpha$	1260
$(3-2)$	$(y_\alpha y_\beta)x_i-(x_iy_\alpha)y_\beta$	1200
$(3-3)$	$(y_\alpha y_\beta)y_\gamma-(y_\alpha y_\gamma)y_\beta$	780
$(4-2)$	$(y_\alpha y_\gamma-x_iy_\beta-x_i^2)x_j-(x_jy_\alpha)y_\gamma+(x_ix_j)(x_i+y_\beta)$	660
$(4-3)$	$(y_\alpha y_\gamma-x_iy_\beta-x_i^2)y_\delta-(y_\alpha y_\delta)y_\gamma+(x_iy_\delta)(x_i+y_\beta)$	860
$(4-3)$	$(y_\alpha y_\gamma-x_iy_\beta-x_i^2)y_\delta-(y_\alpha y_\delta-x_jy_\beta-x_j^2)y_\gamma$ $+(x_iy_\delta)(x_i+y_\beta)-(x_jy_\gamma)(x_j+y_\beta)$	40
$(4-4)$	relations from matrices	60

We use the equations and relations to compute infinitesimal deformations. The computations are similar to the case of the tetrahedron of degree 12. To illustrate our methods we prove that the dodecahedron X has no nontrivial extensions. This statement means that X is only a hyperplane section of the projective cone over it. To prove this we have to show that the affine cone $C(X)$ has no deformations of negative degree.

(5.4) Proposition. *$T^1_{C(X)}(-\nu)=0$ for $\nu>0$.*

Proof. As all quadratic equations occur in linear relations we cannot perturb the equations with constants, so $T^1(-2)=0$.

Now we consider deformations of degree -1. We start with equations of type (1), which we perturb as follows:

$$x_ix_j+\sum a_{ij}^m x_m+\sum b_{ij}^\alpha y_\alpha\ .$$

The relations $(1-1)$ together with the equations give

$$a_{ij}^k x_k^2+\sum_{\overline{k\alpha}\text{ edge}} b_{ij}^\alpha x_k y_\alpha = a_{ik}^j x_j^2+\sum_{\overline{j\alpha}\text{ edge}} b_{ik}^\alpha x_j y_\alpha\ .$$

This shows that $a_{ij}^k=0$ for $k\notin\{i,j\}$. For each α we can find a $k\notin\{i,j\}$ such that $\overline{k\alpha}$ is an edge, so $b_{ij}^\alpha=0$ and the deformation has the form

$$x_ix_j+a_{ij}^i x_i+a_{ij}^j x_j\ .$$

We now perturb equations of type (2):

$$x_iy_\alpha+\sum a_{i\alpha}^m x_m+\sum b_{i\alpha}^\beta y_\beta$$

and use the relations of type $(2-1)$:

$$a_{i\alpha}^j x_j^2 + \sum_{\overline{j\beta}\text{ edge}} b_{i\alpha}^\beta x_j y_\beta = a_{ij}^j x_j y_\alpha$$

to conclude that $a_{i\alpha}^j = 0$ for all $j \neq i$, $b_{i\alpha}^\beta = 0$ for all $\beta \neq \alpha$ and $b_{i\alpha}^\alpha = a_{ij}^j$ for all j such that $\overline{j\alpha}$ is an edge. It follows that $a_{ij}^j = a_{ik}^k$ for all j and k. Using the coordinate transformation ∂_{x_i} we may therefore assume that the equations of type (1) are not perturbed at all, while those of type (2) have the form $x_i y_\alpha + a_{i\alpha}^i x_i$.

Perturbing equations of type (3) in a similar manner as $y_\alpha y_\beta + \sum a_{\alpha\beta}^m x_m + \sum b_{\alpha\beta}^\gamma y_\gamma$ we find from the relations $(3-2)$ that

$$a_{\alpha\beta}^i x_i^2 + \sum_{\overline{i\gamma}\text{ edge}} b_{\alpha\beta}^\gamma x_i y_\gamma = a_{i\alpha}^i x_i y_\beta \;.$$

So $a_{\alpha\beta}^i = 0$, $b_{\alpha\beta}^\gamma = 0$ for $\gamma \notin \{\alpha, \beta\}$ and $b_{\alpha\beta}^\beta = a_{i\alpha}^i$. The coordinate transformation ∂_{y_α} can be used to eliminate the perturbation of the equations of type (2). Then also those of type (3) are not perturbed.

