THE STABILITY OF RATIONAL APPROXIMATIONS OF ANALYTIC SEMIGROUPS*

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Abstract. It is shown that A-acceptable and, more generally, $A(\theta)$-acceptable rational approximations of bounded analytic semigroups in Banach space are stable. The result applies, in particular, to the Crank-Nicolson method.

1. Introduction. Let A be a densely defined, closed linear operator in a Banach space X, which generates a bounded strongly continuous semigroup $E(t) = e^{tA}$, such that

$$\|E(t)\| \le C_0, \qquad \text{for } t \ge 0.$$

Then the initial-value problem

$$u' = Au, \quad \text{for } t \ge 0, \qquad u(0) = v, \tag{1.1}$$

has the unique (generalized) solution $u(t) = E(t)v$, for any $v \in X$, and

$$\|u(t)\| \le C_0\|v\|, \qquad \text{for } t \ge 0.$$

Let k be a time step and $t_n = nk$. An approximate solution $U_n \approx u(t_n)$ of (1.1) may be defined by

$$U_{n+1} = r(kA)U_n, \quad \text{for } n \ge 0, \qquad U_0 = v, \tag{1.2}$$

where r is a rational function that satisfies certain conditions. For example, $r(z) = 1/(1-z)$ and $r(z) = (1+z/2)/(1-z/2)$ correspond to the backward Euler and Crank-Nicolson methods, respectively. The scheme (1.2) then provides an approximation $r(kA)^n$ of $E(t_n)$ and one may ask the following two questions:

1. Stability: Is $r(kA)^n$ bounded for $n \ge 0$?
2. Convergence: Does $r(kA)^n v$ converge to $E(t)v$ as $k \to 0$, $nk = t$?

We say that r is A-acceptable if

$$|r(z)| \le 1, \qquad \text{for } \operatorname{Re} z \le 0,$$

and accurate of order p, where $p \ge 1$, if

$$r(z) = e^z + O(z^{p+1}), \qquad \text{as } z \to 0. \tag{1.3}$$

We quote the following two results from Brenner and Thomée [2]:

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Theorem 1. *There is a constant C depending on r such that if r is A-acceptable, then*

$$\|r(kA)^n\| \le CC_0\sqrt{n}, \qquad \text{for } n \ge 0,\ k > 0.$$

Theorem 2. *There is a constant C depending on r such that if r is A-acceptable and accurate of order p, then*

$$\|r(kA)^n v - E(t_n)v\| \le CC_0 t_n k^p \|A^{p+1}v\|, \qquad \text{for } n \ge 0,\ k > 0,\ v \in \mathcal{D}(A^{p+1}).$$

As indicated in [2], the result of Theorem 1 is best possible in the sense that the $\sqrt{n}$ cannot be removed, in general. The proof of this fact will be given in Section 2 below.

In the case of an analytic semigroup stronger results may be obtained. We assume without loss of generality that A is the generator of a bounded analytic semigroup $E(t) = e^{tA}$ and that $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. This is equivalent to saying that there is an angle $\alpha \in (0, \pi/2)$ such that

$$\rho(A) \supset \overline{\Sigma}_\alpha, \qquad \text{where } \Sigma_\alpha = \{z = \rho e^{i\varphi} : \rho > 0,\ |\varphi| < \pi - \alpha\}, \tag{1.4}$$

and a constant $M \ge 1$ such that, with $R(z; A) = (zI - A)^{-1}$ denoting the resolvent of A,

$$\|R(z; A)\| \le \frac{M}{|z|}, \qquad \text{for } z \in \Sigma_\alpha, \tag{1.5}$$

cf. Pazy [11, Theorem 2.5.2]. We may now quote the following results (see, for example, Sobolevskij and Huang Van Lai [14], Le Roux [9], Larsson, Thomée and Wahlbin [8], Fujita and Suzuki [7, Theorem 17.1]):

Theorem 3. *There is a constant C depending on r and α such that if A satisfies (1.4) and (1.5) and r is A-acceptable and accurate of order p, then*

$$\|r(kA)^n v - E(t_n)v\| \le CMk^p \|A^p v\|, \qquad \text{for } t_n \ge 0,\ k > 0,\ v \in \mathcal{D}(A^p).$$

