LONG TIME BEHAVIOR OF BACKWARD DIFFERENCE TYPE METHODS FOR PARABOLIC EQUATIONS WITH MEMORY IN BANACH SPACE

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Abstract. We show stability in a Banach space framework of backward Euler and second order backward difference timestepping methods for a parabolic equation with memory. The results are applied to derive maximum norm stability estimates for piecewise linear finite element approximations in a plane spatial domain, which is accomplished by a new resolvent estimate for the discrete Laplacian. Error estimates are also given.

1. Introduction. In this work we study the behavior of the solutions of the initial value problem

$$u_t + Au = \int_0^t b(t,\tau)Au(\tau)\,d\tau + f(t), \quad \text{for } t>0, \quad \text{with } u(0)=u_0, \tag{1.1}$$

and of certain discrete analogues. The problem is considered in a Banach space X, $u=u(t)$, $u_t = du/dt$, the kernel $b(t,\tau)$ is a smooth scalar function, and A is a linear not necessarily bounded operator such that $-A$ generates a bounded holomorphic semigroup $E(t)=e^{-tA}$ on X. More precisely we assume that A is (α,φ)-sectorial for some $\alpha>0$ and $\varphi\in(0,\pi/2)$ in the sense that there is $M\ge 1$ such that the resolvent $R(\lambda)=(\lambda I - A)^{-1}$ satisfies

$$\|R(\lambda)\| \le \frac{M}{|\lambda-\alpha|}, \quad \text{for } \lambda\in\mathbf{C}\setminus\Sigma_{\alpha\varphi}, \quad \Sigma_{\alpha\varphi}=\{\lambda\in\mathbf{C} : |\arg(\lambda-\alpha)|<\varphi\}. \tag{1.2}$$

The data of the problem, u_0 and $f(t)$, are assumed to be such that (1.1) has a solution for $t\ge 0$ with the regularity properties needed in our analysis. Such problems occur in modelling heat conduction in materials with memory; see [10].

It is well-known that the semigroup $E(t)$ may be defined by (cf. [5])

$$E(t) = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda}R(\lambda)\,d\lambda, \quad \text{for } t>0, \quad \text{with } \Gamma=\Gamma_{\gamma\varphi}=\partial\Sigma_{\gamma\varphi}, \quad \gamma<\alpha,$$

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where the contour Γ is oriented so that $\operatorname{Im}\lambda$ decreases along Γ. (Note that the domain of definition $\mathcal{D}(A)$ need not be dense in X [3].) It is also well-known that

$$\|E(t)\| \le CMe^{-\alpha t}, \quad \text{for } t \ge 0, \tag{1.3}$$

and, hence, in the purely parabolic case ($b(t,\tau) \equiv 0$), solutions of (1.1) satisfy

$$\|u(t)\| \le CM\Big(e^{-\alpha t}\|u_0\| + \int_0^t e^{-\alpha(t-s)}\|f(s)\|\,ds\Big), \quad \text{for } t \ge 0. \tag{1.4}$$

The main purpose of the present work is to prove similar bounds for solutions of (1.1), under suitable assumptions on the kernel $b(t,\tau)$, and of the discrete counterparts studied.

In order to achieve this we will use the condition that for some $\gamma \in [0,\alpha)$, with $\Gamma_{\alpha\varphi} = \{\lambda \in \mathbf{C} : |\arg(\lambda - \alpha)| = \varphi\}$,

$$\sup_{t\ge 0} \int_0^t e^{\gamma(t-\tau)}|b(t,\tau)|\,d\tau < \Lambda_{\alpha\varphi\gamma} := \inf_{\lambda\in\Gamma_{\alpha\varphi}} \frac{\operatorname{Re}\lambda - \gamma}{|\lambda|}. \tag{1.5}$$

The main result of Section 2, Theorem 2.1, shows that if (1.5) holds together with

$$\sup_{t\ge\tau\ge 0} (e^{\gamma(t-\tau)}|b(t,\tau)|) < \infty, \tag{1.6}$$

then the solution of (1.1) satisfies (1.4) with α replaced by γ. In the special case when A is selfadjoint positive definite in a Hilbert space it suffices to assume (1.5) with $\varphi = 0$ and α equal to the smallest eigenvalue of A (we then have $\Lambda_{\alpha 0\gamma} = 1 - \gamma/\alpha$) to obtain (1.4) with $CM = 1$ and γ instead of α, see Theorem 2.2. Note that the left side of (1.5) is the norm of the operator $\mathcal{B}$, defined by $(\mathcal{B}w)(t) = \int_0^t b(t,\tau)w(\tau)\,d\tau$, with respect to the weighted norm $|w|_\gamma = \sup_{t>0}(e^{\gamma t}\|w(t)\|)$, which will be used in our analysis.

In Section 3 we show similar properties for a discrete counterpart of (1.1) of backward Euler type. Let $k > 0$ be a time step, let $t_n = nk,\ n = 0,1,2,\dots$, and let U_n denote the approximation of $u(t_n)$. We then replace the time derivative in (1.1) by the backward difference quotient $\bar\partial U_n = (U_n - U_{n-1})/k$ and the integral by a quadrature formula with nonnegative weights $\omega_{nj} = \omega_{nj}(k)$,

$$(Q_k g)_n = \sum_{j=0}^{n-1} \omega_{nj} g(t_j) \approx \int_0^{t_n} g(t)\,dt. \tag{1.7}$$

Setting $f_n = f(t_n)$ our time discretization of (1.1) is then

$$\bar\partial U_n + AU_n = (\mathcal{B}_k AU)_n + f_n, \quad \text{for } n \ge 1, \quad \text{with } U_0 = u_0,$$

where, with $b_n(s) = b(t_n,s),\ b_{nj} = b(t_n,t_j)$,

$$(\mathcal{B}_k w)_n = (Q_k(b_n w))_n = \sum_{j=0}^{n-1} \omega_{nj} b_{nj} w_j. \tag{1.8}$$

Under natural assumptions on $b(t,\tau)$ and the quadrature formula (1.7) we show discrete analogues of Theorems 2.1 and 2.2.

In Section 4 we replace the time derivative instead by a second order backward finite difference quotient $\bar{D}U_n = (\frac{3}{2}U_n - 2U_{n-1} + \frac{1}{2}U_{n-2})/k$ and show that the corresponding stability result holds also in this case. In Section 5 we show error estimates of orders $O(k)$ and $O(k^2)$, respectively, for the methods treated in Sections 3 and 4, under the appropriate accuracy assumptions on Q_k.

Our results are applied in Section 6 to derive maximum-norm bounds for piecewise linear finite element approximations of (1.1). In doing so we show a new resolvent estimate of the form (1.2) for the discrete Laplacian in two dimensions, where φ is arbitrary in $(0, \pi/2)$ and M contains a logarithmic factor.

In the literature approximations of equations of the form (1.1) with more general operator integral kernels $B(t,\tau)$ have been considered in, e.g., [10], [11]. Our choice of $B(t,\tau) = b(t,\tau)A$ is motivated by our wish to be able to use spectral representations to reduce our statements to analogous statements for scalar problems. Our results here generalize earlier results in [9] for the case of selfadjoint operators in Hilbert space, and also give more precise information about the exponential decay. For an analysis of backward Euler type methods for (1.1) in the present framework on a finite interval in time, see [1].

2. The continuous problem. Our main purpose here is to show the following analogue of (1.4) for our initial-value problem (1.1).

Theorem 2.1. *Assume that A satisfies (1.2) with $\alpha > 0$ and $\varphi \in (0, \pi/2)$, and let $0 \le \gamma < \alpha$. Assume further that $b(t,s)$ satisfies (1.5) and (1.6). Then*

$$\|u(t)\| \le CM\Big(e^{-\gamma t}\|u_0\| + \int_0^t e^{-\gamma(t-s)}\|f(s)\|\,ds\Big), \quad \textit{for } t \ge 0.$$

With this purpose in mind we consider in addition to (1.1) the problems

$$u_t + Au = \int_s^t b(t,\tau)Au(\tau)\,d\tau, \quad \text{for } t > s, \quad \text{with } u(s) = \phi, \tag{2.1}$$

where $s \ge 0$ is arbitrary. Denoting the solution operators of (2.1) by $F(t,s)$, so that $u(t) = F(t,s)\phi$, it is not difficult to see that (cf. [10], Lemma 2.1) the solution of (1.1) may be written as

$$u(t) = F(t,0)u_0 + \int_0^t F(t,s)f(s)\,ds.$$

To show Theorem 2.1 we shall therefore derive a bound of the form

$$\|F(t,s)\| \le CMe^{-\gamma(t-s)}. \tag{2.2}$$

By introducing the kernel $\tilde{b}(s;t,\tau) = b(t+s,\tau+s)$, we may restrict our proof of (2.2) to the case $s=0$. In fact, the kernel $\tilde{b}(s;t,\tau)$ satisfies (1.5), (1.6) since, e.g.,

$$\int_0^t |\tilde{b}(s;t,\tau)|e^{\gamma(t-\tau)}\,d\tau = \int_s^{t+s} |b(t+s,\tau)|e^{\gamma(t+s-\tau)}\,d\tau \le \sup_{t\ge 0}\int_0^t |b(t,\tau)|e^{\gamma(t-\tau)}\,d\tau.$$

The main part of our analysis will be carried out on the scalar problem obtained formally from (2.1) (with $s=0$) by replacing A by $\lambda \in \mathbf{C}$, and where thus $v = v(\lambda; t)$ is the solution of

$$v_t + \lambda v = \lambda \int_0^t b(t,\tau) v(\tau)\, d\tau, \quad \text{for } t > 0, \quad \text{with } v(0) = 1. \tag{2.3}$$

Before proving Theorem 2.1 we first consider the case that X is a Hilbert space and the operator A is selfadjoint, positive definite with a compact inverse so that there is associated with A an orthonormal basis of eigenvectors $\{\varphi_j\}_{j=1}^\infty$ and corresponding positive eigenvalues $\{\lambda_j\}_{j=1}^\infty$. In this case we shall show the following slightly sharper result.

Theorem 2.2. *Let X be a Hilbert space, let A be selfadjoint positive definite with compact inverse, and let α be its smallest eigenvalue. Let $0 \le \gamma \le \alpha$. Assume that*

$$\sup_{t \ge 0} \int_0^t |b(t,s)| e^{\gamma(t-s)}\, ds \le 1 - \frac{\gamma}{\alpha}. \tag{2.4}$$

Then

$$\|u(t)\| \le e^{-\gamma t}\|u_0\| + \int_0^t e^{-\gamma(t-s)}\|f(s)\|\, ds, \quad \textit{for } t \ge 0.$$

In view of the eigenvector expansion of $u(t)$, the following lemma implies (2.2) with $CM = 1$, which proves Theorem 2.2.