Finally we look at the Del Pezzo equations (4). From the relations $(4-2)$ we conclude as before that the only possible perturbations have the form

$$y_\alpha y_\gamma - x_i y_\beta - x_i^2 + a_{\alpha\gamma}^i x_i \;.$$

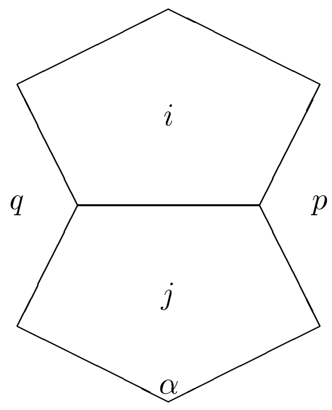
Here $\overline{i\alpha}$, $\overline{i\beta}$ and $\overline{i\gamma}$ all are edges. This means that we can look at each Del Pezzo separately. From the matrix of relations we obtain that $a_{\alpha\gamma}^i = 0$. $\square$

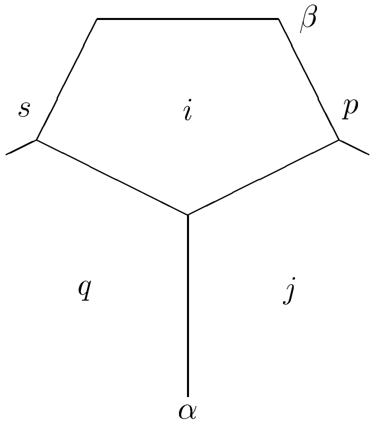
(5.5) Proposition. *The dodecahedron X with icosahedral symmetry is d-semi-stable. The space of locally trivial embedded deformations has dimension* 29.

Proof. We describe a global section of the sheaf $\mathcal{T}_X^1$, without proof. To formulate the result we use the alternative notation y_{ijk} for y_α, if $\overline{i\alpha}$, $\overline{j\alpha}$ and $\overline{k\alpha}$ are the edges involving α.

The equations of type (1) are not deformed, unless $\overline{ij}$ is an edge, in which case we have $x_i x_j + d y_{ijp} y_{ijq}$. An equation of type (2) is perturbed to $x_i y_\alpha + d(x_j y_{ijp} + x_j y_{ijq} + x_j^2)$ if $\alpha = (jkl)$ is opposite the edge $[pq]$; if ijp, ijq and $\alpha = jqr$ are three consecutive vertices, as are $\alpha = jqr$, ijq and iqs, we get $x_i y_\alpha + d(x_j y_{ijq} + x_q y_{ijq} + y_{ijq}^2)$ and in all other cases the equation is not deformed. Perturbation of the equations (3) all vanish except when α and β are nearest possible: one can reach β from α by passing three edges. Suppose that the vertices on this path are $\beta = ipo$, ijp, ijq, $jqr = \alpha$. Then we have $y_\alpha y_\beta + d(x_i y_{ijq} + x_j y_{ijp} + y_{ijp} y_{ijq})$. The Del Pezzo equations are not deformed.

