ON MULTIGRID METHODS FOR PARABOLIC PROBLEMS*

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ABSTRACT. Multigrid methods with nested subspaces and inherited forms are analyzed in an abstract framework that permits application to linear systems of the type that have to be solved at each time level in time-stepping methods for finite element approximations of parabolic problems. Convergence rates that are independent of the space and time steps are obtained in an appropriate time step dependent norm.

1. INTRODUCTION

In this article we discuss the solution of linear systems of equations

$$Au = f \tag{1.1}$$

by iterative methods of multigrid type. We are particularly interested in equations of the kind that arise when a parabolic problem, such as

$$U_t(x,t) - \Delta U(x,t) = g(x,t), \quad x \in \Omega, \ t > 0, \tag{1.2}$$

together with initial and boundary conditions, is discretized with respect to the time variable by a time-stepping method, and with respect to the spatial variable by a finite element method. The operator A is then typically of the form $A = zI - k\Delta_h$, where z is a complex number with $\operatorname{Re} z > 0$, k is a small positive parameter (the local time step) and Δ_h is a discrete version of the Laplacian generated by a finite element method with spatial mesh size h. For instance, if the backward Euler method with time step k is used, then (1.2) is first replaced by

$$(U_n - U_{n-1})/k - \Delta U_n = g_n, \quad U_n \approx U(nk),$$

or

$$(I - k\Delta)U_n = U_{n-1} + kg_n,$$

and the finite element discretization of this elliptic problem has the form (1.1) with $A = I - k\Delta_h$. Analogous equations are obtained in connection with other time-stepping methods such as A-stable onestep or multistep methods, see Section 3 below.

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We first formulate iterative methods of multigrid type for solving (1.1), and demonstrate convergence results within an abstract framework that permits application to the situation described above, where the special feature is the presence of the small parameter k. The framework is essentially that used in [3], where applications to elliptic problems are analyzed under weak assumptions, and our results and their proofs are close to those of earlier work, e.g., [1], [2], [4], and [7]. We restrict our discussion here to the case of nested subspaces and inherited forms, and we make regularity assumptions that are satisfied for convex polygonal domains Ω. This makes it possible to organize the theory in straightforward and compact manner, basing it on three simple assumptions, and to make our paper selfcontained.

Our convergence results for parabolic problems are expressed in a certain k-dependent energy norm and show rates of convergence that are uniform with respect to h and k. They are of the form required in the analysis of incomplete iterations in [10] and [5]. By combining our results with those of [10] and [5] one may obtain estimates of the total error caused both by the discretization and the iterative solution of the algebraic equations.

Various issues concerning multigrid methods for parabolic problems have been addressed in earlier work, for example, in [1], [8], [9], [12], [13], [14], [16], but in most cases (except [1] and [16]) the convergence analysis is restricted to model problems with a uniform mesh, where Fourier methods can be applied.

2. Abstract multigrid analysis

In the first subsection we define the multigrid algorithm and prove some convergence results in the context of symmetric equations in an abstract framework. In the second subsection we extend the analysis to a non-symmetric equation with a special structure.

2.1. Symmetric equations. Let M be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$ and let $A(\cdot,\cdot)$ be a symmetric, positive definite bilinear form on M. With the linear operator $A : M \to M$ defined by

$$(Au, v) = A(u, v), \quad \forall u, v \in M, \tag{2.1}$$

our concern is to solve the equation

$$Au = f, \quad \text{for } f \in M. \tag{2.2}$$

Our multigrid method for (2.2) is then the iterative method

$$u^l = u^{l-1} - B(Au^{l-1} - f), \quad l = 1, 2, \dots, \tag{2.3}$$

where $B : M \to M$ is defined as follows. We assume that we are given a nested sequence of subspaces $M_1 \subset \dots \subset M_j \subset M_{j+1} \subset \dots \subset M_J = M$, and define the local version $A_j : M_j \to M_j$ of A by

$$(A_j u, v) = A(u, v), \quad \forall u, v \in M_j. \tag{2.4}$$

The algorithm then defines $B_j : M_j \to M_j$ recursively for $j = 1, \dots, J$, starting with $B_1 = A_1^{-1}$, and finally sets $B = B_J$.

Assuming that B_{j-1} is at our disposal and $g \in M_j$, we define $B_j g \in M_j$ in three steps, referred to as pre-smoothing, correction, and post-smoothing. The basic ingredient in the first and third steps is a so-called smoothing operator $R_j : M_j \to M_j$, which has the property that the corresponding operator $I - R_j A_j$ reduces particularly well the error components with high frequencies. In the middle step the lower frequencies are reduced by projecting the residual onto M_{j-1} and iterating with the aid of B_{j-1}. With $R_j^{(l)} = R_j$, if l is odd, $R_j^{(l)} = R_j^t$, if l is even, where R_j^t is the adjoint of R_j, and denoting by $Q_j : M \to M_j$ the orthogonal projection, the algorithm is defined more precisely as follows.

