NONSMOOTH DATA ERROR ESTIMATES WITH APPLICATIONS TO THE STUDY OF THE LONG-TIME BEHAVIOR OF FINITE ELEMENT SOLUTIONS OF SEMILINEAR PARABOLIC PROBLEMS

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Abstract. A rather general semilinear parabolic problem is studied together with its spatially semidiscrete finite element approximation. Both problems are formulated within the framework of nonlinear semigroups in the Sobolev space $H^1(\Omega)$. The main result is an error estimate for solutions with initial data in $H^1(\Omega)$, valid during an arbitrary finite time interval. The proof is based on the semigroup formulation and is rather elementary. A completely discrete scheme based on the backward Euler method for time discretization is analyzed in an analogous way. This type of error estimate is a key ingredient in recent work by several authors concerning the long-time behavior of finite element solutions. Two examples of this kind are presented: the upper semicontinuity of attractors and a long-time error bound for exponentially stable solutions.

1. Introduction. Recent work by several authors is concerned with the long-time behavior of numerical approximations of semilinear parabolic problems: long-time error bounds [12], [13], convergence of attractors [8], [9], [15], [5], behavior near stationary points [1], [16]. A common feature of these works is the need for an error bound for solutions with nonsmooth initial data. In order to be specific let us describe the idea in the context of a spatially semidiscrete finite element approximation. The continuous problem defines a nonlinear semigroup $S(t)$ in a subspace V of the Sobolev space $H^1(\Omega)$. The discrete problem defines a nonlinear semigroup $S_h(t)$ in a finite dimensional subspace V_h of V, h being a discretization parameter. A key ingredient of the works mentioned above is an estimate of the distance between two trajectories $S(t)v$ and $S_h(t)v$ with the same initial value $v \in V_h$. More precisely one needs an error bound of the following form: Let B be a bounded subset of $H^1(\Omega)$ and $T > 0$. For any $v \in B \cap V_h$ we have

$$\|S_h(t)v - S(t)v\|_{H^1} \le C(B,T)h^p t^{-\sigma}, \qquad t \in (0,T],$$

for some $p, \sigma > 0$. The singularity $t^{-\sigma}$ arises because the initial regularity is insufficient for the error bound to hold uniformly as $t \to 0$.

In the works mentioned above analogs of this error bound are either postulated or proved, sometimes with sub-optimal order of convergence. The purpose of the present paper is to prove such error bounds in the context of the standard piecewise linear finite element method. We consider both the spatially semidiscrete case and a fully discrete scheme based on the backward Euler method. We obtain results which are optimal both with respect to order of convergence and to regularity requirements. The

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proofs are based entirely on the semigroup formulation and are rather elementary. For completeness we also include error estimates for solutions with smooth initial data.

Error bounds for finite element approximations of solutions of semilinear parabolic problems with nonsmooth initial data have been given before in, for example, [14], [4], [3], [10], but not in the specific form required here. Also, by restricting our attention to piecewise linear finite elements, and to initial data in $H^1(\Omega)$ rather than $L_2(\Omega)$, we are able to give simpler proofs. In [14], [4], [3] it is assumed that the nonlinearity is globally Lipschitz, but here we only require that it satisfies a local Lipschitz condition with a polynomial bound on the constant.

In Section 2 we present the continuous and the spatially semidiscrete problems, and put them into a common abstract framework. In Section 3 we introduce our technique for error estimation in the semidiscrete case, where everything can be presented in a simple way. In Section 4 we review briefly two applications of the error bound from Section 3: a long-time error estimate for exponentially stable solutions (following [12,13]), and upper semicontinuity of attractors (following [8]). Finally in Section 5 we show that the technique from Section 3 can be used also for a completely discrete scheme.

We assume that the reader is familiar with the theory of analytic semigroups, [11, Chapter 1] or [17, Section 2.5], and the corresponding treatment of semilinear parabolic problems, [11, Chapter 3] or [17, Section 6.3].

2. Semigroup formulation. We consider the following semilinear parabolic problem:

$$
\begin{aligned}
u_t + Au &= f(x,u), && x \in \Omega,\ t > 0,\\
u &= 0, && x \in \partial\Omega,\ t > 0,\\
u(x,0) &= u_0, && x \in \Omega,
\end{aligned} \tag{2.1}
$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 1, 2$ or 3, $u = u(x,t)$ and $u_t = \partial u/\partial t$. We assume that $\partial\Omega$ is either smooth or a convex polygon. The operator A is defined by

$$
Au = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} a_i(x)\frac{\partial u}{\partial x_i} + a_0(x)u,
$$

where the coefficients a_{ij}, a_i are assumed to be continuously differentiable in $\overline{\Omega}$ with the matrix a_{ij} symmetric and uniformly positive definite, i.e.,

$$
\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge c_1|\xi|^2, \qquad \forall \xi \in \mathbf{R}^d,\ x \in \overline{\Omega}, \text{ with } c_1 > 0. \tag{2.2}
$$

We assume that f is continuous and satisfies a local Lipschitz condition with respect to u. If $d = 2$ or 3 we assume in addition that the Lipschitz condition is of the form

$$
|f(x,u) - f(x,v)| \le C\Big(1 + |u|^\gamma + |v|^\gamma\Big)|u - v|, \qquad \forall u, v \in \mathbf{R},\ x \in \overline{\Omega}, \tag{2.3}
$$

with $\gamma = 2$ if $d = 3$, and $\gamma \in [0, \infty)$ if $d = 2$.

In the sequel we let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual inner product and norm in the Hilbert space $L_2(\Omega)$. The norms in the Sobolev spaces $H^s(\Omega)$, $s \ge 0$, are denoted by $\|\cdot\|_s$. $V = H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of the functions that vanish on $\partial\Omega$, equipped with the norm $\|\cdot\|_1$. $V' = H^{-1}(\Omega)$ is the dual space of V with norm $\|v\|_{-1} = \sup_{\chi\in V} |(v,\chi)|/\|\chi\|_1$.

With the differential operator A we associate the bilinear form

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \overline{v}}{\partial x_i} + \sum_{i=1}^d a_i \frac{\partial u}{\partial x_i} \overline{v} + a_0 u \overline{v} \Big)\, dx, \qquad u, v \in V.$$

By Gårding's inequality there are constants $c_0 \in \mathbf{R}$ and $c_1 > 0$ such that

$$\operatorname{Re} a(v,v) + c_0 \|v\|^2 \geq c_1 \|v\|_1^2, \qquad \forall v \in V.$$

In fact, c_1 is as in (2.2) and $c_0 = \sup_{x\in\Omega}(\frac{1}{2}\sum_{j=1}^d \partial a_j/\partial x_j - a_0)$, see for example [19, p. 50]. By adding the term $c_0 u$ to both members of the differential equation in (2.1), we see that we may assume that

$$\operatorname{Re} a(v,v) \geq c_1 \|v\|_1^2, \qquad \forall v \in V, \text{ with } c_1 > 0. \tag{2.4}$$

By Green's formula we have $(Au, v) = a(u,v)$ for $u, v \in V \cap H^2(\Omega)$, and the weak form of problem (2.1) is to find $u(t) \in V$ such that

$$\begin{aligned} &(u_t, \chi) + a(u, \chi) = (f(u), \chi), \qquad \forall \chi \in V, \ t > 0, \\ &u(0) = u_0, \end{aligned}$$

where now, with a slight abuse of notation, f denotes the nonlinear operator $u \mapsto f(\cdot, u(\cdot))$.