We compute locally near a triple point and look at the chart $y_\alpha = 1$. All variables can be eliminated except y_β, y_γ and y_δ such that $\overline{\alpha\beta}$, $\overline{\alpha\gamma}$ and $\overline{\alpha\delta}$ are





edges, and x_i, x_j and x_k, where α, β and γ lie on the face k, etc. We have nine equations left, of three types: $x_i x_j + dy_\delta$, $x_k y_\delta + d(1 + x_i + x_j)$ and the Del Pezzo equation $y_\beta y_\gamma - x_k - x_k^2$. The last one shows that even x_k can be eliminated, as the double curve lies in $x_k = 0$. By multiplying the Del Pezzo equation with y_δ and using the other equations we get

$$y_\beta y_\gamma y_\delta + d(1 + x_i + x_j + x_k) \; .$$

This show that our d-deformation indeed represents the class $[1] \in H^0(\mathcal{O}_D) = H^0(\mathcal{T}^1_X)$. $\square$

The general theory gives the dimension of the space of locally trivial deformations, but it can also computed directly. For each Del Pezzo we have 5 deformations by multiplying the x_i in a given column of the defining matrix with a unit of the form $1+\varepsilon_{ij}$. These deformations are trivial and can also be obtained by multiplying the y_α by suitable factors. In total we have 60 such deformations, but globally we have only 31 diagonal coordinate transformations (32 variables, but we have to subtract one for the Euler vector field).

(5.6) Theorem. *There exists a semistable degeneration of $K3$ surfaces of degree 60 with icosahedral symmetry, whose special fibre is our dodecahedron.*

(5.7) The rotation group $G_{60} \cong A_5$ of the icosahedron acts symplectically on the general fibre $\mathcal{X}_t$ and the quotient $\mathcal{X}_t/G_{60}$ is again a $K3$ surface, with 2 A_4, 3 A_2 and 4 A_1 singularities [X]. The locus of such surfaces has dimension two in moduli, so together with a polarisation there is only a curve of such surfaces. It would be interesting to know this curve. A deformation computation as above only gives a parametrisation with power series; anyway, the computation is too complicated.

We can take the quotient of the special fibre, which is our dodecahedron.

Invariants for the icosahedral reflection group are

$$\begin{aligned} X &= \sum x_i \\ Y_1 &= \sum y_\alpha \\ Y_2 &= \sum_{\overline{\alpha\beta}\ \text{edge}} y_\alpha y_\beta\ , \end{aligned}$$

and a skew invariant Z is obtained by taking the G_{60} orbit of $x_i y_\alpha y_\beta (y_\alpha - y_\beta)$. After a coordinate transformation $X \mapsto \frac{1}{5}X$, $Y_2 \mapsto Y_2 + \frac{1}{5}XY_1 + \frac{1}{5}X^2$ the quotient is given by the equation

$$\begin{aligned} Z^2 = -X^2 \big(& 5Y_2^2(4Y_2 + 8X^2 + 12XY_1 - Y_1^2) \\ & + (30Y_1 + 20X)Y_2 X(X^2 + XY_1 - Y_1^2)) + (3X^2 + 4XY_1)(X^2 + XY_1 - Y_1^2)^2 \big)\ . \end{aligned}$$

This is a surface of degree 8 in the weighted projective space $\mathbb{P}(1,1,2,4)$. These numbers are in Reid's list of famous 95 and the general $X_8 \subset \mathbb{P}(1,1,2,4)$ is a $K3$ surface with 2 A_1 singularities. Our surface has a double line and two A_4 singularities, at $Y_2 = X^2 + XY_1 - Y_1^2 = 0$.

(5.8) Finite groups acting symplectically on $K3$ surfaces have been classified by Mukai [Mu], see also [X]. Mukai gives an example of a $K3$ surface with even an action of the symmetric group S_5:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 0 \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 &= 0\ . \end{aligned}$$

(5.9) The topology of the special fibre. One can compute the homology $H_*(X,\mathbb{Z})$ with a Mayer-Vietoris spectral sequence [P, Prop. 2.5.1] with E^1-term $E^1_{p,q} = H_p(X^{[q]},\mathbb{Z})$, where $X^{[0]} = \coprod X_i$, $X^{[1]} = \coprod D_{ij}$ and $X^{[2]}$ the set of triple points $P_{ijk} = X_i \cap X_j \cap X_k$.