Lemma 2.3. *Assume that $0 \le \gamma \le \alpha$, and that (2.4) holds. Then*

$$|v(\lambda; t)| \le e^{-\gamma t}, \quad \textit{for } t > 0, \quad \lambda \ge \alpha.$$

Proof. If $\gamma = \alpha$ there is no integral, and the result is obvious. We may therefore assume $\gamma < \alpha$. By integration (2.3) may be written as the Volterra integral equation

$$v(t) = e^{-\lambda t} + \int_0^t K(t,\tau) v(\tau)\, d\tau, \quad \text{where } K(t,\tau) = \lambda \int_\tau^t e^{-\lambda(t-y)} b(y,\tau)\, dy. \tag{2.5}$$

We consider first the case $b(t,\tau) \ge 0$. Then $K(t,\tau) \ge 0$, from which it easily follows that $v \ge 0$. Introducing the nonnegative function $w(t) = e^{\gamma t} v(t)$ we want to show that $w(t) \le 1$ for $t \ge 0$. Multiplying (2.3) by $e^{\gamma t}$ we find

$$w_t + (\lambda - \gamma) w = \lambda \int_0^t b(t,\tau) e^{\gamma(t-\tau)} w(\tau)\, d\tau.$$

Since $w_t(0) = -(\lambda - \gamma) < 0$, w decreases in an interval to the right of 0 so that $w(t) < 1$ there. In order to show that $w(t) \le 1$ for $t \ge 0$, assume to the contrary that $w(t) = 1$ for some $t > 0$ and let t_0 be the first such point. We then have either $b(t_0, \tau) = 0$ for $\tau \in (0, t_0)$, or, with strict inequality,

$$w_t(t_0) + \lambda - \gamma < \lambda \int_0^{t_0} b(t_0,\tau) e^{\gamma(t_0-\tau)}\, d\tau \le \lambda\big(1 - \frac{\gamma}{\alpha}\big).$$

In the first case $w_t(t_0) = -(\lambda - \gamma) < 0$, and in the second $w_t(t_0) < \gamma(1 - \lambda/\alpha) \leq 0$, which contradicts the assumption that t_0 is the first such point.

We now turn to the case of a general $b(t,\tau)$. Then (2.5) implies

$$|v(t)| \leq e^{-\lambda t} + \int_0^t \tilde{K}(t,\tau)|v(\tau)|\,d\tau, \quad \text{where } \tilde{K}(t,\tau) = \lambda \int_\tau^t e^{-\lambda(t-y)}|b(y,\tau)|\,dy.$$

Defining $V(t)$ by $V(t) = e^{-\lambda t} + \int_0^t \tilde{K}(t,\tau)V(\tau)\,d\tau$, we know from above that $V(t) \leq e^{-\gamma t}$ for $t \geq 0$, and to complete the proof it remains to show $|v(t)| \leq V(t)$. By subtraction this is equivalent to showing that if

$$z(t) \geq \int_0^t \tilde{K}(t,\tau)z(\tau)\,d\tau = (\mathcal{K}z)(t), \quad \text{for } t \geq 0, \quad \text{with } z(0) = 0,$$

then $z(t) \geq 0$. But by repeated application $z(t) \geq (\mathcal{K}^n z)(t) \to 0$, as $n \to \infty$. □

We now turn to Theorem 2.1 in Banach space. We thus want to show (2.2) under the assumptions of Theorem 2.1, and as indicated above, it suffices to do this for $s = 0$. Considering (2.1) (with $s = 0$) as a parabolic equation with the integral as a forcing term, Duhamel's principle shows

$$u(t) = E(t)u_0 + \int_0^t E(t-s) \int_0^s b(s,\tau)Au(\tau)\,d\tau\,ds. \tag{2.6}$$

The first term is immediately bounded as desired by (1.3), so it remains to consider the integral. For this purpose we shall first derive an estimate for the solution of the scalar problem (2.3).

Lemma 2.4. *Under the assumptions of Theorem 2.1, the solution $v(\lambda;t)$ of (2.3) is an entire holomorphic function of λ for fixed $t > 0$, and there are $\alpha_1 \in (\gamma, \alpha)$ and C such that*

$$|v(\lambda;t) - e^{-\lambda t}| \leq \frac{C}{|\lambda|}e^{-\gamma t}, \quad \lambda \in \Sigma_{\alpha_1\varphi},\ t > 0.$$

Proof. The solution $v(\lambda;t)$ of the Volterra equation (2.5) may be represented by a series whose terms are holomorphic in λ for fixed t. The series converges uniformly on compact sets of λ; hence $v(\lambda;t)$ is holomorphic in λ.

With $v_\lambda(t) = v(\lambda;t)$, $e_\lambda(t) = e^{-\lambda t}$, we write (2.3) as

$$v_\lambda = e_\lambda + \mathcal{E}_\lambda \mathcal{B} v_\lambda, \tag{2.7}$$

where the operators $\mathcal{E}_\lambda$ and $\mathcal{B}$ are defined by

$$(\mathcal{E}_\lambda w)(t) = \lambda \int_0^t e^{-\lambda(t-s)}w(s)\,ds, \quad \text{and} \quad (\mathcal{B}w)(t) = \int_0^t b(t,s)w(s)\,ds.$$

With $|\cdot|_\gamma$ denoting the weighted norm $|w|_\gamma = \sup_{t\geq 0}\big(e^{\gamma t}|w(t)|\big)$, as well as the corresponding operator norm, we first note that assumption (1.5) shows that

$$|\mathcal{B}|_\gamma = \sup_{t\geq 0} \int_0^t |b(t,s)|e^{\gamma(t-s)}\,ds \leq \beta\Lambda_{\alpha\varphi\gamma}, \quad \text{with } \beta < 1. \tag{2.8}$$

Straightforward calculations give

$$|\mathcal{E}_\lambda|_\gamma = |\lambda| \int_0^\infty |e^{-(\lambda-\gamma)s}|\, ds = \frac{|\lambda|}{\operatorname{Re}\lambda - \gamma}, \quad \text{for } \operatorname{Re}\lambda > \gamma, \tag{2.9}$$

and, using also (1.6),

$$|\mathcal{B}e_\lambda|_\gamma \le \sup_{t\ge 0} \int_0^t |b(t,s)| e^{\gamma(t-s)} |e^{-(\lambda-\gamma)s}|\, ds \le \frac{C}{\operatorname{Re}\lambda - \gamma}, \quad \text{for } \operatorname{Re}\lambda > \gamma. \tag{2.10}$$

It follows from (2.8) and (2.9) that there are $\alpha_1 \in (\gamma, \alpha)$ and β_1 such that, since $\Lambda_{\alpha\varphi\gamma}$ depends continuously on α,

$$|\mathcal{E}_\lambda B|_\gamma \le \beta \frac{\Lambda_{\alpha\varphi\gamma}}{\Lambda_{\alpha_1\varphi\gamma}} \le \beta_1 < 1, \quad \text{for } \lambda \in \Sigma_{\alpha_1\varphi}, \tag{2.11}$$

so that $\mathcal{E}_\lambda \mathcal{B}$ is a contraction with respect to $|\cdot|_\gamma$ for such λ. Hence, since from (2.7) we have $\mathcal{E}_\lambda \mathcal{B} v_\lambda = \mathcal{E}_\lambda \mathcal{B} e_\lambda + (\mathcal{E}_\lambda\mathcal{B})^2 v_\lambda$, it follows from (2.11), (2.9) and (2.10) that

$$|\mathcal{E}_\lambda \mathcal{B} v_\lambda|_\gamma \le \frac{1}{1-\beta_1} |\mathcal{E}_\lambda \mathcal{B} e_\lambda|_\gamma \le \frac{1}{1-\beta_1} \frac{1}{\Lambda_{\alpha_1\varphi\gamma}} \frac{C}{\Lambda_{\alpha_1\varphi\gamma}|\lambda|} = \frac{C}{|\lambda|}, \quad \text{for } \lambda \in \Sigma_{\alpha_1\varphi}.$$

This completes the proof. □

We need the following well-known result which we include for completeness.

Lemma 2.5. *Let $\alpha_1 \in (\gamma, \alpha)$. Assume that f, g are holomorphic in $\Sigma_{\alpha_1\varphi}$ and that*

$$|f(\lambda)| + |g(\lambda)| \le \frac{C}{|\lambda|}, \quad \textit{for } \lambda \in \Sigma_{\alpha_1\varphi}.$$

Define operators F, G, and H by

$$F = \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda)\, d\lambda, \quad \textit{where } \Gamma = \Gamma_{\alpha_1\varphi},$$

and similarly with f replaced by g and $h = fg$. Then $H = FG$. Moreover,

$$AE(t) = \frac{1}{2\pi i} \int_\Gamma \lambda e^{-\lambda t} R(\lambda)\, d\lambda. \tag{2.12}$$

Proof. Take $\alpha' \in (\alpha_1, \alpha)$ and let $\Gamma' = \Gamma_{\alpha'\varphi}$. Then, by the resolvent identity $R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$, Fubini's theorem, (1.2), and Cauchy's integral formula,

$$\begin{aligned}
FG &= \frac{1}{(2\pi i)^2} \int_{\Gamma'} \int_\Gamma f(\mu) g(\lambda) R(\mu) R(\lambda)\, d\lambda\, d\mu \\
&= \frac{1}{2\pi i} \int_\Gamma g(\lambda) \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\mu)}{\mu - \lambda}\, d\mu\, R(\lambda)\, d\lambda \\
&\quad - \frac{1}{2\pi i} \int_{\Gamma'} f(\mu) \frac{1}{2\pi i} \int_\Gamma \frac{g(\lambda)}{\mu - \lambda}\, d\lambda\, R(\mu)\, d\mu \\
&= \frac{1}{2\pi i} \int_\Gamma g(\lambda) f(\lambda) R(\lambda)\, d\lambda = H.
\end{aligned}$$

The identity (2.12) follows from $AR(\lambda) = \lambda R(\lambda) - I$ and (1.2). □

Proof of Theorem 2.1. With $v(\lambda; t)$ the solution of (2.3) and $\Gamma = \Gamma_{\alpha_1 \varphi}$ we define an operator $V(t)$ by

$$V(t) = \frac{1}{2\pi i} \int_\Gamma v(\lambda; t) R(\lambda)\, d\lambda = E(t) + \frac{1}{2\pi i} \int_\Gamma \big(v(\lambda; t) - e^{-\lambda t}\big) R(\lambda)\, d\lambda. \tag{2.13}$$

Using the bound in Lemma 2.4 we readily prove that

$$\|V(t)\| \le CMe^{-\gamma t}, \quad \text{for } t \ge 0.$$

We need to show that $V(t)$ is equal to the solution operator $F(t, 0)$ of (2.1) with $s = 0$. By uniqueness and (2.6) this follows if we prove that $V(t)$ satisfies

$$V(t) = E(t) + \int_0^t \int_\tau^t E(t - \sigma) A b(\sigma, \tau)\, d\sigma\, V(\tau)\, d\tau =: E(t) + J(t). \tag{2.14}$$

In order to do this we substitute (2.13) into $J(t)$ and use Lemma 2.5 with $f(\lambda) = \lambda e^{-\lambda(t-\sigma)}$, $g(\lambda) = v(\lambda; t)$ to show

$$J(t) = \int_0^t \int_\tau^t \frac{1}{2\pi i} \int_\Gamma \lambda e^{-\lambda(t-\sigma)} b(\sigma, \tau) v(\lambda; \tau) R(\lambda)\, d\lambda\, d\sigma\, d\tau.$$

Since the norm of the integrand is bounded by $Ce^{-\operatorname{Re}\lambda(t-\sigma)}(e^{-\operatorname{Re}\lambda\tau} + |\lambda|^{-1})$ (cf. Lemma 2.4), we may use Fubini's theorem and finally (2.5) to obtain

$$\begin{aligned} J(t) &= \frac{1}{2\pi i} \int_\Gamma \int_0^t \int_\tau^t \lambda e^{-\lambda(t-\sigma)} b(\sigma, \tau) v(\lambda; \tau)\, d\sigma\, d\tau\, R(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma \big(v(\lambda; t) - e^{-\lambda t}\big) R(\lambda)\, d\lambda = V(t) - E(t). \end{aligned}$$

This proves (2.14) and thus completes the proof of the theorem. □

3. Backward Euler type methods. We now turn to the stability of the time discrete problem

$$\bar\partial U_n + AU_n = (\mathcal{B}_k AU)_n + f_n, \quad \text{for } n \ge 1, \quad \text{with } U_0 = u_0, \tag{3.1}$$

where $\mathcal{B}_k = Q_k b_n$ is defined in (1.7) and (1.8). In addition to our previous assumptions on the kernel $b(t, \tau)$ we shall now assume

$$\sup_{t \ge 0} \int_0^t e^{\gamma(t-\tau)} |b'_\tau(t, \tau)|\, d\tau < \infty. \tag{3.2}$$

We also assume that there are $q > 0$ and $k_0 > 0$ such that the quadrature operator Q_k satisfies

$$0 \le \omega_{nj} \le Ck^q, \quad \text{for } 0 \le j \le n,\ k \le k_0, \tag{3.3}$$

and

$$\Big|(Q_k g)_n - \int_0^{t_n} g\, dt\Big| \le Ck^q \int_0^{t_n} \big(|g| + |g'|\big)\, dt, \quad \text{for } n \ge 1,\ k \le k_0. \tag{3.4}$$

We first show a discrete variant of (1.5).