In view of the elliptic regularity estimate

$$\|A^{-1} f\|_2 \leq C \|f\|, \qquad f \in L_2(\Omega), \tag{2.5}$$

which holds for smooth or convex polygonal domains, we may also consider the operator A as an unbounded operator on $L_2(\Omega)$ with $\mathcal{D}(A) = V \cap H^2(\Omega)$. Problem (2.1) then becomes

$$u_t + Au = f(u), \quad t > 0; \qquad u(0) = u_0. \tag{2.6}$$

We now turn to the discretization of (2.1) with respect to the spatial variables by means of the standard piecewise linear finite element method. We thus let $\{V_h\}_{h>0}$ be a family of finite dimensional subspaces of V, where each V_h consists of the continuous piecewise polynomials of degree ≤ 1 with respect to a "triangulation" of the domain Ω with maximum mesh size h. The semidiscrete solution $u_h(t) \in V_h$ satisfies

$$\begin{aligned} &(u_{h,t}, \chi) + a(u_h, \chi) = (f(u_h), \chi), \qquad \forall \chi \in V_h, \ t > 0, \\ &u_h(0) = u_{0h}. \end{aligned} \tag{2.7}$$

Introducing the linear operator $A_h : V_h \to V_h$ defined by the relation

$$(A_h \psi, \chi) = a(\psi, \chi), \qquad \forall \psi, \chi \in V_h,$$

and the orthogonal projection $P_h : L_2(\Omega) \to V_h$, we may write (2.7) as

$$u_{h,t} + A_h u_h = P_h f(u_h), \quad t > 0; \qquad u_h(0) = u_{0h}. \tag{2.8}$$

The V-ellipticity (2.4) implies that both operators $-A$ and $-A_h$ are sectorial on $L_2(\Omega)$ (uniformly in h), i.e., there are $C > 0$ and $\theta \in (\frac{1}{2}\pi, \pi)$ such that

$$\|(\lambda I + A)^{-1} f\| + \|(\lambda I + A_h)^{-1} P_h f\| \leq \frac{C}{|\lambda|}, \qquad \lambda \in \Sigma_\theta, \ f \in L_2(\Omega), \tag{2.9}$$

where Σ_θ denotes the sector $\{\lambda \in \mathbf{C} : \lambda = \rho e^{i\varphi},\ \rho > 0,\ 0 \le |\varphi| \le \theta\}$, see [7, Sections 6–7]. This in turn implies that $-A$ and $-A_h$ are the infinitesimal generators of bounded analytic semigroups $E(t) = \exp(-tA)$ on $L_2(\Omega)$ and $E_h(t) = \exp(-tA_h)$ on V_h. From the analyticity follows that ($D_t = \partial/\partial t$)

$$\|D_t^l E(t)v\| = \|A^l E(t)v\| \le C_l t^{-l}\|v\|, \qquad t > 0,\ v \in L_2(\Omega),\ l \ge 0, \tag{2.10}$$

and similarly for $E_h(t)$. It also follows that the fractional powers of A and A_h are well-defined and we shall use the following equivalences of norms:

$$\|A^{1/2}v\| \approx \|v\|_1,\ v \in \mathcal{D}(A^{1/2}) = V; \quad \|Av\| \approx \|v\|_2,\ v \in \mathcal{D}(A) = V \cap H^2(\Omega),$$

and the ensuing smoothing property

$$\|D_t^l E(t)v\|_\beta \le Ct^{-l-(\beta-\alpha)/2}\|v\|_\alpha, \qquad t > 0,\ v \in \mathcal{D}(A^{\alpha/2}),\quad 0 \le \alpha \le \beta \le 2,\ l = 0, 1, \tag{2.11}$$

For the discrete operator we have

$$\|A_h^{1/2}v\| \approx \|v\|_1, \qquad v \in V_h, \tag{2.12}$$

and the smoothing property

$$\|D_t^l E_h(t)v\|_\beta \le Ct^{-l-(\beta-\alpha)/2}\|v\|_\alpha, \qquad t > 0,\ v \in V_h,\quad 0 \le \alpha \le \beta \le 1,\ l = 0, 1. \tag{2.13}$$

In order to see that these hold, we write $A = A_1 + A_2$, where A_1 is the self-adjoint part of A and we let $a_1(\cdot,\cdot)$ be the corresponding bilinear form, that is, $a_1(u,v) = \sum_{i,j=1}^d (a_{ij}\partial u/\partial x_i, \partial v/\partial x_j) + (a_0 u, v)$. This leads to a corresponding splitting $A_h = A_{1h} + A_{2h}$. It is well-known that

$$\|A_1^{1/2}v\|^2 = a_1(v,v), \quad v \in V; \qquad \|A_{1h}^{1/2}v\|^2 = a_1(v,v), \quad v \in V_h,$$

and that the stated norm equivalences hold for A_1 and A_{1h}. Since $\|A_2 v\| \le C\|v\|_1$ for $v \in V$ and $\|A_{2h}v\| \le C\|v\|_1$ for $v \in V_h$, the general case follows from [11, Theorem 1.4.6].

By Duhamel's principle we may represent solutions of (2.6) by the integral equation

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds, \qquad t \ge 0. \tag{2.14}$$

Similarly, for solutions of (2.8) we have

$$u_h(t) = E_h(t)u_{0h} + \int_0^t E_h(t-s)P_h f(u_h(s))\,ds, \qquad t \ge 0. \tag{2.15}$$

In order to put equations (2.14) and (2.15) into the framework of [11, Chapter 3] or [17, Section 6.3] we must verify that the operator $f : V \to L_2(\Omega)$ satisfies a local Lipschitz condition. This is the first result of the following lemma. The second result will be used in the error analysis below.

Lemma 2.1. *For each bounded set $B \subset V$ there is a constant $C(B)$ such that*

$$\|f(u)-f(v)\| \le C(B)\|u-v\|_1, \qquad \forall u,v \in B, \tag{2.16}$$

$$\|f(u)-f(v)\|_{-1} \le C(B)\|u-v\|, \qquad \forall u,v \in B. \tag{2.17}$$

Proof. We have in view of (2.3) and the Hölder and Sobolev inequalities

$$\begin{aligned}\|f(u)-f(v)\|_{L_2} &\le C\Big(1+\|u\|_{L_q}^{\gamma}+\|v\|_{L_q}^{\gamma}\Big)\|u-v\|_{L_p}\\ &\le C\Big(1+\|u\|_1^{\gamma}+\|v\|_1^{\gamma}\Big)\|u-v\|_1,\end{aligned}$$

where $\frac{1}{p}+\frac{\gamma}{q}=\frac{1}{2}$ with $p=q=6$ if $d=3$, and with arbitrary $p\in(1,\infty)$ if $d\le 2$. This proves (2.16). Moreover,

$$\|f\|_{-1}=\sup_{v\in V}\frac{|(f,v)|}{\|v\|_1}\le \sup_{v\in V}\frac{\|f\|_{L_r}\|v\|_{L_p}}{\|v\|_1}\le C\|f\|_{L_r},$$

where $\frac{1}{p}+\frac{1}{r}=1$ with $p=6$, $r=\frac{6}{5}$ if $d=3$, and with arbitrary $r\in(1,\infty)$ if $d\le 2$, and

$$\begin{aligned}\|f(u)-f(v)\|_{L_r} &\le C\Big(1+\|u\|_{L_q}^{\gamma}+\|v\|_{L_q}^{\gamma}\Big)\|u-v\|_{L_2}\\ &\le C\Big(1+\|u\|_1^{\gamma}+\|v\|_1^{\gamma}\Big)\|u-v\|,\end{aligned}$$

where $\frac{1}{2}+\frac{\gamma}{q}=\frac{1}{r}$ (thus $q=6$ if $d=3$). This proves (2.17). $\square$

We may now apply a standard argument to obtain local existence and uniqueness, see [11, Theorem 3.3.3] or [17, Theorem 6.3.1]. The proof is based on applying the contraction mapping principle in the topology of the Banach space $C([0,t^*],H^1(\Omega))$ to the integral equations (2.14), (2.15).