Theorem 4. *There is a constant C depending on r and α such that if A satisfies (1.4) and (1.5) and r is A-acceptable and accurate of order p, with $|r(\infty)| < 1$, then*

$$\|r(kA)^n v - E(t_n)v\| \le CMk^p t_n^{-p} \|v\|, \qquad \text{for } t_n > 0,\ k > 0,\ v \in X.$$

In particular, Theorem 4 shows stability for A-acceptable rational functions r with $|r(\infty)| < 1$. This condition is not satisfied, e.g., for the Crank-Nicolson method, for which $r(\infty) = -1$. Moreover, by considering the case of a selfadjoint operator A in a Hilbert space, it is easy to see that the error bound of Theorem 4 does not hold for the Crank-Nicolson method. On the other hand, if the operator A is replaced by a family of bounded operators A_h, satisfying (1.4) and (1.5) uniformly in h, then in Theorem 4 the condition $|r(\infty)| < 1$ can be replaced by the mesh restriction that $k\|A_h\| \le C$ for some $C > 0$ (cf. [7, loc. cit.] and Piskarev [12]).

The main purpose of the present work is to show that these conditions are not necessary for stability in the case of an analytic semigroup. In doing so we shall also be able to weaken the assumption on r to $A(\theta)$-acceptability with $\theta > \alpha$. Also, we do not assume (1.3).

Theorem 5. *There is a constant C depending on r and α such that if A satisfies (1.4) and (1.5), and r is $A(\theta)$-acceptable, i.e.,*

$$|r(z)| \le 1, \qquad \text{for } z \in \mathbf{C} \setminus \Sigma_\theta, \tag{1.6}$$

for some angle $\theta \in (\alpha, \pi/2]$, then

$$\|r(kA)^n\| \le CM, \qquad \text{for } n \ge 0,\ k > 0.$$

As usual the assumption that A has a bounded inverse is made only for convenience of the proof. In fact, if A satisfies (1.4)–(1.5) except that $0 \notin \rho(A)$, then the operator $A_\epsilon = A - \epsilon I$, $\epsilon > 0$, satisfies (1.4)–(1.5) with $0 \in \rho(A_\epsilon)$ and M replaced by $M/\sin\alpha$. Hence Theorem 5 shows that $r(kA_\epsilon)^n$ is bounded independently of ϵ. Expanding $r(z)$ in partial fractions, we may write $r(kA_\epsilon)^n$ as a sum of powers of resolvents, and it follows that $r(kA_\epsilon)^n \to r(kA)^n$ as $\epsilon \to 0$. This shows the stability of $r(kA)^n$. Analogous remarks apply to Theorems 3 and 4.

The results of Theorems 3 and 4 are also valid under the assumption of $A(\theta)$-acceptability [7, loc. cit.]. Crouzeix [4] proved the analogs of Theorems 1–4 for multistep methods. It is likely that the techniques of the present work can also be used to show stability in the case of $A(\theta)$-acceptable multistep methods.

After the completion of this work we learnt that Palencia [10] recently proved the stability of A-acceptable approximations of bounded analytic semigroups. His technique is different from ours. Our proof, however, seems to be more elementary.

2. The sharpness of Theorem 1. We shall first show by an example proposed in [2] that the $\sqrt{n}$ in Theorem 1 cannot be removed, in general. The proof is a modification of that of a result concerning maximum-norm stability of difference schemes given in Brenner and Thomée [1].

