POINTWISE A POSTERIORI ERROR ANALYSIS FOR AN ADAPTIVE PENALTY FINITE ELEMENT METHOD FOR THE OBSTACLE PROBLEM

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Abstract. Finite element approximations based on a penalty formulation of the elliptic obstacle problem are analyzed in the maximum norm. *A posteriori* error estimates, which involve a residual of the approximation and a spatially variable penalty parameter, are derived in the cases of both smooth and rough obstacles. An adaptive algorithm is suggested and implemented in one dimension.

1. Introduction

We consider finite element approximations of the obstacle problem

$$\begin{aligned} -\Delta u(x) + \beta(u(x) - \psi(x)) &\ni f(x), \qquad && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where ψ and f are given functions with $\psi \leq 0$ on $\partial\Omega$, and where β is the maximal monotone graph defined by

$$\beta(s) = \begin{cases} \{0\}, & s > 0, \\ (-\infty, 0], & s = 0, \\ \emptyset, & s < 0. \end{cases} \tag{1.2}$$

Our analysis and our finite element method are based on the following penalized (or regularized) form of (1.1): find u_ϵ such that

$$\begin{aligned} -\Delta u_\epsilon(x) + \epsilon(x)^{-1}(u_\epsilon(x) - \psi(x))^- &= f(x), \qquad && x \in \Omega, \\ u_\epsilon(x) &= 0, && x \in \partial\Omega, \end{aligned} \tag{1.3}$$

where $\epsilon = \epsilon(x)$ is a positive function on Ω and $s^- = \min\{0, s\}$. This can also be written

$$\begin{aligned} -\Delta u_\epsilon + \beta_\epsilon(u_\epsilon - \psi) &= f, \qquad && \text{in } \Omega, \\ u_\epsilon &= 0, && \text{in } \partial\Omega, \end{aligned} \tag{1.4}$$

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where β_ϵ is an approximation of β defined by

$$\beta_\epsilon(s) = \begin{cases} 0, & s \ge 0, \\ s/\epsilon, & s \le 0. \end{cases} \tag{1.5}$$

The weak formulation of (1.4) reads

$$u_\epsilon \in H_0^1(\Omega): \quad (\nabla u_\epsilon, \nabla v) + (\beta_\epsilon(u_\epsilon - \psi), v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{1.6}$$

Our finite element method is based on discretizing (1.6). Let $\mathcal{T} = \{K\}$ be a member of a regular family of partitions of Ω into simplices K and let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise polynomials of degree $< r$ ($r \ge 2$) with respect to the mesh $\mathcal{T}$. Our finite element problem is:

$$U_\epsilon \in V_h: \quad (\nabla U_\epsilon, \nabla \chi) + (\beta_\epsilon(U_\epsilon - \psi), \chi) = (f, \chi), \quad \forall \chi \in V_h. \tag{1.7}$$

Detailed assumptions about the domain Ω, the mesh $\mathcal{T}$, the penalty function ϵ, and the data f and ψ are stated in Section 2 below.

The obstacle problem is often considered as a prototype for a class of problems that involve free boundaries, modelling many phenomena such as phase transitions, jet flow, and gas expansion in a porous medium; see Friedman [8]. Thus this is a natural beginning problem to consider when extending the rapidly growing body of research on *a posteriori* error estimates for adaptive finite element algorithms. There are already several papers on this subject; see Ainsworth et al. [1], Hoppe and Kornhuber [12], Kornhuber [15], [16], [17], Johnson [13], Chen and Nochetto [3], Nochetto, Siebert, and Veeser [22], and Veeser [25]. We note also the related work by Nochetto, Paolini, and Verdi [21] on the Stefan problem.

The *a posteriori* estimate that we provide is based on the splitting

$$U_\epsilon - u = (U_\epsilon - u_\epsilon) + (u_\epsilon - u),$$

where the first term is the discretization error and the second is the penalty error.