Theorem 2.2. *For any bounded set $B_0\subset V$ there is $t^*=t^*(B_0)$ such that equation (2.14) has a unique solution $u\in C([0,t^*],V)$ for any $u_0\in B_0$, and equation (2.15) has a unique solution $u_h\in C([0,t^*],V_h)$ for any $u_{0h}\in B_0\cap V_h$.*

Remark. We may replace the Dirichlet boundary condition in (2.1) by a homogeneous Neumann or Robin boundary condition. The resulting problems can be put into the same abstract framework by a different choice of spaces V, V_h and bilinear form $a(\cdot,\cdot)$, cf. [7]. $\square$

3. Error estimates. We introduce the elliptic projection operator $R_h : V\to V_h$ defined by

$$a(R_hv,\chi)=a(v,\chi), \qquad \forall \chi\in V_h. \tag{3.1}$$

Under the usual regularity assumptions on the triangulation, and in view of the V-ellipticity (2.4) and the elliptic regularity estimate (2.5), we have the error bounds

$$\|R_hv-v\|+h\|R_hv-v\|_1\le Ch^s\|v\|_s, \qquad v\in V\cap H^s(\Omega),\ s=1,2, \tag{3.2}$$

see [2]. We may now prove an *a priori* estimate of the difference between $u_h(t)$ and $u(t)$ for t in an arbitrary finite interval. That is, we estimate this difference under the assumption that both $u_h(t)$ and $u(t)$ exist and stay bounded in V during a time interval $[0,\tau]$.

Theorem 3.1. *Let $0 < \tau \le T$ and let $B \subset V$ be a bounded set. Let $u(t)$ and $u_h(t)$ be solutions of (2.14) and (2.15), respectively, such that $u(t), u_h(t) \in B$ for $t \in [0,\tau]$. Then*

$$\|u_h(t) - u(t)\|_1 \le C(B,T)t^{-1/2}\Big(\|u_{0h} - u_0\| + h\Big), \qquad t \in (0,\tau]. \tag{3.3}$$

If, in addition, u_0 belongs to a bounded set $B_1 \subset H^2(\Omega)$, then

$$\|u_h(t) - u(t)\|_1 \le C(B,B_1,T)\Big(\|u_{0h} - u_0\|_1 + h\Big), \qquad t \in [0,\tau]. \tag{3.4}$$

Note that the error bound in (3.3) is of optimal order for $t > 0$, if u_{0h} is chosen so that $\|u_{0h} - u_0\| \le Ch\|u_0\|_1$, for example, if $u_{0h} = R_h u_0$ or $u_{0h} = P_h u_0$. The bound in (3.4) is optimal if $\|u_{0h} - u_0\|_1 \le Ch\|u_0\|_2$, for example, if $u_{0h} = R_h u_0$.

The situation in (3.4), where $u_0 \in V \cap H^2(\Omega)$, is referred to as the *smooth data* case, since the regularity is sufficient for the proof of an optimal order error bound on the interval $[0,\tau]$. The bound (3.3), on the other hand, is an example of a *nonsmooth data* error estimate, where the initial regularity is insufficient for optimal order convergence at $t = 0$. Nevertheless we have optimal order convergence for $t > 0$ due to the smoothing properties of the discrete and continuous solution operators.

In the proof of this we shall need the following regularity estimates.

Theorem 3.2. *Let $B \subset V$ be a bounded set and $0 < \tau \le T$. If $u(t) \in B$ for $t \in [0,\tau]$, then*

$$\|u(t)\|_s \le C(B,T)t^{-(s-1)/2}, \qquad t \in (0,\tau],\ s = 1,2, \tag{3.5–a}$$

$$\|u_t(t)\|_s \le C(B,T)t^{-1-(s-1)/2}, \qquad t \in (0,\tau],\ s = 0,1. \tag{3.5–b}$$

If, in addition, u_0 belongs to a bounded set $B_1 \subset H^2(\Omega)$, then

$$\|u(t)\|_2 \le C(B,B_1,T), \qquad t \in [0,\tau], \tag{3.6–a}$$

$$\|u_t(t)\|_s \le C(B,B_1,T)t^{-s/2}, \qquad t \in (0,\tau],\ s = 0,1. \tag{3.6–b}$$

Proof. In view of (2.16) we have $\|f(u(t))\| \le C(B)$ for $t \in [0,\tau]$ (the constant is not necessarily the same as in (2.16)). The proof of (3.5) can now obtained by tracing the constants in [11, Theorem 3.5.2]. The argument is based on a generalization of Gronwall's lemma: if $\phi(t)$ satisfies

$$0 \le \phi(t) \le K_1 t^{-1+\alpha} + K_2 \int_0^t (t-s)^{-1+\beta}\phi(s)\,ds, \qquad t \in (0,\tau], \tag{3.7}$$

then $\phi(t) \le C(K_2,\tau,\alpha,\beta)K_1 t^{-1+\alpha}$ for $t \in (0,\tau]$, provided that $K_1, K_2 \ge 0$, $\alpha, \beta > 0$. See [11, Section 7.1] or [5, Lemmas 6.3 and 7.1]; it is clear that the constant $C(K_2,\tau,\alpha,\beta)$ is nondecreasing in τ. The case (3.6) of smooth initial data can be treated in a similar way. We omit the details. $\square$

Proof of Theorem 3.1. Following a standard practice we divide the error into two parts:

$$e(t) \equiv u_h(t) - u(t) = \Big(u_h(t) - R_h u(t)\Big) + \Big(R_h u(t) - u(t)\Big) \equiv \theta(t) + \rho(t).$$

In view of (3.2) and (3.5) we have, for $j = 0, 1$ and $s = 1, 2$,

$$\|\rho(t)\|_j \le Ch^{s-j}\|u(t)\|_s \le C(B,T)h^{s-j}t^{-(s-1)/2}, \qquad t \in (0,\tau], \tag{3.8–a}$$

$$\|\rho_t(t)\| \le Ch\|u_t(t)\|_1 \le C(B,T)ht^{-1}, \qquad t \in (0,\tau]. \tag{3.8–b}$$

It remains to estimate $\theta(t)$, which belongs to V_h. In view of (2.8), the identity $A_hR_h = P_hA$ and (2.6), we find that

$$\theta_t + A_h\theta = P_h\Big(f(u_h) - f(u) - \rho_t\Big). \tag{3.9}$$

Hence

$$\theta(t) = E_h(t)\theta(0) + \int_0^t E_h(t-\sigma)P_h\Big(f(u_h(\sigma)) - f(u(\sigma)) - D_\sigma\rho(\sigma)\Big)\,d\sigma, \tag{4.1}$$

where $D_\sigma = \partial/\partial\sigma$. Integration by parts yields

$$\begin{aligned}-\int_0^{t/2} E_h(t-\sigma)P_hD_\sigma\rho(\sigma)\,d\sigma &= E_h(t)P_h\rho(0) - E_h(t/2)P_h\rho(t/2)\\ &\quad + \int_0^{t/2}\Big(D_\sigma E_h(t-\sigma)\Big)P_h\rho(\sigma)\,d\sigma.\end{aligned}$$

Hence

$$\begin{aligned}\theta(t) = E_h(t)P_he(0) - E_h(t/2)P_h\rho(t/2) + \int_0^{t/2}\Big(D_\sigma E_h(t-\sigma)\Big)P_h\rho(\sigma)\,d\sigma\\ - \int_{t/2}^t E_h(t-\sigma)P_hD_\sigma\rho(\sigma)\,d\sigma + \int_0^t E_h(t-\sigma)P_h\Big(f(u_h(\sigma)) - f(u(\sigma))\Big)\,d\sigma.\end{aligned}$$

Using the smoothing property (2.13) of $E_h(t)$, the error estimates (3.8) with $j = 0$, $s = 1$, and the Lipschitz condition (2.16) we obtain

$$\begin{aligned}\|\theta(t)\|_1 &\le Ct^{-1/2}\Big(\|P_he(0)\| + \|\rho(t/2)\|\Big) + C\int_0^{t/2}(t-\sigma)^{-3/2}\|\rho(\sigma)\|\,d\sigma\\
&\quad + C\int_{t/2}^t (t-\sigma)^{-1/2}\|D_\sigma\rho(\sigma)\|\,d\sigma\\
&\quad + C\int_0^t (t-\sigma)^{-1/2}\|f(u_h(\sigma)) - f(u(\sigma))\|\,d\sigma\\
&\le C(B,T)t^{-1/2}\Big(\|e(0)\| + h\Big)\\
&\quad + C(B,T)h\bigg(\int_0^{t/2}(t-\sigma)^{-3/2}\,d\sigma + \int_{t/2}^t (t-\sigma)^{-1/2}\sigma^{-1}\,d\sigma\bigg)\\
&\quad + C(B)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma\\
&\le C(B,T)t^{-1/2}\Big(\|e(0)\| + h\Big) + C(B)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma,\end{aligned}$$

for $t \in (0,\tau]$. Together with (3.8) and $e = \theta + \rho$ this yields

$$\|e(t)\|_1 \le C(B,T)t^{-1/2}\Big(\|e(0)\| + h\Big) + C(B)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma, \ t \in (0,\tau],$$

and the desired bound follows by the generalized Gronwall lemma, cf. (3.7). The smooth data error bound is proved in a similar way by an estimation based on equation (4.1) without integration by parts. We omit the details. □

