A SHADOWING RESULT WITH APPLICATIONS TO FINITE ELEMENT APPROXIMATION OF REACTION-DIFFUSION EQUATIONS

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ABSTRACT. A shadowing result is formulated in such a way that it applies in the context of numerical approximations of semilinear parabolic problems. The qualitative behavior of temporally and spatially discrete finite element solutions of a reaction-diffusion system near a hyperbolic equilibrium is then studied. It is shown that any continuous trajectory is approximated by an appropriate discrete trajectory, and vice versa, as long as they remain in a sufficiently small neighborhood of the equilibrium. Error bounds of optimal order in the L_2 and H^1 norms hold uniformly over arbitrarily long time intervals.

1. INTRODUCTION

The purpose of this article is to compare the dynamical system arising from a semilinear parabolic evolution problem with the dynamical systems that arise from its temporal and spatial discretizations. The long-time behavior of a dynamical system is governed by its invariant sets such as fixed points, periodic orbits, attractors, etc. It is therefore important to investigate whether the discretized dynamical systems have the same kinds of invariant sets and whether their orbits have the same qualitative behavior near these sets. Our aim here is to do so for the special case of a hyperbolic fixed point.

The inspiration for this work came from an article of Beyn, [3], on multi-step approximations of systems of nonlinear ordinary differential equations, $u' = f(u)$. Beyn showed that if the continuous problem has a hyperbolic fixed point $\bar{u}$, then there is a neighborhood O of $\bar{u}$ such that the following conclusion holds: for each initial value $u_0 \in O$ there is $U_0 \in O$ such that the approximate orbit U starting from U_0 is close to the exact orbit u starting from u_0 as long as the latter orbit stays in O. The error $u - U$ satisfies an estimate, which is uniform with respect to $u_0 \in O$ and of optimal order of convergence. Note, in particular, that the error bound is thus uniform over arbitrarily long time intervals. The converse statement is also true: for each $U_0 \in O$ there is $u_0 \in O$ such that the corresponding orbits U and u are optimally close as long as they remain in O. We emphasize that $U_0 \neq u_0$

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in general, because the initial value problem is typically unstable near a hyperbolic fixed point.

Beyn's result was extended to infinite dimensional spaces by Alouges and Debussche [1], who were thus able to cover pure time-discretization of semilinear parabolic equations. Another extension to semilinear parabolic problems was made by the present authors in [12], where we considered spatial semi-discretization by a standard finite element method. We proved a result analogous to that of Beyn including optimal order error bounds in both the L_2 and H^1 norms.

Noting that the analysis in [12] is rather *ad hoc*, and that the more general framework in [1] does not readily apply to spatial discretizations, we decided to reconsider this problem. In §2 below we provide an abstract framework for the long-time aspects of the analysis, which is based on carefully chosen assumptions to be checked in each application by proving rather standard finite-time error and perturbation estimates. More precisely, in §2.1 we prove a shadowing result for discrete dynamical systems of the form $u_{n+1} = S(u_n)$, where S is a nonlinear operator in a Banach space X. We assume that $S = L + N$, where the bounded linear operator L is hyperbolic, and the nonlinear remainder N has a small Lipschitz constant on a subset $D \subset X$. This is an adaptation of the classical shadowing lemma of Anosov [2] and Bowen [4].

In §2.2 the mapping S is studied together with a family of approximations $S_h = L_h + N_h$, where L_h is linear, and it is assumed that we have access to bounds for $L_h - L$ and $S_h - S$, as well as estimates of the Lipschitz constant of N_h. The main result of §2.2 is a theorem analogous to that of Beyn concerning the behavior of the orbits of S and S_h near a hyperbolic fixed point of S.

If $S(t, \cdot)$ is a continuous dynamical system (nonlinear semigroup) with orbits $u(t) = S(t, u_0)$, $t \geq 0$, $u(0) = u_0$, then, for fixed T, the mapping $S = S(T, \cdot)$ defines a discrete dynamical system with orbits $u_n = u(nT)$, $n = 0, 1, 2, \dots$, satisfying the assumptions of §2.1 in a neighborhood D of a hyperbolic fixed point.

In §3 we apply the abstract framework in the context of a system of reaction-diffusion equations discretized in the spatial variables by a standard finite element method, and in the time variable by means of the backward Euler method. The assumptions on $L_h - L$ and $S_h - S$ are verified by application of rather standard error estimates over the finite time interval $[0, T]$, which we quote from Larsson [11].

Our framework is similar to that of [1] but more flexible. First of all it admits applications with both time and space discretization. In the applications discussed in §3 it also allows us to obtain error bounds of optimal order in both the L_2 and H^1 norms. Moreover, it avoids the assumption that $S_h - S$ is small in $C^1(D, X)$ that was used in [1], but which we found inconvenient. Note, in this connection, that we do not assume that L is a derivative of S. This is important, because even in a situation where L is formally a linearization of S, it may not be a Fréchet derivative with respect to the norms that we use, see Remark 3 below.

If X, Y are Banach spaces, then $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X into Y, $\mathcal{L}(X) = \mathcal{L}(X, X)$, and $B_X(x, \rho)$ denotes the closed ball in X with center x and radius ρ.

2. A general framework

2.1. **A basic shadowing result.** We consider a mapping $S : D \subset X \to X$, where X is a Banach space and D a nonempty subset of X. It is assumed that S can be decomposed in the form

$$S = L + N \tag{2.1}$$

in a such a way that, for some constants $\mu \geq 1$ and $\kappa \in (0,1)$, the following hypotheses (HL) and (HN) are fulfilled.

Hypothesis (HL). $L \in \mathcal{L}(X)$, i.e., L is a bounded linear operator in X. Furthermore, X can be decomposed as a direct sum $X = X_1 \oplus X_2$ of closed subspaces X_1, X_2 that are invariant by L, i.e., $LX_i \subset X_i$, $i = 1, 2$. If $L_i \in \mathcal{L}(X_i)$, $i = 1, 2$, denotes the restriction of L to X_i, then L_1 is invertible and

$$\|L_1^{-1}\|_{\mathcal{L}(X_1)} \leq \kappa, \quad \|L_2\|_{\mathcal{L}(X_2)} \leq \kappa. \tag{2.2}$$

Moreover, the projections P_i, $i = 1, 2$, associated with the decomposition $X = X_1 \oplus X_2$ (i.e., $P_i x = x_i$, $i = 1, 2$, for $x = x_1 + x_2$, $x_i \in X_i$) satisfy

$$\|P_i\|_{\mathcal{L}(X)} \leq \mu. \tag{2.3}$$

Hypothesis (HN). The mapping $N : D \to X$ is Lipschitz continuous with a Lipschitz constant that satisfies

$$\mathrm{Lip}(N) \leq \frac{1-\kappa}{4\mu}. \tag{2.4}$$

Note that the boundedness of the projections P_1, P_2 is a consequence of the closedness of subspaces X_1, X_2; this is a well-known consequence of the closed graph theorem, see [10, p. 167].

We now state the main result of this section.

Theorem 2.1. *(i) Assume that for the mapping S in (2.1) the hypotheses (HL), (HN) are satisfied and set*

$$\sigma := \frac{4\mu}{1-\kappa}. \tag{2.5}$$

Let i and f be integers, $i < f$, and let $\{\tilde{x}_n\}_{n=i}^f \subset D$ be a sequence. If $\{x_n\}_{n=i}^f \subset D$ is an orbit of S, i.e., $x_{n+1} = S(x_n)$, $n = i, \dots, f-1$, which satisfies the boundary conditions

$$P_2 x_i = P_2 \tilde{x}_i, \quad P_1 x_f = P_1 \tilde{x}_f, \tag{2.6}$$

then

$$\sup_{i \leq n \leq f} \|x_n - \tilde{x}_n\| \leq \sigma \sup_{i \leq n \leq f-1} \|\tilde{x}_{n+1} - S(\tilde{x}_n)\|. \tag{2.7}$$

(ii) Assume, in addition to (HL), (HN), that the domain D of S contains a closed ball $B_X(z,\rho)$ and that

$$\|z - S(z)\| \leq \rho/\sigma. \tag{2.8}$$

Then, for any sequence $\{\tilde{x}_n\}_{n=i}^f \subset B_X(z, \rho/(\mu\sigma))$, there exists an orbit $\{x_n\}_{n=i}^f \subset B_X(z,\rho)$ of S for which (2.6) and hence (2.7) hold.