We have chosen to conduct our analysis using the penalty formulation (1.4). This approach is motivated both by the partial differential equations analysis (see Friedman [8]) and by the desire to regularize the original non-smooth problem before computation. Finite element error analysis in the penalty formulation was first done by Scholz [23], who provided an *a priori* estimate in the energy norm using constant ϵ. Johnson [13] proved an *a posteriori* estimate in the energy norm and introduced the possibility of letting ϵ vary with x. Due to the monotonicity of the nonlinearity, the results obtained in the energy norm are essentially the same as for the corresponding linear problem. However, the standard Aubin-Nitsche duality argument does not go through here, because the linearized adjoint problem lacks the required regularity in L_2. Thus one does not obtain the usual rates of convergence that would be expected in the L_2-norm.

The aim of the present work is twofold. We first exploit the fact that the linearized adjoint problem does essentially have the necessary smoothing property in L_1, leading by duality to an error bound in the maximum norm. This bound is of the form

$$\|U_\epsilon - u_\epsilon\|_{L_\infty(\Omega)} \le C |\log h_{\max}| \, \|h^2 R_\infty\|_{L_\infty(\Omega)},$$

where h is the piecewise constant mesh function defined by $h|_K = \operatorname{diam}(K)$ and R_∞ is a computable function derived from the residual $R = -\Delta U_\epsilon + \beta_\epsilon(U_\epsilon - \psi) - f \in H^{-1}(\Omega)$ of the computed solution; see Theorem 3.4.

The second important feature of the present work is that we allow the penalty parameter ϵ to vary with x. Our result for the penalty error in the case of a smooth obstacle function, $\psi \in W^2_\infty(\Omega)$, is

$$\|u - u_\epsilon\|_{L_\infty(\Omega)} \le \|\epsilon(f + \Delta\psi)\|_{L_\infty(\hat\Omega)},$$

where $\hat\Omega = \{x \in \Omega : u(x) - \psi(x) = 0,\ u_\epsilon(x) - \psi(x) \le 0\}$ is the "contact set"; see Lemma 4.1. We also consider the case of a rough obstacle where ψ is merely Hölder continuous, but the resulting estimate is more involved; see Lemma 5.3.

The time-dependent obstacle problem is analyzed in a similar way by Boman [2]. For *a posteriori* error estimates in the maximum norm for linear elliptic problems we refer to Eriksson [6], Nochetto [20], and Dari, Durán, and Padra [4].

This paper is organized as follows. In Section 2 we state our assumptions on the continuous problem (1.1) and the finite element method and we introduce some notation. In Section 3 we provide the *a posteriori* estimate of the discretization error. In Section 4 we analyze the penalty error in the case of a smooth obstacle. Section 5 is devoted to the case of a rough obstacle. Finally in Section 6 we present the results of our numerical experiments with an adaptive algorithm in one dimension.

2. Notation and assumptions

We assume that $\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$, is a bounded polyhedral domain. Let $\omega \subset \Omega$ with $|\omega| = \text{meas}(\omega)$ and write

$$(u, v)_\omega = \int_\omega uv\,dx, \quad (u, v) = (u, v)_\Omega.$$

We use the standard Lebesgue spaces $L_p(\omega)$ with the convention that $L_p = L_p(\Omega)$, and the corresponding Sobolev spaces $W^k_p(\omega)$, $W^k_p = W^k_p(\Omega)$, $H^k = W^k_2$. For $k = 1, 2$ we let $\mathring{W}^k_p = \{v \in W^k_p : v|_{\partial\Omega} = 0\}$ with dual space W^{-k}_p and $H^1_0 = \mathring{W}^1_2$ with dual space H^{-1}. Moreover, for $f \in (L_p)^d$ and $v \in W^k_p$ we write

$$\|f\|_{L_p} = \left(\sum_{i=1}^d \|f_i\|^p_{L_p}\right)^{1/p}, \quad \|D^k v\|_{L_p} = \left(\sum_{|\alpha|=k} \|D^\alpha v\|^p_{L_p}\right)^{1/p}. \tag{2.2}$$

Then $\|D^k v\|_{L_p}$ is equivalent to the standard norm in $\mathring{W}^k_p$ and we define the dual norm

$$\|f\|_{W^{-k}_p} = \sup_{v \in \mathring{W}^k_{p'}} \frac{|\langle f, v\rangle|}{\|D^k v\|_{L_{p'}}}, \tag{2.3}$$

where $\langle f, v\rangle$ is the duality pairing, and p, p' are dual exponents, $\frac{1}{p} + \frac{1}{p'} = 1$.