The Multigrid Algorithm. Let p, r, s be integers with $p \geq 1$ and $r + s \geq 1$. Set $B_1 = A_1^{-1}$ and let $2 \leq j \leq J$. Assume that $B_{j-1} : M_{j-1} \to M_{j-1}$ has been defined, and let $g \in M_j$. Then $B_j g \in M_j$ is defined by

(i) r pre-smoothing iterations: set $x^0 = 0$ and

$$x^l = x^{l-1} - R_j^{(l+r)}(A_j x^{l-1} - g), \qquad \text{for } l = 1, \dots, r,$$

(ii) p correction iterations: set $q^0 = 0$ and

$$q^i = q^{i-1} - B_{j-1}\Big(A_{j-1} q^{i-1} - Q_{j-1}(A_j x^r - g)\Big), \qquad \text{for } i = 1, \dots, p,$$

(iii) s post-smoothing iterations: set $y^0 = x^r - q^p$ and

$$y^l = y^{l-1} - R_j^{(l)}(A_j y^{l-1} - g), \qquad \text{for } l = 1, \dots, s.$$

Finally, set $B_j g = y^s$.

If instead of $x^0 = 0$ we set $x^0 = w$ in step (i) of the algorithm, then we obtain $y^s = w - B_j(A_j w - g)$, which is of the form required in step (ii) and in (2.3).

Note thus that in the second step we first project onto M_{j-1} the residual $\rho = A_j x^r - g$ of the error remaining after the pre-smoothing in M_j, and then apply the analog of the iteration (2.3) in M_{j-1} to the equation $A_{j-1} q = Q_{j-1}\rho$ to solve for a correction q^p, which in the final step is interpreted as an element of M_j.

The reduction of error in each step of the iteration (2.3) is determined by the operator $D = I - BA$, and the purpose of the convergence analysis is thus to estimate some norm of D. In order to do so we introduce some more notation. Since $A(\cdot, \cdot)$ is symmetric, positive definite we may define an inner product $[\cdot, \cdot] = A(\cdot, \cdot)$ and a corresponding norm $|\cdot| = [\cdot, \cdot]^{1/2}$. We use the same notation for the induced operator norm, and let T^* denote the adjoint of a linear operator T with respect to $[\cdot, \cdot]$. Moreover, we let $P_j : M \to M_j$ denote the orthogonal projection with respect to $[\cdot, \cdot]$ and $\lambda_j = \lambda_{\max}(A_j)$ the largest eigenvalue of A_j.

Noting that the error reduction operators of the smoothing iterations alternate between $K_j = I - R_j A_j$ and $K_j^* = I - R_j^t A_j$, we now show that $D_j = I - B_j A_j$ satisfies the recursion

$$D_j = K_{j,s}^*(I - P_{j-1} + D_{j-1}^p P_{j-1}) K_{j,r}, \quad j = 2, \dots, J, \tag{2.5}$$

where

$$K_{j,m} = \begin{cases} (K_j^* K_j)^n, & \text{if } m = 2n, \\ K_j^*(K_j K_j^*)^n, & \text{if } m = 2n+1. \end{cases} \tag{2.6}$$

In fact, with $g = A_j w$ in the definition of B_j, we have by (iii)

$$D_j w = w - B_j A_j w = w - y^s = (I - R_j^{(s)} A_j)(w - y^{s-1}) = \cdots = K_{j,s}^*(w - y^0),$$

where $w - y^0 = q^p + w - x^r$. Using the fact that $Q_{j-1} A_j = A_{j-1} P_{j-1}$, we get from (ii)

$$q^p = D_{j-1} q^{p-1} + B_{j-1} Q_{j-1} A_j (x^r - w) = D_{j-1} q^{p-1} - B_{j-1} A_{j-1} P_{j-1}(w - x^r),$$

and hence, since $q^0 = 0$,

$$q^p + P_{j-1}(w - x^r) = D_{j-1}(q^{p-1} + P_{j-1}(w - x^r)) = \cdots = D_{j-1}^p P_{j-1}(w - x^r),$$

so that $w - y^0 = (I - P_{j-1} + D_{j-1}^p P_{j-1})(w - x^r)$. Finally, by (i) $w - x^r = K_{j,r}(w - x^0) = K_{j,r} w$, which completes the proof of (2.5).

Our analysis is based on the following three assumptions: there are positive constants C_1, C_2, C_3 such that

$$\|(I - P_j)v\| \le C_1 \lambda_j^{-1/2} |(I - P_j)v|, \quad v \in M, \ j = 1, \ldots, J, \tag{H$_1$}$$

$$\lambda_j / \lambda_{j-1} \le C_2, \quad j = 2, \ldots, J, \tag{H$_2$}$$

$$|K_j v|^2 \le |v|^2 - C_3 \lambda_j^{-1} \|A_j v\|^2, \quad v \in M_j, \ j = 2, \ldots, J. \tag{H$_3$}$$

The assumption (H_1) is an error estimate. In typical finite element applications it is proved by means of the Aubin-Nitsche duality argument and expresses the fact that the error in the elliptic projection P_j is smaller in the L_2 norm than in the energy norm. Assumption (H_2) means that the change from M_{j-1} to M_j is not too rapid. Assumption (H_3) expresses the smoothing action of K_j: if v is an eigenvector of A_j with eigenvalue λ, then (H_3) implies $|K_j v|^2 \le (1 - C_3 \lambda / \lambda_j)|v|^2$. High frequency eigenmodes are thus attenuated more by K_j than low frequency modes. Note that $C_3 \le 1$ with $C_3 = 1$ for the "perfect smoother" $R_j = A_j^{-1}$ and for $R_j = \lambda_j^{-1} I$, cf. Lemma 2 below. Assumption (H_3) is satisfied in finite element applications by other smoothers of practical interest, for example, the point and block Jacobi and Gauss-Seidel iterations, cf. [3].

