BACKWARD EULER TYPE METHODS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract. Time discretization by backward Euler type methods for a parabolic equation with memory is studied. Stability and error estimates are proved under conditions that permit quadrature rules for approximation of the memory term that have reduced storage requirements. The analysis takes place in a Banach space framework, and the results are used to derive error estimates in the L_2 and maximum norms for piecewise linear finite element discretization in two space dimensions.

Résumé. On étudie la discrétisation en temps d'une équation parabolique avec mémoire par des méthodes de type Euler rétrograde. On montre la stabilité et on donne des estimations d'erreur sous des hypothèses qui permettent d'utiliser des formules de quadrature peu exigeantes en stockage pour l'approximation du terme de mémoire. L'analyse est effectuée dans le cadre des espaces de Banach. Appliqués en dimension deux, ces résultats permettent d'obtenir des estimations d'erreur L_2 et uniforme pour une discrétisation utilisant des éléments finis linéaires par morceaux.

1. Introduction. We consider the initial value problem

$$u_t + Au = \int_0^t B(t,s)u(s)\,ds + f(t), \quad \text{for } t \in [0,T], \quad \text{with } u(0) = v, \tag{1.1}$$

in a Banach space X, where A is a closed linear operator with dense domain $D(A)$, and $B(t,s)$ is a smooth linear operator with $D(B(t,s)) \supset D(A)$ and such that $Q(t,s) = A^{-1}B(t,s)$ and $Q_t(t,s)$ are uniformly bounded for $0 \le s \le t \le T$.

We assume that $-A$ generates a bounded analytic semigroup $E(t) = e^{-tA}$, so that

$$\|E(t)\| + t\|AE(t)\| \le M, \quad \text{for } t > 0. \tag{1.2}$$

It then follows, by Gronwall's lemma (see Theorem 2.1 below), that for the solution of (1.1),

$$\|u(t)\| \le e^{C(T)M} M\Big(\|v\| + \int_0^t \|f\|\,ds\Big), \quad \text{for } t \le T. \tag{1.3}$$

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We shall consider the time discretization of (1.1). Let k be a time step, set $t_n = nk$, $n = 0, 1, 2, \ldots$, and denote by U^n the approximation of $u(t_n)$. We shall replace the time derivative by the backward difference quotient $\bar{\partial} U^n = (U^n - U^{n-1})/k$ and approximate the memory term by a quadrature formula

$$\sigma^n(\varphi) = \sum_{j=0}^{n-1} \omega_{nj} \varphi^j \approx \int_0^{t_n} \varphi(t)\, dt, \quad \text{where } \varphi^j = \varphi(t_j). \tag{1.4}$$

The backward Euler discretization of (1.1) is then

$$\begin{aligned} \bar{\partial} U^n + AU^n &= \sum_{j=0}^{n-1} \omega_{nj} B(t_n, t_j) U^j + f(t_n) \\ &= \sigma^n(B_n U) + f^n, \quad \text{for } n \geq 1, \quad \text{with } U^0 = v. \end{aligned} \tag{1.5}$$

Our aim is to extend stability properties such as (1.3) to this discrete problem, and to use these to derive error estimates. In doing so we need to make the following more specific assumptions about the choice of the quadrature formula in (1.4), namely, for some positive integer p,

$$|\epsilon^n(\varphi)| \leq Ck \int_0^{t_n} \sum_{l=1}^{p} |\varphi^{(l)}(t)|\, dt, \quad \text{where } \epsilon^n(\varphi) = \sigma^n(\varphi) - \int_0^{t_n} \varphi(t)\, dt, \tag{1.6}$$

and, for some positive number q,

$$\sum_{s=0}^{j-1} |\omega_{js} - \omega_{ns}| \leq Ck^q, \quad \text{for } 0 < t_j \leq t_n \leq T. \tag{1.7}$$

Under these hypotheses we shall show (Theorem 2.3), that, for small k,

$$\|U^n\| \leq Ce^{C(T)M} M \Big(\|v\| + k \sum_{j=1}^{n} \|f^j\| \Big), \quad \text{for } t_n \leq T. \tag{1.8}$$

One example of a quadrature formula satisfying (1.6) (with $p = 1$) and (1.7) (trivially) is the left-side rectangle rule, corresponding to $\omega_{nj} = k$ for $j < n$. Since this rule requires the storage of all previous U^j, sparse quadrature rules have been proposed, e.g., in [4], [6]. A short discussion of such rules is given in Section 5 below, where it is shown that our present assumptions on σ^n are different from those made in the earlier work and do not require so called "dominated weights".

Assume that we want to apply the above result to the case when $X = C_0(\bar{\Omega})$ equipped with the maximum norm, where Ω is a smooth domain in $\mathbf{R}^2$, and where A is a discrete analogue of the Laplacian $-\Delta$ based on piecewise linear finite element spaces S_h defined by a family of quasi-uniform triangulations and where $B(t, s)$ is appropriate. In this case it is known, cf. [3], that (1.2) holds with $M = C \log(1/h)$. This means that (1.3) and (1.8) contain the stability factor $e^{C \log(1/h)} = h^{-C}$, which is unbounded as h tends to 0, and these stability estimates are therefore of little value.

In order to find a remedy for this we shall assume that, in addition to (1.2), we also have

$$\|E(t)\| + t\|AE(t)\| \le M_\delta t^{-\delta}, \quad \text{for } t > 0, \text{ for any } \delta \in (0,1). \tag{1.9}$$

This is the case in the above finite element application with M_δ independent of h, see Lemma 4.1.

Under these assumptions we shall show that, for the solution of (1.1) (Theorem 2.2),

$$\|u(t)\| \le C(T, M_\delta, \delta) M \Big(\|v\| + \int_0^t \|f\|\, ds \Big), \quad \text{for } t \le T, \tag{1.10}$$

and, for the solution of (1.5) (Theorem 2.4), if δ and k are sufficiently small,

$$\|U^n\| \le C(T, M_\delta, \delta) M \Big(\|v\| + k \sum_{j=1}^{n} \|f^j\| \Big), \quad \text{for } t_n \le T. \tag{1.11}$$

In the above finite element application the bound now contains a single factor $\log(1/h)$.