We assume further that Ω has the property that there are constants C and $p^* \in (1, 2]$ such that

$$\|D^2 v\|_{L_p} \le \frac{C}{p-1}\|\Delta v\|_{L_p}, \quad \forall v \in \mathring{W}^2_p,\ 1 < p \le p^*. \tag{2.4}$$

This holds with $p^* = 2$ if Ω is bounded and convex. To see this, let $v = Tf$ be the solution of the Dirichlet problem $-\Delta v = f$ in Ω, $v = 0$ on $\partial\Omega$, and let D^2_{ij} denote a partial derivative of second order. It is well known [10] that the operator $D^2_{ij}T$ is bounded on L_2, i.e., it is strong type (2,2); this is the case $p = 2$ of (2.4). Moreover, $D^2_{ij}T$ is weak type (1,1); this is an unpublished result of Dahlberg, Verchota, and

Wolff, but a proof can be found in [9] and a generalization in [7]. An application of the Marcinkiewicz interpolation theorem now yields (2.4). The inequality (2.4) also holds if $\partial\Omega$ is smooth by the Calderón-Zygmund theory of singular integrals.

It is plausible that (2.4) is true even for nonconvex polyhedral domains for some p^* near 1, because the regularity implied by it then holds for $p < 4/3$; see [5], [10], [11], [18]; however, we do not know if the constant behaves like $(p-1)^{-1}$ as $p \to 1$ in this case. Related estimates are derived in [4], [20] for general polyhedral domains under the additional assumption that Δv has a small support. We formulate (2.4) as an assumption rather than a statement about convex domains, because we do not want to rule out this possibility; in any case we are not primarily interested in dealing with corner singularities in this work.

We further assume that the data satisfy $f \in L_\infty(\Omega)$ and $\psi \in H^1(\Omega) \cap C^\alpha(\bar{\Omega})$ is Hölder continuous with some exponent $\alpha \in (0,1)$ and also $\psi \le 0$ on $\partial\Omega$. Then (1.1) has a unique bounded weak solution $u \in H_0^1(\Omega) \cap L_\infty(\Omega)$, which is also Hölder continuous, perhaps with a different exponent α. In our theorems below we make additional regularity assumptions about ψ.

Let $\mathcal{F} = \{\mathcal{T}\}$ denote a regular family of partitions $\mathcal{T} = \{K\}$ of Ω into simplices K with diameters h_K, i.e., there is $C_0 > 0$ such that, for all $\mathcal{T} \in \mathcal{F}$,

$$\max_{K \in \mathcal{T}} \frac{h_K}{\rho_K} \le C_0, \tag{2.5}$$

where ρ_K denotes the radius of the largest ball contained in $\bar{K}$. Let $\epsilon \in C(\bar{\Omega})$ be a positive function and set

$$h_{\min} = \min_{K \in \mathcal{T}} h_K, \quad h_{\max} = \max_{K \in \mathcal{T}} h_K, \quad \epsilon_{\min} = \min_{x \in \bar{\Omega}} \epsilon(x), \quad \epsilon_{\max} = \max_{x \in \bar{\Omega}} \epsilon(x).$$

We also define for each $\mathcal{T} \in \mathcal{F}$ a piecewise constant mesh function $h = h(x)$ by $h|_K = h_K$. We assume that there are constants $\gamma \ge 1$ and $C_1 > 0$ such that, for all $\mathcal{T} \in \mathcal{F}$,

$$h_{\min} \ge C_1 h_{\max}^\gamma, \quad \epsilon_{\min} \ge C_1 h_{\max}^\gamma. \tag{2.7}$$

Let $r \ge 2$ be an integer and for each $\mathcal{T} \in \mathcal{F}$ let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise polynomials of degree $< r$ with respect to the mesh $\mathcal{T}$.