Remark 1. With the terminology used in shadowing theory (see, e.g., [6]) we say that $\{\tilde{x}_n\}_{n=i}^f$ is a *δ-pseudo-orbit* of S if $\sup_{i\le n\le f-1}\|\tilde{x}_{n+1}-S(\tilde{x}_n)\|\le\delta$. If this is the case, then the estimate (2.7) means that $\{\tilde{x}_n\}_{n=i}^f$ is *ϵ-shadowed* by the true orbit $\{x_n\}_{n=i}^f$ with *shadowing distance* $\epsilon\le\sigma\delta$. Part (ii) ensures the existence of a shadow orbit.

Remark 2. Analogous results hold also for infinite sequences $\{\tilde{x}_n\}_{n=-\infty}^f$, $\{\tilde{x}_n\}_{n=i}^\infty$, or $\{\tilde{x}_n\}_{n=-\infty}^\infty$ with obvious modifications. For instance, for a sequence $\{\tilde{x}_n\}_{n=-\infty}^f$, the first condition in (2.6) is absent and the ranges in (2.7) become $n\le f$ and $n\le f-1$. The proof is essentially the same as that given below for finite sequences. Note in this connection that the stability constant σ in (2.7) does not depend on the initial and final indices i and f.

The theorem is proved by using the following lemmas.

Lemma 2.2. *Assume that L satisfies hypothesis (HL), and that i,f are integers with $i<f$. Set $\nu=f-i$, $\mathbf{X}=X^{\nu+1}$, $\mathbf{Y}=X_2\times X^\nu\times X_1$, and define a linear operator $\mathbf{L}:\mathbf{X}\to\mathbf{Y}$ by $\mathbf{L}:(x_i,\dots,x_f)\mapsto(y_i,\dots,y_{f+1})$, where*

$$y_i=P_2x_i,\ y_{f+1}=P_1x_f;\quad y_{n+1}=x_{n+1}-Lx_n,\quad n=i,\dots,f-1. \tag{2.9}$$

Then $\mathbf{L}$ is invertible and, with respect to the supremum norm of the product spaces $\mathbf{X},\mathbf{Y}$,

$$\|\mathbf{L}^{-1}\|_{\mathcal{L}(\mathbf{Y},\mathbf{X})}\le\frac{2\mu}{1-\kappa}. \tag{2.10}$$

Proof. Given an element $\mathbf{y}=(y_i,\dots,y_{f+1})\in\mathbf{Y}$, we define $\mathbf{x}=(x_i,\dots,x_f)\in\mathbf{X}$, by the relations $x_n=P_1x_n+P_2x_n$, where

$$P_1x_n=(L_1^{-1})^{f-n}y_{f+1}-\sum_{j=n+1}^{f}(L_1^{-1})^{j-n}P_1y_j,$$

$$P_2x_n=(L_2)^{n-i}y_i+\sum_{j=i+1}^{n}(L_2)^{n-j}P_2y_j.$$

It is a simple matter to check that (2.9) holds. This proves that $\mathbf{L}$ is onto. To see that $\mathbf{L}$ is one-to-one, assume that y_n, $n=i,\dots,f+1$, in (2.9) vanish. Then, by (2.9), $P_2x_i=0$ and $P_1x_f=0$. Recursion in (2.9) reveals that $P_2x_n=0$ for $n=i+1,\dots,f$. Similarly, a descending recursion in (2.9) shows that $P_1x_n=0$ for $n=f-1,\dots,i$, so that the kernel of $\mathbf{L}$ is trivial.

To derive (2.10), use (2.2)–(2.3) in the definition of x_n, $n=i,\dots,f$,

$$\begin{aligned}\|x_n\|&\le\kappa^{f-n}\|\mathbf{y}\|+\sum_{j=n+1}^{f}\kappa^{j-n}\mu\|\mathbf{y}\|+\kappa^{n-i}\|\mathbf{y}\|+\sum_{j=i+1}^{n}\kappa^{n-j}\mu\|\mathbf{y}\|\\&\le\Big(1+\sum_{j=n+1}^{f}\kappa^{j-n}+\kappa^{n-i}+\sum_{j=i+1}^{n}\kappa^{n-j}\Big)\mu\|\mathbf{y}\|\\&\le\Big(\sum_{j=n}^{\infty}\kappa^{j-n}+\sum_{j=-\infty}^{n}\kappa^{n-j}\Big)\mu\|\mathbf{y}\|=\frac{2\mu}{1-\kappa}\|\mathbf{y}\|.\end{aligned}$$

$\square$

Lemma 2.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be Banach spaces and let* $\mathbf{L} : \mathbf{X} \to \mathbf{Y}$ *be a linear bijection with bounded inverse* $\mathbf{L}^{-1}$. *Assume that the mapping* $\mathbf{N} : \mathbf{D} \subset \mathbf{X} \to \mathbf{Y}$, *defined in a nonempty subset* $\mathbf{D}$ *of* $\mathbf{X}$, *is Lipschitz continuous with*

$$\alpha := \|\mathbf{L}^{-1}\|_{\mathcal{L}(\mathbf{Y},\mathbf{X})} \operatorname{Lip}(\mathbf{N}) < 1.$$

Define $\mathbf{S} = \mathbf{L} + \mathbf{N}$ *and set* $\sigma := \|\mathbf{L}^{-1}\|_{\mathcal{L}(\mathbf{Y},\mathbf{X})}/(1-\alpha)$. *Then*

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathbf{X}} \le \sigma \|\mathbf{S}(\mathbf{x}_1) - \mathbf{S}(\mathbf{x}_2)\|_{\mathbf{Y}}, \quad \textit{for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}. \tag{2.11}$$

Proof. The bound (2.11) follows readily from the identity

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{L}^{-1}\big(\mathbf{S}(\mathbf{x}_1) - \mathbf{S}(\mathbf{x}_2)\big) - \mathbf{L}^{-1}\big(\mathbf{N}(\mathbf{x}_1) - \mathbf{N}(\mathbf{x}_2)\big).$$

□

Lemma 2.4. *In addition to the hypotheses of Lemma 2.3, assume that the domain* $\mathbf{D}$ *of* $\mathbf{N}$ *contains a closed ball* $B_{\mathbf{X}}(\mathbf{z}, \rho)$. *Then, for each* $\mathbf{y} \in \mathbf{Y}$ *with*

$$\|\mathbf{y} - \mathbf{S}(\mathbf{z})\|_{\mathbf{Y}} \le \rho/\sigma, \tag{2.12}$$

the equation $\mathbf{S}(\mathbf{x}) = \mathbf{y}$ *has a unique solution* $\mathbf{x} \in B_{\mathbf{X}}(\mathbf{z}, \rho)$.

Proof. This is a simple consequence of the contraction mapping theorem. In fact, if we define $\mathbf{T}(\mathbf{x}) = \mathbf{L}^{-1}(\mathbf{y} - \mathbf{N}(\mathbf{x}))$, then $\operatorname{Lip}(\mathbf{T}) = \alpha < 1$. Moreover, the identity

$$\mathbf{T}(\mathbf{x}) - \mathbf{z} = \mathbf{L}^{-1}\big(\mathbf{y} - \mathbf{S}(\mathbf{z})\big) - \mathbf{L}^{-1}\big(\mathbf{N}(\mathbf{x}) - \mathbf{N}(\mathbf{z})\big)$$

and (2.12) imply that $\mathbf{T}$ maps $B_{\mathbf{X}}(\mathbf{z}, \rho)$ into itself. □

Proof of Theorem 2.1. Given i and f, we construct the spaces $\mathbf{X}$ and $\mathbf{Y}$ and the linear operator $\mathbf{L}$ as in Lemma 2.2. We further consider the mapping $\mathbf{N} : \mathbf{D} = D^{\nu+1} \subset \mathbf{X} \to \mathbf{Y}$ defined by $\mathbf{N}(\mathbf{x}) = \mathbf{y}$, where

$$y_i = y_{f+1} = 0; \quad y_{n+1} = -N(x_n), \quad n = i, \dots, f-1. \tag{2.13}$$

The assumption (HN) implies that $\operatorname{Lip}(\mathbf{N}) \le (1-\kappa)/(4\mu)$ and (2.10) leads to $\|\mathbf{L}^{-1}\| \operatorname{Lip}(\mathbf{N}) \le 1/2$. We can therefore apply Lemma 2.3 with $\alpha = 1/2$; this yields a value of $\sigma = \|\mathbf{L}^{-1}\|/(1-\alpha)$ that coincides with σ in (2.5). Note also that $\mathbf{S}$ is defined by $\mathbf{S}(\mathbf{x}) = \mathbf{y}$, where (see (2.1), (2.9), and (2.13))

$$y_i = P_2 x_i, \; y_{f+1} = P_1 x_f; \quad y_{n+1} = x_{n+1} - S(x_n), \quad n = i, \dots, f-1.$$

The estimate (2.7) is then a straightforward consequence of (2.11).