We note that, in the case of one space dimension, (1.2) holds in maximum norm for finite elements of any order with a constant independent of h, cf. [1], so that (1.8) shows a uniform bound for T bounded.

The above stability estimates are proved in Section 2 below, and in Section 3 we give corresponding error estimates. In Section 4 we discuss the application to the finite element case in more detail, and Section 5 is concerned with sparse quadrature rules.

2. Stability estimates.

We begin with the basic stability result (1.3) for the continuous equation.

Theorem 2.1. *Assume that (1.2) holds. Then, for the solution of (1.1),*

$$\|u(t)\| \le e^{C(T)M} M \Big(\|v\| + \int_0^t \|f\|\, ds \Big), \quad \textit{for } t \le T.$$

Proof. Using Duhamel's principle, we have

$$\begin{aligned} u(t) &= E(t)v + \int_0^t E(t-s)f(s)\, ds + \int_0^t E(t-y) \int_0^y B(y,s)u(s)\, ds\, dy \\ &= F(t) + \int_0^t G(t,s)u(s)\, ds, \end{aligned} \tag{2.1}$$

where $F(t) = E(t)v + \int_0^t E(t-s)f(s)\, ds$, and, since $AE(t) = -E'(t)$,

$$\begin{aligned} G(t,s) &= \int_s^t AE(t-y)Q(y,s)\, dy \\ &= \big(I - E(t-s)\big)Q(t,s) + \int_s^t AE(t-y)(Q(y,s) - Q(t,s))\, dy, \end{aligned}$$

where $Q(t,s) = A^{-1}B(t,s)$. Since by our assumption

$$\|Q(t,s)\| + \|Q_t(t,s)\| \le C, \quad 0 \le s \le t \le T, \tag{2.2}$$

we may use (1.2) to conclude that

$$\|G(t,s)\| \le C\big(1 + \|E(t-s)\|\big) + C\int_s^t (t-y)\|AE(t-y)\|\,dy \le C(T)M, \tag{2.3}$$

and we hence obtain from (2.1)

$$\|u(t)\| \le M\Big(\|v\| + \int_0^t \|f\|\,ds\Big) + C(T)M\int_0^t \|u(s)\|\,ds, \quad \text{for } t \le T,$$

from which the desired result follows by Gronwall's lemma. □

We now turn to the modified stability estimate (1.10) for the continuous equation.

Theorem 2.2. *Assume that (1.2) and (1.9) hold. Then for the solution of (1.1) we have, for any $\delta \in (0,1)$,*

$$\|u(t)\| \le C(T, M_\delta, \delta)M\Big(\|v\| + \int_0^t \|f\|\,ds\Big), \quad \textit{for } t \le T.$$

Proof. This time we use (1.9) in the first inequality of (2.3) to obtain

$$\|G(t,s)\| \le C + CM_\delta(t-s)^{-\delta} + CM_\delta\int_s^t (t-y)^{-\delta}\,dy \le C(T,M_\delta)(t-s)^{-\delta}.$$

Hence, by (2.1),

$$\|u(t)\| \le M\Big(\|v\| + \int_0^t \|f\|\,ds\Big) + C(T,M_\delta)\int_0^t (t-s)^{-\delta}\|u(s)\|\,ds, \tag{2.4}$$

and our result follows by a variant of Gronwall's lemma (cf. [2, Lemma 5.6.7]; it also follows easily from the time-continuous version of Lemma 2.2 below). □

We note that under assumption (1.9) one may also show for the solution of (1.1)

$$\|u(t)\| \le C(T, M_\delta, \delta)\Big(t^{-\delta}\|v\| + \int_0^t (t-s)^{-\delta}\|f(s)\|\,ds\Big), \quad \text{for } t \le T. \tag{2.5}$$

In fact, instead of (2.4) one has, with $C = C(M_\delta)$,

$$\|u(t)\| \le C\Big(t^{-\delta}\|v\| + \int_0^t (t-s)^{-\delta}\|f\|\,ds\Big) + C\int_0^t (t-s)^{-\delta}\|u(s)\|\,ds,$$

from which (2.5) follows by a variant of Gronwall's lemma.

In order to prove the discrete analogues (1.8) and (1.11) we introduce the backward Euler one step evolution operator $E_k = (I+kA)^{-1}$ and first show the following discrete analogues of (1.2) and (1.9) (with the former contained as a special case with $M_0 = M$).

Lemma 2.1. *Assume that, for some $\delta \in [0,1)$,*

$$\|E(t)\| + t\|AE(t)\| \le M_\delta t^{-\delta}, \quad \text{for } t > 0.$$

Then

$$\|E_k^n\| + t_n\|AE_k^n\| \le CM_\delta t_n^{-\delta}, \quad \text{for } t_n > 0.$$

Proof. We have, cf., e.g., [2, p. 21],

$$E_k^n = (I + kA)^{-n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-t}E(kt)\,dt, \quad \text{for } n \ge 1.$$

Hence, by our assumption on $E(t)$, and since $\Gamma(n-\delta)/\Gamma(n) \le Cn^{-\delta}$,

$$\|E_k^n\| \le M_\delta \frac{k^{-\delta}}{(n-1)!}\int_0^\infty t^{n-1-\delta}e^{-t}\,dt = M_\delta k^{-\delta}\frac{\Gamma(n-\delta)}{\Gamma(n)} \le CM_\delta t_n^{-\delta}.$$

The estimation of AE_k^n is similar for $n \ge 2$; for $n = 1$ we use $kAE_k = I - E_k$. □

Our proofs in the discrete case will depend on the following version of Gronwall's lemma. We remark that, in the earlier work [4], [6], instead of (1.7) the analysis was based on the assumption that the weights ω_{ns} are "dominated" in the sense that there are weights ω_s, independent of n and such that $\omega_{ns} \le \omega_s$ for $0 \le t_s < t_n \le T$ with $\sum_{s=0}^{n-1}\omega_s \le C$, which makes it possible to apply a more standard Gronwall lemma.