Remark. The above argument also gives an error bound in the L_2 norm:

$$\|u_h(t) - u(t)\| \le C(B,T)\Big(\|u_{0h} - P_h u_0\| + h^2 t^{-1/2}\Big), \qquad t \in (0,\tau]. \tag{3.11}$$

This bound is of optimal order for $t > 0$, if and only if $u_{0h} = P_h u_0$. The proof requires an additional regularity estimate, namely,

$$\|u_t(t)\|_2 \le C(B,T)t^{-3/2}, \qquad t \in (0,\tau],$$

and one uses the Lipschitz condition (2.17) instead of (2.16) together with the inequality

$$\|A_h^{-1/2} P_h f\| \le C\|f\|_{-1}, \qquad f \in L_2(\Omega), \tag{3.12}$$

which follows from

$$\begin{aligned}
\|A_h^{-1/2} P_h f\| &= \sup_{v_h \in V_h} \frac{|(A_h^{-1/2} P_h f, v_h)|}{\|v_h\|} = \sup_{v_h \in V_h} \frac{|(f, (A_h^*)^{-1/2} v_h)|}{\|v_h\|} \\
&= \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|(A_h^*)^{1/2} w_h\|} \le C \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|w_h\|_1} \\
&\le C \sup_{w \in V} \frac{|(f,w)|}{\|w\|_1} = C\|f\|_{-1}.
\end{aligned}$$

The interested reader is referred to Section 5, where the details are presented in the context of a fully discrete scheme. □

Remark. A remark about higher order methods is in order. The argument that we used in the proof of the nonsmooth data estimate (3.3) gives optimal order convergence essentially only for piecewise linear finite elements. Optimal error bounds for higher order methods have been proved in [14], [4] by a more complicated technique and in a slightly different setup. However, due to nonlinear effects, there is a limitation on the order of convergence. For example, it was shown in [4] that, in general, $\|u_h(t) - u(t)\|$ is no smaller than $O(h^4)$ for $t > 0$, if $u_0 \in V$, no matter what the formal order of the finite element method is. □

4. Long-time behavior. We let $S(t) : V \to V$ denote the (local) solution operator of the nonlinear problem (2.14) defined through the relation $u(t) = S(t)u_0$. Similarly, the nonlinear operator $S_h(t) : V_h \to V_h$ is defined by $u_h(t) = S_h(t)u_{0h}$, where u_h is a local solution of equation (2.15).

We first formulate an immediate consequence of the nonsmooth data error estimate of Theorem 3.1. In the terminology of [8] the corollary says that $S_h(t)$ conditionally approximates $S(t)$ on B uniformly on compact subsets of $\mathbf{R}^+ = (0,\infty)$.

Corollary 4.1. *Let $0 < T_1 < \tau \le T_2$ and let $B \subset V$ be a bounded set. Then*

$$\|S_h(t)v_h - S(t)v_h\|_1 \le C(B, T_1, T_2)h, \qquad t \in [T_1, \tau],$$

provided that $v_h \in V_h$ is such that $S_h(t)v_h, S(t)v_h \in B$ for $t \in [0,\tau]$.

Before we discuss long-time behavior we must address the question of global existence. For this purpose we assume that solutions of (2.14) are *a priori* bounded:

for any bounded set $B_0 \subset V$ we have $\|u(t)\|_1 \leq C(B_0)$ for $t \in [0, \infty)$, if $u_0 \in B_0$, or equivalently

$$\text{for any bounded set } B_0 \subset V \text{ there is a bounded set } B_1 \subset V \\ \text{such that } S(t)B_0 \subset B_1 \text{ for } t \in [0, \infty). \tag{4.1}$$

By a standard argument this implies that solutions of (2.14) exist for all time. The solution operator $S(t)$ is thus a nonlinear semigroup on V.

It is sometimes possible to prove directly that the solutions of the discrete problem (2.15) are *a priori* bounded too. But this is not necessary. In fact we shall show by means of the error estimate in Corollary 4.1 that an *a priori* bound of the form (4.1) for u implies an *a priori* bound for u_h on a finite time interval.

Corollary 4.2. *Assume that (4.1) holds and let $T > 0$ and $B_0 \subset V$ be a bounded set. Then there are $h_0 = h_0(B_0, T) > 0$ and a bounded set $B_2 \subset V$, depending only on B_0, such that for $h < h_0$ we have $S_h(t)(B_0 \cap V_h) \subset B_2$ for $t \in [0, T]$.*

Proof. The local existence result of Theorem 2.2 implies that there are $T_1 > 0$ and a bounded set $B \subset V$ such that $S_h(t)(B_0 \cap V_h) \subset B$ for $t \in [0, T_1]$. Let $\delta > 0$ be arbitrary and let B_2 be the closed δ-neighborhood of $B_1 \cup B$. Let $\tau \in [0, T]$ be the largest time such that $S_h(t)(B_0 \cap V_h) \subset B_2$ for $t \in [0, \tau]$. We must show that $\tau = T$. Assume that $\tau < T$. Clearly $\tau > T_1$. Moreover, by Corollary 4.1, for any $v_h \in B_0 \cap V_h$ we have $\|S_h(t)v_h - S(t)v_h\|_1 \leq C(B_2, T_1, T)h$ for $t \in [T_1, \tau]$. Choosing h_0 so that $C(B_2, T_1, T)h_0 \leq \delta/2$ and recalling that $S(t)v_h \in B_1$ for $t \in [0, T]$, we conclude that $S_h(\tau)v_h$ is in the interior of B_2. By local existence this contradicts the maximality of τ. $\square$

We may now replace the assumption $S(t)v_h, S_h(t)v_h \in B$ in Corollary 4.1 by a bound on the initial value v_h.

Corollary 4.3. *Assume that (4.1) holds and let $0 < T_1 < T_2$ and let $B_0 \subset V$ be a bounded set. Then there is $h_0 = h_0(B_0, T_2) > 0$ such that for any $v_h \in B_0 \cap V_h$ and $h < h_0$ we have*

$$\|S_h(t)v_h - S(t)v_h\|_1 \leq C(B_0, T_1, T_2)h, \qquad t \in [T_1, T_2].$$

Using also the estimate (3.11) we can prove $\|S_h(t)v_h - S(t)v_h\| \leq C(B_0, T_1, T_2)h^2$. Similar comments apply to the results below.

In our first application of Corollary 4.3 we follow [12, 13]. Let u be a solution of (2.6). We say that v is a *perturbed solution starting at* t_0, if $v(t) = S(t - t_0)v_0$, $t \geq t_0$, with v_0 near $u(t_0)$. The solution u is *exponentially stable*, if there are numbers $\delta, T > 0$ such that any perturbed solution $v(t) = S(t - t_0)v_0$ with $\|v_0 - u(t_0)\|_1 < \delta$ satisfies

$$\|v(t) - u(t)\|_1 \leq \tfrac{1}{2}\|v_0 - u(t_0)\|_1, \qquad t \in [t_0 + T, \infty).$$

Thus T is the time it takes for the perturbation $v - u$ to decay to half its initial size. This would hold, for instance, if $u(t)$ is in the neighborhood of a stable hyperbolic stationary solution. In [12, 13] this property was assumed to hold for certain solutions of the Navier-Stokes equations on the basis of experimental evidence.