Lemma 3.1. *Assume that (1.5), (3.2), and (3.4) hold. Then*

$$\sup_{n\geq 1}\sum_{j=0}^{n-1}\omega_{nj}|b_{nj}|e^{\gamma t_{n-j}}<\Lambda_{\alpha\varphi\gamma},\quad \textit{for } k\leq k_0.$$

Proof. This follows from (1.5), (3.2), and (3.4) with $g(s)=|b(t_n,s)|e^{\gamma(t_n-s)}$. □

We next introduce some notation that will be used in the sequel. The discrete analogue of the homogeneous problem (2.1) is

$$\text{(3.5)}\quad \bar\partial U_n+AU_n=(\mathcal{B}_{s,k}AU)_n=\sum_{j=s}^{n-1}\omega_{nj}b_{nj}AU_j,\quad \text{for } n\geq s+1,\quad \text{with } U_s=\phi,$$

and we denote its solution operator by $F_{ns}=F_{ns}(kA)$ so that $U_n=F_{ns}\phi$. It is then easily verified that the solution of (3.1) may be represented as

$$\text{(3.6)}\qquad U_n=F_{n0}u_0+kE_k\sum_{j=1}^{n}F_{nj}f_j,\quad \text{where } E_k=(I+kA)^{-1}.$$

In order to show stability we thus want to bound F_{n0} and E_kF_{nj} for $1\leq j\leq n$, and in doing so we may restrict the consideration to $j=0$. The main part of our analysis will be carried out on the scalar analogue of (3.5), which for $s=0$ reads

$$\text{(3.7)}\qquad \bar\partial v_n+\lambda v_n=\lambda(\mathcal{B}_k v)_n=\lambda\sum_{j=0}^{n-1}\omega_{nj}b_{nj}v_j,\quad \text{for } n\geq 1,\quad \text{with } v_0=1.$$

We begin with the following result in the case that A is a selfadjoint operator in a Hilbert space.

Theorem 3.2. *Let X be a Hilbert space, let A be selfadjoint positive definite with compact inverse, and let α be its smallest eigenvalue. Let $0\leq\gamma<\alpha$. Assume that (1.5), (3.2), and (3.4) hold. Then there is $k_0>0$ such that, for the solution of (3.1),*

$$\|U_n\|\leq e^{-\gamma t_n}\|u_0\|+k\sum_{j=1}^{n}e^{-\gamma t_{n-j}}\|f_j\|,\quad \textit{for } n\geq 0,\ k\leq k_0.$$

Proof. Since $\|E_k\|\leq 1$ it suffices to show that $\|F_{ns}\|\leq e^{-\gamma t_{n-s}}$, and by above it suffices to bound $F_n=F_{n0}$. By spectral representation, this is achieved if we can show that for the solution of (3.7) we have $|v_n|\leq e^{-\gamma t_n}$ for $\lambda\geq\alpha$. For this purpose we write (3.7) as

$$\text{(3.8)}\qquad v_n=r(\mu)v_{n-1}+\mu r(\mu)(\mathcal{B}_k v)_n,\quad \text{with } r(\mu)=(1+\mu)^{-1},\ \mu=k\lambda,$$

and, after multiplication by $e^{\gamma t_n}$,

$$\text{(3.9)}\qquad e^{\gamma t_n}v_n=e^{\gamma k}r(\mu)e^{\gamma t_{n-1}}v_{n-1}+\mu r(\mu)\sum_{j=0}^{n-1}\omega_{nj}b_{nj}e^{\gamma t_{n-j}}(e^{\gamma t_j}v_j).$$

We note that as a result of Lemma 3.1 (with $\varphi = 0$) there are δ and k_0 such that

$$\sum_{j=0}^{n-1} \omega_{nj}|b_{nj}|e^{\gamma t_{n-j}} \le \delta < 1 - \frac{\gamma}{\alpha}, \quad \text{for } n \ge 1,\ k \le k_0.$$

Further, $e^{\gamma k} \le 1 + \gamma_1 k$ for k small, if $\gamma_1 > \gamma$, and hence, with $\gamma_1 = \alpha(1-\delta)$,

$$e^{\gamma k} + \mu\delta \le 1 + \alpha(1-\delta)k + k\lambda\delta \le 1 + k\lambda = 1 + \mu, \quad \text{for } \lambda \ge \alpha,$$

so that $(e^{\gamma k} + \mu\delta)r(\mu) \le 1$. Hence by (3.9)

$$|e^{\gamma t_n} v_n| \le \max_{0 \le j \le n-1} |e^{\gamma t_j} v_j| \le \cdots \le |v_0| = 1,$$

which completes the proof. □

We now turn to the Banach space case.

Theorem 3.3. *Assume that (1.2) holds, and let $0 \le \gamma < \alpha$. Assume that (1.5), (1.6), and (3.2) hold, and further that the quadrature rule satisfies (3.3) and (3.4). Then there are $k_0 > 0$ and $C > 0$ such that for the solution of (3.1) we have*

$$\|U_n\| \le CM\Big(e^{-\gamma t_n}\|u_0\| + k\sum_{j=1}^{n} e^{-\gamma t_{n-j}}\|f_j\|\Big), \quad \textit{for } n \ge 0,\ k \le k_0.$$

Proof. In view of (3.6) we now want to show

$$\|F_{n0}\| \le CMe^{-\gamma t_n} \quad \text{and} \quad \|E_k F_{nj}\| \le CMe^{-\gamma t_{n-j}}, \quad \text{for } 1 \le j \le n. \tag{3.10}$$

As above it suffices to consider $F_n = F_{n0}$. We shall bound F_n by using the Dunford-Taylor functional calculus, in which for a rational function $w(\lambda)$, holomorphic in a neighborhood of some $\Sigma_{\alpha_1\varphi}$ with $\alpha_1 < \alpha$ and with $w(\infty) = 0$, we may write

$$w(A) = \frac{1}{2\pi i}\int_\Gamma w(\lambda)R(\lambda)\,d\lambda, \quad \text{where } \Gamma = \Gamma_{\alpha_1\varphi}. \tag{3.11}$$

With $\mu = k\lambda$ we introduce the sequences $\boldsymbol{v}_\mu = (v_n(\mu))_{n=0}^\infty$ and $\boldsymbol{r}_\mu = (r(\mu)^n)_{n=0}^\infty$. After iteration (3.8) may then be written

$$\boldsymbol{v}_\mu = \boldsymbol{r}_\mu + \mathcal{E}_{k,\mu}\mathcal{B}_k\boldsymbol{v}_\mu, \tag{3.12}$$

where the operator $\mathcal{B}_k$ is defined by (1.8) with $(\mathcal{B}_k\boldsymbol{w})_0 = 0$ and $\mathcal{E}_{k,\mu}$ by

$$(\mathcal{E}_{k,\mu}\boldsymbol{w})_n = \tilde r_\mu \sum_{j=1}^{n} r(\mu)^{n-j} w_j, \quad \text{where } \tilde r_\mu = \mu r(\mu), \quad \text{for } n \ge 1,$$

with $(\mathcal{E}_{k,\mu}\boldsymbol{w})_0 = 0$. Further, letting $\mu \to \infty$, we see that (3.12) holds with $\mu = \infty$, where $(\mathcal{E}_{k,\infty}\boldsymbol{w})_n = w_n$ for $n \ge 1, (\mathcal{E}_{k,\infty}\boldsymbol{w})_0 = 0$ and $\boldsymbol{r}_\infty = (1, 0, 0, \ldots)$. It follows by induction on n that $\boldsymbol{v}_\mu$ is a rational function of μ. We now define the sequence $\boldsymbol{w}_\mu$ by

$$\boldsymbol{w}_\mu = \boldsymbol{v}_\mu - \tilde{r}_\mu \boldsymbol{v}_\infty - \boldsymbol{r}_\mu + \tilde{r}_\mu \boldsymbol{r}_\infty. \tag{3.13}$$

Since $\boldsymbol{w}_{\mu,n} = w_n(\mu) = v_n(\mu) - \tilde{r}_\mu v_n(\infty) - r(\mu)^n$ is a rational function for $n \geq 1$ which vanishes at $\mu = \infty$, we have by (3.11)

$$F_n = v_n(kA) = kAE_k v_n(\infty) + E_k^n + \frac{1}{2\pi i}\int_\Gamma w_n(k\lambda)\, R(\lambda)\, d\lambda. \tag{3.14}$$

Similarly

$$E_k F_n = kAE_k^2 v_n(\infty) + E_k^{n+1} + \frac{1}{2\pi i}\int_\Gamma r(k\lambda) w_n(k\lambda)\, R(\lambda)\, d\lambda. \tag{3.15}$$

Since $kAE_k = I - E_k$ and $kAE_k^2 = E_k - E_k^2$, the desired bounds in (3.10) will follow from the following two lemmas.

Lemma 3.4. *For any $\gamma \in [0, \alpha)$ there are C and k_0 such that, for $k \leq k_0$,*

$$\|E_k^n\| \leq CMe^{-\gamma t_n}, \quad \textit{for } n \geq 0.$$

Proof. We have

$$E_k^n = k^{-n}(k^{-1}I + A)^{-n} = k^{-n}\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-t/k}E(t)\, dt.$$

Using (1.3) with $\gamma_1 \in (\gamma, \alpha)$ we obtain, since $(1 + \gamma_1 k)^{-1} \leq e^{-\gamma k}$ for $k \leq k_0$,

$$\|E_k^n\| \leq CMk^{-n}\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-(k^{-1}+\gamma_1)t}\, dt = CM(1 + k\gamma_1)^{-n} \leq CMe^{-\gamma t_n}.$$

□

Lemma 3.5. *Under the assumptions of Theorem 3.3, there are $\alpha_1 \in (\gamma, \alpha)$, $k_0 > 0$, and C such that, for $k \leq k_0$, $n \geq 1$,*

$$|v_n(\infty)| \leq Ce^{-\gamma t_n}, \tag{3.16}$$

$$|w_n(k\lambda)| \leq Ce^{-\gamma t_n}\Big(\frac{1}{|\lambda|} + \frac{k^q}{1 + k|\lambda|}\Big), \quad \textit{for } \lambda \in \Sigma_{\alpha_1 \varphi}. \tag{3.17}$$

Before we give the proof of this lemma we complete the proof of of Theorem 3.3 by noting that since $r(k\lambda)$ is bounded on Γ it implies that the integrals in (3.14) and (3.15) are bounded by CM times

$$\int_\alpha^\infty \Big(\frac{1}{y} + \frac{k^q}{1 + ky}\Big)\frac{dy}{y} \leq C(1 + k^q \log\frac{1}{k}) \leq C.$$