Let $X = C_0(\mathbf{R})$ be the space of continuous functions, which vanish at ∞, normed by $\|v\| = \sup_x |v(x)|$, and consider the contraction semigroup $(E(t)v)(x) = v(x+t)$, so that $A = \partial/\partial x$. Then for a rational function $q(z)$, bounded for $\operatorname{Re} z \le 0$, we have, with $\mathcal{F}$ denoting Fourier transformation,

$$\mathcal{F}(q(A)v)(\xi) = q_I(\xi)\mathcal{F}v(\xi), \qquad \text{where } q_I(\xi) = q(i\xi). \tag{2.1}$$

In fact, using Lemmas 1–3 in [2], we find that, if $q(\infty) = 0$, then the inverse Fourier transform $\check{q} = \mathcal{F}^{-1}q$ is in L_1 and has its support in $[0,\infty)$ and

$$q(z) = \int_0^\infty e^{zt}\check{q}(t)\,dt.$$

Further,

$$(q(A)v)(x) = \int_0^\infty e^{tA}v(x)\check{q}(t)\,dt = \int_0^\infty v(x+t)\check{q}(t)\,dt$$

and, with $\hat{v} = \mathcal{F}v$,

$$\mathcal{F}(q(A)v)(\xi) = \int_{-\infty}^\infty e^{-ix\xi}\int_0^\infty v(x+t)\check{q}(t)\,dt\,dx = \int_0^\infty e^{it\xi}\check{q}(t)\,dt\,\hat{v}(\xi) = q(i\xi)\hat{v}(\xi).$$

If $q(\infty) \ne 0$, then we may write $q(z) = q(\infty) + \tilde{q}(z)$ with $\tilde{q}(\infty) = 0$, and (2.1) follows easily. Hence, with m denoting the Fourier multiplier norm on $C_0(\mathbf{R})$ (cf. [3]), we have

$$\|q(kA)\| = m(q_I(k\cdot)) = m(q_I).$$

Considering, in particular, the Crank-Nicolson method defined by the rational function $r(z) = (1 + z/2)/(1 - z/2)$, we shall demonstrate

$$m(r_I^n) \geq c\sqrt{n}, \qquad \text{with } c > 0, \tag{2.2}$$

which will thus show that Theorem 1 is sharp.

To prove (2.2), note that since $|r(i\xi)| = 1$, for $\xi \in \mathbf{R}$, we may write $r_I(\xi) = e^{i\psi(\xi)}$ with ψ real-valued. Choose $\eta \in C_0^\infty(\mathbf{R})$ such that $\psi''(\xi) \neq 0$ in $\operatorname{supp} \eta$. By Parseval's relation we have

$$2\pi\|\eta\|_{L_2}^2 = 2\pi\|r_I^n\eta\|_{L_2}^2 = \|\mathcal{F}(r_I^n\eta)\|_{L_2}^2 \leq \|\mathcal{F}(r_I^n\eta)\|_{L_1}\|\mathcal{F}(r_I^n\eta)\|_{L_\infty}.$$

Here

$$\|\mathcal{F}(r_I^n\eta)\|_{L_1} \leq m(r_I^n)\|\hat{\eta}\|_{L_1},$$

and we shall show below that

$$\|\mathcal{F}(r_I^n\eta)\|_{L_\infty} \leq Cn^{-1/2}. \tag{2.3}$$

Hence

$$Cn^{-1/2}m(r_I^n)\|\hat{\eta}\|_{L_1} \geq \|\eta\|_{L_2}^2,$$

which is (2.2). It remains to prove (2.3). Recalling that $r_I(\xi)^n = r(i\xi)^n = e^{in\psi(\xi)}$ with $\psi'' \neq 0$ in $\operatorname{supp} \eta$, this follows by the following lemma from [1].

Lemma 1. *Let $\eta \in C_0^1(\mathbf{R})$ and let $\psi \in C^2(\mathbf{R})$ be real-valued with $|\psi''| \geq \delta > 0$ in $\operatorname{supp} \eta$. Then*

$$\|\mathcal{F}(e^{i\psi}\eta)\|_{L_\infty} \leq \frac{8}{\sqrt{\delta}}\|\eta'\|_{L_1}.$$

Proof. We have

$$\mathcal{F}(e^{i\psi}\eta)(\xi) = \int_{-\infty}^{\infty} e^{i\psi(x)-i\xi x}\eta(x)\,dx = \int_{-\infty}^{\infty}\eta'(x)\int_{x_0}^{x} e^{i(\psi(y)-\xi y)}\,dy\,dx,$$

for some x_0. By van der Corput's lemma (cf. [3]) we have $|\int_{x_0}^{x} e^{i(\psi(y)-\xi y)}\,dy| \leq 8/\sqrt{\delta}$, which completes the proof. □