Under this assumption we may prove a uniform long-time error estimate.

Theorem 4.4. *Assume that (4.1) holds and that u is an exponentially stable solution of (2.6). Assume that $u_0 \in V \cap H^2(\Omega)$ and let u_h be a solution of (2.8) with $\|u_{0h} - u_0\|_1 \leq Ch\|u_0\|_2$. Then there are $C, h_0 > 0$ such that for $h < h_0$ we have*

$$\|u_h(t) - u(t)\|_1 \leq Ch, \qquad t \in [0, \infty). \tag{4.2}$$

Proof. By (4.1) and Corollary 4.2 there is a bounded set $B_0 \subset V$ such that $u(t) \in B_0$ for $t \in [0, \infty)$, and $u_h(t) \in B_0$ for $t \in [0, T]$, if h is small. Let B be a δ-neighborhood of B_0 in V and let $C_1 = C_1(B, B_1, T)$ (where $u_0 \in B_1 \subset H^2(\Omega)$) be the error constant in the smooth data estimate (3.4) of Theorem 3.1, and $C_2 = C_2(B, T, 2T)$ the error constant in Corollary 4.3. Set $C = \max(C_1, 2C_2)$ and let $h_0 \leq \delta/(2C)$ be so small that Corollaries 4.2, 4.3 are applicable. Then

$$\|u_h(t) - u(t)\|_1 \leq C_1 h \leq Ch, \qquad t \in [0, T], \tag{4.5}$$

$$\|S_h(t)v_h - S(t)v_h\|_1 \leq C_2 h \leq \tfrac{1}{2}Ch, \qquad t \in [T, 2T], \tag{4.6}$$

for $v_h \in B \cap V_h$, $h < h_0$.

We shall prove (4.2) by induction. By (4.5) we have the desired bound on the interval $[0, T]$. To complete the proof it suffices to show that if, for some integer $n \geq 1$,

$$\sup_{t \in [0, T_n]} \|u_h(t) - u(t)\|_1 \leq Ch, \qquad h < h_0, \tag{4.7}$$

where $T_n = nT$, then

$$\sup_{t \in [T_n, T_{n+1}]} \|u_h(t) - u(t)\|_1 \leq Ch, \qquad h < h_0. \tag{4.8}$$

In order to show this we use the perturbed solution $v(t) = S(t - T_{n-1})u_h(T_{n-1})$ starting at T_{n-1}. We first note that, by our choice of h_0, (4.7) implies that

$$\|u_h(t) - u(t)\|_1 \leq \delta/2, \qquad t \in [0, T_n],\ h < h_0. \tag{4.9}$$

Hence by exponential stability and (4.7)

$$\begin{aligned} \|v(t) - u(t)\|_1 &\leq \tfrac{1}{2}\|v(T_{n-1}) - u(T_{n-1})\|_1 = \tfrac{1}{2}\|u_h(T_{n-1}) - u(T_{n-1})\|_1 \\ &\leq \tfrac{1}{2}Ch, \qquad t \in [T_n, T_{n+1}],\ h < h_0. \end{aligned}$$

Moreover, (4.9) implies that $u_h(T_{n-1}) \in B$ and hence by (4.6) we have

$$\begin{aligned} \|u_h(t) - v(t)\|_1 &\leq \|S_h(t - T_{n-1})u_h(T_{n-1}) - S(t - T_{n-1})u_h(T_{n-1})\|_1 \\ &\leq \tfrac{1}{2}Ch, \qquad t \in [T_n, T_{n+1}],\ h < h_0. \end{aligned}$$

Taken together these estimates show (4.8). $\square$

Remark. All error estimates in [12, 13] are for initial data in $V \cap H^2(\Omega)$. Thus non-smooth data error bounds are used only for higher order methods, but not for piecewise linears. While we compare $u_h(t)$ to $v(t) = S(t - T_{n-1})v_0$ with $v_0 = u_h(T_{n-1}) \in V_h$, the argument in [12, 13] uses a smoother initial value $v_0 = A^{-1}A_h u_h(T_{n-1}) \in V \cap H^2(\Omega)$. Instead one needs to prove the *a priori* bound $\|Au(t)\| \leq C$, $t \in [0, \infty)$. $\square$

In our second application we give a (somewhat simplified) presentation of a result of [8], see also [18, Section I.1.4]. We assume that $S(t)$ has a global attractor $\mathcal{A}$, i.e., $\mathcal{A}$ is a compact invariant subset of V, which attracts the bounded sets of V. Thus, for any bounded set $B \subset V$ and any $\epsilon > 0$ there is $T > 0$ such that

$$S(t)B \subset \mathcal{N}(\mathcal{A}, \epsilon), \qquad t \in [T, \infty),$$

where $\mathcal{N}(\mathcal{A}, \epsilon)$ denotes the ϵ-neighborhood of $\mathcal{A}$ in V. Or equivalently,

$$\delta(S(t)B, \mathcal{A}) \to 0 \qquad \text{as } t \to \infty,$$

where $\delta(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_1$ denotes the unsymmetric semidistance between two subsets A, B of V. For simplicity we assume also that $S_h(t)$ has a global attractor $\mathcal{A}_h$ in V_h, although this is actually a consequence of the results of [8].

Theorem 4.5. *Under the above assumptions we have* $\delta(\mathcal{A}_h, \mathcal{A}) \to 0$ *as* $h \to 0$.

The result means that for any $\epsilon > 0$ there is $h_0 > 0$ such that $\mathcal{A}_h \subset \mathcal{N}(\mathcal{A}, \epsilon)$ if $h < h_0$. In the terminology of [8] we say that $\mathcal{A}_h$ is upper semicontinuous at $h = 0$. A similar result was proved in [15] by a different method. It is interesting to note that the error bound of Corollary 4.3 is one of the basic assumptions of [15]. The lower semicontinuity was proved in [9] under much stronger assumptions.

Proof. Let $\epsilon > 0$ and B_0 be a bounded set. We must show that there are $T = T(B_0, \epsilon) > 0$ and $h_0 = h_0(B_0, \epsilon) > 0$ such that

$$S_h(t)(B_0 \cap V_h) \subset \mathcal{N}(\mathcal{A}, \epsilon), \qquad t \in [T, \infty),\ h < h_0. \tag{4.10}$$

By (4.1) we have $S(t)B_0 \subset B_1$ for $t \in [0, \infty)$. Let $B = \mathcal{N}(B_1, \epsilon)$ and choose T such that

$$S(t)B \subset \mathcal{N}(\mathcal{A}, \epsilon/2), \qquad t \in [T, \infty). \tag{4.11}$$

By Corollary 4.3 we may choose h_0 such that

$$\|S_h(t)v_h - S(t)v_h\|_1 \le C(B, T, 2T)h \le \epsilon/2, \qquad t \in [T, 2T],\ h < h_0, \tag{4.12}$$

for all $v_h \in B \cap V_h$.