This proves Theorem 3.3. □

Proof of Lemma 3.5. The proof is modeled after the proof of Lemma 2.4. Using (3.12) we find that $\boldsymbol{w}_\mu$ satisfies

$$\begin{aligned}\boldsymbol{w}_\mu &= \mathcal{E}_{k,\mu}\mathcal{B}_k\boldsymbol{w}_\mu + \tilde{r}_\mu(\mathcal{E}_{k,\mu} - \mathcal{E}_{k,\infty})\mathcal{B}_k(\boldsymbol{v}_\infty - \boldsymbol{r}_\infty) \\ &\quad + (\mathcal{E}_{k,\mu} - \tilde{r}_\mu\mathcal{E}_{k,\infty})\mathcal{B}_k\boldsymbol{r}_\mu + \tilde{r}_\mu\mathcal{E}_{k,\infty}\mathcal{B}_k(\boldsymbol{r}_\mu - \boldsymbol{r}_\infty) \\ &= \mathcal{E}_{k,\mu}\mathcal{B}_k\boldsymbol{w}_\mu + I_1 + I_2 + I_3.\end{aligned} \tag{3.18}$$

We shall first show that $\mathcal{E}_{k,\mu}\mathcal{B}_k$ is a contraction with respect to the weighted norm $|\boldsymbol{w}|_\gamma = \sup_{n\geq 0}(e^{\gamma t_n}|w_n|)$. To this end we shall show that for the corresponding operator norm (cf. (2.9))

$$|\mathcal{E}_{k,\mu}|_\gamma \leq \frac{|\lambda|}{\operatorname{Re}\lambda - \gamma_1}, \quad \text{for } \operatorname{Re}\lambda \geq \gamma_1 > \gamma. \tag{3.19}$$

Since by Lemma 3.1 there is $\beta \in (0,1)$ such that (cf. (2.8))

$$|\mathcal{B}_k|_\gamma \leq \sup_{n\geq 1}\sum_{j=0}^{n-1}\omega_{nj}|b_{nj}|e^{\gamma t_{n-j}} \leq \beta\Lambda_{\alpha\varphi\gamma} \leq C, \quad \text{for } k \leq k_0, \tag{3.20}$$

it follows that, for $\lambda \in \Sigma_{\alpha_1\varphi}$, $\gamma < \gamma_1 < \alpha_1 < \alpha$, $k \leq k_0$,

$$|\mathcal{E}_{k,\mu}\mathcal{B}_k|_\gamma \leq \frac{|\lambda|}{\operatorname{Re}\lambda - \gamma_1}\beta\Lambda_{\alpha\varphi\gamma} \leq \beta\frac{\Lambda_{\alpha\varphi\gamma}}{\Lambda_{\alpha_1\varphi\gamma_1}}. \tag{3.21}$$

We may then choose α_1, γ_1, and then possibly a new k_0, so that (cf. (2.11))

$$|\mathcal{E}_{k,\mu}\mathcal{B}_k|_\gamma \leq \beta_1 < 1, \quad \text{for } \lambda \in \Sigma_{\alpha_1\varphi},\ k \leq k_0. \tag{3.22}$$

Hence from (3.18)

$$|\boldsymbol{w}_\mu|_\gamma \leq \frac{1}{1-\beta_1}(|I_1|_\gamma + |I_2|_\gamma + |I_3|_\gamma).$$

In order to bound the right hand side we shall show that, for $\lambda \in \Sigma_{\alpha_1\varphi}$, $k \leq k_0$,

$$|\mathcal{E}_{k,\mu} - \tilde{r}_\mu\mathcal{E}_{k,\infty}|_\gamma + |\mathcal{E}_{k,\mu} - \mathcal{E}_{k,\infty}|_\gamma \leq \frac{C}{1+k|\lambda|}, \tag{3.23}$$

$$|\boldsymbol{v}_\infty - \boldsymbol{r}_\infty|_\gamma \leq Ck^q, \tag{3.24}$$

$$|\mathcal{B}_k\boldsymbol{r}_\mu|_\gamma \leq C\Big(\frac{1}{|\lambda|} + k^q\Big), \tag{3.25}$$

$$|\tilde{r}_\mu\mathcal{B}_k(\boldsymbol{r}_\mu - \boldsymbol{r}_\infty)|_\gamma \leq \frac{Ck^q}{1+k|\lambda|}, \tag{3.26}$$

Together these estimates yield

$$|I_1|_\gamma + |I_2|_\gamma + |I_3|_\gamma \leq C\Big(\frac{1}{|\lambda|} + \frac{k^q}{1+k|\lambda|}\Big),$$

which completes the proof of (3.17). Moreover, (3.16) follows from (3.24).

It remains to prove (3.19) and (3.23)–(3.26). For (3.19) we write

$$e^{\gamma t_n}(\mathcal{E}_{k,\mu}\boldsymbol{w})_n = \tilde{r}_\mu \sum_{j=1}^{n} \big(e^{\gamma k} r(\mu)\big)^{n-j} e^{\gamma t_j} w_j.$$

Here

$$e^{\gamma k}|r(\mu)| \le \frac{e^{\gamma k}}{1+k\operatorname{Re}\lambda} \le \frac{1}{1+k(\operatorname{Re}\lambda-\gamma_1)}, \quad \text{for } \operatorname{Re}\lambda > \gamma_1,\ k \le k_0, \tag{3.27}$$

and hence

$$|\mathcal{E}_{k,\mu}|_\gamma = |\tilde{r}_\mu| \sum_{j=0}^{\infty} \big(e^{\gamma k}|r(\mu)|\big)^j = |\mu| \frac{e^{\gamma k}|r(\mu)|}{1-e^{\gamma k}|r(\mu)|} \le \frac{|\lambda|}{\operatorname{Re}\lambda-\gamma_1}. \tag{3.28}$$

We now prove (3.23). For $n \ge 2$ we have

$$\begin{aligned}(\mathcal{E}_{k,\mu}\boldsymbol{w})_n &= \tilde{r}_\mu w_n + \tilde{r}_\mu \sum_{j=1}^{n-1} r(\mu)^{n-j} w_j \\ &= \tilde{r}_\mu(\mathcal{E}_{k,\infty}\boldsymbol{w})_n + r(\mu)\tilde{r}_\mu \sum_{j=2}^{n} r(\mu)^{n-j} w_{j-1},\end{aligned}$$

or $(\mathcal{E}_{k,\mu} - \tilde{r}_\mu \mathcal{E}_{k,\infty})\boldsymbol{w} = r(\mu)\mathcal{E}_{k,\mu}\bar{\boldsymbol{w}}$, where $\bar{w}_j = w_{j-1}$ for $j \ge 2, \bar{w}_1 = \bar{w}_0 = 0$. Hence, since $\operatorname{Re}\lambda \ge c|\lambda|$ and $|\mathcal{E}_{k,\mu}|_\gamma \le C$,

$$|(\mathcal{E}_{k,\mu} - \tilde{r}_\mu\mathcal{E}_{k,\infty})\boldsymbol{w}|_\gamma \le |r(k\lambda)|\,|\mathcal{E}_{k,\mu}\bar{\boldsymbol{w}}|_\gamma \le \frac{C}{1+k|\lambda|}|\boldsymbol{w}|_\gamma, \quad \text{for } \lambda \in \Sigma_{\alpha_1\varphi}.$$

This proves the first bound in (3.23). The second bound follows from this and $1 = \tilde{r}_\mu + r(\mu)$, which implies $\mathcal{E}_{k,\mu} - \mathcal{E}_{k,\infty} = \big(\mathcal{E}_{k,\mu} - \tilde{r}_\mu\mathcal{E}_{k,\infty}\big) - r(\mu)\mathcal{E}_{k,\infty}$.

To prove (3.24) we use (3.12), (3.22), (3.3), and (1.6) to get

$$|\boldsymbol{v}_\infty - \boldsymbol{r}_\infty|_\gamma = |\mathcal{E}_{k,\infty}\mathcal{B}_k\boldsymbol{v}_\infty|_\gamma \le \frac{1}{1-\beta_1}|\mathcal{E}_{k,\infty}\mathcal{B}_k\boldsymbol{r}_\infty|_\gamma = C\sup_{n\ge1}\big(e^{\gamma t_n}\omega_{n0}|b_{n0}|\big) \le Ck^q.$$

To prove (3.25) we use assumption (1.6) to get

$$|\mathcal{B}_k\boldsymbol{r}_\mu|_\gamma \le \sup_{n\ge1}\sum_{j=0}^{n-1}\omega_{nj}|b_{nj}|e^{\gamma t_{n-j}}\big(e^{\gamma k}|r(\mu)|\big)^j \le C\sup_{n\ge1}\sum_{j=0}^{n-1}\omega_{nj}\big(e^{\gamma k}|r(\mu)|\big)^j. \tag{3.29}$$

With $\nu = \operatorname{Re}\lambda - \gamma_1$ (3.27) shows $e^{\gamma k}|r(\mu)| \le 1/2$ for $k\nu \ge 1$ and hence by (3.3)

$$\sum_{j=0}^{n-1}\omega_{nj}(e^{\gamma k}|r(\mu)|)^j \le Ck^q\sum_{j=0}^{n-1}2^{-j} \le Ck^q. \tag{3.30}$$

For $k\nu \le 1$, $(e^{\gamma k}|r(\mu)|)^j \le e^{-c\nu t_j}$ with $c<1$, and by (3.4)

$$\begin{aligned}(3.31)\qquad \sum_{j=0}^{n-1} \omega_{nj}(e^{\gamma k}|r(\mu)|)^j &\le \sum_{j=0}^{n-1}\omega_{nj}e^{-c\nu t_j}\\ &\le \int_0^{t_n} e^{-c\nu t}\,dt + Ck^q\int_0^{t_n}(1+\nu)e^{-c\nu t}\,dt \le C\big(\frac{1}{\nu}+k^q\big) \le C\big(\frac{1}{|\lambda|}+k^q\big).\end{aligned}$$

Finally, by (1.6) and (3.3), and using (3.28), since $(\boldsymbol{r}_\mu - \boldsymbol{r}_\infty)_n = r(\mu)^n$ for $n \ge 1$ and $(\boldsymbol{r}_\mu - \boldsymbol{r}_\infty)_0 = 0$,

$$|\tilde r_\mu \mathcal{B}_k(\boldsymbol{r}_\mu - \boldsymbol{r}_\infty)|_\gamma \le Ck^q|r(\mu)|\,|\tilde r_\mu|\sum_{j=0}^{\infty}\big(e^{\gamma k}|r(\mu)|\big)^j \le Ck^q|r(\mu)| \le \frac{Ck^q}{1+k|\lambda|}.$$

This proves (3.26) and thus completes the proof of Lemma 3.5. ☐

4. Second order backward difference methods. In this section we shall treat second order backward difference methods of the form

$$(4.1)\qquad \bar D U_n + AU_n = (\mathcal{B}_k AU)_n + f_n, \quad \text{for } n \ge 2,$$

where $\bar D U_n = (\frac{3}{2}U_n - 2U_{n-1} + \frac12 U_{n-2})/k$ and where again $\mathcal{B}_k = Q_k b_n$ is defined in (1.7) and (1.8), with start-up values U_0 and U_1 given by

$$(4.2)\qquad \bar\partial U_1 + AU_1 = (\mathcal{B}_k AU)_1 + f_1, \quad U_0 = u_0.$$