To prove part (ii) of the theorem, we apply Lemma 2.4 in the ball $B(\mathbf{z}, \rho) = B(z, \rho)^{\nu+1}$, where $\mathbf{z} = (z, \dots, z)$. Given $\{\tilde{x}_n\}_{i=1}^{f} \subset B(z, \rho/(\mu\sigma))$, we put $\mathbf{y} = (P_2\tilde{x}_i, 0, \dots, 0, P_1\tilde{x}_f)$. The condition (2.12) is satisfied. In fact, the first component of $\mathbf{y} - \mathbf{S}(\mathbf{z})$ is the vector $P_2\tilde{x}_i - P_2 z$, whose norm can be estimated by $\|P_2\| \, \|\tilde{x}_i - z\| \le \rho/\sigma$. Similarly, the last component of $\mathbf{y} - \mathbf{S}(\mathbf{z})$ has norm $\le \rho/\sigma$. The remaining components of $\mathbf{y} - \mathbf{S}(\mathbf{z})$ equal $0 - (z - S(z))$ and, in view of (2.8), are also bounded in norm by ρ/σ. Since (2.12) is satisfied, the equation $\mathbf{S}(\mathbf{x}) = \mathbf{y}$ has a solution $\mathbf{x} = (x_i, \dots, x_f)$. The choice of $\mathbf{y}$ ensures that $\{x_n\}_{n=i}^{f}$ is the sequence required. □

2.2. **Shadowing and approximation.** We now consider, along with the mapping S in (2.1), a family of approximations $\{S_h\}$.

Let $\mathcal{H}$ be a set of positive numbers with $\inf \mathcal{H} = 0$. For each $h \in \mathcal{H}$, let X_h be a subspace of the Banach space X, in such a way that there exist bounded projections $Q_h : X \to X_h$ and a number $\gamma \geq 1$ with

$$\|Q_h\|_{\mathcal{L}(X)} \leq \gamma. \tag{2.14}$$

We assume that the spaces X_h approximate X in the sense that

$$\lim_{h \to 0} Q_h u = u, \quad \text{for all } u \in X. \tag{2.15}$$

For $h \in \mathcal{H}$, we consider mappings $S_h : D_h \subset X_h \to X_h$ with domains $D_h = D \cap X_h$, that approximate S in the sense that a continuous positive function $\epsilon(h)$ exists such that

$$\lim_{h \to 0} \epsilon(h) = 0 \tag{2.16}$$

and

$$\|S_h(Q_h u) - S(u)\| \leq \epsilon(h), \quad \text{for all } u \in D \text{ such that } Q_h u \in D_h. \tag{2.17}$$

Finally, we assume that S_h can be decomposed as

$$S_h = L_h + N_h, \tag{2.18}$$

and that (HL) and (HN) hold for this decomposition. More precisely, this means that $L_h \in \mathcal{L}(X_h)$; X_h can be decomposed as a direct sum $X_h = X_{1h} \oplus X_{2h}$ of closed subspaces invariant by L_h; the restrictions L_{ih}, $i = 1, 2$, of L_h to X_{ih} satisfy

$$\|L_{1h}^{-1}\|_{\mathcal{L}(X_{1h})} \leq \kappa, \quad \|L_{2h}\|_{\mathcal{L}(X_{2h})} \leq \kappa; \tag{2.19}$$

the associated projections satisfy

$$\|P_{ih}\|_{\mathcal{L}(X)} \leq \mu, \quad i = 1, 2; \tag{2.20}$$

and $N_h : D_h \to X_h$ with Lipschitz constant

$$\mathrm{Lip}(N_h) \leq \frac{1 - \kappa}{4\mu}.$$

Note that, X_{ih} is in general different from $X_i \cap X_h$; the latter may well be the trivial subspace $\{0\}$.

Theorem 2.5. *(i) Assume that the subspaces X_h of the Banach space X possess the approximation properties (2.14)–(2.15), the mappings S in (2.1) and S_h in (2.18) satisfy (HL) and (HN), and the S_h approximate S as in (2.17). Let i and f be integers with $i < f$. Then the following results hold.*

(i.a) Let $\{u_{h,n}\}_{n=i}^{f} \subset D_h$ be an orbit of S_h. If $\{u_n\}_{n=i}^{f}$ is an orbit of S with

$$P_2 u_i = P_2 u_{h,i}, \quad P_1 u_f = P_1 u_{h,f}, \tag{2.21}$$

then

$$\sup_{i \leq n \leq f} \|u_n - u_{h,n}\| \leq \sigma \epsilon(h), \tag{2.22}$$

where σ is the constant in (2.5).

(i.b) Let $\{u_n\}_{n=i}^{f} \subset D$ be an orbit of S with

$$Q_h u_n \in D_h, \quad n = i, \dots, f. \tag{2.23}$$

If $\{u_{h,n}\}_{n=i}^{f} \subset D_h$ *is an orbit of* S_h *with*

$$P_{2h} u_{h,i} = P_{2h} Q_h u_i, \quad P_{1h} u_{h,f} = P_{1h} Q_h u_f, \tag{2.24}$$

then

$$\sup_{i \le n \le f} \|Q_h u_n - u_{h,n}\| \le \gamma\sigma\epsilon(h). \tag{2.25}$$

(ii) Assume that the hypotheses in (i) hold and that S *has a fixed point* $\bar{u}$ *such that* $B_X(\bar{u}, \rho) \subset D$ *for some* $\rho > 0$. *Then the following conclusions hold.*

(ii.a) For any orbit $\{u_{h,n}\}_{n=i}^{f} \subset B_X(\bar{u}, \rho_0)$, $\rho_0 = \rho/(\mu\sigma) < \rho$, *of* S_h *there is an orbit* $\{u_n\}_{n=i}^{f}$ *of* S *for which (2.21) and (2.22) are true.*

(ii.b) Let h *be small enough for the inequalities*

$$\|\bar{u} - Q_h \bar{u}\| \le \rho/2, \quad 2\gamma\sigma\epsilon(h) \le \rho \tag{2.26}$$

to hold (cf. (2.15)–(2.16)). Then, for any orbit $\{u_n\}_{n=i}^{f} \subset B_X(\bar{u}, \rho_0)$, $\rho_0 = \rho/(2\gamma\mu\sigma)$, *of* S, *the relation (2.23) is satisfied and there is an orbit* $\{u_{h,n}\}_{n=i}^{f}$ *of* S_h *for which (2.24) and (2.25) hold.*

(ii.c) For h *chosen as in (ii.b), the mapping* S_h *has a fixed point that is unique in the ball* $B_{X_h}(Q_h \bar{u}, \rho/2)$. *Furthermore,*

$$\|Q_h \bar{u} - \bar{u}_h\| \le \gamma\sigma\epsilon(h). \tag{2.27}$$

Proof. To prove (i.a) we apply part (i) of Theorem 2.1 with $\tilde{x}_n = u_{h,n} \in D_h \subset D$, and $x_n = u_n$. Then (2.7) yields

$$\sup_{i \le n \le f} \|u_n - u_{h,n}\| \le \sigma \sup_{i \le n \le f-1} \|u_{h,n+1} - S(u_{h,n})\|.$$

Since $u_{h,n+1} - S(u_{h,n}) = S_h(Q_h u_{h,n}) - S(u_{h,n})$, the bound (2.22) is a consequence of (2.17).

For part (i.b) we again resort to part (i) of Theorem 2.1, but this time with S_h playing the role of S, and $\tilde{x}_n = Q_h u_n \in D_h$, $x_n = u_{h,n} \in D_h$. The estimate (2.7) reads

$$\sup_{i \le n \le f} \|u_{h,n} - Q_h u_n\| \le \sigma \sup_{i \le n \le f-1} \|Q_h u_{n+1} - S_h(Q_h u_n)\|,$$

and (2.25) is a consequence of (2.14), (2.17), in view of the identity

$$Q_h u_{n+1} - S_h(Q_h u_n) = Q_h[S(u_n) - S_h(Q_h u_n)].$$

Part (ii.a) is a direct consequence of part (ii) of Theorem 2.1 with $z = \bar{u}$.