Lemma 2.2. *Let u_n, $n = 0, 1, \ldots,$ be nonnegative numbers such that*

$$u_n \le K + \sum_{s=0}^{n-1}\mu_{ns}u_s, \quad \text{for } t_n = nk \ge 0. \tag{2.6}$$

Assume that the coefficients $\mu_{ns} = \mu_{ns}(k)$ are nonnegative and that there exist positive numbers γ, τ and k_0, independent of k, such that

$$\sum_{s=m}^{l-1}\mu_{ns} \le \gamma < 1, \quad \text{for } t_{l-1} - t_m \le \tau,\ 0 \le t_m < t_l \le t_n,\ k \le k_0. \tag{2.7}$$

Then, with $C = \log\big((1-\gamma)^{-1}\big)$,

$$u_n \le Ke^{C(t_n/\tau+1)}, \quad \text{for } t_n \ge 0,\ k \le k_0. \tag{2.8}$$

Proof. Let $I_j = [(j-1)\tau, j\tau]$. Then, by (2.6) and (2.7),

$$\max_{t_n\in I_j} u_n \le K + \gamma\sum_{l=1}^{j}\max_{t_n\in I_l} u_n,$$

from which we easily conclude that $\max_{t_n\in I_j} u_n \le K(1-\gamma)^{-j}$, which implies (2.8) since $j \le t_n/\tau + 1$ for $t_n \in I_j$. □

In the following we let T be fixed (but arbitrary) and we let C denote various constants that may depend on T.

We now show the following stability result for (1.5). For the purpose of our error estimates below it is phrased in a more general way than (1.8), which latter is contained for $g^j = 0$.

Theorem 2.3. *Assume (1.2) and let $\{g^j\} \subset X$ be arbitrary. Then we have, for the solution of (1.5), for $t_n \le T,\ k \le k_0(M)$,*

$$\|U^n\| \le Ce^{CM} M\Big(\|v\| + k\sum_{j=1}^{n} \|f^j - g^j\| + \|A^{-1}g^1\| + k\sum_{j=2}^{n} \|A^{-1}\bar\partial g^j\|\Big).$$

Proof. The proof is modeled on the proof of Theorem 2.1. We have by Duhamel's principle and a change of the order of summation in the double sum, cf. (2.1),

$$U^n = E_k^n v + k\sum_{j=1}^{n} E_k^{n-j+1} f^j + k\sum_{j=1}^{n} E_k^{n-j+1} \sum_{s=0}^{j-1} \omega_{js} B_{js} U^s = F^n + \sum_{s=0}^{n-1} G_{ns} U^s, \tag{2.9}$$

where, since $kAE_k^n = E_k^{n-1} - E_k^n$, we may write

$$F^n = E_k^n v + k\sum_{j=1}^{n} E_k^{n-j+1}(f^j - g^j) + A^{-1}g^n - E_k^n A^{-1} g^1 - k\sum_{j=2}^{n} E_k^{n-j+1} A^{-1}\bar\partial g^j,$$

and, with $Q_{js} = A^{-1}B_{js}$, we have $G_{ns} = k\sum_{j=s+1}^{n} AE_k^{n-j+1}\omega_{js}Q_{js}$. Here, by (1.2) and Lemma 2.1, for small k and $t_n \le T$,

$$\begin{aligned}\|F^n\| &\le CM\Big(\|v\| + k\sum_{j=1}^{n} \|f^j - g^j\| + \max_{j=1,n} \|A^{-1}g^j\| + k\sum_{j=2}^{n} \|A^{-1}\bar\partial g^j\| \\ &\le CM\Big(\|v\| + k\sum_{j=1}^{n} \|f^j - g^j\| + \|A^{-1}g^1\| + k\sum_{j=2}^{n} \|A^{-1}\bar\partial g^j\|\Big).\end{aligned} \tag{2.10}$$

In view of Lemma 2.2, the result therefore follows once we have shown that, for small k,

$$\sum_{s=m}^{l-1} \|G_{ns}\| \le CM\Big(t_{l-m-1} + k^q \log\frac{1}{k}\Big), \quad \text{for } 0 \le t_m < t_l \le t_n \le T, \tag{2.11}$$

so that (2.7) holds for $\mu_{ns} = \|G_{ns}\|$, with $\gamma = \frac{1}{2}$, $\tau = (4CM)^{-1}$ and $k_0 = k_0(M)$.

In order to prove (2.11) we write

$$\begin{aligned}G_{ns} &= k\sum_{j=s+1}^{n} AE_k^{n-j+1}\omega_{ns}Q_{ns} + k\sum_{j=s+1}^{n} AE_k^{n-j+1}\omega_{ns}(Q_{js} - Q_{ns}) \\ &\quad + k\sum_{j=s+1}^{n} AE_k^{n-j+1}(\omega_{js} - \omega_{ns})Q_{js} \equiv G_{ns}^1 + G_{ns}^2 + G_{ns}^3.\end{aligned}$$

Here $G_{ns}^1 = (I - E_k^{n-s})\omega_{ns}Q_{ns}$, so that by (1.2) and Lemma 2.1 (cf. (2.3))

$$\|G_{ns}^1 + G_{ns}^2\| \le C\omega_{ns}\Big(1 + k\sum_{j=s+1}^{n} t_{n-j}\|AE_k^{n-j+1}\|\Big) \le CM\omega_{ns}.$$

Also

$$\|G_{ns}^3\| \le CMk \sum_{j=s+1}^{n} t_{n-j+1}^{-1} |\omega_{js} - \omega_{ns}|.$$

We shall show

$$\sum_{s=m}^{l-1} \omega_{ns} \le t_{l-m} + Ck^q, \tag{2.12}$$

and, for small k and $t_n \le T$,

$$k \sum_{s=m}^{l-1} \sum_{j=s+1}^{n} t_{n-j+1}^{-1} |\omega_{js} - \omega_{ns}| \le Ck^q \log \frac{1}{k}, \tag{2.13}$$

which together show (2.11). In order to prove (2.12) we note that the quadrature formula is exact for constants (cf. (1.6)) and use (1.7) to get

$$\sum_{s=m}^{l-1} \omega_{ns} = \sum_{s=0}^{l-1} \omega_{ls} - \sum_{s=0}^{m-1} \omega_{ms} - \sum_{s=0}^{l-1} (\omega_{ls} - \omega_{ns}) + \sum_{s=0}^{m-1} (\omega_{ms} - \omega_{ns}) \le t_{l-m} + Ck^q.$$