(5.10) Proposition. *Let $X = \cup_{i=1}^k X_i$ be a d-semistable $K3$ surface of type III in (-1)-form, with k components. Then*

$$\begin{aligned} \dim H^0(X,\mathbb{Z}) &= 1 \\ \dim H^2(X,\mathbb{Z}) &= k + 19 \\ \dim H^4(X,\mathbb{Z}) &= k \end{aligned}$$

Proof. The E^1-term of the spectral sequence looks like

$$\begin{array}{ccc} \oplus H_4(X_i,\mathbb{Z}) & & \\ 0 & & \\ \oplus H_2(X_i,\mathbb{Z}) & \oplus H_2(D_{ij},\mathbb{Z}) & \\ 0 & 0 & \\ \oplus H_0(X_i,\mathbb{Z}) & \oplus H_0(D_{ij},\mathbb{Z}) & \oplus H_0(T_{ijk},\mathbb{Z}) \end{array}$$

To prove that the map $\oplus H_2(D_{ij},\mathbb{Z}) \to \oplus H_2(X_i,\mathbb{Z})$ is injective one can use the same type of arguments occurring in the computation of the cotangent cohomology, for details see [S2]. The dimension formulas now follow from the spectral sequence, using Euler's formula. $\square$

(5.11) We describe the non-algebraic homology class in more detail. Each double curve contains two triple points, which are homologous, so the boundary of an interval. On a component X_i these intervals make up a closed polygon (with e_i edges), which itself is the boundary of a topological disc. For the case of $\mathbb{P}^2$ blown up in 4 points this is illustrated in Fig. 4: after blowing up we have a pentagon, which is the boundary of the strict transform of the shaded area. With the given coordinates this strict transform consists of all points on the Del Pezzo surface with positive coordinates. Finally the discs glue together to a real polyhedron with the same dual graph as the complex surface X.

(5.12) A nice construction for studying the homology the general fibre is given by [A'C]. Let $\sigma_i\colon \mathcal{Z}_i \to \mathcal{X}$ be the oriented real blow-up of $X_i \subset \mathcal{X}$. This is a manifold with boundary, whose boundary $\partial\mathcal{Z}_i = \sigma_i^{-1}(X_i)$ is isomorphic to the boundary of a tubular neighbourhood of X_i in $\mathcal{X}$. The fibred product $\sigma\colon \mathcal{Z} \to \mathcal{X}$ of the σ_i is a manifold with corners. Its boundary $\mathcal{N} := \partial\mathcal{Z}$ comes with a map to X. It also fibres over S^1: the composed map $\mathcal{Z} \to \mathcal{X} \to S \ni 0$ extends to a map from $\mathcal{Z}$ to the real oriented blow-up of S in 0 (polar coordinates!). A fibre of $\mathcal{N} \to S^1$ is then a topological model of the general fibre.

This model is not sufficient to describe the monodromy. One has first to replace X by the geometric realisation of the simplicial object $X^{[\cdot]}$: one replaces each double point by an interval, and each triple point by a 2-simplex. A final fibred product then gives the new model. For details see [A'C, §2].

(5.13) The Stanley-Reisner ring of the icosahedron. A different family of $K3$ surfaces with icosahedral symmetry is obtained by smoothing the Stanley-Reisner ring of the icosahedron. The infinitesimal deformations can be found from [A-C, Sect. 4] or computed directly with the methods above. All deformations are unobstructed ($T^2_X = 0$). We have $T^1_X(\nu) = 0$ for $\nu < 0$ and $\dim T^1_X(0) = 30$. Furthermore the dimension of $H^0(\Theta_X)$ equals 11, which fits with the fact that X deforms to smooth $K3$ surfaces $(30 - 11 = 19)$.

We number the vertices as in Fig. 5. Then we have two types of equations, depending on the distance between vertices. The infinitesimal deformations are

$$x_0x_6 + \varepsilon_{06}x_2x_3, \qquad x_0x_{11}$$

By taking all ε_{ij} equal we get an icosahedral invariant deformation. The lift to a one-parameter deformation seems to involve power series of the deformation variable (I computed up to order 7). Anyway, equations for a $K3$ of degree 20 are not very illuminating.