For (ii.b) we use part (ii) of Theorem (2.1) with S_h playing the role of S and $z = Q_h \bar{u}$. If $v \in B_{X_h}(Q_h \bar{u}, \rho/2)$, then the assumption (2.26) implies

$$\|v - \bar{u}\| \le \|v - Q_h \bar{u}\| + \|Q_h \bar{u} - \bar{u}\| \le \rho;$$

this shows that $B_{X_h}(Q_h \bar{u}, \rho/2)$ is contained in $B_X(\bar{u}, \rho)$, which in turn is assumed to be contained in D. Hence $B_{X_h}(Q_h \bar{u}, \rho/2) \subset D_h$. Furthermore, by (2.14), (2.17) and (2.26),

$$\|Q_h \bar{u} - S_h(Q_h \bar{u})\| = \|Q_h[S(\bar{u}) - S_h(Q_h \bar{u})]\| \le \gamma\epsilon(h) \le \frac{\rho/2}{\sigma}, \tag{2.28}$$

so that (2.8) holds with the role of $B_X(z, \rho)$ played by $B_{X_h}(Q_h \bar{u}, \rho/2)$. If $\{u_n\}_{n=i}^{f} \subset B_X(\bar{u}, \rho_0)$, then, by (2.14), $\{Q_h u_n\}_{n=i}^{f} \subset B_{X_h}(Q_h \bar{u}, \rho/(2\mu\sigma)) \subset B_{X_h}(Q_h \bar{u}, \rho/2)$ and (2.23) holds. The existence of an orbit $\{u_{h,n}\}_{n=i}^{f}$ of S_h satisfying (2.24) now follows from Theorem 2.1.

For (ii.c) we apply Lemma 2.4 with $\mathbf{X} = \mathbf{Y} = X_h$, $\mathbf{L} = L_h - I$, $\mathbf{N} = N_h$, $\mathbf{y} = 0$ and the role of $B_{\mathbf{X}}(\mathbf{z}, \rho)$ played by $B_{X_h}(Q_h\bar{u}, \rho/2)$, which we know to be contained in D_h. By Lemma 2.8, $\|(L_h - I)^{-1}\| \leq 2\mu/(1-\kappa)$ leading in Lemma 2.3 to $\alpha = 1/2$ and a value of σ that agrees with the value in (2.5). The condition (2.12) is fulfilled in view of (2.28), because $\mathbf{y} - \mathbf{S}(\mathbf{z}) = -S_h(Q_h\bar{u}) + Q_h\bar{u}$. It only remains to prove the bound (2.27). We apply again Lemma 2.4, but this time in the smaller ball $B_{X_h}(Q_h\bar{u}, \sigma\gamma\epsilon(h)) \subset B_{X_h}(Q_h\bar{u}, \rho/2)$; the condition (2.12) is still fulfilled in view (2.28), so that the unique fixed point $\bar{u}_h$ of S_h in $B_{X_h}(Q)_h\bar{u}, \rho/2)$ also lies in $B_{X_h}(Q_h\bar{u}, \sigma\gamma\epsilon(h))$. □

Our next theorem gives a condition under which the "hyperbolicity" (HL) of the operator L carries over to L_h.

Theorem 2.6. *Assume that the operator $L \in \mathcal{L}(X)$, X a Banach space, satisfies hypothesis (HL) and choose $\tilde{\kappa} \in (\kappa, 1)$, $\tilde{\mu} > \mu$. Assume that the subspaces X_h with corresponding projections Q_h are such that (2.14) holds and that the operators $L_h \in \mathcal{L}(X_h)$ approximate L in the sense that*

$$\|L - L_h Q_h\|_{\mathcal{L}(X)} \leq \epsilon(h) \tag{2.29}$$

with $\epsilon(h)$ as in (2.16). Then there exists $h_0 > 0$ (depending only on γ, κ, μ, $\tilde{\kappa}$, $\tilde{\mu}$, and the function ϵ) such that, uniformly for $h < h_0$, the operators L_h satisfy (HL) with constants $\tilde{\kappa}$, $\tilde{\mu}$.

For the proof of the theorem we need two simple lemmas.

Lemma 2.7. *Let $A, B \in \mathcal{L}(X)$, X a Banach space, with $A^{-1} \in \mathcal{L}(X)$ and $\|B\| \leq \frac{1}{2}\|A^{-1}\|^{-1}$. Then $(A+B)^{-1} \in \mathcal{L}(X)$ and*

$$\|(A+B)^{-1} - A^{-1}\|_{\mathcal{L}(X)} \leq 2\|A^{-1}\|^2_{\mathcal{L}(X)}\|B\|_{\mathcal{L}(X)}.$$

Proof. We have $\|(A+B)^{-1}\|_{\mathcal{L}(X)} = \|A^{-1}\sum_{n=0}^{\infty}(-A^{-1}B)^n\|_{\mathcal{L}(X)} \leq 2\|A^{-1}\|_{\mathcal{L}(X)}$ and $(A+B)^{-1} - A^{-1} = -(A+B)^{-1}BA^{-1}$. □

Lemma 2.8. *Assume that the operator $L \in \mathcal{L}(X)$, X a Banach space, satisfies hypothesis (HL) and let ω be a complex number with $|\omega| = 1$. Then $(\omega I - L)^{-1} \in \mathcal{L}(X)$ with*

$$\|(\omega I - L)^{-1}\|_{\mathcal{L}(X)} \leq \frac{2\mu}{1-\kappa}.$$

Proof. Let $|\omega| = 1$. Assumption (2.2) implies that

$$(\omega I - L_1)^{-1} = -L_1\sum_{n=0}^{\infty}(\omega L_1^{-1})^n, \quad (\omega I - L_2)^{-1} = \omega^{-1}\sum_{n=0}^{\infty}(\omega^{-1}L_2)^n,$$

and hence

$$\|(\omega I - L_1)^{-1}\|_{\mathcal{L}(X_1)} \leq \frac{\kappa}{1-\kappa}, \quad \|(\omega I - L_2)^{-1}\|_{\mathcal{L}(X_2)} \leq \frac{1}{1-\kappa}.$$

Using also (2.3) and $(\omega I - L)^{-1} = (\omega I - L_1)^{-1}P_1 + (\omega I - L_2)^{-1}P_2$, we obtain

$$\|(\omega I - L)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1+\kappa}{1-\kappa}\mu \leq \frac{2\mu}{1-\kappa}.$$

□

Proof of Theorem 2.6. Define $\tilde{L}_h = L_h Q_h \in \mathcal{L}(X)$ and let $|\omega| = 1$. We begin by showing that $(\omega I - \tilde{L}_h)^{-1} \in \mathcal{L}(X)$. In order to do so we shall apply Lemma 2.7 with $A = \omega I - L$ and $B = L - \tilde{L}_h$. From Lemma 2.8 we know that $\|(\omega I - L)^{-1}\| \leq 2\mu/(1-\kappa)$. Hence, for h sufficiently small, we have

$$\|L - \tilde{L}_h\|_{\mathcal{L}(X)} \leq \epsilon(h) \leq \frac{1-\kappa}{4\mu} \leq \frac{1}{2\|(\omega I - L)^{-1}\|}.$$

Now Lemma 2.7 applies and gives $(\omega I - \tilde{L}_h)^{-1} \in \mathcal{L}(X)$ and

$$\|(\omega I - \tilde{L}_h)^{-1} - (\omega I - L)^{-1}\|_{\mathcal{L}(X)} \leq 2\Big(\frac{2\mu}{1-\kappa}\Big)^2 \|L - \tilde{L}_h\|_{\mathcal{L}(X)} \leq K\epsilon(h), \tag{2.30}$$

where $K = 2(2\mu/(1-\kappa))^2$.

We next show that $(\omega I - L_h)^{-1} \in \mathcal{L}(X_h)$ and that

$$(\omega I - L_h)^{-1} Q_h = (\omega I - \tilde{L}_h)^{-1} Q_h. \tag{2.31}$$

In fact, if $f \in X$, then $u = (\omega I - \tilde{L}_h)^{-1} Q_h f \in X$ satisfies

$$Q_h f = (\omega I - \tilde{L}_h)(Q_h u + (I - Q_h)u) = (\omega I - L_h) Q_h u + \omega(I - Q_h)u.$$

We conclude that $(I - Q_h)u = 0$, and $(\omega I - L_h)Q_h u = Q_h f$, so that $u = Q_h u = (\omega I - L_h)^{-1} Q_h f$, which implies (2.31).