For the proof of (2.13) we use (1.7) as follows:

$$\begin{aligned} k \sum_{s=m}^{l-1} \sum_{j=s+1}^{n} t_{n-j+1}^{-1} |\omega_{js} - \omega_{ns}| &= k \sum_{j=m+1}^{n} t_{n-j+1}^{-1} \sum_{s=m}^{\min(l-1,j-1)} |\omega_{js} - \omega_{ns}| \\ &\le Ck^{q+1} \sum_{j=m+1}^{n} t_{n-j+1}^{-1} \le Ck^q \big(1 + \log \frac{t_n - t_m}{k}\big) \le Ck^q \log \frac{t_n}{k}. \quad \Box \end{aligned}$$

The following is a modified stability result for the time stepping method.

Theorem 2.4. *Assume (1.2) and (1.9), let $\{g^j\} \subset X$ and $r \in (1, \infty]$ be arbitrary. Then we have, for the solution of (1.5), for some $\delta \in (0,1)$, for $t_n \le T,\ k \le k_0(M_\delta)$,*

$$\|U^n\| \le C(T, M_\delta, \delta) \Big\{ M \Big(\|v\| + k \sum_{j=1}^{n} \|f^j - g^j\| \Big) + M_\delta \Big(k \sum_{j=1}^{n} \|g^j\|^r \Big)^{1/r} \Big\}.$$

Proof. Again we have (2.9) and instead of (2.10) we use (1.9) and Lemma 2.1 to get, for δ sufficiently small,

$$\Big\| k \sum_{j=1}^{n} E_k^{n-j+1} g^j \Big\| \le CM_\delta k \sum_{j=1}^{n} t_{n-j+1}^{-\delta} \|g^j\| \le C(T,\delta) M_\delta \Big(k \sum_{j=1}^{n} \|g^j\|^r \Big)^{1/r}.$$

It remains to bound $\sum_{s=m}^{l-1} \|G_{ns}\|$, so that Lemma 2.2 may be applied with $\mu_{ns} = \|G_{ns}\|$ and with τ depending on M_δ instead of M. For this purpose we use (1.9) and Lemma 2.1 to get, similarly to the proof of Theorem 2.3, (this is a correction of an error in the version the paper printed in *RAIRO M^2AN*)

$$\|G^1_{ns}+G^2_{ns}\| \le C\omega_{ns}\Big(1+\|E_k^{n-s}\|+k\sum_{j=s+1}^{n} t_{n-j+1}\|AE_k^{n-j+1}\|\Big) \le CM_\delta t_{n-s}^{-\delta}\omega_{ns},$$

and

$$\|G^3_{ns}\| \le CM_\delta k\sum_{j=s+1}^{n} t_{n-j+1}^{-1-\delta}|\omega_{js}-\omega_{ns}|.$$

Estimating $\sum_{s=m}^{l-1}\|G^3_{ns}\|$ we have, uniformly in δ, (cf. the proof of (2.13))

$$\begin{aligned} k\sum_{s=m}^{l-1}\sum_{j=s+1}^{n} t_{n-j+1}^{-1-\delta}|\omega_{js}-\omega_{ns}| &\le k\sum_{j=m+1}^{n} t_{n-j+1}^{-1-\delta}\sum_{s=0}^{j-1}|\omega_{js}-\omega_{ns}| \\ &\le Ck^{q-\delta+1}\sum_{j=m+1}^{n} t_{n-j+1}^{-1} \le Ck^{q-\delta}\log\frac{1}{k}. \end{aligned}$$

In order to bound $\sum_{s=m}^{l-1}\|G^1_{ns}+G^2_{ns}\|$ in terms of t_{l-m} and k, we argue as in the proof of (2.12). We have

$$\begin{aligned} \sum_{s=m}^{l-1} t_{n-s}^{-\delta}\omega_{ns} &= \sum_{s=0}^{l-1} t_{n-s}^{-\delta}\omega_{ls} - \sum_{s=0}^{m-1} t_{n-s}^{-\delta}\omega_{ms} \\ &\quad - \sum_{s=0}^{l-1} t_{n-s}^{-\delta}(\omega_{ls}-\omega_{ns}) + \sum_{s=0}^{m-1} t_{n-s}^{-\delta}(\omega_{ms}-\omega_{ns}) \\ &= \int_{t_m}^{t_l}(t_n-s)^{-\delta}\,ds + \epsilon^l - \epsilon^m - \eta^l + \eta^m, \end{aligned}$$

where, uniformly for small δ, and using the elementary inequality $(x+y)^\gamma \le x^\gamma + y^\gamma$ for $x, y \ge 0$, $\gamma \in (0,1)$,

$$\int_{t_m}^{t_l}(t_n-s)^{-\delta}\,ds = \frac{(t_n-t_m)^{1-\delta}-(t_n-t_l)^{1-\delta}}{1-\delta} \le \frac{t_{l-m}^{1-\delta}}{1-\delta} \le Ct_{l-m}^{1-\delta}.$$

Moreover, according to (1.6) we have

$$|\epsilon^i| = |\epsilon^i((t_n-\cdot)^{-\delta})| \le Ck\int_0^{t_i}\sum_{l=0}^{p}|D_t^l(t_n-t)^{-\delta}|\,dt \le Ckt_{n-i}^{-p},$$

again uniformly for small δ, where for simplicity we have replaced δ by 1 in the final step. Also, according to (1.7),

$$|\eta^i| = \Big|\sum_{s=0}^{i-1} t_{n-s}^{-\delta}(\omega_{is}-\omega_{ns})\Big| \le k^{-\delta}\sum_{s=0}^{i-1}|\omega_{is}-\omega_{ns}| \le Ck^{q-\delta}.$$