Note that in the above counter-example X is not a Hilbert space. In fact, if A generates a contraction semigroup $E(t)$ in a Hilbert space, so that, with $u(t) = E(t)v$, we have $\|u(t)\| \leq \|u(s)\|$, for $s \leq t$. Then

$$2\operatorname{Re}(Au, u) = (Au, u) + (u, Au) = \frac{d}{dt}\|u\|^2 \leq 0,$$

so that $\operatorname{Re} A \leq 0$. But then $\{z : \operatorname{Re} z \leq 0\}$ is a spectral set for A in the sense of von Neumann (cf. [13]). This means that, for a rational function f, $|f(z)| \leq 1$ for $\operatorname{Re} z \leq 0$ implies $\|f(A)\| \leq 1$. In particular, if r is A-acceptable, then $\|r(kA)^n\| \leq 1$.

For special choices of r better results are possible. Assume that

$$|r_I(\xi)| < 1, \quad \text{for } 0 \neq \xi \in \mathbf{R}, \qquad \text{and } |r(\infty)| < 1, \tag{2.4}$$

and that there exist positive integers p, q with q even, $q \geq p + 1$ and a real number $\gamma > 0$ such that

$$r_I(\xi) = e^{i\xi + \psi(\xi)}, \qquad \text{with } \psi(\xi) = O(\xi^{p+1}), \quad \xi \to 0, \tag{2.5}$$

$$\operatorname{Re}\psi(\xi) \leq -\gamma\xi^q, \qquad \text{for } |\xi| \leq 1. \tag{2.6}$$

Note that (1.3) implies (2.5). But then, by analyticity and (2.4), $\operatorname{Re}\psi(\xi) = \gamma_0\xi^q(1 + o(1))$ as $\xi \to 0$ for some $q \geq p + 1$, with $\gamma_0 < 0$ and q even, so that (2.6) follows.

Under the assumptions (2.4)–(2.6) we may conclude that

$$\|r(kA)^n\| \leq Cn^{\frac{1}{2}(1-\frac{p+1}{q})},$$

cf. [2]. This result is also best possible.

3. Proof of Theorem 5. We may assume that $|r(z)| \not\equiv 1$, so that, by $A(\theta)$-acceptability and the maximum principle, $|r(z)| < 1$ in the interior of the sector $\mathbf{C} \setminus \Sigma_\theta$. Let $\beta = r(0)$ and $\gamma = r(\infty)$. The proof is based on using asymptotic expansions of $r(z)$ at $z = 0$ and $z = \infty$. We formulate these in the following lemma.

Lemma 2. *There exist non-zero complex numbers a, b and positive integers p, q such that, if $|\beta| = 1$,*

$$r(z) = \beta e^{-a(-z)^p} + O(z^{p+1}), \quad \text{as } z \to 0, \qquad |\arg a| + p\theta \le \frac{\pi}{2}, \tag{3.1}$$

and, if $|\gamma| = 1$,

$$r(z) = \gamma e^{-b(-w)^q} + O(w^{q+1}), \quad \text{as } w = z^{-1} \to 0, \qquad |\arg b| + q\theta \le \frac{\pi}{2}. \tag{3.2}$$

Proof. It suffices to demonstrate (3.1). The function $r(z)$ is analytic at $z = 0$ and $|r(z)| \not\equiv 1$. Hence a Taylor expansion gives (3.1) for some complex number $a \neq 0$ and integer $p \ge 1$. Using (1.6) we see that $\operatorname{Re}(-a(-z)^p) \le 0$, for all $z \in \mathbf{C} \setminus \Sigma_\theta$. This implies that $|\arg a| + p\theta \le \pi/2$. In particular, $\operatorname{Re} a > 0$. We remark that if $r(z)$ satisfies (1.3), then (3.1) holds with $\beta = a = p = 1$. □