Under these assumptions it follows that problems (1.6) and (1.7) have unique solutions u_ϵ and U_ϵ, respectively.

3. The Discretization Error

In this section we compare the solutions u_ϵ and U_ϵ. We first define the errors

$$e = U_\epsilon - u_\epsilon, \quad e_\beta = \beta_\epsilon(U_\epsilon - \psi) - \beta_\epsilon(u_\epsilon - \psi),$$

and the residual $R = -\Delta U_\epsilon + \beta_\epsilon(U_\epsilon - \psi) - f \in H^{-1}$, i.e.,

$$\langle R, v\rangle = (\nabla U_\epsilon, \nabla v) + (\beta_\epsilon(U_\epsilon - \psi), v) - (f, v), \quad \forall v \in H_0^1, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between H_0^1 and H^{-1}. In view of (1.6) we thus have

$$e \in H_0^1: \quad (\nabla e, \nabla v) + (e_\beta, v) = \langle R, v\rangle, \quad \forall v \in H_0^1, \tag{3.3}$$

and (1.7) means that R is orthogonal to V_h, i.e.,

$$\langle R, \chi\rangle = 0, \quad \forall \chi \in V_h. \tag{3.4}$$

As a consequence of this orthogonality we have the following *a posteriori* bound of the residual with respect to the dual norms (2.3).

Lemma 3.1. *For each $p \in [1,\infty]$ and $l = 1,2$ there is a constant C such that,*

$$|\langle R, v\rangle| \le C\|h^l R_p\|_{L_p}\|D^l v\|_{L_{p'}}, \quad \forall v \in \mathring{W}^l_{p'},$$

where the functions $h = h(x)$ and $R_p = R_p(x)$ are defined piecewise by

$$\begin{aligned} &h|_K = h_K, \\ &R_p|_K = |-\Delta U_\epsilon + \beta_\epsilon(U_\epsilon - \psi) - f| + h_K^{-1/p'}|K|^{-1/p}\|[\partial_n U_\epsilon]\|_{L_p(\partial K\setminus\partial\Omega)}. \end{aligned} \tag{3.6}$$

Here $[\partial_n U_\epsilon]$ is the jump across ∂K in the exterior normal derivative $\partial_n U_\epsilon = n\cdot\nabla U_\epsilon$.

Proof. Write $\beta_\epsilon = \beta_\epsilon(U_\epsilon - \psi)$. Elementwise integration by parts in (3.2) gives

$$\begin{aligned} \langle R, v\rangle &= \sum_K \Big((\nabla U_\epsilon, \nabla v)_K + (\beta_\epsilon - f, v)_K\Big) \\ &= \sum_K \Big((-\Delta U_\epsilon + \beta_\epsilon - f, v)_K + (\partial_n U_\epsilon, v)_{\partial K}\Big) \\ &= \sum_K \Big((-\Delta U_\epsilon + \beta_\epsilon - f, v)_K - \tfrac{1}{2}([\partial_n U_\epsilon], v)_{\partial K\setminus\partial\Omega}\Big), \quad \forall v \in H^1_0. \end{aligned} \tag{3.7}$$

The orthogonality (3.4) implies that $\langle R, v\rangle = \langle R, v - \chi\rangle$ for any $\chi \in V_h$, so that

$$\begin{aligned} |\langle R, v\rangle| \le \sum_K \Big(&\|-\Delta U_\epsilon + \beta_\epsilon - f\|_{L_p(K)}\|v - \chi\|_{L_{p'}(K)} \\ &+ \tfrac{1}{2}\|[\partial_n U_\epsilon]\|_{L_p(\partial K\setminus\partial\Omega)}\|v - \chi\|_{L_{p'}(\partial K)}\Big). \end{aligned} \tag{3.8}$$