Together these estimates show

$$\sum_{s=m}^{l-1} t_{n-s}^{-\delta}\omega_{ns} \le Ct_{l-m}^{1-\delta} + Ckt_{n-l}^{-p} + Ck^{q-\delta}. \tag{2.14}$$

Since our estimate of ϵ^l may be as large as Ck^{1-p} (when $t_l = t_{n-1}$), we have to make a refined estimation. Let $\epsilon > 0$ be arbitrary. The contribution of the terms in $\sum_{s=m}^{l-1} t_{n-s}^{-\delta}\omega_{ns}$ with $t_{n-s} \ge (k/\epsilon)^{1/p}$ is then bounded by $Ct_{l-m}^{1-\delta} + C\epsilon + Ck^{q-\delta}$ according to (2.14). On the other hand, the contribution of the terms with $t_{n-s} < (k/\epsilon)^{1/p}$ can be bounded, using (2.12), by

$$k^{-\delta} \sum_{t_{n-s}<(k/\epsilon)^{1/p}} \omega_{ns} \le k^{-\delta}\big((k/\epsilon)^{1/p} + Ck^q\big).$$

Thus, putting these bounds together we have (cf. (2.11))

$$\sum_{s=m}^{l-1} \|G_{ns}\| \le CM_\delta\Big(t_{l-m}^{1-\delta} + \epsilon + k^{-\delta+1/p}\epsilon^{-1/p} + k^{q-\delta}\log\frac{1}{k}\Big),$$

so that (2.7) holds for $\mu_{ns} = \|G_{ns}\|$, if $0 < \delta < \min(q, 1/p)$, by choosing $\epsilon = k^{(1-p\delta)/(1+p)}$, with $k \le k_0(M_\delta)$ sufficiently small, and $\tau = (4CM_\delta)^{-1/(1-\delta)}$. $\square$

In analogy with (2.5) one may also show that under assumption (1.9) we have for the solution of (1.5)

$$\|U^n\| \le C(T, M_\delta, \delta)\Big(t_n^{-\delta}\|v\| + k\sum_{j=1}^{n} t_{n-j+1}^{-\delta}\|f^j\|\Big), \quad \text{for } t_n \le T.$$

Since this will not be used below, we refrain from the details.

3. Error estimates. We show the following error estimate in our abstract framework.

Theorem 3.1. *Assume that (1.2) holds. Then we have, for the solutions of (1.5) and (1.1), with $C = C(T)$, for $t_n \le T$*

$$\|U^n - u(t_n)\| \le Ce^{CM}Mk\int_0^{t_n}\Big(\|u_{tt}\| + \sum_{l=1}^{p}\|Au^{(l)}\|\Big)\,dt.$$

If also (1.9) holds, then the result holds with the stability factor $C(T, M_\delta)M$, for some (sufficiently small) $\delta > 0$.

Proof. Set $e^n = U^n - u^n$. Then

$$\bar\partial e^n + Ae^n = \sigma(B_n e) + \tau_1^n + \tau_2^n,$$

where $\tau_1^n = -(\bar\partial u^n - u_t(t_n))$ and $\tau_2^n = \epsilon^n(B_n u)$. Here, $\|\tau_1^n\| \le \int_{t_{n-1}}^{t_n}\|u_{tt}\|\,dt$, and using (1.6),

$$\|\tau_2^n\| \le Ck\int_0^{t_n}\sum_{l=1}^{p}\|Au^{(l)}\|\,dt.$$

(Note that, by duality, the assumption (1.6) for scalar functions implies the corresponding statement for vector valued functions.) By the stability result of Theorem 2.3, we now have

$$\|e^n\| \le Ce^{CM} Mk \sum_{j=1}^{n} \big(\|\tau_1^j\| + \|\tau_2^j\|\big),$$

from which the first result follows. The proof of the second is analogous, using Theorem 2.4 instead of Theorem 2.3. □

4. Application to piecewise linear finite elements. In this section we consider the case of the initial value problem (1.1) when A is a self-adjoint positive definite elliptic operator, and $B(t,s)$ is a second order partial differential operator with smooth coefficients, in a plane convex domain Ω and with Dirichlet boundary conditions. Together with this problem we shall consider its spatial discretization in piecewise linear finite element spaces S_h. With $(\cdot,\cdot)$ the standard L_2-inner product on Ω, the spatially discrete analogue of (1.1) is to find $u_h(t) \in S_h$ for $t \ge 0$ such that

$$\begin{aligned} (u_{h,t},\chi) + A(u_h,\chi) &= \int_0^t B(t,s;u_h(s),\chi)\,ds + (f,\chi), \quad \forall \chi \in S_h,\ t\in[0,T], \\ u_h(0) &= v_h, \end{aligned} \tag{4.1}$$

where $A(\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ are the standard bilinear forms associated with A and $B(t,s)$. Introducing the discrete operators A_h and $B_h(t,s) : S_h \to S_h$ by

$$(A_h\psi,\chi) = A(\psi,\chi), \quad (B_h(t,s)\psi,\chi) = B(t,s;\psi,\chi), \quad \forall \psi,\chi \in S_h,$$

the problem (4.1) may be expressed as (1.1), with A and $B(t,s)$ replaced by their discrete analogues A_h and $B_h(t,s)$ in S_h. It is to this spatially discrete problem that we now apply the backward Euler discretization (1.5), which yields the completely discrete problem to find $U^n \in S_h$ such that, with $B_n(s;\cdot,\cdot) = B(t_n,s;\cdot,\cdot)$,

$$\begin{aligned} (\bar\partial U^n,\chi) + A(U^n,\chi) &= \sigma^n(B_n(U,\chi)) + (f^n,\chi), \quad \forall \chi\in S_h,\ n\ge 1, \\ U^0 &= v_h. \end{aligned} \tag{4.2}$$

We shall begin by considering this problem in the Hilbert space $L_2(\Omega)$. In order to apply the above theory to this problem, we recall the well known fact that for $E_h(t) = e^{-A_h t}$, we have, with respect to the L_2-norm,

$$\|E_h(t)\| + t\|A_hE_h(t)\| \le C, \quad \text{for } t>0,$$

so that (1.2) is valid. We also need to assume that, uniformly in h,

$$\|A_h^{-1}B_h(t,s)\| + \|A_h^{-1}B_{h,t}(t,s)\| + \|B_h(t,s)A_h^{-1}\| \le C \tag{4.3}$$

(cf. (2.2)). This is the case, e.g., if the triangulation underlying the definition of S_h is quasi-uniform, or if the principal part of $B(t,s)$ equals a scalar function $b(t,s)$ times the principal part of A, see [4], [5], [6].