By a spectral argument it follows from (H_3) that $|K_j| \le (1 - C_3/\kappa(A_j))^{1/2}$, where in typical applications the condition number $\kappa(A_j) = \lambda_{\max}(A_j)/\lambda_{\min}(A_j) \to \infty$ as $j \to \infty$. The convergence rate of the smoothing iteration may thus deteriorate as j grows large. By contrast we shall show that $|D_j| \le \delta < 1$ independently of j. Hence the multigrid iteration (2.3) has a uniform rate of convergence.

The assumptions (H_1), (H_2), and (H_3) enter our analysis combined into the inequality

$$|(I - P_{j-1})v|^2 \le C\Big(|v|^2 - |K_j v|^2\Big), \quad v \in M_j, \quad C = C_1^2 C_2 / C_3. \tag{2.7}$$

To prove this inequality we first use (H_1) to get

$$\begin{aligned}|(I-P_{j-1})v|^2 &= [(I-P_{j-1})v, v] = ((I-P_{j-1})v, A_j v)\\ &\le \|(I-P_{j-1})v\|\,\|A_j v\| \le C_1\lambda_{j-1}^{-1/2}|(I-P_{j-1})v|\,\|A_j v\|,\end{aligned}$$

and hence

$$|(I-P_{j-1})v|^2 \le C_1^2\lambda_{j-1}^{-1}\|A_j v\|^2,$$

from which (2.7) follows in view of (H_2) and (H_3).

We begin by stating and proving a convergence result for the so-called V-cycle algorithm, i.e., for $p=1$, with one pre- and/or one post-smoothing iteration. Its proof may be considered as an adaptation to our simple situation of an argument in [4].

Theorem 1. *Assume that (H_1), (H_2), and (H_3) hold. If $r=s=p=1$ then*

$$|I-BA| \le \delta_0 = 1-1/C, \quad \textit{where } C = C_1^2 C_2/C_3.$$

If $p=1$ and $r+s=1$ then $|I-BA|\le\sqrt{\delta_0}$.

Proof. If $r=s=p=1$ then the recursion (2.5) becomes

$$D_j = K_j(I-P_{j-1}+D_{j-1}P_{j-1})K_j^*. \tag{2.8}$$

This is an identity in M_j for $j=2,\dots,J$. We extend its scope to all of M by setting

$$\begin{aligned}\tilde D_j &= I-P_j+D_jP_j = I-B_jA_jP_j,\\ \tilde K_j &= I-P_j+K_jP_j = I-R_jA_jP_j,\\ \tilde K_j^* &= I-P_j+K_j^*P_j = I-R_j^tA_jP_j,\end{aligned}$$

and find that

$$\tilde D_j = \tilde K_j\tilde D_{j-1}\tilde K_j^*.$$

In fact, restricted to M_j this is the same as (2.8), and restricted to the orthogonal complement of M_j with respect to $[\cdot,\cdot]$ its left and right sides reduce to the identity operator. Since $\tilde D_1 = I-P_1$, we hence have

$$\tilde D_J = \tilde K_J\cdots\tilde K_2(I-P_1)^2\tilde K_2^*\cdots\tilde K_J^*,$$

and with $E_1 = I-P_1$, and $E_j = \tilde K_j E_{j-1}$ for $j=2,\dots,J$, this yields

$$I-BA = \tilde D_J = E_J E_J^*. \tag{2.9}$$

Note that E_J^* and E_J are the error reduction operators of the non-symmetric algorithms with $r=1,\ s=0$, and $r=0,\ s=1$, respectively. Since $|I-BA| = |E_JE_J^*| = |E_J^*|^2$ and

$|E_J^*| = |E_J|$ it therefore suffices, in all the cases considered, to estimate the latter norm. In order to do so we take $v \in M$ and consider the expressions

$$|E_{j-1}v|^2 - |E_j v|^2 = |E_{j-1}v|^2 - |\tilde{K}_j E_{j-1}v|^2, \quad \text{for } j = 2, \dots, J.$$