We choose $\chi = \Pi v$, where $\Pi : H^1_0 \to V_h$ is an interpolation operator such that

$$\|D^m(v - \Pi v)\|_{L_{p'}(K)} \le C h_K^{l-m}\|D^l v\|_{L_{p'}(S_K)}, \quad m = 0,1,\ l = 1,2, \tag{3.9}$$

where S_K is the union of all simplices adjacent to K; see [24]. Using also the (scaled) trace inequality

$$\|w\|_{L_{p'}(\partial K)} \le C\Big(h_K^{-1/p'}\|w\|_{L_{p'}(K)} + h_K^{1-1/p'}\|Dw\|_{L_{p'}(K)}\Big), \tag{3.10}$$

we obtain

$$\|v - \Pi v\|_{L_{p'}(\partial K)} \le C h_K^{l-1/p'}\|D^l v\|_{L_{p'}(S_K)}, \quad l = 1,2.$$

Hence

$$\begin{aligned} |\langle R, v\rangle| &\le C\sum_K h_K^l\Big(\|-\Delta U_\epsilon + \beta_\epsilon - f\|_{L_p(K)} \\ &\qquad + h_K^{-1/p'}\|[\partial_n U_\epsilon]\|_{L_p(\partial K\setminus\partial\Omega)}\Big)\|D^l v\|_{L_{p'}(S_K)} \\ &\le C\|h^l R_p\|_{L_p}\|D^l v\|_{L_{p'}}, \end{aligned} \tag{3.12}$$

where the constant depends only on the constants in (3.9) and (3.10), which in turn depend only on the constant in (2.5). $\square$

As a result of the inequality

$$(s^- - t^-)(s - t) \ge (s^- - t^-)^2, \quad \forall s,t \in \mathbf{R}, \tag{3.13}$$

we have

$$\big(\beta_\epsilon(v - \psi) - \beta_\epsilon(w - \psi), v - w\big) \ge \|\epsilon^{1/2}(\beta_\epsilon(v - \psi) - \beta_\epsilon(w - \psi))\|^2_{L_2}. \tag{3.14}$$

Taking $v = e$ in (3.3) we immediately conclude from (3.14) and Lemma 3.1 with $p = 2$, $l = 1$, that

$$\|\nabla e\|_{L_2}^2 + 2\|\epsilon^{1/2} e_\beta\|_{L_2}^2 \le C\|hR_2\|_{L_2}^2,$$

which is Johnson's result [13].

In order to estimate $\|e\|_{L_p}$ we argue by duality. We first note that (3.3) may be written as

$$e \in H_0^1: \quad (\nabla e, \nabla v) + (be, v) = \langle R, v\rangle, \quad \forall v \in H_0^1, \tag{3.16}$$

where

$$b(x) = \begin{cases} e_\beta(x)/e(x), & \text{if } e(x) \neq 0, \\ 0, & \text{if } e(x) = 0. \end{cases}$$

Clearly, cf. (3.13), we have

$$0 \le b(x) \le \epsilon(x)^{-1} \le \epsilon_{\min}^{-1}, \quad \forall x \in \bar\Omega. \tag{3.18}$$

This suggests the introduction of the adjoint problem

$$G \in H_0^1: \quad (\nabla w, \nabla G) + (w, bG) = (w, g), \quad \forall w \in H_0^1, \tag{3.19}$$

with data $g \in L_{p'}$. Combining (3.16) and (3.19), and using Lemma 3.1, we get

$$|(e,g)| = |(\nabla e, \nabla G) + (be, G)| = |\langle R, G\rangle| \le C\|h^2 R_p\|_{L_p}\|D^2 G\|_{L_{p'}}. \tag{3.20}$$

Here we need to bound $\|D^2G\|_{L_{p'}}/\|g\|_{L_{p'}}$. Trying first $p = p' = 2$, we are only able to show, using (2.4), assuming temporarily that Ω is convex,

$$\|D^2 G\|_{L_2} \le C\|\Delta G\|_{L_2} \le C\big(\|g\|_{L_2} + \|bG\|_{L_2}\big) \le C\big(1 + \epsilon_{\min}^{-1/2}\big)\|g\|_{L_2};$$

see (3.32) below. This bound is too large to be useful, reflecting the fact that the nonlinearity is "too strong" to be controlled in the L_2-norm. On the other hand we have $\|bG\|_{L_1} \le \|g\|_{L_1}$, see (3.32), and we therefore use $p = \infty$, $p' = 1$, in (3.20). However, $\|D^2G\|_{L_1}/\|g\|_{L_1}$ cannot be estimated by (2.4) for general data g and we therefore proceed as in [20] and choose g of a special form.