Under these assumptions we have the following. We assume for simplicity that the discrete initial value is R_hv, where $R_h : H_0^1(\Omega) \to S_h$ denotes the Ritz projection, i.e., the orthogonal projection with respect to the inner product $A(\cdot,\cdot)$.

Theorem 4.1. *We have, for the solutions of (1.1) and (4.2) with* $v_h = R_h v$,

$$\|U^n - u(t_n)\| \le C(T,u)(h^2 + k), \quad \textit{for } t_n \le T.$$

Proof. We write

$$e^n = U^n - u^n = (U^n - R_h u^n) + (R_h u^n - u^n) = \theta^n + \rho^n.$$

It is well known that

$$\|\rho^n\| \le Ch^2 \|u^n\|_{H^2}. \tag{4.4}$$

For $\theta^n \in S_h$ we note that, with $B_{h,n}(s) = B_h(t_n, s)$,

$$\bar\partial \theta^n + A_h \theta^n = \sigma^n(B_{h,n}\theta) + \tau^n, \tag{4.5}$$

where, with ϵ^n defined in (1.6),

$$\begin{aligned}\tau^n &= -\bar\partial(R_h u^n - u^n) + (\bar\partial u^n - u_t^n) + \epsilon^n(B_{h,n} R_h u) + \int_0^{t_n} (B_{h,n} R_h - P_h B_n) u \, dt \\ &= \tau_1^n + \tau_2^n + \tau_3^n + \tau_4^n.\end{aligned}$$

Here, by standard estimates,

$$\|\tau_1^n\| + \|\tau_2^n\| \le Ch^2 \frac{1}{k} \int_{t_{n-1}}^{t_n} \|u_t^n\|_{H^2} \, dt + \int_{t_{n-1}}^{t_n} \|u_{tt}\| \, dt \le C(u)(h^2 + k).$$

Further, since $\|B_{h,n} A_h^{-1}\| \le C$ and $R_h = A_h^{-1} P_h A$,

$$\|\tau_3^n\| = \|\epsilon^n(B_{h,n} R_h u)\| \le Ck \int_0^{t_n} \sum_{l=1}^{p} \|Au^{(l)}\| \, dt \le C(u)k.$$

For τ_4^n we have, with $Z_{h,n} = B_{h,n} R_h - P_h B_n$,

$$k\bar\partial \tau_4^n = \int_0^{t_n} Z_{h,n} u \, dt - \int_0^{t_{n-1}} Z_{h,n-1} u \, dt = \int_{t_{n-1}}^{t_n} Z_{h,n} u \, dt + k \int_0^{t_{n-1}} \bar\partial Z_{h,n} u \, dt.$$

We shall show presently that

$$\|A_h^{-1} Z_{h,n} u\| + \|A_h^{-1} \bar\partial Z_{h,n} u\| \le C(u) h^2. \tag{4.6}$$

We now apply Theorem 2.3 to (4.5), with $f^n = \sum_{i=1}^4 \tau_i^n$, $g^n = \tau_4^n$, to obtain, since $\theta^0 = 0$,

$$\begin{aligned}\|\theta^n\| &\le C(T)\Big(k \sum_{j=1}^{n} (\|\tau_1^j\| + \|\tau_2^j\| + \|\tau_3^j\|) + \|A_h^{-1}\tau_4^1\| + k \sum_{j=2}^{n} \|A_h^{-1} \bar\partial \tau_4^j\|\Big) \\ &\le C(T,u)(h^2 + k).\end{aligned}$$

Together with (4.4) this completes the proof.

It remains thus to show (4.6). We have, taking the supremum over all $\chi \in S_h$ with $\|\chi\| = 1$,

$$\begin{aligned}\|A_h^{-1} Z_{h,n} u\| &= \sup_\chi (A_h^{-1} Z_{h,n} u, \chi) = \sup_\chi (A_h^{-1}(B_{h,n} R_h - P_h B_n) u, \chi) \\ &= \sup_\chi B_n(\rho, A_h^{-1}\chi) \le \sup_\chi B_n(\rho, (A_h^{-1} - A^{-1})\chi) + \sup_\chi B_n(\rho, A^{-1}\chi) \\ &\le C\|\rho\|_{H^1} h + C\|\rho\| \sup_{\chi \in S_h} \|B_n^* A^{-1}\chi\| \le C(u) h^2,\end{aligned}$$

which completes the proof of the first part. The second part follows similarly from

$$\|A_h^{-1} \bar\partial Z_{h,n} u\| \le C k^{-1} \sup_\chi \big(B_n(\rho, A_h^{-1}\chi) - B_{n-1}(\rho, A_h^{-1}\chi) \big) \le C(u) h^2. \quad \square$$

We now turn to a discussion of the above problem in the Banach space $C_0(\bar\Omega)$ and throughout the rest of this section we now use the maximum norm $\|v\| = \sup_{x \in \Omega} |v(x)|$. We note that in the general case $A^{-1}B(t,s)$ is then not a bounded operator and (4.3) cannot be expected to hold. We therefore now restrict the considerations to the case that $A = -\Delta$, $B = -b(t,s)\Delta$, where Δ is the Laplacian and $b(t,s)$ is a smooth scalar function. In this case $A(\cdot,\cdot) = (\nabla\cdot, \nabla\cdot)$ and $B(t,s;\cdot,\cdot) = b(t,s)(\nabla\cdot,\nabla\cdot)$, and the discrete analogues of A and $B(t,s)$ are defined by $A_h = -\Delta_h$, $B_h(t,s) = -b(t,s)\Delta_h$, where

$$-(\Delta_h \psi, \chi) = (\nabla\psi, \nabla\chi), \quad \forall \psi, \chi \in S_h.$$

In order to apply our abstract theory in this case, we note that $A_h^{-1}B_h(t,s) = b(t,s)I$ is bounded together with its derivatives. We also need to know to what extent the assumptions (1.2) and (1.9) are satisfied for $A_h = -\Delta_h$. We first recall from [3] that, if the family of triangulations underlying the definition of S_h is quasi-uniform, then for $E_h(t) = e^{-A_h t}$ we have, with respect to the maximum norm,

$$\|E_h(t)\| + t\|A_h E_h(t)\| \le C \log\frac{1}{h}, \quad \text{for } t > 0, \tag{4.7}$$

so that (1.2) is satisfied with $M = C\log(1/h)$. We also want to show that (1.9) is satisfied with M_δ independent of h.