We have now proved that, for each sufficiently small h, L_h has no spectrum on the unit circle. By a standard theorem, see, e.g., [10, Theorem III–6.17, p. 178], this implies the existence of a splitting $X_h = X_{1h} \oplus X_{2h}$ as required in assumption (HL). It only remains to prove that the corresponding inequalities (2.19) and (2.20) with constants $\tilde{\kappa}$, $\tilde{\mu}$ hold uniformly with respect to h.

In order to obtain a bound for $\|P_{2h}\|_{\mathcal{L}(X_h)}$ we first estimate $\|(P_2 - P_{2h})Q_h\|_{\mathcal{L}(X)}$. Using the representations

$$P_2 = \frac{1}{2\pi i}\int_\Gamma (\omega I - L)^{-1}\, d\omega, \quad P_{2h} = \frac{1}{2\pi i}\int_\Gamma (\omega I - L_h)^{-1}\, d\omega,$$

where Γ denotes the unit circle with positive orientation, together with (2.31), (2.30), and (2.14) we obtain

$$\|(P_2 - P_{2h})Q_h\|_{\mathcal{L}(X)} = \frac{1}{2\pi}\Big\|\int_\Gamma \big((\omega I - L)^{-1} - (\omega I - \tilde{L}_h)^{-1}\big)\, d\omega\ Q_h\Big\|_{\mathcal{L}(X)} \leq \gamma K\epsilon(h).$$

This implies that, for $x \in X_h$,

$$\|P_{2h}x\| \leq \|P_2 x\| + \|(P_2 - P_{2h})x\| \leq \big(\mu + \gamma K\epsilon(h)\big)\|x\|,$$

so that

$$\|P_{2h}\|_{\mathcal{L}(X_h)} \leq \mu + \gamma K\epsilon(h) \leq \tilde{\mu},$$

provided that h is sufficiently small. Since $(P_1 - P_{1h})Q_h = (P_{2h} - P_2)Q_h$, we also have $\|P_{1h}\|_{\mathcal{L}(X_h)} \leq \tilde{\mu}$.

We now turn to the bound for $\|L_{2h}\|_{\mathcal{L}(X_{2h})}$. Since

$$L_2 P_2 = \frac{1}{2\pi i}\int_\Gamma \omega(\omega I - L)^{-1}\, d\omega, \quad L_{2h}P_{2h} = \frac{1}{2\pi i}\int_\Gamma \omega(\omega I - L_h)^{-1}\, d\omega,$$

we have

$$\begin{aligned}\|(L_2P_2 - L_{2h}P_{2h})Q_h\|_{\mathcal{L}(X)} &= \frac{1}{2\pi}\Big\|\int_\Gamma \omega\big((\omega I - L)^{-1} - (\omega I - \tilde{L}_h)^{-1}\big)\, d\omega\ Q_h\Big\|_{\mathcal{L}(X)} \\ &\leq \gamma K\epsilon(h).\end{aligned}$$

Hence, for $x \in X_{2h}$,

$$\begin{aligned} \|L_{2h}x\| = \|L_{2h}P_{2h}x\| &\le \|(L_{2h}P_{2h} - L_2P_2)Q_hx\| + \|L_2P_2x\| \\ &\le \gamma K\epsilon(h)\|x\| + \kappa\|P_2x\|. \end{aligned}$$

Here, since $Q_hx = P_{2h}x = x$,

$$\|P_2x\| \le \|(P_2 - P_{2h})Q_hx\| + \|P_{2h}x\| \le \gamma K\epsilon(h)\|x\| + \|x\|,$$

so that

$$\|L_{2h}x\| \le \big(\kappa + 2\gamma K\epsilon(h)\big)\|x\|,$$

and we conclude that, for small h,

$$\|L_{2h}\|_{\mathcal{L}(X_{2h})} \le \kappa + 2\gamma K\epsilon(h) \le \tilde{\kappa}.$$

The required bound for $\|L_{1h}^{-1}\|_{\mathcal{L}(X_{1h})}$ is obtained in the same way, using the representations

$$L^{-1}P_1 = -\frac{1}{2\pi i}\int_\Gamma \omega^{-1}(\omega I - L)^{-1}\,d\omega, \quad L_{1h}^{-1}P_{1h} = -\frac{1}{2\pi i}\int_\Gamma \omega^{-1}(\omega I - L_h)^{-1}\,d\omega.$$

□

3. Application to a system of reaction-diffusion equations

The purpose of this section is to illustrate how the theory of the previous section can be applied in the context of a standard finite element approximation of a system of reaction-diffusion equations.

3.1. **The continuous problem.** We consider the model problem

$$\begin{aligned} u_t - D\Delta u &= \tilde{f}(u), \quad && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) &= u_0, && x \in \Omega, \end{aligned} \tag{3.1}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$, $u = u(x,t) \in \mathbf{R}^s$, $u_t = \partial u/\partial t$, $\Delta u = \sum_i \partial^2 u/\partial x_i^2$, $D = \operatorname{diag}(d_1, \dots, d_s)$ is a diagonal matrix of constant coefficients $d_i > 0$, and $\tilde{f} : \mathbf{R}^s \to \mathbf{R}^s$ is continuously differentiable. We assume that Ω is either a convex polygon or has a smooth boundary. If $d = 2, 3$ we assume, in addition, that the Jacobian of $\tilde{f}$ satisfies the growth condition

$$|\tilde{f}'(\xi)| \le C(1 + |\xi|^\delta), \quad \xi \in \mathbf{R}^s,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^s$ and the induced matrix norm, and where $\delta = 2$ if $d = 3$, $\delta \in [0, \infty)$ if $d = 2$.

In the sequel we use the Hilbert space $H = (L_2(\Omega))^s$, with its standard norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. The norms in the Sobolev spaces $(H^m(\Omega))^s$, $m \ge 0$ are denoted by $\|\cdot\|_m$. The space $V = (H_0^1(\Omega))^s$, with norm $\|\cdot\|_1$, consists of the functions in $(H^1(\Omega))^s$ that vanish on $\partial\Omega$. We define the operator $A = -D\Delta$ with domain $\mathcal{D}(A) = (H^2(\Omega) \cap H_0^1(\Omega))^s$. Then A is a closed, densely defined and selfadjoint operator in H with compact inverse. Moreover, our assumptions guarantee that the mapping $\tilde{f}$ induces an operator $f : V \to H$ through $f(v)(x) = \tilde{f}(v(x))$, see Lemma 3.1 below. The initial-boundary value problem (3.1) may then be formulated as an initial value problem in V:

$$u' + Au = f(u), \quad t > 0; \qquad u(0) = u_0. \tag{3.2}$$

We assume further that (3.2) has a stationary solution $\bar{u}$ with $\bar{u} \in \mathcal{D}(A)$, $A\bar{u} = f(\bar{u})$; by standard embedding results $\bar{u}$ is continuous in the closure of Ω. The formula $(Bv)(x) = \tilde{f}'(\bar{u}(x))v(x)$ clearly defines an operator $B \in \mathcal{L}(H)$. The operator $\mathcal{A} = A - B$ is the linearization of $A - f$ at $\bar{u}$ and, being a bounded perturbation of A, it is a sectorial operator in H (see [9, Theorem 1.3.2]). Hence $-\mathcal{A}$ is the generator of an analytic semigroup $e^{-t\mathcal{A}}$. We assume that $\bar{u}$ is "hyperbolic", i.e., that the spectrum of $\mathcal{A}$ does not intersect the imaginary axis. Let P_1 and P_2 the projections respectively associated with the sets $\sigma_1 = \sigma(\mathcal{A}) \cap \{\operatorname{Re} z < 0\}$ and $\sigma_2 = \sigma(\mathcal{A}) \cap \{\operatorname{Re} z > 0\}$ that partition the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ and let H_1 and H_2 be the ranges of P_1 and P_2. It follows that H is a direct sum $H = H_1 \oplus H_2$; the subspaces H_i are invariant under $\mathcal{A}$ and, if $\mathcal{A}_i$, $i = 1, 2$, denotes the restriction of $\mathcal{A}$ to H_i, then $\mathcal{A}_1 \in \mathcal{L}(H_1)$, $\mathcal{D}(\mathcal{A}_2) = \mathcal{D}(\mathcal{A}) \cap H_2$. Furthermore, there are $M \geq 1$ and $\alpha > 0$, such that,

$$
\begin{aligned}
&\|e^{-t\mathcal{A}_1}v\|_m \leq Me^{\alpha t}\|v\|, && t \leq 0,\ v \in H_1,\ m = 1, 2,\\
&\|e^{-t\mathcal{A}_2}v\|_m \leq Mt^{-m/2}e^{-\alpha t}\|v\|, && t > 0,\ v \in H_2,\ m = 1, 2,\\
&\|e^{-t\mathcal{A}_2}v\|_1 \leq Me^{-\alpha t}\|v\|_1, && t \geq 0,\ v \in H_2 \cap V.
\end{aligned} \tag{3.3}
$$

We refer to [9, §1.5] for these facts.