We shall prove (4.10) by induction. Take $u_{0h} \in B_0 \cap V_h$ and let $u_h(t) = S_h(t)u_{0h}$. (4.11) and (4.12) imply that $u_h(t) \in \mathcal{N}(\mathcal{A}, \epsilon)$ for $t \in [T, 2T]$, $h < h_0$. Assume that, for some integer $n \ge 2$,

$$u_h(t) \in \mathcal{N}(\mathcal{A}, \epsilon), \qquad t \in [T, T_n],\ h < h_0,$$

where $T_n = nT$. In particular, this implies that $u_h(T_{n-1}) \in B$. Let $v(t) = S(t - T_{n-1})u_h(T_{n-1})$. By (4.12) we then have

$$\begin{aligned} \|u_h(t) - v(t)\|_1 &\le \|S_h(t - T_{n-1})u_h(T_{n-1}) - S(t - T_{n-1})u_h(T_{n-1})\|_1 \\ &\le \epsilon/2, \qquad t \in [T_n, T_{n+1}],\ h < h_0. \end{aligned}$$

Since $v(t) \in \mathcal{N}(\mathcal{A}, \epsilon/2)$ for $t \in [T_n, T_{n+1}]$ by (4.11), we may conclude that $u_h(t) \in \mathcal{N}(\mathcal{A}, \epsilon)$ for $t \in [T_n, T_{n+1}]$. □

5. A completely discrete scheme. In this section we shall show that the above program can be carried through also for a completely discrete scheme based on the backward Euler method. We replace the time derivative in (2.8) by a backward difference quotient $\bar{\partial}_t U_n = (U_n - U_{n-1})/k$, where k is a time step and U_n is the approximation of $u_n = u(t_n)$ and $t_n = nk$. The discrete solution $U_n \in V_h$ thus satisfies

$$\bar{\partial}_t U_n + A_h U_n = P_h f(U_n), \quad t_n > 0; \qquad U_0 = u_{0h}.$$

Duhamel's principle yields

$$U_n = E_{kh}^n u_{0h} + k \sum_{j=1}^{n} E_{kh}^{n-j-1} P_h f(U_j), \qquad t_n \ge 0, \tag{5.1}$$

where $E_{kh} = (I + kA_h)^{-1}$. Since $-A_h$ is sectorial uniformly in h, see (2.9), we have the inequality

$$\|\bar{\partial}_t^l E_{kh}^n v\| = \|A_h^l E_{kh}^n v\| \le C_l t_n^{-l} \|v\|, \qquad t_n > t_l \ge 0,\ v \in V_h,$$

which is the discrete analog of (2.10) and can be proved by means of the Dunford-Taylor integral, see [6]. In view of the norm equivalence in (2.12) this leads to a smoothing property analogous to (2.13):

$$\|\bar{\partial}_t^l E_{kh}^n v\|_\beta \le C t_n^{-l-(\beta-\alpha)/2}\|v\|_\alpha, \qquad t_n > t_{l+(\beta-\alpha)/2},\ v \in V_h,\quad 0 \le \alpha \le \beta \le 1,\ l = 0,1. \tag{5.2}$$

In this section we need some regularity estimates in addition to those of Theorem 3.2. In order to prove these we assume that $f(x,u)$ is twice differentiable with respect to u and once with respect to x, and that

$$\begin{aligned} |f'_u(x,u)| + |f''_{xu}(x,u)| &\le C\Big(1+|u|^\gamma\Big),\\ |f''_{uu}(x,u)| &\le C\Big(1+|u|^{\gamma-1}\Big), \end{aligned} \qquad \forall u \in \mathbf{R},\ x \in \overline{\Omega}, \tag{5.3}$$

with $\gamma = 2$ if $d = 3$, and $\gamma \in [0,\infty)$ if $d = 2$. Note that this implies (2.3). In addition to the bounds in Lemma 2.1 we then have the following:

Lemma 5.1. *For each bounded set $B \subset V$ there is a constant $C(B)$ such that for all $u \in B \cap H^2(\Omega)$, $v \in V \cap H^2(\Omega)$ we have $f'_u(u)v \in V$ with*

$$\|f'_u(u)v\|_1 \le C(B)\Big(\|v\|_2 + \|u\|_2\|v\|_1\Big). \tag{5.4}$$

We omit the straightforward proof. This will enable us to obtain the additional regularity estimates that we need in the error analysis.

Theorem 5.2. *Let $B \subset V$ be a bounded set and $0 < \tau \le T$. If $u(t) \in B$ for $t \in [0,\tau]$, then*

$$\|u_{tt}(t)\| + \|u_t(t)\|_2 \le C(B,T)t^{-3/2}, \qquad t \in (0,\tau]. \tag{5.5}$$

If, in addition, u_0 belongs to a bounded set $B_1 \subset H^2(\Omega)$, then

$$\|u_{tt}(t)\| + \|u_t(t)\|_2 \le C(B,B_1,T)t^{-1}, \qquad t \in (0,\tau]. \tag{5.6}$$

If we further assume that $Au_0 - f(u_0)$ belongs to a bounded set $B_2 \subset V$, then

$$\|D_t^l u(t)\|_s \le C(B,B_1,B_2,T)t^{-\sigma/2}, \qquad t \in (0,\tau],\ l = 0,1,2,\ s = 1,2, \tag{5.7}$$

where $\sigma = \max(2l+s-3, 0)$.

Proof. We only sketch the proof of (5.5), which can be based on the fact that $v(t) = tu_t(t)$ satisfies

$$v_t + Av = u_t + f'_u(u)v, \quad t > 0; \qquad v(0) = 0,$$

so that

$$v(t) = \int_0^t E(t-s)\Big(u_s(s) + f'_u(u(s))v(s)\Big)\,ds.$$

Hence, in view of (2.11), (5.4) and (3.5),

$$\begin{aligned} \|v(t)\|_2 &\le C\int_0^t (t-s)^{-3/4}\|u_s(s)\|_{1/2} + C\int_0^t (t-s)^{-1/2}\|f'_u(u(s))v(s)\|_1\,ds \\ &\le C(B,T)t^{-1/2} + C(B)\int_0^t (t-s)^{-1/2}\|v(s)\|_2\,ds, \qquad t \in (0,\tau]. \qquad \square \end{aligned}$$

We now present our main result. It is evident that the error estimates (5.8), (5.9) can be used in the study of long-time behavior in exactly the same way as in Section 4. The smooth data results are included for completeness.

Theorem 5.3. *Let $0 < \tau \le T$ and let $B \subset V$ be a bounded set. Let $u(t)$ and U_n be solutions of (2.14) and (5.1), respectively, such that $u(t), U_n \in B$ for $t, t_n \in [0,\tau]$. Then, for $k \le k_0(B)$, we have*

$$\|U_n - u(t_n)\| \le C(B,T)\Big(\|u_{0h} - P_h u_0\| + h^2 t_n^{-1/2} + k t_n^{-1/2}\Big),\ t_n \in (0,\tau], \tag{5.8}$$

$$\|U_n - u(t_n)\|_1 \le C(B,T)\Big(\|u_{0h} - u_0\| t_n^{-1/2} + h t_n^{-1/2} + k t_n^{-1}\Big),\ t_n \in (0,\tau]. \tag{5.9}$$

If, in addition, u_0 belongs to a bounded set $B_1 \subset H^2(\Omega)$, then, for $k \le k_0(B)$, we have

$$\|U_n - u(t_n)\| \le C(B,B_1,T)\Big(\|u_{0h} - u_0\| + h^2 + k\Big), \qquad t_n \in [0,\tau], \tag{5.10}$$

$$\|U_n - u(t_n)\|_1 \le C(B,B_1,T)\Big(\|u_{0h} - u_0\|_1 + h + k t_n^{-1/2}\Big),\ t_n \in (0,\tau]. \tag{5.11}$$

If we further assume that $Au_0 - f(u_0)$ belongs to a bounded set $B_2 \subset V$, then, for $k \le k_0(B)$, we have

$$\|U_n - u(t_n)\|_1 \le C(B,B_1,B_2,T)\Big(\|u_{0h} - u_0\|_1 + h + k\Big), \qquad t_n \in [0,\tau]. \tag{5.12}$$

Proof. The proof is modeled after the proof of Theorem 3.1. Again we write

$$e_n \equiv U_n - u_n = (U_n - R_h u_n) + (R_h u_n - u_n) \equiv \theta_n + \rho_n,$$

and the required estimates for ρ_n are given in (3.8–a). Moreover, by the error bound (3.2) and the regularity estimates (3.5–b), (5.5),

$$\begin{aligned} \|\bar\partial_t \rho_n\| &\le \max_{t_{n-1}\le t\le t_n} \|\rho_t(t)\| \le C(B,T) h^s t_{n-1}^{-1-(s-1)/2} \\ &\le C(B,T) h^s t_n^{-1-(s-1)/2}, \qquad t_n \in [t_2,\tau],\ s = 1,2, \end{aligned} \tag{5.13}$$

because $\frac{1}{2} t_n \le t_{n-1} \le t_n$ for $n \ge 2$, a fact that will be used frequently in the following.