In analogy with (2.1) and (3.5) we consider the homogeneous problem

$$(4.3)\qquad \bar D U_n + AU_n = (\mathcal{B}_{s,k}AU)_n, \quad \text{for } n \ge s+2, \quad \text{with } U_s = \phi_0,\ U_{s+1} = \phi_1,$$

and denote its solution operator by F_{ns} so that $U_n = F_{ns}(\phi_0,\phi_1)$. The solution of (4.1) may then be represented as

$$(4.4)\qquad U_n = F_{n0}(U_0,U_1) + k\tilde E_k \sum_{s=2}^{n} F_{n,s-1}(0,f_s), \quad \text{where } \tilde E_k = (\tfrac32 I + kA)^{-1},$$

and our aim is to bound F_{n0} and $\tilde E_k F_{nj}$ with $1 \le j \le n-1$. As before we may set $s=0$ and bound $F_n = F_{n0}$ and $\tilde E_k F_n$, and then analyze the scalar version of (4.3) with $s=0$, where $v_n = v_n(k\lambda)$ satisfies

$$(\tfrac32 + k\lambda)v_n - 2v_{n-1} + \tfrac12 v_{n-2} = k\lambda\big(\mathcal{B}_k\boldsymbol{v}\big)_n, \quad \text{for } n \ge 2,$$

with v_0, v_1 given and $\boldsymbol{v} = (v_n)_{n=0}^\infty$, or, with $\mu = k\lambda$,

$$(4.5)\qquad v_n - \frac{4}{3+2\mu}v_{n-1} + \frac{1}{3+2\mu}v_{n-2} = \frac{2\mu}{3+2\mu}\big(\mathcal{B}_k\boldsymbol{v}\big)_n, \quad \text{for } n \ge 2.$$

We associate with this equation the characteristic polynomial

$$r^2 - \frac{4}{3+2\mu}r + \frac{1}{3+2\mu} = (r - r_+(\mu))(r - r_-(\mu)),$$

with roots $r_\pm = r_\pm(\mu) = (2\pm\sqrt{1-2\mu})/(3+2\mu)$. The solution of (4.5) then satisfies

$$v_n = \hat\rho_n(\mu)v_0 + \bar\rho_n(\mu)v_1 + \tilde r_\mu \sum_{j=2}^{n} \rho_{n-j}(\mu)\big(\mathcal{B}_k \boldsymbol{v}\big)_n, \tag{4.6}$$

where

$$\rho_n = \frac{r_+^{n+1}-r_-^{n+1}}{r_+-r_-}, \quad \bar\rho_n = \rho_{n-1}, \quad \hat\rho_n = -r_+r_-\frac{r_+^{n-1}-r_-^{n-1}}{r_+-r_-}, \quad \tilde r_\mu = \frac{2\mu}{3+2\mu},$$

with the obvious modification when $r_+ = r_-$, i.e., when $\mu = 1/2$.

In order to state our stability result we shall need to introduce the quantity

$$m_\varphi = \sup_{|\arg\mu|\le\varphi} \Big|\frac{2\mu}{3+2\mu}\Big| \sum_{j=0}^{\infty} |\rho_j(\mu)|.$$

In the case $\varphi = 0$, this quantity occurred in the stability analysis in [9], where it was shown by computer calculation that $1 < m_0 < 1.05$.

Here we shall go directly to the Banach space case; with $\varphi = 0$ and $M = 1$ it covers the case of a selfadjoint operator in a Hilbert space as a special case.

Theorem 4.1. *Assume that (1.2) holds, and let* $0 \le \gamma < \alpha$. *Assume further that* $b(t,\tau)$ *satisfies (1.6) and (3.2), and that*

$$\sup_{t\ge0}\int_0^t |b(t,s)|e^{\gamma(t-s)}\,ds < \min(\Lambda_{\alpha\varphi\gamma}, m_\varphi^{-1}).$$

Assume finally that the quadrature rule satisfies (3.3) and (3.4). Then there are $k_0 > 0$ *and* $C > 0$ *such that for the solution of (4.1) and (4.2) we have*

$$\|U_n\| \le CM\Big(e^{-\gamma t_n}\|U_0\| + k\sum_{j=1}^{n} e^{-\gamma t_{n-j}}\|f_j\|\Big), \quad \textit{for } n \ge 0,\ k \le k_0.$$

Proof. In view of (4.4) and with $F_n = F_{n0}$ the result will follow from

$$\|F_n(\phi_0,\phi_1)\| + \|\tilde E_k F_n(\phi_0,\phi_1)\| \le CMe^{-\gamma t_n}\big(\|\phi_0\| + \|\phi_1\|\big),$$

together with the obvious estimate for U_1 (which follows from Theorem 3.3). Analogously to (3.12) we write for the solution (4.6) of the scalar problem (4.5)

$$\boldsymbol{v}_\mu = \hat{\boldsymbol{\rho}}_\mu v_0 + \bar{\boldsymbol{\rho}}_\mu v_1 + \mathcal{E}_{k,\mu}\mathcal{B}_k\boldsymbol{v}_\mu, \tag{4.8}$$

where $\boldsymbol{v}_\mu = (v_n(\mu))_{n=0}^\infty$, $\hat{\boldsymbol{\rho}}_\mu = (\hat\rho_n(\mu))_{n=0}^\infty$, $\bar{\boldsymbol{\rho}}_\mu = (\bar\rho_n(\mu))_{n=0}^\infty$, and where now

$$\big(\mathcal{E}_{k,\mu}\boldsymbol{w}\big)_n = \tilde r_\mu \sum_{j=2}^{n} \rho_{n-j} w_j, \quad \text{for } n \ge 2, \tag{4.9}$$

with $(\mathcal{E}_{k,\mu}\boldsymbol{w})_n = 0$ for $n = 0, 1$. Letting $\mu \to \infty$ (4.8) remains valid with $\boldsymbol{v}_\infty = (v_n(\infty))_{n=0}^{\infty}$, $\hat{\boldsymbol{\rho}}_\infty = (1, 0, 0, \dots)$, $\bar{\boldsymbol{\rho}}_\infty = (0, 1, 0, 0, \dots)$, and $(\mathcal{E}_{k,\infty}\boldsymbol{w})_n = w_n$ for $n \geq 2$, $(\mathcal{E}_{k,\infty}\boldsymbol{w})_n = 0$ for $n = 0, 1$. We define analogously to (3.13)

$$\boldsymbol{w}_\mu = \boldsymbol{v}_\mu - \tilde{r}_\mu \boldsymbol{v}_\infty - (\hat{\boldsymbol{\rho}}_\mu - \tilde{r}_\mu \hat{\boldsymbol{\rho}}_\infty) v_0 - (\bar{\boldsymbol{\rho}}_\mu - \tilde{r}_\mu \bar{\boldsymbol{\rho}}_\infty) v_1. \tag{4.10}$$

Since $\boldsymbol{w}_{\mu,n} = w_n(\mu) = v_n(\mu) - \tilde{r}_\mu v_n(\infty) - \hat{\rho}_n(\mu) v_0 - \bar{\rho}_n(\mu) v_1$ is a rational function of μ for $n \geq 2$, which vanishes at $\mu = \infty$, we may apply (3.11). We obtain for $n \geq 2$

$$\begin{aligned} F_n(v_0, v_1) = v_n(kA) &= kA\tilde{E}_k v_n(\infty) \\ &\quad + \bar{\rho}_n(kA) v_1 + \hat{\rho}_n(kA) v_0 + \frac{1}{2\pi i} \int_\Gamma w_n(k\lambda)\, R(\lambda)\, d\lambda; \end{aligned}$$

cf. (3.14), and for $\tilde{E}_k F_n(v_0, v_1)$ we have the corresponding result with the integrand multiplied by $(\frac{3}{2} + k\lambda)^{-1}$. Since one easily sees that $kA\tilde{E}_k$ and $kA\tilde{E}_k^2$ are bounded as claimed, the desired bound (4.7) will follow once we have proved the appropriate bounds for $v_n(\infty)$, $w_n(k\lambda)$, $\hat{\rho}_n(kA)$, and $\bar{\rho}_n(kA)$. These are the contents of the following two lemmas which are analogous to Lemmas 3.5 and 3.4, respectively. $\square$

Lemma 4.2. *Under the assumptions of Theorem 4.1 there are $\alpha_1 \in (\gamma, \alpha)$, $k_0 > 0$, and C such that, for $\lambda \in \Sigma_{\alpha_1 \varphi}$, $k \leq k_0$, $n \geq 1$,*

$$\begin{aligned} &|v_n(\infty)| \leq C e^{-\gamma t_n}, \\ &|w_n(k\lambda)| \leq C e^{-\gamma t_n} \Big(\frac{1}{|\lambda|} + \frac{k^q}{\sqrt{1 + k|\lambda|}} \Big) (|v_0| + |v_1|), \end{aligned} \tag{4.11}$$

Proof. The proof follows the lines of that of Lemma 3.5. Multiplying the case $\mu = \infty$ of (4.8) by $\tilde{r}_\mu$ and subtracting the result from (4.8), using (4.10) shows

$$\begin{aligned} \boldsymbol{w}_\mu &= \mathcal{E}_{k,\mu}\mathcal{B}_k \boldsymbol{w}_\mu + \tilde{r}_\mu (\mathcal{E}_{k,\mu} - \mathcal{E}_{k,\infty}) \mathcal{B}_k (\boldsymbol{v}_\infty - \hat{\boldsymbol{\rho}}_\infty v_0 - \bar{\boldsymbol{\rho}}_\infty v_1) \\ &\quad + (\mathcal{E}_{k,\mu} - \tilde{r}_\mu \mathcal{E}_{k,\infty}) \mathcal{B}_k (\hat{\boldsymbol{\rho}}_\mu v_0 + \bar{\boldsymbol{\rho}}_\mu v_1) \\ &\quad + \tilde{r}_\mu \mathcal{E}_{k,\infty} \mathcal{B}_k \big((\hat{\boldsymbol{\rho}}_\mu - \hat{\boldsymbol{\rho}}_\infty) v_0 + (\bar{\boldsymbol{\rho}}_\mu - \bar{\boldsymbol{\rho}}_\infty) v_1 \big) \\ &= \mathcal{E}_{k,\mu}\mathcal{B}_k \boldsymbol{w}_\mu + I_1 + I_2 + I_3. \end{aligned}$$

We shall show that $\mathcal{E}_{k,\mu}\mathcal{B}_k$ is a contraction with respect to $|\cdot|_\gamma$. For this purpose we show that for any $\delta > 0, \gamma_1 > \gamma$, and $\eta \in (\gamma/\gamma_1, 1)$ there are $\varepsilon > 0$ and k_0 such that for $\lambda \in \Sigma_{\gamma_1,\varphi}, k \leq k_0$, we have

$$|\mathcal{E}_{k,\mu}|_\gamma = |\tilde{r}_\mu| \sum_{j=0}^{\infty} e^{\gamma t_j} |\rho_j| \leq \begin{cases} \eta^{-2} \dfrac{|\lambda|}{\operatorname{Re}\lambda - \gamma_1} + \delta, & \text{for } k|\lambda| \leq \varepsilon \\ m_\varphi + \delta, & \text{for } k|\lambda| \geq \varepsilon. \end{cases} \tag{4.12}$$

As earlier, Lemma 3.1 implies (3.20) so that, in particular, using also (4.12) we may choose α_1, γ_1, and then k_0, so that $|\mathcal{E}_{k,\mu}\mathcal{B}_k|_\gamma \leq \beta_1 < 1$ for $\lambda \in \Sigma_{\alpha_1 \varphi}$, $k \leq k_0$ (cf. (3.22)). To show (4.11) it remains to show

$$|I_1|_\gamma + |I_2|_\gamma + |I_3|_\gamma \leq C \Big(\frac{1}{|\lambda|} + \frac{k^q}{\sqrt{1 + k|\lambda|}} \Big) (|v_0| + |v_1|), \quad \text{for } \lambda \in \Sigma_{\alpha_1 \varphi}.$$