Since $H_1 \subset \mathcal{D}(\mathcal{A})$, we see that we also have a direct sum $V = V_1 \oplus V_2$, where $V_1 = H_1$ and $V_2 = H_2 \cap V$, with associated projections $P_1|_V$ and $P_2|_V$. By the closed graph theorem, we may select a constant $\mu \geq 1$ such that

$$\|P_i\|_{\mathcal{L}(H)} \leq \mu, \quad \|P_i\|_{\mathcal{L}(V)} \leq \mu, \quad i = 1, 2. \tag{3.4}$$

By combination of these with (3.3) we have

$$
\begin{aligned}
&\|e^{-t\mathcal{A}}v\|_1 \leq Ct^{-1/2}e^{\alpha t}\|v\|, && t > 0,\ v \in H,\\
&\|e^{-t\mathcal{A}}v\|_1 \leq Ce^{\alpha t}\|v\|_1, && t \geq 0,\ v \in V.
\end{aligned} \tag{3.5}
$$

With $F(v) = f(v) - Bv$, we may rewrite (3.2) as

$$u' + \mathcal{A}u = F(u), \quad t > 0; \qquad u(0) = u_0, \tag{3.6}$$

and clearly we also have

$$\mathcal{A}\bar{u} = F(\bar{u}). \tag{3.7}$$

As shown by the following lemma, whose proof is similar to that of Lemma 2.2 in [12], the nonlinear operator $F : V \to H$ is Lipschitz continuous with a Lipschitz constant that may be rendered arbitrarily small by restricting the attention to a sufficiently small neighborhood of $\bar{u}$.

Lemma 3.1. *If $v, w \in B_V(\bar{u}, \rho)$, then*

$$\|F(v) - F(w)\| \leq k(\rho)\|v - w\|_1, \tag{3.8}$$

where $k(\rho) = O(\rho)$ as $\rho \to 0$.

The initial value problem (3.6) (or (3.2)) has a unique local solution for any initial datum $u_0 \in V$, see [9, Theorem 3.3.3]. We denote by $S(t, \cdot)$ the corresponding (local) solution operator, so that $u(t) = S(t, u_0)$ is the solution of (3.6). The following lemma shows that the local solutions can be extended in time, if they start sufficiently near $\bar{u}$.

Lemma 3.2. *For each $\rho_1 > 0$ and $T > 0$ there is $\rho > 0$ such that, if $u_0 \in B_V(\bar{u}, \rho)$, then $S(t, u_0)$ is defined and belongs to $B_V(\bar{u}, \rho_1)$ for $t \in [0, T]$.*

Proof. Let $\rho_1, T > 0$ be given. For $\rho > 0$ let $\tau \in [0, T]$ be the largest time such that $u_0 \in B_V(\bar{u}, \rho)$ implies that $u(t) = S(t, u_0)$ exists and belongs to $B_V(\bar{u}, \rho_1 + 1)$ for $t \in [0, \tau]$. We must choose ρ such that $\tau = T$.

Let $z(t) = u(t) - \bar{u}$. Forming the difference between (3.6) and (3.7) and using the variation of constants formula, we obtain

$$z(t) = e^{-t\mathcal{A}} z(0) + \int_0^t e^{-(t-s)\mathcal{A}} \big(F(u(s)) - F(\bar{u})\big)\, ds.$$

Invoking (3.5) and (3.8), we therefore have, for $t \in [0, \tau]$,

$$\begin{aligned} \|z(t)\|_1 &\le C e^{\alpha t} \|z(0)\|_1 + C \int_0^t (t-s)^{-1/2} e^{\alpha(t-s)} \|F(u(s)) - F(\bar{u})\|\, ds \\ &\le C e^{\alpha T} \Big(\rho + k(\rho_1 + 1) \int_0^t (t-s)^{-1/2} \|z(s)\|_1\, ds\Big). \end{aligned}$$

Gronwall's lemma (see [13, Lemma 5.6.7] or [9, Exercise 4 of §6.1]) now yields

$$\|z(t)\|_1 \le C(\rho_1, T)\rho, \quad t \in [0, \tau],$$

so that if we choose $\rho = \rho_1 / C(\rho_1, T)$, then

$$\|z(t)\|_1 \le \rho_1, \quad t \in [0, \tau].$$

If $\tau < T$, then by local existence we obtain a contradiction with the maximality of τ. Hence, $S(t, u_0)$ is defined and belongs to $B_V(\bar{u}, \rho_1)$ for $t \in [0, T]$. □

The following lemma provides a bound for the H^2 norm of the solution found in Lemma 3.2.

Lemma 3.3. *Let $\rho_1 > 0$, $T > 0$ and assume that $u(t) = S(t, u_0)$ exists and belongs to $B_V(\bar{u}, \rho_1)$ for $t \in [0, T]$. Then there exists $C(\rho_1, T)$ such that*

$$\|u(t)\|_2 \le C(\rho_1, T)\, t^{-1/2}, \quad t \in (0, T].$$

Proof. In view of (3.8) we have $\|F(u(t))\| \le C(\rho_1)$ for $t \in [0, T]$. The proof is now obtained by tracing the constants in [9, Theorem 3.5.2]. □

In order to set the present problem in the framework of §2, we choose T such that

$$\kappa := M e^{-\alpha T} < 1, \tag{3.9}$$

where M and α are the constants in (3.3). We then define $S = S(T, \cdot)$, $L = e^{-T\mathcal{A}}$, $N = S - L$. It is clear from the above that assumption (HL) is satisfied, both with $X = H$ and $X = V$. In order to choose the domain D so that (HN) holds we need the following result.

Lemma 3.4. *For each $\epsilon > 0$ there is $\rho > 0$ such that $v_1, v_2 \in B_V(\bar{u}, \rho)$ implies*

$$\|N(v_1) - N(v_2)\|_m \le \epsilon \|v_1 - v_2\|_m, \quad m = 0, 1.$$

Proof. Let T be as in (3.9) and let $\rho_1 > 0$. We first carry out an *a priori* estimation under the assumption that $u_i = S(t, v_i)$, $i = 1, 2$, exist and belong to $B_V(\bar{u}, \rho_1)$ for $t \in [0, T]$. From the variation of constants formula

$$u_i(t) = e^{-t\mathcal{A}} v_i + \int_0^t e^{-(t-s)\mathcal{A}} F(u_i(s))\, ds, \quad t \in [0, T],$$

so that $N(v_i) = w_i(T)$, where

$$w_i(t) = \int_0^t e^{-(t-s)\mathcal{A}} F(u_i(s))\, ds.$$

Using (3.5) and (3.8), we obtain

$$\begin{aligned} \|w_1(t) - w_2(t)\|_1 &\le Ce^{\alpha t} \int_0^t (t-s)^{-1/2} \|F(u_1(s)) - F(u_2(s))\|\, ds \\ &\le C(T)k(\rho_1) \int_0^t (t-s)^{-1/2} \|u_1(s) - u_2(s)\|_1\, ds. \end{aligned}$$

Here $u_1(s) - u_2(s) = e^{-s\mathcal{A}}(v_1 - v_2) + w_1(s) - w_2(s)$, so that another application of (3.5) yields, for $t \in [0,T]$,

$$\begin{aligned} \|w_1(t) - w_2(t)\|_1 &\le C(T)k(\rho_1)\|v_1 - v_2\| \\ &\quad + C(T,\rho_1) \int_0^t (t-s)^{-1/2} \|w_1(s) - w_2(s)\|_1\, ds. \end{aligned}$$