From the definition of $\tilde{K}_j$ and (2.7) it follows that

$$\begin{aligned} |\tilde{K}_j w|^2 &= |(I - P_j)w|^2 + |K_j P_j w|^2 \\ &\le |(I - P_j)w|^2 + |P_j w|^2 - C^{-1}|(I - P_{j-1})P_j w|^2 \\ &= |w|^2 - C^{-1}|(P_j - P_{j-1})w|^2, \end{aligned}$$

for $w \in M$, $j = 2, \dots, J$. Hence, setting $w = E_{j-1}v$,

$$C(|E_{j-1}v|^2 - |E_j v|^2) \ge |(P_j - P_{j-1})E_{j-1}v|^2 = |(P_j - P_{j-1})v|^2,$$

where in the last step we used the fact that $(I - E_{j-1})v \in M_{j-1}$ so that $(P_j - P_{j-1})(I - E_{j-1})v = 0$. This follows by induction and the recursion relation $I - E_1 = P_1$, $I - E_j = I - E_{j-1} + R_j A_j P_j E_{j-1}$ for $j = 2, \dots, J$. Hence, by summation,

$$C(|E_1 v|^2 - |E_J v|^2) \ge \sum_{j=2}^{J} \Big(|P_j v|^2 - |P_{j-1}v|^2\Big) = |v|^2 - |P_1 v|^2 = |(I - P_1)v|^2,$$

which, since $E_1 v = (I - P_1)v$, yields

$$|E_J v|^2 \le \delta_0 |(I - P_1)v|^2 \le \delta_0 |v|^2.$$

This implies the desired result. $\square$

We now consider a more general case of the multigrid algorithm with an arbitrary but equal number of pre- and post-smoothings, and an arbitrary number of corrections. The result is a generalization of a result of [2] and may be seen as a special case of a more general result in [7].

Theorem 2. *Let $r = s = m \ge 1$, $p \ge 1$. Assume that (H_1), (H_2), and (H_3) hold. Then*

$$|I - BA| \le \delta_m = \frac{C}{C + m}, \quad \textit{where } C = C_1^2 C_2 / C_3.$$

Note that the result of Theorem 1 is slightly better than that of Theorem 2 in the case $r = s = p = 1$ because $\delta_0 = 1 - 1/C < C/(1 + C) = \delta_1$. Note also that the convergence is faster if $m > 1$, but the present argument does not show that the performance is better for $p > 1$.

Proof. We shall prove, by induction on j, that $|D_j| \le \delta_m$, for $j = 1, \dots, J$. Since $D_1 = 0$ this is clearly true for $j = 1$. For $j \ge 2$ we use the recursion (2.5) which now takes the form

$$D_j = K_{j,m}^*(I - P_{j-1} + D_{j-1}^p P_{j-1})K_{j,m}.$$

Taking $v \in M_j$, setting temporarily $u = K_{j,m}v$, we have

$$[D_j v, v] = |(I - P_{j-1})u|^2 + [D^p_{j-1} P_{j-1}u, P_{j-1}u].$$

It follows by induction that D_j is selfadjoint and positive semidefinite with respect to $[\cdot,\cdot]$ so that $|D_j| = \sup_{v\in M_j}[D_j v, v]/|v|^2$. Moreover, assuming that $|D_{j-1}| \le \delta_m < 1$ we have

$$[D_j v, v] \le |(I - P_{j-1})u|^2 + \delta_m|P_{j-1}u|^2 = (1-\delta_m)|(I - P_{j-1})u|^2 + \delta_m|u|^2.$$

The desired result then follows from the following lemma. $\square$

Lemma 1. *Assume that K_j satisfies (2.7). Let $m \ge 1$ and $\delta_m = C/(C+m)$. Then*

$$(1-\delta_m)|(I - P_{j-1})K_{j,m}v|^2 + \delta_m|K_{j,m}v|^2 \le \delta_m|v|^2, \quad v \in M_j.$$

Proof. Let $\delta \in [0,1]$. Setting $u = K_{j,m}v$ we first use (2.7) to obtain

$$(1-\delta)|(I - P_{j-1})u|^2 + \delta|u|^2 \le C(1-\delta)(|u|^2 - |K_j u|^2) + \delta|u|^2.$$

We now want to express the norms on the right in terms of v. Recalling the definition of $K_{j,m}$ in (2.6) we find that

$$K^*_{j,m}K_{j,m} = \begin{cases} (K^*_j K_j)^{2n} = (K^*_j K_j)^m, & \text{if } m = 2n, \\ (K_j K^*_j)^n K_j K^*_j (K_j K^*_j)^n = (K_j K^*_j)^m, & \text{if } m = 2n+1, \end{cases}$$

and, similarly,

$$(K_j K_{j,m})^*(K_j K_{j,m}) = \begin{cases} (K^*_j K_j)^{m+1}, & \text{if } m \text{ is even}, \\ (K_j K^*_j)^{m+1}, & \text{if } m \text{ is odd}, \end{cases}$$

so that with

$$\hat{K}_j = \begin{cases} K^*_j K_j, & \text{if } m \text{ is even}, \\ K_j K^*_j, & \text{if } m \text{ is odd}, \end{cases}$$

we have $|u|^2 = [\hat{K}^m_j v, v]$, and $|K_j u|^2 = [\hat{K}^{m+1}_j v, v]$. Hence

$$(1-\delta)|(I - P_{j-1})u|^2 + \delta|u|^2 \le |C(1-\delta)(I - \hat{K}_j)\hat{K}^m_j + \delta\hat{K}^m_j|\,|v|^2.$$