Lemma 4.1. *Under the present assumptions we have, for any $\delta \in (0,1)$ and for $h \le h_\delta$,*

$$\|E_h(t)\| + t\|A_h E_h(t)\| \le C_\delta t^{-\delta}, \quad \textit{for } t > 0.$$

Proof. We use techniques from [3, Theorem 3.3]. It is easy to show that $\|E_h(t)\|$ and $\|A_h E_h(t)\|$ are bounded for $t \ge t_0 > 0$ and decay exponentially (uniformly in h) as $t \to \infty$, so it suffices to consider $0 < t \le 1$, say. By the maximum principle, $\|E(t)\| \le 1$. We shall show that, with P_h the L_2-projection onto S_h,

$$\|E_h(t)P_h v - E(t)v\| \le C_\delta t^{-\delta}\|v\|, \quad \text{for } 0 < t \le 1, \ h \le h_\delta, \tag{4.8}$$

which, applied with $v \in S_h$ shows the desired estimate for $E_h(t)$.

From [3] we quote that, for any $\epsilon > 0$,

$$\|E_h(t)P_h - E(t)\| \le C_\epsilon h^{2-3\epsilon} t^{-1+\epsilon}, \tag{4.9}$$

which implies (4.8) for any $\delta \ge \frac{1}{3}$. To consider smaller δ, we use (4.9) with $\epsilon = \frac{1}{2}$, say, together with (4.7), to obtain, for $0 < \delta \le \frac{1}{2}$ and $h \le h_\delta$,

$$\|E_h(t)P_h - E(t)\| \le \Big(C \log \frac{1}{h}\Big)^{1-2\delta} \Big(Ch^{1/2}t^{-1/2}\Big)^{2\delta} \le C_\delta t^{-\delta}.$$

We now turn to the estimate for $A_h E_h(t) = -E_h'(t)$. As before the statement is valid for the continuous analogue $E'(t)$, so it suffices to show

$$t\|E_h'(t)P_h v - E'(t)v\| \le C_\delta t^{-\delta}\|v\|.$$

With $u_h(t) = E_h(t)P_h v$ and $u(t) = E(t)v$ we write

$$u_h(t) - u(t) = \big(u_h(t) - P_h u(t)\big) + \big(P_h u(t) - u(t)\big) = \eta + \zeta.$$

Here, since P_h is bounded in maximum norm,

$$t\|\zeta_t(t)\| = t\|P_h u_t(t) - u_t(t)\| \le Ct\|u_t(t)\| \le C\|v\| \le C_\delta t^{-\delta}\|v\|.$$

For η we note, with $\rho = R_h u - u$,

$$\eta_t + A_h \eta = A_h(R_h - P_h)u = A_h P_h \rho.$$

Differentiating and setting $\omega = t\eta_t$ we have

$$\omega_t + A_h \omega = A_h P_h(t\rho_t) + \eta_t,$$

so that, since $\omega(0) = 0$ and $\eta(0) = 0$,

$$\begin{aligned}\omega(t) &= \int_0^t E_h(t-s)\big(A_h P_h(s\rho_t(s)) + \eta_t(s)\big)\,ds \\ &= \eta(t) + \int_0^t A_h E_h(t-s)\big(P_h(s\rho_t(s)) - \eta(s)\big)\,ds.\end{aligned}$$

We recall from [3] that, since $(\alpha+\beta)^{-1} \le \alpha^{-1+\gamma}\beta^{-\gamma}$, for $\alpha, \beta > 0$, $0 < \gamma < 1$, we have

$$\|A_h E_h(t)\| \le C \log \frac{1}{h}(t + h^2)^{-1} \le Ch^{-2(1-\gamma)} \log \frac{1}{h} t^{-\gamma}.$$

Further, using the stability of P_h, the logarithmic stability of R_h, and the analyticity of $E(t)$ in $L_p(\Omega)$, we have for a suitable $\nu < \infty$,

$$\begin{aligned}s\|P_h \rho_t(s)\| &\le Cs\|\rho_t(s)\| \le C \log \frac{1}{h} s \inf_{\chi \in S_h} \|u_t(s) - \chi\| \\ &\le C \log \frac{1}{h} h^{2-3\epsilon} s\|u_t(s)\|_{W_\infty^{2-3\epsilon}} \le C \log \frac{1}{h} h^{2-3\epsilon} s\|u_t(s)\|_{W_\nu^{2-2\epsilon}} \\ &\le C \log \frac{1}{h} h^{2-3\epsilon} s^{-1+\epsilon}\|v\|_{L_\nu} \le C \log \frac{1}{h} h^{2-3\epsilon} s^{-1+\epsilon}\|v\|.\end{aligned}$$

It is easy to show $\eta(t) \le C_\delta t^{-\delta}$ and, moreover, from [3, (3.17)] we have

$$\|\eta(t)\| \le Ch^{2-3\epsilon}s^{-1+\epsilon}\|v\|,$$

and we conclude

$$\begin{aligned}\|\omega(t)\| &\le C_\delta t^{-\delta}\|v\| + Ch^{2\gamma-3\epsilon}\Big(\log \frac{1}{h}\Big)^2 \int_0^t (t-s)^{-\gamma}s^{-1+\epsilon}\,ds\|v\| \\ &= C_\delta t^{-\delta}\|v\| + Ch^{2\gamma-4\epsilon}t^{\epsilon-\gamma}\|v\| \le C_\delta t^{-\delta}\|v\|,\end{aligned}$$

by the choice $\epsilon = \delta$, $\gamma = 2\delta$. This completes the proof. $\square$

We now show the analogue of Theorem 4.1 in the case of the maximum norm.