Gronwall's lemma now shows that

$$\|w_1(t) - w_2(t)\|_1 \le C(T,\rho_1)k(\rho_1)\|v_1 - v_2\|, \quad t \in [0,T]. \tag{3.10}$$

This is the required *a priori* bound and we may now complete the proof. Let $\epsilon > 0$. Since $k(\rho_1) = O(\rho_1)$ and $C(T,\rho_1) = O(1)$ as $\rho_1 \to 0$, we may choose ρ_1 so that $C(T,\rho_1)k(\rho_1) \le \epsilon$. Lemma 3.2 provides ρ such that $v_1, v_2 \in B_V(\bar{u}, \rho)$ implies $u_1(t), u_2(t) \in B_V(\bar{u}, \rho_1)$ for $t \in [0,T]$, and (3.10) then yields

$$\|w_1(T) - w_2(T)\|_1 \le \epsilon \|v_1 - v_2\|,$$

which implies both the required estimates. □

Lemma 3.4 shows that there is ρ such that, if we set $D = B_V(\bar{u}, \rho)$, then N satisfies (HN) with both $X = H$ and $X = V$. Moreover, we have found a larger radius $\rho_1 > \rho$ such that

$$u_0 \in D = B_V(\bar{u}, \rho) \Rightarrow S(t, u_0) \in B_V(\bar{u}, \rho_1), \quad t \in [0,T]. \tag{3.11}$$

In summary, we have chosen the parameters so as to make sure that $S = L + N$ satisfies (HL) and (HN) in both $X = H$ and $X = V$.

Remark 3. Note that L is the linearization of S at $\bar{u}$. In fact, the mapping $S : D \subset V \to V$ is Fréchet differentiable with $L = S'(\bar{u}) \in \mathcal{L}(V)$. However, the mapping $S : D \subset H \to H$ is not differentiable, because $D = B_V(\bar{u}, \rho)$ is not a neighborhood of $\bar{u}$ with respect to the topology of H.

3.2. **The discrete problem.** In this section we first discretize the initial-boundary value problem (3.1) with respect to the spatial variables by means of a standard piecewise linear finite element method and apply the shadowing results of §2.2. At the end of the section we then briefly discuss completely discrete approximations obtained by means of the backward Euler time-stepping.

Let $\{V_h\}_{0<h<1}$ be a family of finite dimensional subspaces of V, where each V_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to a triangulation of Ω with maximal mesh size h, see [5]. The approximate solution $u_h(t) \in V_h$ of (3.1) is defined by

$$(3.12)\qquad \begin{aligned} (u_h', \chi) + (D\nabla u_h, \nabla \chi) &= (\tilde{f}(u_h), \chi), \quad \forall \chi \in V_h,\ t > 0, \\ u_h(0) &= u_{0h} \end{aligned},$$

where $= u_{0h} \in V_h$ is an approximation of u_0.

We want to set this problem in the framework of §2.2 with $X_h = V_h$, and both $X = H$ and $X = V$. Let $Q_h : H \to V_h$ be the orthogonal projection. Then Q_h satisfies (2.14) (with $\delta = 1$) if $X = H$. In order to satisfy (2.14) with $X = V$ we assume that Q_h is bounded (uniformly in h) with respect to the H^1 norm. It is easy to see that this is true if the spaces V_h satisfy an inverse assumption. For a more general discussion of the H^1 boundedness of Q_h we refer to [7].

It is well known that (in view of standard interpolation error bounds and the L_2 and H^1 boundedness of Q_h)

$$(3.13)\qquad \|Q_h v - v\|_m \le C h^{2-m} \|v\|_2, \quad v \in D(A),\ m = 0, 1.$$

Since $D(A)$ is dense in H it follows that (2.15) holds with $X = H$ and $X = V$.

Introducing the linear operator $A_h : V_h \to V_h$ defined by

$$(A_h \psi, \chi) = (D\nabla \psi, \nabla \chi), \quad \forall \psi, \chi \in V_h,$$

and with $f : V \to H$ defined as before, we may write (3.12) as

$$(3.14)\qquad u_h' + A_h u_h = Q_h f(u_h), \quad t > 0; \qquad u_h(0) = u_{0h},$$

which is the discrete analogue of (3.2). In the same way as for the continuous problem we can show that there is a local solution operator $S_h(t, \cdot)$, such that $u_h(t) = S_h(t, u_{0h})$ is the unique local solution of (3.14). Just as in the continuous case the proof is based on the variation of constants formula, the analyticity of the semigroup $\exp(-tA_h)$, and the local Lipschitz condition for the mapping $f : V \to H$, see [11].

With $\mathcal{A}_h = A_h - Q_h B$ and $F(v) = f(v) - Bv$ as before, we rewrite (3.14) as

$$u_h' + \mathcal{A}_h u_h = Q_h F(u_h), \quad t > 0; \qquad u_h(0) = u_{0h},$$

which is the discrete version (3.6). Since A_h is selfadjoint, positive definite (uniformly in h), and $Q_h B$ is bounded, we deduce that $\mathcal{A}_h$ is sectorial (uniformly in h), so that for some $c > 0$

$$(3.15)\qquad \begin{aligned} \|e^{-t\mathcal{A}_h} v\|_1 &\le C e^{ct} \|v\|_1, &\quad t \ge 0,\ v \in V_h, \\ \|e^{-t\mathcal{A}_h} v\|_1 &\le C t^{-1/2} e^{ct} \|v\|, &\quad t > 0,\ v \in V_h, \end{aligned}$$

which are discrete versions of (3.5). Here we have employed the equivalence of norms $\|v\|_1 \approx \|A_h^{1/2} v\|$ for $v \in V_h$. The inequalities (3.15) can also be proved by noting that $u_h(t) = e^{-t\mathcal{A}_h} u_{0h}$ satisfies (3.14) with $f(u_h)$ replaced by Bu_h, and by making estimations based on the variation of constants formula.

From [11] we quote the following *a priori* error estimates.

Lemma 3.5. *Let $0 < \tau \le T$ and assume that $S(t, u_0), S_h(t, u_{0h}) \in B_V(\bar{u}, \rho)$ for $t \in [0, \tau]$. Then, for $t \in [0, \tau]$, we have*

$$\|S_h(t, u_{0h}) - S(t, u_0)\| \le C(\rho, T)\Big(\|u_{0h} - Q_h u_0\| + h^2 t^{-1/2}\Big),$$

$$\|S_h(t, u_{0h}) - S(t, u_0)\|_1 \le C(\rho, T) t^{-1/2}\Big(\|u_{0h} - u_0\| + h\Big).$$

The following is a discrete analogue of Lemma 3.2.

Lemma 3.6. *For each $\rho_1 > 0$, $T > 0$ there are $\rho > 0$, $h_0 > 0$ such that, if $u_{0h} \in B_V(\bar u, \rho) \cap V_h$ and $h < h_0$, then $S_h(t, u_{0h})$ exists and belongs to $B_V(\bar u, \rho_1) \cap V_h$ for $t \in [0, T]$.*

Proof. Let $\rho_1, T > 0$ be given. For $\rho > 0$ let $\tau \in [0, T]$ be the largest time such that $u_{0h} \in B_V(\bar u, \rho) \cap V_h$ implies that $S_h(t, u_{0h})$ exists and belongs to $B_V(\bar u, \rho_1 + 1) \cap V_h$ for $t \in [0, \tau]$. By local existence there are $t_0 > 0$ and $\rho > 0$ such that $u_{0h} \in B_V(\bar u, \rho) \cap V_h$ implies that $S_h(t, u_{0h})$ exists and belongs to $B_V(\bar u, \rho_1) \cap V_h$ for $t \in [0, t_0]$. Moreover, the second error estimate of Lemma 3.5 gives the *a priori* estimate

$$\begin{aligned}\|S_h(t, u_{0h}) - \bar u\|_1 &\le C(\rho_1, T) t^{-1/2} \Big(\|u_{0h} - \bar u\| + h \Big) \\ &\le C(\rho_1, T) t_0^{-1/2} (\rho + h_0) \le \rho_1, \quad t \in [t_0, \tau],\end{aligned}$$

provided that ρ and h_0 are sufficiently small. If $\tau < T$, then by local existence we obtain a contradiction with the maximality of τ. Hence, $S_h(t, u_0)$ is defined and belongs to $B_V(\bar u, \rho_1) \cap V_h$ for $t \in [0, T]$. □

We will also use the error bounds

$$\begin{aligned}\|e^{-t\mathcal{A}_h} Q_h v - e^{-t\mathcal{A}} v\| &\le C h^2 t^{-1} e^{\alpha t} \|v\|, \qquad t > 0,\ v \in H, \\ \|e^{-t\mathcal{A}_h} Q_h v - e^{-t\mathcal{A}} v\|_1 &\le C h t^{-1/2} e^{\alpha t} \|v\|_1, \quad t > 0,\ v \in V,\end{aligned} \tag{3.16}$$

which can be proved by the using the techniques of [11].