The operator $\hat{K}_j$ is selfadjoint and positive semidefinite. Moreover, (2.7) implies that $|K_j| \le 1$ so that $|\hat{K}_j| \le |K^*_j||K_j| = |K_j|^2 \le 1$ and the largest eigenvalue of $\hat{K}_j$ is therefore ≤ 1. Hence, by the spectral theorem we have

$$|C(1-\delta)(I - \hat{K}_j)\hat{K}^m_j + \delta\hat{K}^m_j| \le \max_{x\in[0,1]}(C(1-\delta)(1-x)x^m + \delta x^m).$$

Since $(1-x)x^m \le m^{-1}(1-x^m)$ for $x \in [0,1]$, we have

$$0 \le C(1-\delta)(1-x)x^m + \delta x^m \le \frac{C}{m}(1-\delta)(1-x^m) + \delta x^m = \delta,$$

provided that $C(1-\delta)/m = \delta$, that is, $\delta = \delta_m = C/(C+m)$. This proves the lemma.

This choice of δ is optimal. In fact, the above maximum is

$$f(\delta) = C(1-\delta)(1-x(\delta))x(\delta)^m + \delta x(\delta)^m, \quad \text{where } x(\delta) = \frac{m}{m+1}\frac{C(1-\delta)+\delta}{C(1-\delta)},$$

and it can be shown that $\delta_m = C/(C+m)$ is the smallest fixed point of f. □

Next we discuss the non-symmetric multigrid algorithms without pre-smoothing or without post-smoothing.

Theorem 3. *Let $p \ge 1$, $r+s=m \ge 1$ and either $s=0$ or $r=0$ and assume that (H_1), (H_2), and (H_3) hold. Then*

$$|I - BA| \le \sqrt{\delta_m}, \qquad \textit{where } \delta_m = C/(C+m) \textit{ with } C = C_1^2 C_2/C_3.$$

Proof. We first consider the case when $r = m \ge 1$, $s = 0$. The recurrence relation (2.5) then becomes

$$D_j = (I - P_{j-1} + D_{j-1}^p P_{j-1})K_{j,m},$$

and D_j is no longer selfadjoint. But, if $|D_{j-1}| \le \sqrt{\delta_m} < 1$, then we have for $u = K_{j,m}v$

$$\begin{aligned}|D_j v|^2 &= |(I-P_{j-1})u|^2 + |D_{j-1}^p P_{j-1}u|^2 \le |(I-P_{j-1})u|^2 + \delta_m |P_{j-1}u|^2 \\ &= (1-\delta_m)|(I-P_{j-1})u|^2 + \delta_m |u|^2.\end{aligned}$$

Hence Lemma 1 implies $|D_j| \le \sqrt{\delta_m}$, which is the desired result in this case.

Finally when $r=0$, $s=m\ge 1$ the recursion (2.5) is

$$D_j = K_{j,m}^*(I - P_{j-1} + D_{j-1}^p P_{j-1}).$$

Taking adjoints we obtain

$$D_j^* = (I - P_{j-1} + (D_{j-1}^*)^p P_{j-1})K_{j,m},$$

and since $|D_j| = |D_j^*|$ the desired result follows from the above. □

Finally we remark that when $p=1$ we have $I - B^S A = (I - B^N A)^*(I - B^N A)$, where B^S and B^N, respectively, are the operators associated with the symmetric algorithm considered in Theorem 2, and the non-symmetric algorithm considered in the first part of Theorem 3, see (2.9). Hence $|I - B^S A| = |I - B^N A|^2$ when $p = 1$. Moreover, the convergence rate for $p > 1$ is always bounded by the convergence rate for $p = 1$, see [6]. Thus it actually suffices to prove either Theorem 2 or Theorem 3 for $p = 1$ in order to obtain the results of this subsection.

2.2. A non-symmetric equation. In this subsection we assume that M is a complex Hilbert space of finite dimension and consider a special non-symmetric bilinear form

$$A(u,v) = S(u,v) + i\beta(u,v), \quad u,v \in M,$$

where $i^2 = -1$, $\beta \in \mathbf{R}$ and $S(\cdot,\cdot)$ is Hermitian, positive definite with

$$S(u,u) \geq \alpha\|u\|^2, \quad \alpha > 0. \tag{2.10}$$

Defining again the operators A, A_j by (2.1), (2.4) we want to solve equation (2.2) by the iteration (2.3) defined by the multigrid algorithm described in the previous subsection.

In order to analyze this algorithm we now use the inner product $[\cdot,\cdot] = S(\cdot,\cdot)$ and corresponding norm $|\cdot| = [\cdot,\cdot]^{1/2}$. We also define operators $S_j : M_j \to M_j$ and $P_j : M \to M_j$ by the equations

$$\begin{aligned}(S_j u, v) &= S(u,v), \quad \forall u,v \in M_j,\\ A(P_j u, v) &= A(u,v), \quad \forall u \in M, v \in M_j,\end{aligned} \tag{2.11}$$

and let λ_j denote the largest eigenvalue of S_j. Note that A_j is invertible so that P_j exists with $P_j = A_j^{-1}Q_jA$. Note also that $A_j = S_j + i\beta I$, where S_j is Hermitian and thus A_j is normal with respect to $[\cdot,\cdot]$, and that λ_j is the largest of the real parts of the eigenvalues of A_j.