For $\theta_n \in V_h$ we have similarly to (3.9)

$$\bar\partial_t \theta_n + A_h \theta_n = P_h\Big(f(U_n) - f(u_n) - \bar\partial_t \rho_n + \omega_n\Big), \tag{5.14}$$

where $\omega_n = u_t(t_n) - \bar\partial_t u_n$. Hence

$$\theta_n = E_{kh}^n \theta_0 + k\sum_{j=1}^{n} E_{kh}^{n-j+1} P_h\Big(f(U_j) - f(u_j) - \bar\partial_t \rho_j + \omega_j\Big). \tag{5.15}$$

We first estimate $\|\theta_n\|$. This estimate will then be employed in the estimation of $\|\theta_n\|_1$. Let n_2 be the integer part of $n/2$. Summation by parts gives

$$-k\sum_{j=1}^{n_2} E_{kh}^{n-j+1} P_h \bar\partial_t \rho_j = E_{kh}^n P_h \rho_0 - E_{kh}^{n-n_2} P_h \rho_{n_2} + k\sum_{j=1}^{n_2}\Big(\bar\partial_t E_{kh}^{n-j+1}\Big) P_h \rho_j,$$

so that

$$\begin{aligned}
\theta_n = {}& E_{kh}^n P_h e_0 - E_{kh}^{n-n_2} P_h \rho_{n_2} + k \sum_{j=1}^{n_2} \Big(\bar\partial_t E_{kh}^{n-j+1} \Big) P_h \rho_j \\
&- k \sum_{j=n_2+1}^{n} E_{kh}^{n-j+1} P_h \bar\partial_t \rho_j + k \sum_{j=1}^{n} A_h^{1/2} E_{kh}^{n-j+1} A_h^{-1/2} P_h \Big(f(U_j) - f(u_j) \Big) \\
&+ k \sum_{j=1}^{n_2} A_h E_{kh}^{n-j+1} \Big(A_h^{-1} P_h - A^{-1} \Big) \omega_j + k \sum_{j=1}^{n_2} A_h E_{kh}^{n-j+1} P_h A^{-1} \omega_j \\
&+ k \sum_{j=n_2+1}^{n} E_{kh}^{n-j+1} P_h \omega_j \equiv \sum_{i=1}^{8} R_i .
\end{aligned}$$

The first five terms on the right are handled in a similar way as the corresponding terms in the proof of Theorem 3.1. In this connection we shall use the inequalities

$$\begin{aligned}
&k \sum_{j=1}^{n} t_{n-j+1}^{-1+\alpha} t_j^{-1+\beta} \le C_{\alpha,\beta} t_n^{-1+\alpha+\beta}, \qquad \alpha, \beta > 0, \\
&k \sum_{j=1}^{n_2} t_{n-j+1}^{-\alpha} t_j^{-1+\beta} \le t_{n-n_2+1}^{-\alpha} k \sum_{j=1}^{n_2} t_j^{-1+\beta} \le C_{\alpha,\beta} t_n^{-\alpha+\beta}, \qquad \alpha, \beta > 0,
\end{aligned}$$

which follow by comparison with the integral $\int_0^t (t-s)^{-1+\alpha} s^{-1+\beta}\, ds$.

Thus, using the smoothing property (5.2), the error bounds (3.8–a), (5.13), and also (3.12), (2.17) in the fifth term, we have for $s = 1, 2$

$$\begin{aligned}
\sum_{i=1}^{5} \|R_i\| \le {}& C\Big(\|P_h e_0\| + \|\rho_{n_2}\| \Big) + Ck \sum_{j=1}^{n_2} t_{n-j+1}^{-1} \|\rho_j\| \\
&+ Ck \sum_{j=n_2+1}^{n} \|\bar\partial_t \rho_j\| + Ck \sum_{j=1}^{n} t_{n-j+1}^{-1/2} \|f(U_j) - f(u_j)\|_{-1} \\
\le {}& C\|P_h e_0\| + C(B,T) h^s t_n^{-(s-1)/2} \\
&+ C(B,T) h^s \left(k \sum_{j=1}^{n_2} t_{n-j+1}^{-1} t_j^{-(s-1)/2} + k \sum_{j=n_2+1}^{n} t_j^{-1-(s-1)/2} \right) \\
&+ C(B) k \sum_{j=1}^{n} t_{n-j+1}^{-1/2} \|e_j\| \\
\le {}& C\|P_h e_0\| + C(B,T) h^s t_n^{-(s-1)/2} + C(B) k \sum_{j=1}^{n} t_{n-j+1}^{-1/2} \|e_j\|, \ t_n \in (0, \tau].
\end{aligned}$$

(Note that the above argument works also when $n = 1$, $n_2 = 0$.) For the terms involving ω_j we need the following inequalities:

(5.16–a) $\qquad \|A^{-l}\omega_j\|_s \le C(B,T) t_j^{l-1-(s-1)/2}, \qquad t_j \in (0,\tau],\ l = 0, 1,$

(5.16–b) $\qquad \|A^{-l}\omega_j\| \le C(B,T) k t_j^{l-3/2}, \qquad t_j \in (0,\tau],\ l = 0, 1,$

where we allow $s = 0$ if $l = 0$, and $s = 1, 2$ if $l = 1$. In fact, using regularity estimates from Theorems 3.2, 5.2,

$$\|A^{-l}\omega_j\|_s \le \max_{t_{j-1} \le t \le t_j} \|A^{-l} u_t(t)\|_s \le C(B,T) t_{j-1}^{l-1-(s-1)/2} \le C(B,T) t_j^{l-1-(s-1)/2},$$

for $t_j \in [t_2, \tau]$, since, by equation (2.6) and (2.16),

$$\|A^{-1}u_t\|_s \le \|u\|_s + \|A^{-1}f(u)\|_s \le \|u\|_s + C\|f(u)\| \le \|u\|_s + C(B).$$

Similarly,

$$\begin{aligned}\|A^{-l}\omega_j\| &= \Big\|k^{-1}\int_{t_{j-1}}^{t_j}(t-t_{j-1})A^{-l}u_{tt}(t)\,dt\Big\| \le k \max_{t_{j-1}\le t\le t_j}\|A^{-l}u_{tt}(t)\|\\ &\le C(B,T)kt_{j-1}^{l-3/2} \le C(B,T)kt_j^{l-3/2},\end{aligned}$$

for $t_j \in [t_2, \tau]$, since

$$\|A^{-1}u_{tt}\| \le \|u_t\| + \|A^{-1}f'_u(u)u_t\| \le \|u_t\| + C\|f'_u(u)u_t\|_{-1} \le C(B)\|u_t\|,$$

cf. (3.12) and (2.17). Moreover, under the stated restrictions on s,

$$\begin{aligned}\|A^{-l}\omega_1\|_s &\le k^{-1}\int_0^k t\|A^{-l}u_{tt}(t)\|_s\,dt \le C(B,T)k^{-1}\int_0^k t^{l-1-(s-1)/2}\,dt\\ &= C(B,T)t_1^{l-1-(s-1)/2} = C(B,T)kt_1^{l-2-(s-1)/2},\end{aligned}$$

and we conclude that the above estimates of ω_j hold also for $j = 1$.