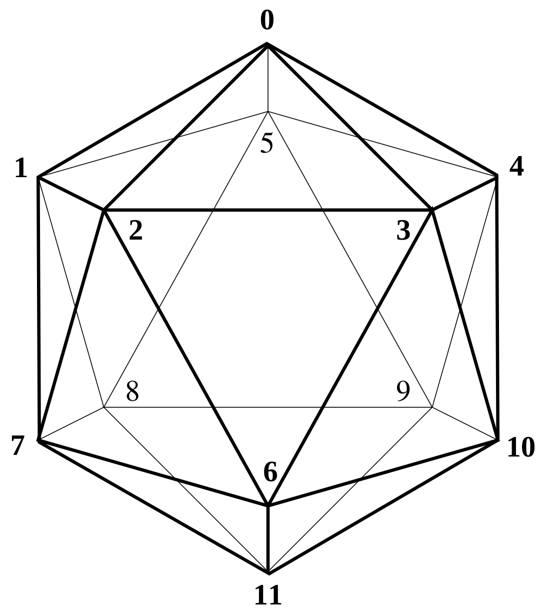


Figure 5: Icosahedron.

As before, this deformation is not semistable, because the total space has singularities. Each vertex of the icosahedron gives a singularity, which is the cone over a pentagon. It is smoothed negatively, with total space the cone over a Del Pezzo of degree 5. We resolve these singularities by blowing up. We introduce 12 Del Pezzo surfaces. The sides are blown up in three points, giving hexagons. The dual graph of the central fibre is now a stellated dodecahedron. The object itself consists of pentagons and hexagons. It contains a real homology class, as described in (5.11), which looks like a football, so our special fibre is a complexified football.

(5.14) A degeneration of degree 12. The existence of the two degenerations above with icosahedral symmetry follows from a deformation argument, but it is too complicated to give explicit equations. In the football case pentagons arise because of the singularities of the total space. This suggests that one can get a degeneration of low degree by blowing down components of the special fibre. Blowing down means removing vertices from the dual graph.

We start from the icosahedron (Fig. 5) and remove non-adjacent vertices, say those numbered 0, 7 and 10. This means breaking the symmetry. The resulting dual graph is shown in Fig. 6. Of the double curves on the components six are triangles and four are rectangles. We realise them on planes resp. quadric surfaces. The picture also shows a realisation (as a real polyhedron). We cannot take the Stanley-Reisner ideal, as the realisation contains rectangles. For those we take an equation of the form $xy - zt$. One may think that a rectangle can be triangulated in two ways, each giving a monomial, which are forced to be equal. The result is a surface $X \subset \mathbb{P}^7$ of degree 12. With the numbering in the figure

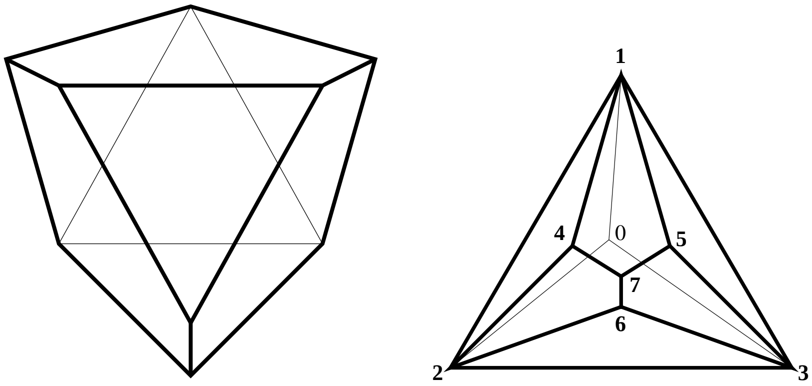


Figure 6: Dual graph and its realisation for X of degree 12.

we get the S_3-invariant ideal

$$\begin{gathered} x_0x_7, \\ x_0x_4, \quad x_0x_5, \quad x_0x_6, \\ x_1x_6, \quad x_2x_6, \quad x_3x_4, \\ x_1x_7 - x_4x_5, \quad x_2x_7 - x_4x_6, \quad x_3x_7 - x_5x_6, \\ x_1x_2x_3 \ . \end{gathered}$$