We still want to use the assumptions (H_1), (H_2), and (H_3) for P_j, λ_j, and K_j. Concerning K_j we note that it is no longer true in general that $I - R_j^t A_j$ is equal to K_j^*, so that alternating products of factors K_j and K_j^* would not occur as in (2.5). We therefore restrict our discussion now to operators of the form $K_j = I - \mu A_j$, corresponding to smoothing operators $R_j = \mu I$, where μ is a positive parameter. In this case K_j is a normal operator with respect to $[\cdot,\cdot]$ and, as we shall see in the convergence proof below, the spectral argument in Lemma 1 can still be applied. In the following lemma we first show that (H_3) and (2.7) are satisfied with the proper choice of μ.

Lemma 2. *Let $R_j = \mu I$ and $\gamma = (1 + \beta^2/\alpha^2)^{-1}$. If $|\mu\lambda_j - \gamma| \leq \epsilon < \gamma$, then ($H_3$) holds with $C_3 = \gamma^2 - \epsilon^2$. If, in addition, ($H_1$) and ($H_2$) hold, then (2.7) follows.*

The optimal choice $\epsilon = 0$ gives $\mu = \gamma\lambda_j^{-1}$, $C_3 = \gamma^2$ and, in particular, $\mu = \lambda_j^{-1}$, $C_3 = 1$ if $A(\cdot,\cdot)$ is symmetric. This is difficult to achieve exactly because λ_j is unknown, but the lemma permits λ_j to be replaced by an estimate.

Proof. We have

$$|v|^2 - |K_j v|^2 = 2\mu \operatorname{Re}[A_j v, v] - \mu^2|A_j v|^2.$$

Here $|A_j v|^2 \leq \lambda_j\|A_j v\|^2$ and $\operatorname{Re}[A_j v, v] = \|S_j v\|^2$. Since

$$\|A_j v\|^2 = \|S_j v\|^2 + \beta^2\|v\|^2 \leq (1 + \beta^2/\alpha^2)\|S_j v\|^2,$$

we conclude

$$|v|^2 - |K_j v|^2 \geq (2\gamma\mu - \mu^2\lambda_j)\|A_j v\|^2 = (2\gamma\mu\lambda_j - (\mu\lambda_j)^2)\lambda_j^{-1}\|A_j v\|^2.$$

Here $2\gamma\mu\lambda_j - (\mu\lambda_j)^2 = \gamma^2 - (\mu\lambda_j - \gamma)^2 \geq \gamma^2 - \epsilon^2$ from which the first part of the lemma follows. In order to prove (2.7) we first use the definitions of P_j, A_j in (2.11), (2.4) together with (H_1) to get

$$\begin{aligned}|(I-P_{j-1})v|^2 &= \operatorname{Re}\Big(A((I-P_{j-1})v,(I-P_{j-1})v) - i\beta((I-P_{j-1})v,(I-P_{j-1})v)\Big)\\ &= \operatorname{Re} A((I-P_{j-1})v, v) = \operatorname{Re}((I-P_{j-1})v, A_j^t v)\\ &\leq \|(I-P_{j-1})v\|\,\|A_j^t v\| \leq C_1\lambda_{j-1}^{-1/2}|(I-P_{j-1})v|\,\|A_j v\|,\end{aligned}$$

since $A_j^t = S_j - i\beta I$, so that $\|A_j^t v\| = \|A_j v\|$. Using also (H_2) and (H_3) we obtain (2.7). □

We now show, using an argument of [1], that if the number of correction iterations p is greater than 1, and the number of smoothing iterations is sufficiently large, then we have a uniform rate of convergence.

Theorem 4. *Assume that (H_1), (H_2) hold and that $R_j = \mu I$ with μ chosen according to the condition in Lemma 2. Let $p \geq 2$, $r = m \geq 1$, $s = 0$, $C = C_1^2 C_2/C_3$, and $\sigma = 1 + |\beta|/\alpha$. Then*

$$|I - BA| \leq 2\sqrt{\frac{C}{m}}, \quad \text{if } m \geq 4C(2\sigma)^{2/(p-1)}.$$

Proof. We argue by induction as in the proof of Theorem 3. The recurrence relation for D_j is now

$$D_j = (I - P_{j-1} + D_{j-1}^p P_{j-1})K_j^m,$$

which is proved in the same way as (2.5). If $|D_{j-1}| \leq \delta < 1$, then since, as is easily seen, $|P_{j-1}| \leq 1 + |\beta|/\alpha = \sigma$ and, by (H_3), $|K_j| \leq 1$, we have

$$|D_j v| \leq |(I - P_{j-1})K_j^m v| + \sigma\delta^p |v|.$$

Using (2.7) and the fact that K_j is normal we get with $\hat{K}_j = K_j^* K_j$

$$|(I-P_{j-1})K_j^m v|^2 \leq C\Big(|K_j^m v|^2 - |K_j^{m+1} v|^2\Big) = C[(I - \hat{K}_j)\hat{K}_j^m v, v].$$