Let $x_0 \in \Omega$ and let $g = g_{x_0}$ be a regularized δ-function such that

$$\int_{\mathbf{R}^d} g\,dx = 1; \quad \operatorname{supp} g \subset \mathcal{B}(x_0;\rho); \quad 0 \le g(x) \le C\rho^{-d}. \tag{3.22}$$

Here $\mathcal{B}(x_0;\rho)$ denotes the closed ball with center x_0 and small radius ρ to be chosen.

Lemma 3.2. *Let e satisfy (3.16) and $x_0 \in \Omega$ be such that $\|e\|_{L_\infty} = |e(x_0)|$. Let $g = g_{x_0}$ be as in (3.22). There are constants $h^* > 0$ and $\sigma^* > 0$, depending only on Ω and the constants in Section 2, such that, if $\rho \le h_{\max}^\sigma$ with $\sigma > \sigma^*$ and $h_{\max} \le h^*$, then*

$$\|e\|_{L_\infty} \le |(e,g)| + \|h^2 R_\infty\|_{L_\infty}. \tag{3.23}$$

Proof. In this proof we extend all functions by 0 outside Ω. By the mean value theorem there is $x_1 \in \mathcal{B}(x_0;\rho) \cap \bar\Omega$ such that $(e,g) = e(x_1)$. In order to estimate $e(x_0) - e(x_1)$ we use a classical Hölder estimate of DeGiorgi and Nash: there are constants $K > 0$ and $\alpha \in (0,1)$ such that if $v \in H_0^1$ is a weak solution of $-\Delta v = \nabla \cdot F + G$, where $F \in (L_p)^d$, $G \in L_{p/2}$ for some $p > d$, then for all $x_0 \in \Omega$, $\rho > 0$

$$\sup_{x,y \in \mathcal{B}(x_0;\rho)\cap\bar\Omega} |v(x) - v(y)| \le K\rho^\alpha\big(\|F\|_{L_p} + \|G\|_{L_{p/2}}\big).$$

The constants K and α depend only on p and Ω; see [14, Theorem C.2].

We want to apply this result to equation (3.16) written in the form $-\Delta e = R - be$. By a well known characterization of the dual space W_p^{-1} we have

$$\|R\|_{W_p^{-1}} = \inf_F \|F\|_{L_p},$$

where the norms are defined in (2.2) and (2.3), and the infimum is taken over all $F \in (L_p)^d$ such that $R = \nabla \cdot F$, i.e., $\langle R, v\rangle = -\int_\Omega F \cdot \nabla v \, dx$ for all $v \in \mathring{W}^1_{p'}$. Hence, with $p = 4 > d$,

$$|e(x_0) - e(x_1)| \le K\rho^\alpha \big(\|R\|_{W_4^{-1}} + \|be\|_{L_2}\big).$$

Inequality (3.18) and a standard energy argument, based on taking $v = e$ in (3.16), imply

$$\|be\|_{L_2} \le \epsilon_{\min}^{-1/2} \|b^{1/2} e\|_{L_2} \le \epsilon_{\min}^{-1/2} \|R\|_{W_2^{-1}}.$$

Using Hölder's inequality $\|Dv\|_{L_1} \le |\Omega|^{1/p} \|Dv\|_{L_{p'}}$ in (2.3) and then Lemma 3.1 we get

$$\|R\|_{W_p^{-1}} \le |\Omega|^{1/p} \|R\|_{W_\infty^{-1}} \le C\|hR_\infty\|_{L_\infty} \le C h_{\min}^{-1} \|h^2 R_\infty\|_{L_\infty},$$

and we conclude

$$|e(x_0) - e(x_1)| \le C\rho^\alpha \epsilon_{\min}^{-1/2} h_{\min}^{-1} \|h^2 R_\infty\|_{L_\infty}.$$