For this we shall show that, for $\lambda \in \Sigma_{\alpha_1\varphi}$ and $k \le k_0$,

$$|\mathcal{E}_{k,\mu} - \tilde r_\mu \mathcal{E}_{k,\infty}|_\gamma + |\mathcal{E}_{k,\mu} - \mathcal{E}_{k,\infty}|_\gamma \le \frac{C}{\sqrt{1+k|\lambda|}}, \tag{4.13}$$

$$|\boldsymbol{v}_\infty - \hat{\boldsymbol{\rho}}_\infty v_0 - \bar{\boldsymbol{\rho}}_\infty v_1|_\gamma \le Ck^q\big(|v_0| + |v_1|\big), \tag{4.14}$$

$$|\mathcal{B}_k \hat{\boldsymbol{\rho}}_\mu|_\gamma + |\mathcal{B}_k \bar{\boldsymbol{\rho}}_\mu|_\gamma \le C\Big(\frac{1}{|\lambda|} + k^q\Big), \tag{4.15}$$

$$|\mathcal{B}_k(\hat{\boldsymbol{\rho}}_\mu - \hat{\boldsymbol{\rho}}_\infty)|_\gamma + |\mathcal{B}_k(\bar{\boldsymbol{\rho}}_\mu - \bar{\boldsymbol{\rho}}_\infty)|_\gamma \le \frac{Ck^q}{\sqrt{1+k|\lambda|}}. \tag{4.16}$$

To prove (4.12) we first note that the equality follows at once from the definition of $\mathcal{E}_{k,\mu}$ in (4.9). Since $r_+ - r_- = 2\sqrt{1-2\mu}/(3+2\mu)$ we conclude

$$|\mathcal{E}_{k,\mu}|_\gamma \le \frac{|\mu|}{|\sqrt{1-2\mu}|} \sum_{j=0}^{\infty} e^{\gamma t_j}\big(|r_+(\mu)|^{j+1} + |r_-(\mu)|^{j+1}\big).$$

We consider first small $|\mu|$ and begin with the contribution to the sum from $r_-(\mu)$. Since $|r_-(\mu)| \le \kappa < 1$ for $|\mu| \le \varepsilon$ we have for such μ and k small

$$\frac{|\mu|}{|\sqrt{1-2\mu}|} \sum_{j=0}^{\infty} e^{\gamma t_j} |r_-(\mu)|^{j+1} \le Ce^{-\gamma k}|\mu| \sum_{j=1}^{\infty} (e^{\gamma k}\kappa)^j \le C|\mu| \le C\varepsilon, \quad \text{for } |\mu| \le \varepsilon,$$

which is as small as desired for ε small enough.

In order to consider the contribution from $r_+(\mu)$, let $\gamma_1 > \gamma$. We note that $r_+(\mu) = 1 - \mu + O(\mu^2)$ as $\mu \to 0$, so that the bound in (3.27) holds also for $r_+(\mu)$ when $\mu \in \Sigma_{0\varphi}$ and $|\mu| \le \varepsilon$. Reducing ε further, if necessary, we also have $|(1-2\mu)^{-1/2}| \le \delta^{-1}$ for $|\mu| \le \varepsilon$, and hence, as in (3.28),

$$\Big|\frac{\mu r_+(\mu)}{\sqrt{1-2\mu}}\Big| \sum_{j=0}^{\infty} (e^{\gamma k}|r_+(\mu)|)^j \le \frac{1}{\delta} \frac{|\lambda|}{\operatorname{Re}\lambda - \gamma_1}, \quad \text{if } \lambda \in \Sigma_{\gamma_1\varphi},\ k|\lambda| \le \varepsilon.$$

For $|\mu| \ge \varepsilon$ we write

$$|\mathcal{E}_{k,\mu}|_\gamma = \Big|\frac{2\mu}{3+2\mu} \sum_{j=0}^{\infty} (1 + (e^{\gamma t_j} - 1))\rho_j\Big| \le m_\varphi + \Big|\frac{2\mu}{3+2\mu}\Big| \sum_{j=1}^{\infty} (1 - e^{-\gamma t_j}) e^{\gamma t_j} |\rho_j|.$$

Here, since $\rho_j = \sum_{l=0}^{j} r_+^{j-l} r_-^l$, and $e^{\gamma k}|r_\pm(\mu)| \le \kappa < 1$, we have $e^{\gamma t_j}|\rho_j| \le (j+1)\kappa^j$, so that

$$\sum_{j=1}^{\infty} (1 - e^{-\gamma t_j}) e^{\gamma t_j} |\rho_j| \le \sum_{j=1}^{\infty} \gamma j k (j+1) \kappa^j \le Ck,$$

which shows the desired result in this case.

We now show (4.13). Since $|\mathcal{E}_{k,\mu}|_\gamma$ is bounded it suffices to consider large μ. We have

$$((\mathcal{E}_{k,\mu} - \tilde r_\mu \mathcal{E}_{k,\infty})\boldsymbol{w})_n = \tilde r_\mu \sum_{j=2}^{n-1} \rho_{n-j} w_j,$$

and hence

$$|\mathcal{E}_{k,\mu}-\tilde r_\mu\mathcal{E}_{k,\infty}|_\gamma=|\tilde r_\mu|\sum_{j=1}^\infty e^{\gamma t_j}|\rho_j|\le\sum_{j=1}^\infty e^{\gamma t_j}|\rho_j|\le\frac{C}{\sqrt{|\mu|}}\le\frac{C}{\sqrt{1+k|\lambda|}},$$

where we have used that for large $|\mu|$, $e^{\gamma k}|r_\pm(\mu)|\le\min(\frac{C}{\sqrt{|\mu|}},\kappa)$, $\kappa<1$, so that

$$e^{\gamma t_j}|\rho_j|\le C\sum_{l=0}^j(e^{\gamma k}|r_+|)^{j-l}(e^{\gamma k}|r_-|)^l\le\frac{C}{\sqrt{|\mu|}}(j+1)\kappa^{j-1}. \tag{4.17}$$

This proves the first bound in (4.13). The second bound follows from this and $1=\tilde r_\mu+3/(3+2\mu)$, which implies $\mathcal{E}_{k,\mu}-\mathcal{E}_{k,\infty}=\mathcal{E}_{k,\mu}-\tilde r_\mu\mathcal{E}_{k,\infty}-\frac{3}{3+2\mu}\mathcal{E}_{k,\infty}$.

To show (4.14) we use (4.8) with $\mu=\infty$ as in the proof of (3.24).

For (4.15) we have using (1.6), cf. (3.29),

$$|\mathcal{B}_k\hat{\boldsymbol\rho}_\mu|_\gamma+|\mathcal{B}_k\bar{\boldsymbol\rho}_\mu|_\gamma\le C\sup_{n\ge1}\sum_{j=0}^{n-1}\omega_{nj}e^{\gamma t_j}(|\hat\rho_{\mu j}|+|\bar\rho_{\mu j}|).$$

For $|\mu|\le\varepsilon$,

$$e^{\gamma t_j}(|\hat\rho_j(\mu)|+|\bar\rho_j(\mu)|)\le C(e^{\gamma k}|r_+(\mu)|^j+\kappa^j),\quad\text{with }\kappa<1, \tag{4.18}$$

and we may use (3.30) and (3.31) for the desired bound. For $|\mu|\ge\varepsilon$, the left side of (4.18) is bounded by $C\kappa^j$ and again contributes as in (3.30).

We now prove (4.16). Since $(\hat{\boldsymbol\rho}_\mu-\hat{\boldsymbol\rho}_\infty)_n=\hat\rho_n(\mu)$ for $n\ge2$ and $=0$ for $n=0,1$, we may use (3.3), (1.6) and (4.12) to get

$$\begin{aligned}|\mathcal{B}_k(\hat{\boldsymbol\rho}_\mu-\hat{\boldsymbol\rho}_\infty)|_\gamma+|\mathcal{B}_k(\bar{\boldsymbol\rho}_\mu-\bar{\boldsymbol\rho}_\infty)|_\gamma&\le Ck^q|\tilde r_\mu|\sum_{j=2}^\infty e^{\gamma t_j}(|\hat\rho_j(\mu)|+|\bar\rho_j(\mu)|)\\&\le Ck^q|\tilde r_\mu|\Big(|r_+(\mu)r_-(\mu)|+\sum_{j=1}^\infty e^{\gamma t_j}|\rho_j|\Big)\le C\frac{k^q}{\sqrt{1+k|\mu|}}.\end{aligned}$$

The proof is now complete. □

Lemma 4.3. *For any $\gamma\in[0,\alpha)$ there are C and k_0 such that, for $k\le k_0$,*

$$\|\hat\rho_n(kA)\|+\|\bar\rho_n(kA)\|\le CMe^{-\gamma t_n},\quad\textit{for } n\ge0.$$

Proof. Let $F_n(\mu)$ denote either $\hat\rho_n(\mu)$ or $\bar\rho_n(\mu)$. Let $0\le\gamma<\gamma_1<\alpha$, set $A_{\gamma_1}=A-\gamma_1 I$ and note that $\|(\lambda I-A_{\gamma_1})^{-1}\|\le M/|\lambda-\gamma_1|\le CM/|\lambda|$ for $\lambda\notin\Sigma_{0\varphi}$. Writing $F_n(kA)=F_n(kA_{\gamma_1}+k\gamma_1 I)$ we have

$$e^{\gamma t_n}F_n(kA)=\frac{1}{2\pi i}\int_\Gamma e^{\gamma t_n}F_n(k\lambda+k\gamma_1)(\lambda I-A_{\gamma_1})^{-1}\,d\lambda,\quad\text{with }\Gamma=\Gamma_{0\varphi}. \tag{4.19}$$

We shall show that there are $c, C, \varepsilon, k_0 > 0$ such that, for $k \le k_0$,

$$e^{\gamma t_n} |\rho_n(k\lambda + k\gamma_1)| \le \begin{cases} Ce^{ct_n|\lambda|}, & \text{for } k|\lambda| \le \varepsilon, \\ Ce^{-ct_n|\lambda|}, & \text{for } k|\lambda| \le \varepsilon,\ \lambda \in \Sigma_{0\varphi}, \\ C(k|\lambda|)^{-1/2}, & \text{for } k|\lambda| \ge \varepsilon,\ \lambda \in \Sigma_{0\varphi}. \end{cases} \tag{4.20}$$

Since $|r_+(\mu) r_-(\mu)| = |3+2\mu|^{-1} \le C$ for $\operatorname{Re}\mu \ge 0$, this implies the same bounds for $e^{\gamma t_n}|F_n(k\lambda + k\gamma_1)|$ and the proof is then accomplished by deforming the contour Γ in (4.19) into the boundary of $B(0, \varepsilon/t_n) \cup \Sigma_{0\varphi}$ and considering the cases $|\lambda| = \varepsilon/t_n$, $\varepsilon/t_n \le |\lambda| \le \varepsilon/k$, and $|\lambda| \ge \varepsilon/k$ separately.