Examples. If $r(z)$ is A-acceptable, i.e., if $\theta = \pi/2$, then $a, b > 0$ and $p = q = 1$. For example, for the Crank-Nicolson scheme $r(z) = (1 + z/2)/(1 - z/2)$ we have $r(z) = e^z + O(z^3)$ as $z \to 0$ and hence $r(z) = -r(4w) = -e^{4w} + O(w^3)$ as $w = z^{-1} \to 0$. Thus $\beta = 1$, $\gamma = -1$, $a = 1$, $b = 4$ and $p = q = 1$ in this case. Further examples are provided by other Padé approximations of e^z. It is known that the Padé approximant of degree n, m is $A(\theta)$-acceptable for some $\theta \in (0, \pi/2]$ if $n \le m$, and A-acceptable if and only if $m - 2 \le n \le m$, cf. Wanner, Hairer and Nørsett [15]. An example with $|\gamma| = 1$ and $q = 2$ is furnished by the function $r(z) = (1 + z/2 - z^3)/(1 - z/2 - z^3)$. Here $r(z) = e^z + O(z^3)$ as $z \to 0$, and $r(z) = e^{-w^2} + O(w^3)$ as $w = z^{-1} \to 0$. □

Let a, b and p, q be as in Lemma 2, with the addition that $a = p = 1$ if $|\beta| < 1$ and $b = q = 1$ if $|\gamma| < 1$, and set

$$f_n(z) = r(z)^n - \beta^n e^{-na(-z)^p} - \gamma^n e^{-nb(-z)^{-q}}.$$

Note that we use less accurate expansions if $|\beta| < 1$ or $|\gamma| < 1$. We then define an operator $f_n(kA)$ by the contour integral

$$f_n(kA) = \frac{1}{2\pi i} \int_\Gamma f_n(kz) R(z; A)\, dz, \tag{3.3}$$

where Γ is the boundary of Σ_α consisting of the two rays $\Gamma_\pm = \{z = -\rho e^{\pm i\alpha} : \rho \ge 0\}$. The path is oriented so that $\operatorname{Im} z$ increases along Γ. The function $f_n(z)$ was chosen such that $f_n(z) \to 0$ as $z \to \infty$ and as $z \to 0$ along Γ. Recall that A^{-1} is bounded and note also that $f_n(z)$ is analytic to the left of Γ, but has a singularity at the origin. The proof now proceeds in three steps:

1. Show that $f_n(kA)$ is bounded uniformly in n and k.
2. Show that

$$f_n(kA) = r(kA)^n - \beta^n e^{-nak^p(-A)^p} - \gamma^n e^{-nbk^{-q}(-A)^{-q}}, \tag{3.4}$$

where $e^{-t(-A)^p}$ and $e^{-t(-A)^{-q}}$ denote the semigroups generated by $-(-A)^p$ and $-(-A)^{-q}$, respectively.

3. Show that $e^{-t(-A)^p}$ and $e^{-t(-A)^{-q}}$ are bounded uniformly for $|\arg t| \le \frac{\pi}{2} - p\alpha$ and $|\arg t| \le \frac{\pi}{2} - q\alpha$, respectively.

Together these statements imply the boundedness of $r(kA)^n$ and thus complete the proof of Theorem 5.

It is convenient to collect some simple bounds before beginning step 1.

Lemma 3. *There are positive numbers C and c depending on r and α such that, for $z \in \mathbf{C} \setminus \Sigma_\alpha$,*

$$|e^{-a(-z)^p}| \le e^{-c|z|^p}, \tag{3.5}$$

$$|r(z)^n - \beta^n e^{-na(-z)^p}| \le C|z|e^{-cn|z|^p}, \qquad |z| \le 1, \ n \ge 1, \tag{3.6}$$

$$|e^{-b(-z)^{-q}}| \le e^{-c|z|^{-q}}, \tag{3.7}$$

$$|r(z)^n - \gamma^n e^{-nb(-z)^{-q}}| \le C|z|^{-1}e^{-cn|z|^{-q}}, \qquad |z| \ge 1, \ n \ge 1. \tag{3.8}$$