Recalling assumption (2.7) and taking $\rho \le h_{\max}^\sigma$ we finally get

$$|e(x_0) - e(x_1)| \le C h_{\max}^{\sigma\alpha - 3\gamma/2} \|h^2 R_\infty\|_{L_\infty}$$

and the assertion of the lemma follows with $\sigma^* = \frac{3\gamma}{2\alpha}$. $\square$

It remains to bound $\|D^2 G\|_{L_1} / \|g\|_{L_1} = \|D^2 G\|_{L_1}$.

Lemma 3.3. *Let $G = G_{x_0}$ be the solution of* (3.19) *with data $g = g_{x_0}$ as in* (3.22). *For any $\kappa > 0$ there is C such that, if $\rho \le \epsilon_{\min}^\kappa$, then, for all $x_0 \in \Omega$,*

$$\|D^2 G\|_{L_1} \le C|\log \rho|.$$

Proof. Let first g be arbitrary. We shall prove that, for $1 \le p \le 2$,

$$\|bG\|_{L_p} \le (C\epsilon_{\min})^{-1/p'} \|g\|_{L_p}, \quad \forall g \in L_p. \tag{3.32}$$

From (2.4) and (3.19) we then obtain, for $1 < p \le 2$, (note that $p-1 = p/p' > 1/p'$)

$$\|D^2 G\|_{L_p} \le \frac{C}{p-1} \|\Delta G\|_{L_p} \le Cp'\big(\|g\|_{L_p} + \|bG\|_{L_p}\big) \le Cp' \epsilon_{\min}^{-1/p'} \|g\|_{L_p}. \tag{3.33}$$

Now take g as in (3.22). A direct calculation gives

$$\|g\|_{L_p} \le C\rho^{-d(1-1/p)} = C\rho^{-d/p'},$$

and since $\rho \le \epsilon_{\min}^\kappa$ we conclude from (3.33) with $p' = |\log \rho|$ that

$$\|D^2 G\|_{L_1} \le C\|D^2 G\|_{L_p} \le Cp' \rho^{-c/p'} = C|\log \rho|.$$

We now prove (3.32). Taking $w = G/\sqrt{G^2 + \delta}$ with $\delta > 0$ in (3.19), and passing to the limit as $\delta \downarrow 0$, yields

$$\|bG\|_{L_1} \le \|g\|_{L_1}, \quad \forall g \in L_1. \tag{3.36}$$

Taking $w = G$ in (3.19) shows $\|b^{1/2} G\|_{L_2} \le C\|g\|_{L_2}$, so that in view of (3.18),

$$\|bG\|_{L_2} \le \epsilon_{\min}^{-1/2} \|b^{1/2} G\|_{L_2} \le (C\epsilon_{\min})^{-1/2} \|g\|_{L_2}, \quad \forall g \in L_2. \tag{3.37}$$

The Riesz-Thorin theorem applied to the linear operator $g \mapsto bG$ now yields

$$\sup_g \frac{\|bG\|_{L_p}}{\|g\|_{L_p}} \le \Big(\sup_g \frac{\|bG\|_{L_1}}{\|g\|_{L_1}}\Big)^{1-2/p'} \Big(\sup_g \frac{\|bG\|_{L_2}}{\|g\|_{L_2}}\Big)^{2/p'} \le (C\epsilon_{\min})^{-1/p'},$$

which is (3.32). ☐

We now state the main result of this section. Recall that R_∞ is defined in (3.6).