Since $|r_+(\mu)| \le e^{-c\operatorname{Re}\mu}$ for $|\mu| \le 2\varepsilon$, with $c = \gamma/\gamma_1$, we have for $k \le \varepsilon/\gamma_1$ that $e^{\gamma k}|r_+(k\lambda + k\gamma_1)| \le e^{-ck\operatorname{Re}\lambda}$ for $k|\lambda| \le \varepsilon$. Since $e^{\gamma k}|r_-(\mu)| \le \kappa \le e^{-c|\mu|}$ for $|\mu| \le 2\varepsilon$, and $|r_+(\mu) - r_-(\mu)|$ is bounded below for $|\mu| \le 2\varepsilon$, the first two of the bounds in (4.20) follow. The third bound follows readily from (4.17). □

5. Error estimates. So far we have only considered the stability of our discrete schemes for (1.1). We shall now briefly give some associated error estimates, and we then need to assume that the quadrature rules are accurate of orders $O(k)$ and $O(k^2)$ for the backward Euler and the second order backward difference methods, respectively. We assume thus that the quadrature rule defined in (1.7) satisfies, with $m = 1$, or 2 and J a positive integer, and with $\|g\|_J = \sum_{j=0}^{J} \|g^{(j)}\|$,

$$\|q_n(g)\| \le Ck^m \int_0^{t_n} \|g\|_J \, dt, \quad \text{where } q_n(g) = (Q_k g)_n - \int_0^{t_n} g \, dt. \tag{5.1}$$

Theorem 5.1. *Under the assumptions of Theorem 3.3 and with (5.1) satisfied with $m = 1$ we have for the solutions of (3.1) and (1.1), for $n \ge 0,\ k \le k_0$,*

$$\|U_n - u(t_n)\| \le Ck \int_0^{t_n} e^{-\gamma(t_n - s)} \big(\|u_{tt}(s)\| + (t_{n+1} - s)\|Au(s)\|_J \big) \, ds.$$

Proof. Setting $e_n = U_n - u(t_n)$ we have

$$\bar\partial e_n + Ae_n = (\mathcal{B}_k Ae)_n - \tau_n + q_n(b_n Au), \quad \text{where } \tau_n = \bar\partial u(t_n) - u_t(t_n),$$

and hence, by Theorem 3.3, since $e_0 = 0$,

$$\|e_n\| \le CMk \sum_{j=1}^{n} e^{-\gamma t_{n-j}} \big(\|\tau_j\| + \|q_j(b_j Au)\| \big), \quad \text{for } n \ge 0,\ k \le k_0.$$

Here

$$k \sum_{j=1}^{n} e^{-\gamma t_{n-j}} \|\tau_j\| \le k \sum_{j=1}^{n} e^{-\gamma t_{n-j}} \int_{t_{j-1}}^{t_j} \|u_{tt}\| \, ds \le ke^{\gamma k} \int_0^{t_n} e^{-\gamma(t_n - s)} \|u_{tt}(s)\| \, ds.$$

Further, using (5.1) and (1.6),

$$\|q_j(b_j Au)\| \le Ck \int_0^{t_j} \|b(t_j, s) Au(s)\|_J \, ds \le Ck \int_0^{t_j} e^{-\gamma(t_j - s)} \|Au(s)\|_J \, ds,$$

so that

$$k \sum_{j=1}^{n} e^{-\gamma t_{n-j}} \|q_j(b_j Au)\| \le Ck \int_0^{t_n} e^{-\gamma(t_n - s)} (t_{n+1} - s) \|Au(s)\|_J \, ds.$$

Together these estimates complete the proof. □

We only state the corresponding result for the second order method.

Theorem 5.2. *Under the assumptions of Theorem 4.1 and with (5.1) satisfied with $m = 2$ we have for the solutions of (4.1), (4.2), and (1.1), for $n \geq 0,\ k \leq k_0$,*

$$\|U_n - u(t_n)\| \leq Ck^2 \int_0^{t_n} e^{-\gamma(t_n - s)} \big(\|u_{ttt}(s)\| + (t_{n+1} - s)\|Au(s)\|_J\big)\, ds.$$

6. Maximum-norm stability of finite element approximations. In this final section we shall apply our results to the case that A is the discrete analogue of the Laplacian $-\Delta$, with Dirichlet boundary conditions, in a convex plane domain Ω with smooth or polygonal boundary, based on piecewise linear finite elements. We first recall from [4] and [7] that for any $\varphi \in (0, \pi/2)$ there is a positive M such that

$$\|(\lambda I + \Delta)^{-1} v\|_{L_\infty} \leq \frac{M}{|\lambda|}\|v\|_{L_\infty}, \quad \text{for } v \in C(\bar\Omega),\ \lambda \in \mathbf{C} \setminus \Sigma_{0\varphi}.$$

In fact, with $X = C(\bar\Omega)$, $A = -\Delta$ satisfies (1.2) for any $\varphi \in (0, \pi/2)$ with $\alpha > 0$ the smallest eigenvalue of $-\Delta$ (see the argument for the discrete analogue at the end of this section). Note that $\mathcal{D}(A) = \{u \in C(\bar\Omega) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}$ with $\overline{\mathcal{D}(A)} \neq X$. Theorems 2.1 and 3.3 hence apply for any $\gamma \in [0, \alpha)$, and since $\Lambda_{\alpha\varphi\gamma}$ may now be chosen arbitrarily close to $\Lambda_{\alpha 0 \gamma} = 1 - \gamma/\alpha$, condition (1.5) on $b(t, s)$ essentially reduces to condition (2.4) in the selfadjoint case of Theorems 2.2 and 3.2.

Let now S_h denote the standard continuous piecewise linear functions associated with a member of a set of quasiuniform triangulations of Ω, and let $-\Delta_h : S_h \to S_h$ be defined by $(-\Delta_h \psi, \chi) = (\nabla\psi, \nabla\chi)$ for $\psi, \chi \in S_h$. In order to apply our above results we need to know that $-\Delta_h$ satisfies (1.2) for suitable α and φ. It was shown in [6] that, with $\ell_h = \log(1/h),\ c > 0$,

$$\|(\lambda I + \Delta_h)^{-1}\chi\|_{L_\infty} \leq C\ell_h^2 |\lambda|^{-1} \|\chi\|_{L_\infty}, \quad \forall \chi \in S_h,\ \lambda \in \mathbf{C} \setminus \Sigma_{0\varphi},\ \varphi = \tfrac{\pi}{2} - c\ell_h^{-2}.$$

For our purposes it is desirable to have an estimate which is valid also for small values of $|\arg\lambda|$. (In one dimension such a result, without the factor ℓ_h, is shown in [2].) With this in mind we shall show the following.

Lemma 6.1. *For any $\varphi \in (0, \pi/2)$ and with α the smallest eigenvalue of $-\Delta$, there exists $C = C_\varphi$ such that, for $\chi \in S_h$,*

$$\|(\lambda I + \Delta_h)^{-1}\chi\|_{L_\infty} \leq C\ell_h |\lambda - \alpha|^{-1} \|\chi\|_{L_\infty}, \quad \textit{for } \lambda \in \mathbf{C} \setminus \Sigma_{\alpha\varphi}. \tag{6.1}$$

Proof. In the first part of the proof we show (6.1) with $\alpha = 0$. Let $E_h(t) = e^{t\Delta_h}$ be the semigroup on S_h generated by Δ_h. It is clear that $E_h(z) = e^{z\Delta_h}$ is holomorphic for all $z \in \mathbf{C}$ and we shall prove that for any $\varphi_1 \in (0, \pi/2)$ there is C such that

$$\|E_h(z)\chi\|_{L_\infty} \leq C\ell_h \|\chi\|_{L_\infty}, \quad \forall \chi \in S_h,\ |\arg z| \leq \varphi_1. \tag{6.2}$$

Assume for a moment that this has been shown. We have

$$(\lambda I + \Delta_h)^{-1} = -\int_0^\infty e^{\lambda t} E_h(t)\, dt, \quad \text{for } \operatorname{Re}\lambda < 0.$$

For $|\arg\lambda| \in (\varphi,\pi]$ we may rotate the path of integration to any ray $z = te^{i\varphi_1}$ with $|\varphi_1| \le \frac{1}{2}(\pi-\varphi)$, say. In view of (6.2) this shows (6.1) with $\alpha=0$, because $|\arg(\lambda z)|$ may be chosen in $[\frac{1}{2}(\pi+\varphi),\pi]$, so that $\mathrm{Re}(\lambda z) \le -ct|\lambda|$ with $c = \sin(\varphi/2)$.

The inequality (6.2) will follow, if we show that for any $\varepsilon > 0$ there is C_ε such that, for $n \ge 0,\ t > 0$,

$$\|E_h^{(n)}(t)\chi\|_{L_\infty} \le C_\varepsilon \ell_h (1+n^{1/2}) n!\, t^{-n}(1+\varepsilon)^n \|\chi\|_{L_\infty}, \quad \forall \chi \in S_h. \tag{6.3}$$

In fact, (6.3) implies that, for any $\delta \in (0,1)$,

$$\begin{aligned}\|E_h(z)\chi\|_{L_\infty} &= \Big\| \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} E_h^{(n)}(t)\chi \Big\|_{L_\infty} \\ &\le C_\varepsilon \ell_h \|\chi\|_{L_\infty} \sum_{n=0}^{\infty} \Big(\frac{|z-t|(1+\varepsilon)}{t}\Big)^n (1+n^{1/2}) \le C_{\varepsilon,\delta}\ell_h\|\chi\|_{L_\infty},\end{aligned}$$

for all z in the union of the discs $|z-t| \le \delta t/(1+\varepsilon),\ t>0$, i.e., in the sector $|\arg z| \le \arcsin(\delta/(1+\varepsilon)) = \varphi_1$, where φ_1 may be arbitrarily close to $\pi/2$.

The special cases $n = 0,1$ of (6.3) were proved in [6] (cf. also [8]). In the following we extend the techniques of [6] to cover the general case $n \ge 1$. We first define a discrete fundamental solution $\Gamma = \Gamma_h^x(t)$ as $\Gamma_h^x(t) = E_h(t)\delta_h^x$ with $\delta_h^x \in S_h$ the discrete delta-function given by $(\delta_h^x,\chi) = \chi(x),\ \forall\chi \in S_h$. We then have

$$(E_h^{(n)}(t)\chi)(x) = (\Gamma^{(n)}(t),\chi) \le \|\Gamma^{(n)}(t)\|_{L_1}\, \|\chi\|_{L_\infty}.$$

Introducing also the modified distance function $\rho(y) = \rho_h^x(y) = (|x-y|^2+h^2)^{1/2}$, we have (for simplicity we often write $\|\cdot\|$ for $\|\cdot\|_{L_2}$ below)

$$\|\Gamma^{(n)}\|_{L_1} \le \|\rho^{-1}\|\,\|\rho\Gamma^{(n)}\| \le C\ell_h^{1/2}\|\rho\Gamma^{(n)}\|,$$

and the estimate (6.3) will follow from the weighted L_2-bound

$$t^n\|\rho\Gamma^{(n)}(t)\| \le C_\varepsilon \ell_h^{1/2} n!\, n^{1/2}(1+\varepsilon)^n, \quad \text{for } t>0,\ n\ge 1,\ \varepsilon>0. \tag{6.4}$$

To show (6.4) we use the identity

$$(\Gamma^{(n+1)},\chi) + (\nabla\Gamma^{(n)},\nabla\chi) = 0, \quad \forall\chi\in S_h,\ n\ge 0, \tag{6.5}$$

to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&(t^{2n}\|\rho\Gamma^{(n)}\|^2) + t^{2n}\|\rho\nabla\Gamma^{(n)}\|^2 \\ &= t^{2n}(\Gamma^{(n+1)},\rho^2\Gamma^{(n)}-\chi) + t^{2n}(\nabla\Gamma^{(n)},\nabla(\rho^2\Gamma^{(n)}-\chi)) \\ &\qquad - t^{2n}2(\rho\nabla\Gamma^{(n)},\Gamma^{(n)}\nabla\rho) + nt^{2n-1}\|\rho\Gamma^{(n)}\|^2, \quad \forall\chi\in S_h.\end{aligned}$$