Proof. It is sufficient to demonstrate (3.5) and (3.6). Assume first that $|\beta| = 1$. The bound (3.5), with $c = |a|\cos(|\arg a| + p\alpha)$, is obvious, since $|\arg a| + p\alpha < |\arg a| + p\theta \le \pi/2$, and, in view of (3.1), it then follows that

$$|r(z)| \le e^{-c|z|^p}, \qquad |z| \le 1,$$

for some $c > 0$, since $|r(z)| < 1$ away from the origin. Hence

$$\begin{aligned} |r(z)^n - \beta^n e^{-na(-z)^p}| &= \left| \big(r(z) - \beta e^{-a(-z)^p}\big) \sum_{j=0}^{n-1} r(z)^{n-1-j} \beta^j e^{-ja(-z)^p} \right| \\ &\le C|z|^{p+1} n e^{-(n-1)c|z|^p} \le C|z|e^{-cn|z|^p}, \end{aligned}$$

which is (3.6). If $|\beta| < 1$, then $|\beta| \le e^{-c}$, $|r(z)| \le e^{-c}$ and $|r(z) - \beta e^z| \le C|z|$ for $|z| \le 1$, so that $|r(z)^n - \beta^n e^{nz}| \le C|z|e^{-nc}$ and (3.6) follows. $\square$

Lemma 4. *The integral in (3.3) converges in the operator norm and there is a constant C depending on r and α such that*

$$\|f_n(kA)\| \le CM(n^{-1/p} + n^{-1/q}), \qquad n \ge 1, \ k > 0.$$

Proof. Let $z = -\rho e^{i\alpha}$. Using (1.5) we obtain

$$\left\| \int_{\Gamma_+} f_n(kz) R(z; A)\, dz \right\| \le M \int_0^\infty |f_n(kz)| \frac{d\rho}{\rho} = M \int_0^\infty |f_n(z)| \frac{d\rho}{\rho}.$$

Here, in view of (3.6) and (3.7), we have (with $s = n\rho^p$ and $s = n\rho^{-q}$, respectively)

$$\begin{aligned} \int_0^1 |f_n(z)| \frac{d\rho}{\rho} &\le \int_0^1 |r(z)^n - \beta^n e^{-na(-z)^p}| \frac{d\rho}{\rho} + \int_0^1 |e^{-nb(-z)^{-q}}| \frac{d\rho}{\rho} \\ &\le C \int_0^1 \rho e^{-cn\rho^p} \frac{d\rho}{\rho} + \int_0^1 e^{-cn\rho^{-q}} \frac{d\rho}{\rho} \\ &= Cn^{-1/p} p^{-1} \int_0^n s^{1/p} e^{-cs} \frac{ds}{s} + q^{-1} \int_n^\infty e^{-cs} \frac{ds}{s} \\ &\le Cn^{-1/p} \int_0^\infty s^{-1+1/p} e^{-cs}\, ds + \int_n^\infty e^{-cs}\, ds \\ &= Cn^{-1/p} + Ce^{-cn} \le Cn^{-1/p}. \end{aligned}$$

Similarly, by (3.8) and (3.5) and substituting $\sigma = 1/\rho$, the above argument shows

$$\begin{aligned} \int_1^\infty |f_n(z)| \frac{d\rho}{\rho} &\le \int_1^\infty |r(z)^n - \gamma^n e^{-nb(-z)^{-q}}| \frac{d\rho}{\rho} + \int_1^\infty |e^{-na(-z)^p}| \frac{d\rho}{\rho} \\ &\le C \int_1^\infty \rho^{-1} e^{-cn\rho^{-q}} \frac{d\rho}{\rho} + \int_1^\infty e^{-cn\rho^p} \frac{d\rho}{\rho} \\ &\le Cn^{-1/q} + Ce^{-cn} \le Cn^{-1/q}. \quad \square \end{aligned}$$

For the next two steps of the proof we rewrite (3.3) as

$$
\begin{aligned}
f_n(kA) = \frac{1}{2\pi i}\int_\Gamma \big(r(kz)^n - \gamma^n\big) R(z;A)\,dz - \beta^n \frac{1}{2\pi i}\int_\Gamma e^{-nak^p(-z)^p} R(z;A)\,dz \\
- \gamma^n \frac{1}{2\pi i}\int_\Gamma \big(e^{-nbk^{-q}(-z)^{-q}} - 1\big) R(z;A)\,dz.
\end{aligned}
$$