With T as in (3.9) we define $L_h = e^{-T\mathcal{A}_h}$, and (3.16) shows

$$\|L_h Q_h - L\|_{\mathcal{L}(H)} + h \|L_h Q_h - L\|_{\mathcal{L}(V)} \le C(T) h^2.$$

The assumption (2.29) is thus satisfied with both $X = H$ and $X = V$. We conclude that Theorem 2.6 applies, showing that, for small h, L_h satisfies (HL) with slightly larger constants $\tilde\kappa > \kappa$, $\tilde\mu > \mu$. Adjusting κ, μ, we may conclude that (2.19), (2.20) hold.

Finally, we define $S_h = S_h(T, \cdot)$, $N_h = S_h - L_h$, and note that, after these preparations, the analogue of Lemma 3.4 holds with the same proof. As for the continuous problem we may select ρ, h_0 such that, for $h < h_0$, N_h satisfies (HN) with $D_h = D \cap V_h$, $D = B_V(\bar u, \rho)$. The argument also selects ρ_1 such that, in analogy with (3.11),

$$u_{0h} \in D_h \Rightarrow S_h(t, u_{0h}) \in B_V(\bar u, \rho_1), \quad t \in [0, T]. \tag{3.17}$$

Moreover, using Lemma 3.5 together with (3.11) and (3.17), we see that, if $v \in D$, $Q_h v \in D_h$, then

$$\|S_h(Q_h v) - S(v)\| \le C(\rho_1, T) h^2, \quad \|S_h(Q_h v) - S(v)\|_1 \le C(\rho_1, T) h,$$

since $\|Q_h v - v\| \le C h \|v\|_1 \le C h \rho$. We conclude that (2.17) holds with $X = H$ and $\epsilon(h) = C h^2$, and with $X = V$ and $\epsilon(h) = C h$.

We have now checked all the assumptions of Theorem 2.5 and we are ready to apply it.

Theorem 3.7. *There are positive numbers ρ_0, h_0, and C such that, for any $h < h_0$, the following hold:*

(1) If u_h is a solution of (3.12) with $u_h(t) \in B_V(\bar{u}, \rho_0)$ for $t \in [0, \mathcal{T}]$, then there is a solution u of (3.1) such that

$$\|u_h(t) - u(t)\|_m \leq C\big(1 + t^{-1/2}\big)h^{2-m}, \quad t \in (0, \mathcal{T}],\ m = 0, 1. \tag{3.18}$$

(2) Conversely, if u is a solution of (3.1) with $u(t) \in B_V(\bar{u}, \rho_0)$ for $t \in [0, \mathcal{T}]$, then there is a solution u_h of (3.12) such that (3.18) holds.

(3) Equation (3.12) has a stationary solution $\bar{u}_h$ such that

$$\|\bar{u}_h - \bar{u}\|_1 \leq Ch.$$

Proof. Let ρ, ρ_1, h_0, T be as above. Choose ρ_0 and adjust h_0 in such a way that the requirements of parts (ii.a) and (ii.b) of Theorem 2.5 are satisfied with $X = V$.

(1) Let $u_h(t) \in B_V(\bar{u}, \rho_0)$ for $t \in [0, \mathcal{T}]$ and apply part (ii.a) with $X = V$ to the sequence $u_h(nT)$, $nT \in [0, \mathcal{T}]$, which is an orbit of S_h. This gives the existence of an orbit $u(nT)$, $nT \in [0, \mathcal{T}]$, of S, satisfying (2.21) of part (i.a), and hence (2.22) gives the special case $m = 1$ of the inequality

$$\|u_h(nT) - u(nT)\|_m \leq Ch^{2-m}, \quad nT \in [0, \mathcal{T}],\ m = 0, 1. \tag{3.19}$$

Another application of part (i.a), now with $X = H$, proves the case $m = 0$ of (3.19).

From the sequence $u(nT)$ we define $u(t) = S(t - nT, u(nT))$ for $t \in [nT, (n+1)T]$. By uniqueness of solutions this is a solution of (3.1). Error bounds at intermediate times are obtained by combining (3.19) with Lemma 3.5 as follows. For $t \in [0, T]$ we have

$$\|u_h(t) - u(t)\|_1 \leq C(\rho_1, T)t^{-1/2}\Big(\|u_h(0) - u(0)\| + h\Big) \leq C(\rho_1, T)t^{-1/2}h.$$

For $t \in [(n+1)T, (n+2)T]$, $n \geq 0$, we have

$$\|u_h(t) - u(t)\|_1 \leq C(\rho_1, 2T)t^{-1/2}\Big(\|u_h(nT) - u(nT)\| + h\Big) \leq C(\rho_1, T)h.$$

This proves the special case $m = 1$ of (3.18). The case $m = 0$ is obtained similarly.

(2) Let $u(t) \in B_V(\bar{u}, \rho_0)$ for $t \in [0, \mathcal{T}]$ and apply part (ii.b) with $X = V$ to the sequence $u(nT)$, $nT \in [0, \mathcal{T}]$, which is an orbit of S. This gives the existence of an orbit $u_h(nT)$, $nT \in [0, \mathcal{T}]$, of S_h, satisfying (2.24) of part (i.b), and hence (2.25) gives the special case $m = 1$ of the inequality

$$\|Q_h u(nT) - u_h(nT)\|_m \leq Ch^{2-m}, \quad nT \in [0, \mathcal{T}],\ m = 0, 1. \tag{3.20}$$

Another application of part (i.b), now with $X = H$, proves the case $m = 0$ of (3.20). The required error bound (3.18) now follows as in part (1) above, noting that, by (3.13) and Lemma 3.3,

$$\|Q_h u(t) - u(t)\|_m \leq Ch^{2-m}\|u(t)\|_2 \leq C(\rho_1, T)h^{2-m}t^{-1/2}.$$

(3) Part (ii.c) of Theorem 2.5 gives $\bar{u}_h$ and the error bound is an immediate consequence of (2.27) and (3.13). □

We conclude this section by briefly indicating how time discretization by the backward Euler method can be incorporated into the above argument.

After discretization with constant time steps k (3.14) becomes

$$(U_j - U_{j-1})/k + A_h U_j = Q_h f(U_j), \quad t_j = jk > 0; \qquad U_0 = u_{0h}.$$

The local solution operator $S_{h,k}(t_j, u_{0h})$ is readily obtained by using the smoothing property of the corresponding linear evolution operator $E_{h,k}(t_j) = (I - kA_h)^{-j}$, and the local Lipschitz condition for $f : V \to H$, see [11]. This smoothing property carries over to the linearized operator $\mathcal{E}_{h,k}(t_j) = (I - k\mathcal{A}_h)^{-j}$ in the same way as in the semidiscrete case, see (3.15). Error bounds analogous to those of Lemma 3.5 can also be found in [11]. With these ingredients we may prove an analog of Lemma 3.6. Error bounds for $\mathcal{E}_{h,k}(t_j)$ analogous to those in (3.16) may be found in [11], and with a discrete time T suitably chosen we find that $L_{h,k} = \mathcal{E}_{h,k}(T)$ satisfies (HL). Setting $S_{h,k} = S_{h,k}(T, \cdot)$, $N_{h,k} = S_{h,k} - L_{h,k}$ we then prove an analog of Lemma 3.4. Further arguments, parallel to those above, lead to an analog of Theorem 3.7 with an error bound of the form

$$\|U_j - u(t_j)\|_m \leq C\Big(\big(1 + t_j^{-1/2}\big)h^{2-m} + \big(1 + t_j^{-(m+1)/2}\big)k\Big), \quad t_j \in (0, \mathcal{T}],\ m = 0, 1.$$

Remark 4. The framework of §2 applies also in the context of a finite element method for the Cahn-Hilliard equation, for which the finite time analysis was carried out in [8]. We skip the details.

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