Since $\hat{K}_j$ is selfadjoint, and since by (H_3) its spectrum lies in the interval $[0,1]$, we conclude as in Lemma 1, using the inequality $(1-x)x^m \leq 1/m$ for $x \in [0,1]$, that

$$|(I-P_{j-1})K_j^m v|^2 \leq \frac{C}{m}|v|^2.$$

Hence

$$|D_j| \leq \sqrt{\frac{C}{m}} + \sigma\delta^p \leq \delta,$$

if, for example, we choose $\delta = 2\sqrt{C/m}$ and m so large that $\sigma\delta^p \leq \frac{1}{2}\delta$, which is the desired result. □

3. Application to parabolic problems

In this section we illustrate how the abstract results of the previous section can be applied to parabolic problems by considering the heat equation (1.2) with homogeneous Dirichlet boundary conditions written in weak form: find $U(t) \in H_0^1(\Omega)$ for $t \geq 0$ such that $U(0) = U_0$ and

$$(U_t, v) + (\nabla U, \nabla v) = (g, v), \quad \forall v \in H_0^1(\Omega),\ t > 0, \tag{3.1}$$

where $(\cdot,\cdot)$ denotes the standard inner product in $L_2(\Omega)$. We assume that $\Omega \subset \mathbf{R}^2$ is a bounded convex polygonal domain. For the approximation of (3.1) with respect to the spatial variable we introduce a nested sequence $M_1 \subset \ldots \subset M_{j-1} \subset M_j \subset \ldots \subset H_0^1(\Omega)$ of piecewise polynomial finite element spaces. Defining first M_1 by means of a coarse triangulation of Ω, we assume that the triangulation defining M_j with mesh-size h_j is obtained by subdividing each triangle corresponding to M_{j-1} into a fixed finite number N^2 of congruent triangles so that $h_{j-1} = Nh_j$.

We now consider the discretization of the time variable. Let $\Delta_j : M_j \to M_j$ denote the discrete Laplacian defined by

$$-(\Delta_j u, v) = (\nabla u, \nabla v), \quad \forall u, v \in M_j.$$

In an A-stable linear multistep method with time step k one has to solve an equation in M_J of the form

$$\Big((\alpha + i\beta)I - k\Delta_J\Big)u = F, \quad \text{with } \alpha > 0,\ \beta \in \mathbf{R}, \tag{3.2}$$

on each time level, cf. [15]. For the backward Euler method we have $\alpha = 1$, $\beta = 0$, and for the second order backward differentiation method $\alpha = 3/2$, $\beta = 0$, cf. [5]. F depends on g, k and on the approximate solution at one or more previous time levels.

In a onestep method based on an A-stable rational approximation $r(z) = p(z)/q(z)$ of e^{-z}, such as certain Padé approximations, one similarly has to solve the equation

$$q(-k\Delta_J)u_n = p(-k\Delta_J)u_{n-1} + F, \tag{3.3}$$

where F depends on g and k. After factorization of the denominator $q(z)$ this reduces to a sequence of equations of the form (3.2), since, by A-stability, $r(z)$ has all its poles in the left half-plane. For the Crank-Nicolson method we have $\alpha = 2$, $\beta = 0$. At the end of this section we present an example where $\beta \neq 0$.

Setting

$$A(v, w) = k(\nabla v, \nabla w) + (\alpha + i\beta)(v, w), \quad \text{with } \alpha > 0,\ \beta \in \mathbf{R},$$

our aim is now to check that the assumptions of the previous section are satisfied with constants that are independent of j and k. With

$$[v, w] = S(v, w) = k(\nabla v, \nabla w) + \alpha(v, w), \quad |v| = \Big(k\|\nabla v\|^2 + \alpha\|v\|^2\Big)^{1/2},$$

it is first of all clear that $S(\cdot,\cdot)$ is symmetric, positive definite and that (2.10) holds.

Let μ_j be the largest eigenvalue of $-\Delta_j$. Then the largest eigenvalue of S_j is $\lambda_j = \alpha + k\mu_j$, where μ_j is bounded above and below by positive multiples of h_j^{-2}. Since $h_{j-1} = Nh_j$ we thus have

$$\frac{\lambda_j}{\lambda_{j-1}} \le \frac{1 + c_2 k h_j^{-2}}{1 + c_1 k h_{j-1}^{-2}} = \frac{1 + c_2 k h_j^{-2}}{1 + N^2 c_1 k h_j^{-2}} \le C,$$

which is (H_2).