Here the first term on the right is equal to $r(kA)^n - \gamma^n I$, since the rational function $r(kz)^n - \gamma^n$ is equal to 0 at $z = \infty$. The second term may easily be estimated by a standard technique. We obtain

$$
\left\| \frac{1}{2\pi i}\int_\Gamma e^{-t(-z)^p} R(z;A)\,dz \right\| \le CM, \qquad \text{for } |\arg t| \le \frac{\pi}{2} - p\alpha,
$$

cf. (3.5) and (1.5). In fact, the above integral is the bounded analytic semigroup generated by $-(-A)^p$, see [6] and Remark 2 below. Note that Lemma 2 implies that a belongs to the sector where the bound holds. Now (3.4) follows, and the proof will be complete, once we have proved the following lemma.

Lemma 5. *There is a constant C depending on r and α such that*

$$
\|e^{-t(-A)^{-q}}\| \le CM, \qquad \textit{for } |\arg t| \le \frac{\pi}{2} - q\alpha. \tag{3.9}
$$

Moreover, with Γ as in (3.3), we have

$$
\frac{1}{2\pi i}\int_\Gamma \big(e^{-t(-z)^{-q}} - 1\big) R(z;A)\,dz = e^{-t(-A)^{-q}} - I. \tag{3.10}
$$

Proof. We have

$$
e^{-t(-A)^{-q}} = \frac{1}{2\pi i}\int_{\widetilde\Gamma} e^{-t(-w)^q} R(w;A^{-1})\,dw, \tag{3.11}
$$

for a suitable contour $\widetilde\Gamma$ enclosing the spectrum of the bounded operator A^{-1}. We may choose $\widetilde\Gamma = \Gamma_{S,R} = \{w = -Re^{i\varphi} : |\varphi| \le \alpha\} \cup \{w = -\rho e^{\pm i\alpha} : S \le \rho \le R\} \cup \{w = Se^{i\varphi} : |\varphi| \le \pi - \alpha\}$, with counter-clockwise orientation and with R sufficiently large and arbitrary $S > 0$. The contribution to (3.11) from the arc with radius R is 0 in the limit $R \to \infty$. The formula (3.11) therefore holds with $\widetilde\Gamma = \Gamma_S = \{w = -\rho e^{\pm i\alpha} : \rho \ge S\} \cup \{w = Se^{i\varphi} : |\varphi| \le \pi - \alpha\}$.

Using the identity $R(w;A^{-1}) = -w^{-1}AR(w^{-1};A)$, we now have

$$
e^{-t(-A)^{-q}} = -\frac{1}{2\pi i}\int_{\Gamma_S} e^{-t(-w)^q} AR(w^{-1};A)\,\frac{dw}{w}, \tag{3.12}
$$

and, setting $y = \tau w$ with $\tau = |t|^{1/q}$,

$$
e^{-t(-A)^{-q}} = -\frac{1}{2\pi i}\int_{\Gamma_{\tau S}} e^{-\frac{t}{|t|}(-y)^q} AR(\tau y^{-1};A)\,\frac{dy}{y}.
$$

Since $AR(z;A) = -I + zR(z;A)$, we have $\|AR(\tau y^{-1};A)\| \le 1 + M \le 2M$ by (1.5). Hence (3.9) follows, cf. (3.7), if we take $S = \tau^{-1}$, so that $\Gamma_{\tau S}$ is independent of t.