The next thing to do is to compute the T^1 and T^2 for the affine cone $C(X)$ over X. This is conveniently done with a computer algebra program. A computation with *Macaulay* [B-S] gives the following result:

(5.15) Lemma. *As $\mathcal{O}_{C(X)}$-module $T^1_{C(X)}$ is generated by eight elements, represented by the following perturbations of the equations:*

$$\begin{gathered} x_0x_7, \\ x_0x_4 - c_3x_1x_2, \quad x_0x_5 - c_2x_1x_3, \quad x_0x_6 - c_1x_2x_3, \\ x_1x_6 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6, \\ x_2x_5 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6, \\ x_3x_4 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6, \\ x_1x_7 - x_4x_5, \quad x_2x_7 - x_4x_6, \quad x_3x_7 - x_5x_6, \\ x_1x_2x_3 + ax_0 + b_1x_2x_3 + b_2x_1x_3 + b_3x_1x_2, \end{gathered}$$

and $\dim T^2_{C(X)} = 2$, *concentrated in degree* -2.

The quadratic obstruction is given by $a(c_1-c_2) = a(c_1-c_3) = 0$. We conclude that the degree zero deformations are unobstructed. The base space for $C(X)$ in non-positive degrees has two components. As we are mainly interested in S_3-invariant deformations we consider only the component with $c_1 = c_2 = c_3 (=: c)$.

The component will be obtained by substituting polynomials for the deformation variables a, b_i and c. A computation gives the equations

$$\begin{gathered}
x_0x_7 + c(b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac),\\
x_0x_4 - cx_1x_2, \quad x_0x_5 - cx_1x_3, \quad x_0x_6 - cx_2x_3,\\
x_1x_6 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac,\\
x_2x_5 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac,\\
x_3x_4 + b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac,\\
x_1x_7 - x_4x_5, \quad x_2x_7 - x_4x_6, \quad x_3x_7 - x_5x_6,\\
x_1x_2x_3 + ax_0 + b_1x_2x_3 + b_2x_1x_3 + b_3x_1x_2 - b_0(b_0x_7 + b_1x_6 + b_2x_5 + b_3x_6 + ac),
\end{gathered}$$

For $c \neq 0$ we derive the three equations $x_0x_7 - cx_ix_{7-i}$, which show that we have a hypersurface in the cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(5.16) Proposition. *A general one-parameter deformation in degree 0 on the component described above has a minimal model in (-1)-form with the icosahedron as dual graph for the central fibre. In particular, this holds for*

$$\begin{aligned}
a &= c(x_0^2 + x_0(x_1 + x_2 + x_3) + x_1^2 + x_2^2 + x_3^2),\\
b_0 &= cx_7,\\
b_1 &= c(x_2 + x_3 + x_6 + x_7),\\
b_2 &= c(x_1 + x_3 + x_5 + x_7),\\
b_3 &= c(x_1 + x_2 + x_4 + x_7).
\end{aligned}$$

Proof. One first checks that the general fibre is a smooth $K3$ surface. For this it suffices to look at the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

In the particular example the total space has at the origin of the affine chart $x_1 = 1$ a singularity, which is isomorphic to the cone over the Del Pezzo surface of degree 5, as it should be: the point to check is that we indeed have a generic local deformation. Furthermore there are 18 singularities of type A_1. On the (x_1, x_4)-line we have the point $x_1 + x_4 = 0$. On the (x_0, x_1)-line we have two points, given by $x_0^2 + x_0x_1 + x_1^2$, and on the (x_7, x_7)-line the two points $x_6^2 + x_6x_7 + x_7^2$. The other singular points are found by symmetry.

By blowing up the three singularities of multiplicity 5 and making a small resolution of the A_1-points we get a smooth total space. To obtain the (-1)-form one has to place one exceptional curve on either component in case the double line contains two singularities. If there is only one, the exceptional curve should lie on the triangle component. □

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