We next consider (H_1) in the case $k \le h_j^2$. Here $\lambda_j \le 1 + c_2 k h_j^{-2} \le C$ and hence

$$\|(I-P_j)v\| \le \alpha^{-1/2}|(I-P_j)v| \le C\lambda_j^{-1/2}|(I-P_j)v|.$$

For $k \ge h_j^2$ we use an adaptation of the standard duality argument. Let $\psi \in L_2$ and let $w \in H_0^1(\Omega)$ be the solution of

$$A(\phi, w) = (\phi, \psi), \quad \forall \phi \in H_0^1(\Omega). \tag{3.4}$$

Then, for any $\chi \in M_j$, and with $C = 1 + |\beta|/\alpha$,

$$((I-P_j)v, \psi) = A((I-P_j)v, w) = A((I-P_j)v, w-\chi) \le C|(I-P_j)v|\,|w-\chi|. \tag{3.5}$$

Here, with suitable χ,

$$|w-\chi|^2 = k\|\nabla(w-\chi)\|^2 + \alpha\|w-\chi\|^2 \le C(kh_j^2 + h_j^4)\|w\|_{H^2(\Omega)}^2 \le Ckh_j^2\|w\|_{H^2(\Omega)}^2.$$

We now show that $k\|w\|_{H^2(\Omega)} \le C\|\psi\|$ uniformly in k. In fact, since

$$-\Delta w = k^{-1}(\psi - (\alpha + i\beta)w),$$

we have by the standard regularity estimate for elliptic problems

$$k\|w\|_{H^2(\Omega)} \le C(\|w\| + \|\psi\|).$$

But choosing $\phi = w$ in (3.4) we have

$$\alpha\|w\|^2 \le |w|^2 = \operatorname{Re} A(w,w) = \operatorname{Re}(w,\psi) \le \|w\|\,\|\psi\|,$$

so that $\|w\| \le C\|\psi\|$, which completes the proof. Thus, for $\chi \in M_j$ suitable,

$$|w-\chi|^2 \le Ck^{-1}h_j^2\|\psi\|^2 \le C\lambda_j^{-1}\|\psi\|^2,$$

since $\lambda_j \le Ckh_j^{-2}$. Together with (3.5) this shows

$$\|(I-P_j)v\| \le C\lambda_j^{-1/2}|(I-P_j)v|,$$

and thus completes the proof of (H_1).

For $R_j = \mu I$, with μ appropriately chosen, we have thus checked the assumptions of Theorem 4, and if $\beta = 0$ the assumptions of Theorems 1, 2, and 3. In the latter (symmetric) case we may also use smoothing iterations of Jacobi and Gauss-Seidel type, see Chapter 5 of [3], where proofs of A.4, which is equivalent to our (H_3), may be directly adapted to our situation to show that the constant C_3 is independent of the time step k.

In these situations we thus conclude that the multigrid iteration (2.3) has a rate of convergence which is independent of J and k. Since our results are expressed in the k-dependent norm $|\cdot|$ they may be combined with the results of [10] and [5] on incomplete iterations to obtain estimates of the total error caused both by the discretization and the iterative solution of the algebraic equations.

We finish by presenting a detailed example: the piecewise linear discontinuous Galerkin method, which is related to the (2,1)-Padé approximation of the exponential, cf. [11]. In this method the approximate solution of (3.1) is sought in the space of discontinuous, piecewise linear functions of t with coefficients in M_J corresponding to a subdivision of the t-axis into small intervals not necessarily of the same lengths. If $I = (t_1, t_2)$ is such an interval of length $k = t_2 - t_1$, then we introduce

$$\Pi_1(I; M_J) = \left\{ v : v(t) = v_1 \frac{t_2 - t}{k} + v_2 \frac{t - t_1}{k},\ v_1, v_2 \in M_J \right\},$$

and represent functions $v \in \Pi_1(I; M_J)$ by their left and right nodal values $v_1, v_2 \in M_J$. The approximation is determined on I as the solution of the variational problem: find $u \in \Pi_1(I; M_J)$ such that

$$(u_1, v_1) + \int_I \Big((u_t, v) + (\nabla u, \nabla v) \Big)\, dt = (u_0, v_1) + \int_I (g, v)\, dt, \quad \forall v \in \Pi_1(I; M_J),$$

where u_0 is the right nodal value from the previous time interval. In order to be able to progress in time we need to compute the right nodal value u_2 on the present time interval. Simple calculations show that

$$\left(-\frac{k}{6} \Delta_J \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

where $F_1 = Q_J(u_0 + \int_I g(t)(t_2 - t)\, dt/k)$ and $F_2 = Q_J \int_I g(t)(t - t_1)\, dt/k$. By elimination of u_1 we obtain

$$((-k\Delta_J)^2 + 4(-k\Delta_J) + 6) u_2 = -(2(-k\Delta_J) + 6) F_1 + (4(-k\Delta_J) + 6) F_2,$$

which is of the form (3.3). Dividing the solution into partial fractions we find

$$u_2 = \mathrm{Re} \Big(\big(-k\Delta_J + 2 + i\sqrt{2} \big)^{-1} \big((-2 + i5\sqrt{2}) F_1 + (4 - i\sqrt{2}) F_2 \big) \Big),$$

so that $u_2 = \mathrm{Re}\, w$, where

$$(-k\Delta_J + 2 + i\sqrt{2}) w = (-2 + i5\sqrt{2}) F_1 + (4 - i\sqrt{2}) F_2, \tag{3.6}$$

which is of the form (3.2).

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