Using $AR(z;A) = -I + zR(z;A)$ again, we rewrite (3.12) as

$$\begin{aligned} e^{-t(-A)^{-q}} &= \frac{1}{2\pi i}\int_{\Gamma_S} e^{-t(-w)^q}\,\frac{dw}{w} I - \frac{1}{2\pi i}\int_{\Gamma_S} e^{-t(-w)^q} R(w^{-1};A)\,\frac{dw}{w^2} \\ &= I - \frac{1}{2\pi i}\int_{\Gamma_S} e^{-t(-w)^q} R(w^{-1};A)\,\frac{dw}{w^2}. \end{aligned}$$

Setting $z = w^{-1}$ and $\Gamma'_S = \{z = -\rho e^{\pm i\alpha} : 0 \le \rho \le S^{-1}\} \cup \{z = S^{-1}e^{i\varphi} : |\varphi| \le \pi - \alpha\}$ with clockwise orientation, this leads to

$$\begin{aligned} e^{-t(-A)^{-q}} &= I + \frac{1}{2\pi i}\int_{\Gamma'_S} e^{-t(-z)^{-q}} R(z;A)\,dz \\ &= I + \frac{1}{2\pi i}\int_{\Gamma'_S} R(z;A)\,dz + \frac{1}{2\pi i}\int_{\Gamma'_S} \big(e^{-t(-z)^{-q}} - 1\big) R(z;A)\,dz. \end{aligned}$$

Here the next to last term vanishes, since $R(z;A)$ is analytic inside the contour, and the integral over the arc of radius S^{-1} contributes 0 to the last term in the limit $S \to 0$. This proves (3.10). $\square$

Remark 1. The proof of Theorem 4 (for the case $|\gamma| < 1$) in [8] proceeds by showing

$$\begin{aligned} \|r(kA)^n - e^{nkA}\| &\le \|r(kA)^n - e^{nkA} + \gamma^n AR(k^{-1};A)\| + |\gamma|^n \|AR(k^{-1};A)\| \\ &\le CMn^{-p} + CM|\gamma|^n \le CMn^{-p}, \end{aligned}$$

in a similar way as in the proof of Lemma 4 above. $\square$

Remark 2. The boundedness of the semigroup $e^{-t(-A)^{-q}}$ was a crucial part of our argument, and we proved it in Lemma 5 under our general assumption that the inverse of A is bounded ($0 \in \rho(A)$). This can also be derived from more general results of deLaubenfels. In [5] he shows that if A is the generator of a bounded analytic semigroup, and if $R(A)$ is dense in X, then A is one-to-one and A^{-1} (with domain $R(A)$) generates a semigroup, which is bounded and analytic in the same sector of the t-plane as A. He also gives an integral formula for $e^{tA^{-1}}$ in terms of the resolvent of A, which is the same as (3.12) (with $q = 1$). In [6] he shows that if A generates a bounded analytic semigroup, and if $q\alpha < \pi/2$, where α is the angle in the resolvent estimate (1.5) for A, then $-(-A)^q$ generates a semigroup, which is bounded and analytic in the sector $|\arg t| \le \pi/2 - q\alpha$, and he gives an integral formula for $e^{-t(-A)^q}$. Combining these results one obtains (3.12) and the first part of Lemma 5. But we still have to show (3.10). $\square$

Remark 3. A slightly shorter (but less explicit) proof is obtained by replacing the operator $e^{-t(-A)^{-q}}$ in the above argument by its backward Euler approximation $(I + k(-A)^{-q})^{-n}$. More precisely, we replace the function $e^{-b(-z)^{-q}}$ by $1/(1+b(-z)^{-q}) = (-z)^q/(b+(-z)^q)$. In view of Lemma 2 we then have $r(z) = \gamma/(1+b(-w)^q) + O(w^{q+1})$ as $w = z^{-1} \to 0$. The analogs of Lemmas 3 and 4 can now be proved in literally the same way as before. Moreover, since the rational function $r(z)^n - \gamma^n/(1+b(-z)^{-q})^n$ is analytic in $\mathbf{C} \setminus \overline{\Sigma}_\theta$ and equal to 0 at $z = \infty$, we have

$$r(kA)^n - \gamma^n (I + b(-kA)^{-q})^{-n} = \frac{1}{2\pi i}\int_\Gamma \left(r(kz)^n - \frac{\gamma^n}{(1+b(-kz)^{-q})^n} \right) R(z;A)\,dz.$$

By the work of deLaubenfels cited above, we know that $-(-A)^{-q}$ generates a bounded analytic semigroup. The boundedness of $(I + bk^{-q}(-A)^{-q})^{-n}$ for positive b, k and n now follows from Theorem 4. $\square$

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