Taking $\chi = P_h(\rho^2\Gamma^{(n)})$, with P_h the orthogonal L_2-projection onto S_h, then the first term on the right vanishes, and for the second term we obtain by Lemma 1.4 in [6], and the inverse inequality $\|\nabla\chi\| \le Ch^{-1}\|\chi\|$ for $\chi \in S_h$,

$$\begin{aligned}\|\nabla\Gamma^{(n)}\|\,\|\nabla(\rho^2\Gamma^{(n)}-\chi)\| &\le Ch^{-1}\|\Gamma^{(n)}\|\,Ch\big(\|\Gamma^{(n)}\| + \|\rho\nabla\Gamma^{(n)}\|\big) \\ &\le C\big(\|\Gamma^{(n)}\|^2 + \|\Gamma^{(n)}\|\,\|\rho\nabla\Gamma^{(n)}\|\big).\end{aligned} \tag{6.6}$$

Using also $|\nabla\rho| \le 1$, we find

$$\frac{1}{2}\frac{d}{dt}(t^{2n}\|\rho\Gamma^{(n)}\|^2) + t^{2n}\|\rho\nabla\Gamma^{(n)}\|^2 \\ \le Ct^{2n}\big(\|\Gamma^{(n)}\|^2 + \|\Gamma^{(n)}\|\,\|\rho\nabla\Gamma^{(n)}\|\big) + nt^{2n-1}\|\rho\Gamma^{(n)}\|^2.$$

This shows, for any $\varepsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\big(t^{2n}\|\rho\Gamma^{(n)}\|^2\big) + (1-\varepsilon)t^{2n}\|\rho\nabla\Gamma^{(n)}\| \le C_\varepsilon t^{2n}\|\Gamma^{(n)}\|^2 + nt^{2n-1}\|\rho\Gamma^{(n)}\|^2,$$

and hence by integration

$$t^{2n}\|\rho\Gamma^{(n)}(t)\|^2 + 2(1-\varepsilon)\int_0^t s^{2n}\|\rho\nabla\Gamma^{(n)}\|^2\,ds \\ \le C_\varepsilon\int_0^t s^{2n}\|\Gamma^{(n)}\|^2 ds + 2n\int_0^t s^{2n-1}\|\rho\Gamma^{(n)}\|^2 ds. \tag{6.7}$$

We shall now bound the two terms on the right by the square of the right hand side of (6.4) and thus complete the proof.

Using (6.5) with $n-1$ instead of n and $\chi = t^{2n}\Gamma^{(n)}$ we find

$$t^{2n}\|\Gamma^{(n)}\|^2 + \frac{1}{2}\frac{d}{dt}\big(t^{2n}\|\nabla\Gamma^{(n-1)}\|^2\big) = nt^{2n-1}\|\nabla\Gamma^{(n-1)}\|^2,$$

and hence after integration

$$\int_0^t s^{2n}\|\Gamma^{(n)}\|^2\,ds \le n\int_0^t s^{2n-1}\|\nabla\Gamma^{(n-1)}\|^2\,ds.$$

Similarly, using instead $\chi = t^{2n-1}\Gamma^{(n-1)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(t^{2n-1}\|\Gamma^{(n-1)}\|^2\big) + t^{2n-1}\|\nabla\Gamma^{(n-1)}\|^2 = \frac{1}{2}(2n-1)t^{2n-2}\|\Gamma^{(n-1)}\|^2,$$

and after integration

$$\int_0^t s^{2n-1}\|\nabla\Gamma^{(n-1)}\|^2\,ds \le \frac{1}{2}(2n-1)\int_0^t s^{2n-2}\|\Gamma^{(n-1)}\|^2\,ds,$$

so that

$$\int_0^t s^{2n}\|\Gamma^{(n)}\|^2\,ds \le \frac{2n(2n-1)}{4}\int_0^t s^{2n-2}\|\Gamma^{(n-1)}\|^2\,ds.$$

By repeated application, and using $\int_0^t \|\Gamma\|^2\,ds \le C\ell_h$, see (2.8) of [6],

$$\int_0^t s^{2n}\|\Gamma^{(n)}\|^2\,ds \le \frac{(2n)!}{2^{2n}}\int_0^t \|\Gamma\|^2\,ds \le C\frac{(2n)!}{2^{2n}}\ell_h \le C(n!)^2\ell_h. \tag{6.8}$$

This shows the desired estimate for the first term in (6.7).

We now turn to the second term on the right in (6.7). We have

$$
\begin{aligned}
(6.9)\qquad & t^{2n-1}\|\rho\Gamma^{(n)}\|^2 + \frac{1}{2}\frac{d}{dt}\big(t^{2n-1}\|\rho\nabla\Gamma^{(n-1)}\|^2\big) \\
&= t^{2n-1}(\Gamma^{(n)}, \rho^2\Gamma^{(n)} - \chi) + t^{2n-1}(\nabla\Gamma^{(n-1)}, \nabla(\rho^2\Gamma^{(n)} - \chi)) \\
&\quad - 2t^{2n-1}(\rho\nabla\Gamma^{(n-1)}, \Gamma^{(n)}\nabla\rho) + \frac{1}{2}(2n-1)t^{2n-2}\|\rho\nabla\Gamma^{(n-1)}\|^2 \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

Setting $\chi = P_h(\rho^2\Gamma^{(n)})$, we get $I_1 = 0$. By the same argument as in (6.6), the inverse inequality, and the fact that $\|\rho\nabla\Gamma^{(n)}\| \le Ch^{-1}\|\rho\Gamma^{(n)}\|$ (see [6], p. 283), we derive

$$
\begin{aligned}
|I_2| &\le Ct^{2n-1}\|\nabla\Gamma^{(n-1)}\|\, h(\|\Gamma^{(n)}\| + \|\rho\nabla\Gamma^{(n)}\|) \\
&\le Ct^{2n-1}(\|\Gamma^{(n-1)}\|\,\|\Gamma^{(n)}\| + \|\nabla\Gamma^{(n-1)}\|\,\|\rho\Gamma^{(n)}\|), \\
&\le C\big(t^{2n-2}\|\Gamma^{(n-1)}\|^2 + t^{2n}\|\Gamma^{(n)}\|^2\big) + C_\varepsilon t^{2n-1}\|\nabla\Gamma^{(n-1)}\|^2 + \varepsilon t^{2n-1}\|\rho\Gamma^{(n)}\|^2,
\end{aligned}
$$

and, using $|\nabla\rho| \le 1$,

$$
\begin{aligned}
|I_3| + |I_4| &\le 2t^{2n-1}\|\rho\nabla\Gamma^{(n-1)}\|\,\|\Gamma^{(n)}\| + \frac{1}{2}(2n-1)t^{2n-2}\|\rho\nabla\Gamma^{(n-1)}\|^2 \\
&\le Ct^{2n}\|\Gamma^{(n)}\|^2 + nt^{2n-2}\|\rho\nabla\Gamma^{(n-1)}\|^2.
\end{aligned}
$$

Inserting these estimates into (6.9) and integrating yields

$$
\begin{aligned}
(1-\varepsilon)\int_0^t s^{2n-1}\|\rho\Gamma^{(n)}\|^2\,ds &\le C\int_0^t s^{2n}\|\Gamma^{(n)}\|^2\,ds + C\int_0^t s^{2n-2}\|\Gamma^{(n-1)}\|^2\,ds \\
&\quad + C_\varepsilon\int_0^t s^{2n-1}\|\nabla\Gamma^{(n-1)}\|^2\,ds + n\int_0^t s^{2n-2}\|\rho\nabla\Gamma^{(n-1)}\|^2\,ds.
\end{aligned}
$$

To estimate the last term we recall from (6.7) that, for any $\varepsilon > 0$,

$$
\int_0^t s^{2n}\|\rho\nabla\Gamma^{(n)}\|^2\,ds \le C_\varepsilon\int_0^t s^{2n}\|\Gamma^{(n)}\|^2\,ds + n(1+\varepsilon)\int_0^t s^{2n-1}\|\rho\Gamma^{(n)}\|^2\,ds.
$$

Using this with n replaced by $n-1$ we thus find, again for any $\varepsilon > 0$,

$$
\begin{aligned}
\int_0^t s^{2n-1}\|\rho\Gamma^{(n)}\|^2\,ds &\le C_\varepsilon\int_0^t s^{2n}\|\Gamma^{(n)}\|^2\,ds + C_\varepsilon\int_0^t s^{2n-2}\|\Gamma^{(n-1)}\|^2\,ds \\
&\quad + n^2(1+\varepsilon)\int_0^t s^{2n-3}\|\rho\Gamma^{(n-1)}\|^2\,ds.
\end{aligned}
$$

Hence, using also (6.8),

$$
\int_0^t s^{2n-1}\|\rho\Gamma^{(n)}\|^2\,ds \le C\ell_h(n!)^2 + (1+\varepsilon)n^2\int_0^t s^{2n-3}\|\rho\Gamma^{(n-1)}\|^2\,ds.
$$

By repeated application, and using that $\int_0^t s\|\rho\Gamma'\|^2\,ds \le C\ell_h$ (see [6], p. 284),

$$\int_0^t s^{2n-1}\|\rho\Gamma(n)\|^2\,ds \le C\ell_h(n!)^2(1+(1+\varepsilon)+\cdots+(1+\varepsilon)^{n-1}) \le C_\varepsilon\ell_h(n!)^2(1+\varepsilon)^n,$$

which together with (6.8) and (6.7) completes the proof of (6.4).

It remains to show (6.1) with α equal to the smallest eigenvalue of $-\Delta$. Since $|\lambda-\alpha| \le C|\lambda|$ for $\varphi/2 \le |\arg\lambda| \le \pi$, $|\lambda| \ge \varepsilon > 0$, it remains to show the bound for λ in the bounded set $D = \{\lambda : |\arg(\lambda-\alpha)| \ge \varphi,\ |\arg\lambda| \le \varphi/2\} \cup \{\lambda : |\lambda| \le \varepsilon\}$. For this purpose we use the inequality

$$\|\chi\|_{L_\infty} \le C\|\Delta_h\chi\|_{L_2}. \tag{6.10}$$

Setting $T_h = (-\Delta_h)^{-1}$, $T = (-\Delta)^{-1}$, this follows easily with $f = \Delta_h\chi$ from

$$\|T_hf\|_{L_\infty} \le \|P_hTf\|_{L_\infty} + \|(T_h - P_hT)f\|_{L_\infty} \le C\|f\|_{L_2},$$

where we first used the boundedness of P_h in L_∞, then $\|Tf\|_{L_\infty} \le C\|Tf\|_{H^2} \le C\|f\|_{L_2}$, and finally, by standard arguments (cf. [8]),

$$\begin{aligned}\|(T_h - P_hT)f\|_{L_\infty} &\le Ch^{-1}\|(T_h - P_hT)f\|_{L_2}\\ &\le Ch^{-1}\big(\|(T_h - T)f\|_{L_2} + \|(P_h - I)Tf\|_{L_2}\big) \le Ch\|f\|_{L_2}.\end{aligned}$$

Since $\Delta_h(\lambda I + \Delta_h)^{-1} = I - \lambda(\lambda I + \Delta_h)^{-1}$ we find from (6.10)

$$\begin{aligned}\|(\lambda I + \Delta_h)^{-1}\chi\|_{L_\infty} &\le C\|\Delta_h(\lambda I + \Delta_h)^{-1}\chi\|_{L_2}\\ &\le C\big(1 + |\lambda|\,\|(\lambda I + \Delta_h)^{-1}\|_{L_2\to L_2}\big)\|\chi\|_{L_\infty}.\end{aligned}$$

For $\lambda \in D$ we have, since the eigenvalues of $-\Delta_h$ are bounded from below by α,

$$\|(\lambda I + \Delta_h)^{-1}\|_{L_2\to L_2} \le \operatorname{dist}(\lambda, \sigma(-\Delta_h))^{-1} \le C|\lambda - \alpha|^{-1},$$

and since D is bounded this completes the proof. □

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