Theorem 4.2. *With respect to the maximum norm we have under the above assumptions, for the solutions of (1.1) and (4.2) with $v_h = R_h v$, for $h \le h_0$, $k \le k_0$,*

$$\|U^n - u(t_n)\| \le C(T,u) \log \frac{1}{h} \Big(h^2 + k \Big), \quad \text{for } t_n \le T.$$

Proof. We follow the lines of the proof of Theorem 4.1. In maximum norm we have

$$\|\rho^n\| \le Ch^2 \log \frac{1}{h} \|u^n\|_{W^2_\infty}.$$

Similarly to above we now get

$$\|\tau_1^n\| \le C(u) h^2 \log \frac{1}{h}, \quad \|\tau_2^n\| \le C(u)k.$$

Further, since $\Delta_h R_h = P_h \Delta$,

$$\|\tau_3^n\| = \|\epsilon^n(b_n \Delta_h R_h u)\| \le Ck \int_0^{t_n} \sum_{l=1}^{p} \|\Delta u^{(l)}\| \, dt \le C(u)k.$$

In this case $Z_{h,n} = -b(t_n, \cdot)(\Delta_h R_h - P_h \Delta) = 0$ and hence $\tau_4^n = 0$. Theorem 2.4 with $f^n = \sum_{i=1}^4 \tau_i^n$, $g^n = \tau_1^n$, therefore shows (recall that now $M = C \log 1/h$)

$$\|\theta^n\| \le C(T,u) \log \frac{1}{h} \Big(h^2 + k \Big). \quad \square$$

5. Quadrature rules. We now give some examples of quadrature rules satisfying our assumptions. As mentioned in the introduction, the most obvious choice is the rectangle rule, which corresponds to taking all $\omega_{ns} = k$, for $s < n$. Clearly then (1.6) holds with $p = 1$ and the sum in (1.7) vanishes.

A drawback of this method is that all the previously computed values of the solution enter into the equation (1.5), so that all of these have to be stored for future use. Following the philosophy of [4] and [6] we shall now turn to some sparse rules, that reduce the storage requirement.

We begin with a quadrature rule based on the trapezoidal rule on intervals of lengths $O(k^{1/2})$, with a slight modification near t_n. Let $m = [k^{-1/2}]$, set $k_1 = mk$ and $\bar t_j = jk_1$, and let j_n be the largest integer with $\bar t_{j_n} < t_n$. For the interval $(0, t_n)$ we then apply the composite trapezoidal rule with stepsize k_1 on $(0, \bar t_{j_n})$, then the one-interval trapezoidal rule on $(\bar t_{j_n}, t_{n-1})$, and finally the left side rectangle rule on (t_{n-1}, t_n). Thus

$$\sigma^n(\varphi) = \frac{k_1}{2} \sum_{j=1}^{j_n} (\varphi(\bar t_j) + \varphi(\bar t_{j-1})) + \frac{1}{2}(t_{n-1} - \bar t_{j_n})(\varphi(t_{n-1}) + \varphi(\bar t_{j_n})) + k\varphi(t_{n-1}).$$

Since the rule is second order in k_1 over $(0, \bar t_{j_n})$ and $(\bar t_{j_n}, t_{n-1})$, and first order on (t_{n-1}, t_n), (1.6) holds with $p = 2$. Here $\omega_{ns} \le k^{1/2}$ for $s < n$, and it is easy to see that (1.7) holds with $q = 1/2$. The number of time levels that enter the computation is of order $O(k^{-1/2})$ for this rule, as compared with $O(k^{-1})$ for the rectangle rule.

In [4] a similar quadrature rule was used, with the difference that the left side composite rectangle rule with stepsize k was used on $(\bar t_{j_n}, t_n)$. Again (1.6) and (1.7) hold with $p = 2$, $q = 1/2$. The reason for using the rectangle rule on $(\bar t_{j_n}, t_n)$ in this way was that here the ω_{ns} are "dominated weights" in the sense described before Lemma 2.2. In this example, we may take $\omega_s = O(k_1) = O(k^{1/2})$ for s divisible by m and $\omega_s = O(k)$ for all other s. Even though the storage requirement is increased compared to the above method, it is still of the same order $O(k^{-1/2})$. We remark that the first sparse rule described above does not have dominated quadrature weights, since $\omega_{s,s-1} \ge ck^{1/2}$ with $c > 0$ for all t_s in the right hand halves (say) of the intervals $(\bar t_{j-1}, \bar t_j)$, so that $\omega_s \ge \omega_{ns}$ implies $\sum_{s=0}^{n-1} \omega_s \ge \frac{1}{2} ck^{-1/2}$.

Going one step further with the idea of reducing the storage requirement, we may set $m = [k^{-1/4}]$ and $k_2 = m^3 k = O(k^{1/4})$, and do the following. We first use Simpson's rule on as many intervals of lengths $2k_2$ that can be fitted into $[0, t_{n-1})$, and then, on the remaining interval, which is of length at most $O(k^{1/2})$, the composite trapezoidal rule on as many intervals of lengths $k_1 = m^2 k = O(k^{1/2})$ as fit in, thus reaching $\bar t_{j_n}$, then the one-interval trapezoidal rule on the interval $(\bar t_{j_n}, t_{n-1})$, and finally the left rectangle rule on (t_{n-1}, t_n). Similarly to above, (1.6) and (1.7) hold with $p = 4$, $q = 1/4$, and the number of time-levels that need to be stored per unit time is now $O(k_2^{-1}) + O(k_2 k_1^{-1}) + 1 = O(k^{-1/4})$. This rule does not have dominated weights. Thus, our present assumptions allow some advantageous rules that were not covered in [4] or [6].

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