Hence, since $A_h^{-1}P_h - A^{-1} = (R_h - I)A^{-1}$, we have by (3.2) and (5.16), for $s = 1, 2$,

$$\begin{aligned}\|R_6\| &\le Ck\sum_{j=1}^{n_2} t_{n-j+1}^{-1}\|(R_h - I)A^{-1}\omega_j\| \le Ch^s k\sum_{j=1}^{n_2} t_{n-j+1}^{-1}\|A^{-1}\omega_j\|_s\\ &\le C(B,T)h^s k\sum_{j=1}^{n_2} t_{n-j+1}^{-1}t_j^{-(s-1)/2} \le C(B,T)h^s t_n^{-(s-1)/2},\\ \|R_7\| &\le Ck\sum_{j=1}^{n_2} t_{n-j+1}^{-1}\|A^{-1}\omega_j\| \le C(B,T)k^2\sum_{j=1}^{n_2} t_{n-j+1}^{-1}t_j^{-1/2} \le C(B,T)kt_n^{-1/2},\end{aligned}$$

and

$$\|R_8\| \le Ck\sum_{j=n_2+1}^{n}\|\omega_j\| \le C(B,T)k^2\sum_{j=n_2+1}^{n} t_j^{-3/2} \le C(B,T)kt_n^{-1/2}.$$

Taken together these estimates show

$$\|e_n\| \le C(B,T)\Big(\|P_he_0\| + h^s t_n^{-(s-1)/2} + kt_n^{-1/2}\Big) + C(B)k\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\|e_j\|,\ t_n \in (0,\tau],$$

and the generalized discrete Gronwall lemma [5, Lemma 7.1] yields, for $k \le k_0(B)$,

$$\|e_n\| \le C(B,T)\Big(\|P_he_0\| + h^s t_n^{-(s-1)/2} + kt_n^{-1/2}\Big), \qquad t_n \in (0,\tau],\ s = 1,2, \tag{5.17}$$

and, since $P_he_0 = u_{0h} - P_hu_0$, this proves (5.8).

We now turn to the estimation of $\|\theta_n\|_1$. Because of the singularity t_n^{-1} in the final result we multiply equation (5.14) by t_n:

$$\bar\partial_t(t_n\theta_n) + A_h(t_n\theta_n) = \theta_{n-1} + t_nP_h\Big(f(U_n) - f(u_n) - \bar\partial_t\rho_n + \omega_n\Big).$$

Hence

$$t_n\theta_n = k\sum_{j=1}^{n} E_{kh}^{n-j+1}\left(\theta_{j-1} + t_j P_h\Big(f(U_j) - f(u_j) - \bar\partial_t\rho_j + \omega_j\Big)\right),$$

so that

$$t_n\|\theta_n\|_1 \le Ck\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\Big(\|\theta_{j-1}\| + t_j\|f(U_j) - f(u_j)\| + t_j\|\bar\partial_t\rho_j\| + t_j\|\omega_j\|\Big).$$

Here we have

$$\|\theta_{j-1}\| \le C(B,T)\Big(\|e_0\| + h + kt_j^{-1/2}\Big), \qquad t_j \in (0,\tau].$$

For $t_j \in [t_2,\tau]$ this follows from (5.17) (recall that $\frac{1}{2}t_j \le t_{j-1}$) and for $j=1$ we have $\|\theta_0\| \le \|e_0\| + \|\rho_0\| \le \|e_0\| + C(B)h$. Moreover,

$$t_j\|\bar\partial_t\rho_j\| \le C(B,T)h, \qquad t_j \in (0,\tau].$$

For $t_j \in [t_2,\tau]$ this follows from (5.13) and for $j=1$ we have simply $t_1\|\bar\partial_t\rho_1\| \le \|\rho_1\| + \|\rho_0\| \le C(B)h$. Hence, using also (5.16–b) we obtain

$$\begin{aligned} t_n\|e_n\|_1 &\le C(B,T)k\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\Big(\|e_0\| + h + kt_j^{-1/2}\Big) + C(B)k\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\Big(t_j\|e_j\|_1\Big) \\ &\le C(B,T)\Big(\|e_0\|t_n^{1/2} + ht_n^{1/2} + k\Big) + C(B)k\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\Big(t_j\|e_j\|_1\Big),\ t_n \in (0,\tau], \end{aligned}$$

and Gronwall's lemma yields, for $k \le k_0(B)$,

$$t_n\|e_n\|_1 \le C(B,T)\Big(\|e_0\|t_n^{1/2} + ht_n^{1/2} + k\Big), \qquad t_n \in (0,\tau],$$

which is (5.9). The smooth data error bounds can be proved in a similar way using the above ingredients. We omit the details. □

Remark. The above argument works also for the semi-implicit scheme

$$\bar\partial_t U_n + A_h U_n = P_h f(U_{n-1}), \quad t_n > 0; \qquad U_0 = u_{0h}.$$

In this case equation (5.14) is replaced by

$$\bar\partial_t\theta_n + A_h\theta_n = P_h\Big(f(U_{n-1}) - f(u_n) - \bar\partial_t\rho_n + \omega_n\Big),$$

and we make the estimation (for $l = 0,1$)

$$\begin{aligned} \Big\|k\sum_{j=1}^{n} E_{kh}^{n-j+1}P_h\Big(f(U_{j-1}) - f(u_j)\Big)\Big\|_l &\le Ck\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\|U_{j-1} - u_j\|_l \\ &\le Ck\sum_{j=1}^{n} t_{n-j+1}^{-1/2}\Big(\|e_{j-1}\|_l + k\|\bar\partial_t u_j\|_l\Big), \end{aligned}$$

where

$$k\|\bar\partial_t u_j\|_l \le C(B,T)t_j^{-1-(l-1)/2}, \qquad t \in (0,\tau].$$

Note that the condition $k \le k_0(B)$ is now not necessary in the Gronwall argument. □

References

1. F. Alouges and A. Debussche, *On the qualitative behavior of the orbits of a parabolic partial differential equation and its discretization in the neighborhood of a hyperbolic fixed point*, Numer. Funct. Anal. Optim. **12** (1991), 253–269.
2. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.
3. M. Crouzeix and V. Thomée, *On the discretization in time of semilinear parabolic equations with nonsmooth initial data*, Math. Comp. **49** (1987), 359–377.
4. M. Crouzeix, V. Thomée and L. B. Wahlbin, *Error estimates for spatially discrete approximations of semilinear parabolic equations with initial data of low regularity*, Math. Comp. **53** (1989), 25–41.
5. C. M. Elliott and S. Larsson, *Error estimates with smooth and nonsmooth data for a finite element method for the Cahn-Hilliard equation*, Math. Comp. **58** (1992), 603–630.
6. H. Fujita and A. Mizutani, *On the finite element method for parabolic equations, I; approximation of holomorphic semigroups*, J. Math. Soc. Japan **28** (1976), 749–771.
7. H. Fujita and T. Suzuki, *Evolution problems*, Handbook of Numerical Analysis, Vol. II. Finite Element Methods (Part 1) (P. G. Ciarlet and J. L. Lions, eds.), North-Holland, 1991, pp. 789–928.
8. J. K. Hale, X.-B. Lin and G. Raugel, *Upper semicontinuity of attractors for approximations of semigroups and partial differential equations*, Math. Comp. **50** (1988), 89–123.
9. J. K. Hale and G. Raugel, *Lower semicontinuity of attractors of gradient systems and applications*, Ann. Mat. Pura Appl. **154** (1989), 281–326.
10. H.-P. Helfrich, *Error estimates for semidiscrete Galerkin type approximations for semilinear evolution equations with nonsmooth initial data*, Numer. Math. **51** (1987), 559–569.
11. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Vol. 840, Springer-Verlag, Berlin, Heidelberg, New York, 1981.
12. J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem II. Stability of solutions and error estimates uniform in time*, SIAM J. Numer. Anal. **23** (1986), 750–777.
13. J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem III. Smoothing property and higher order estimates for spatial discretization*, SIAM J. Numer. Anal. **25** (1988), 489–512.
14. C. Johnson, S. Larsson, V. Thomée and L. B. Wahlbin, *Error estimates for spatially discrete approximations of semilinear parabolic equations with nonsmooth initial data*, Math. Comp. **49** (1987), 331–357.
15. P. E. Kloeden and J. Lorenz, *Lyapunov stability and attractors under discretization*, Differential Equations, Proceedings of the EQUADIFF Conference (C. M. Dafermos, G. Ladas and G. Papanicolaou, eds.), Marcel Dekker, Inc., 1989, pp. 361–368.
16. S. Larsson and J.-M. Sanz-Serna, *The behavior of finite element solutions of semilinear parabolic problems near stationary points*, preprint #1992–11, Department of Mathematics, Chalmers University of Technology.
17. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.
18. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, 1988.
19. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Mathematics, Vol. 1054, Springer-Verlag, 1984.

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