Theorem 3.4. *There are $C > 0$ and $h^* > 0$ such that, if $h_{\max} \le h^*$, then*

$$\|U_\epsilon - u_\epsilon\|_{L_\infty} \le C|\log h_{\max}|\,\|h^2 R_\infty\|_{L_\infty}.$$

Proof. Take $\rho = h^\sigma_{\max}$ with σ as in Lemma 3.2. In view of (2.7) we may find κ such that $h^\sigma_{\max} = \rho \le \epsilon^\kappa_{\min}$ and Lemma 3.3 applies. The error bound now follows from (3.20) in conjunction with these lemmas. ☐

Remark 3.1. The constant C of Lemma 3.3 enters as a "stability factor" in the error bound. The essence of Lemma 3.3 is that this stability factor is of moderate size; in particular, it is almost independent of ϵ.

4. The Penalization Error: Smooth Obstacle

We now compare the solutions u and u_ϵ of problems (1.1) and (1.4), respectively. In the following lemma we assume that the obstacle ψ is smooth.

Lemma 4.1. *Let u and u_ϵ be the solutions of* (1.1) *and* (1.4)*, respectively. If $\psi \in W^2_\infty(\Omega)$, then*

$$\|u - u_\epsilon\|_{L_\infty} \le \|\epsilon(f + \Delta\psi)\|_{L_\infty(\hat\Omega)}, \tag{4.1}$$

where $\hat\Omega = \hat\Omega_\psi = \{x \in \Omega : u(x) - \psi(x) = 0,\ u_\epsilon(x) - \psi(x) \le 0\}$ is the "contact set".

Proof. We define

$$\begin{aligned} \Omega^- &= \{x \in \Omega : u(x) - \psi(x) = 0\}, &\quad \Omega^+ &= \{x \in \Omega : u(x) - \psi(x) > 0\},\\ \Omega^-_\epsilon &= \{x \in \Omega : u_\epsilon(x) - \psi(x) \le 0\}, &\quad \Omega^+_\epsilon &= \{x \in \Omega : u_\epsilon(x) - \psi(x) > 0\}, \end{aligned}$$

so that $\hat\Omega = \Omega^- \cap \Omega^-_\epsilon$. Let $v = u - u_\epsilon$. We shall show that there is a constant C such that

$$\|v\|_{L_q} \le (Cq')^{1/q}\|\epsilon^{1/q'}(f + \Delta\psi)\|_{L_q(\hat\Omega)}, \tag{4.3}$$

for all even integers $q \ge 2$ (q and q' are conjugate exponents). Letting $q \to \infty$ we then obtain (4.1).

In order to prove (4.3) we define

$$\begin{aligned} B(x) &:= \Delta u(x) + f(x) \in \beta(u(x) - \psi(x)),\\ B_\epsilon(x) &:= \Delta u_\epsilon(x) + f(x) = \beta_{\epsilon(x)}(u_\epsilon(x) - \psi(x)), \end{aligned} \tag{4.4}$$

so that, for any even integer $q \ge 2$,

$$(\nabla v, \nabla v^{q-1}) = (B_\epsilon - B, v^{q-1}), \tag{4.5}$$

where the left side is equal to (note that $q - 1 = q/q'$)

$$(\nabla v, \nabla v^{q-1}) = (q-1)\|v^{-1+q/2}\nabla v\|^2_{L_2} = \tfrac{4}{qq'}\|\nabla(v^{q/2})\|^2_{L_2}. \tag{4.6}$$

We now turn to the right side of (4.5). We first show that

$$(B_\epsilon - B, v^{q-1}) \le (B_\epsilon - B, v^{q-1})_{\hat\Omega}. \tag{4.7}$$

This follows from (4.4) and the monotonicity of the graphs β and $\beta_{\epsilon(x)}$. More precisely, if $x \in \Omega^+$, then $B(x) = 0 = \beta_{\epsilon(x)}(u(x) - \psi(x))$, so that

$$(B_\epsilon - B)v^{q-1} = -\Big(\beta_\epsilon(u_\epsilon - \psi) - \beta_\epsilon(u - \psi)\Big)(u_\epsilon - u)v^{q-2} \leq 0 \quad \text{in } \Omega^+.$$

Here we used the monotonicity $(\beta_\epsilon(s) - \beta_\epsilon(t))(s - t) \geq 0$, and the assumption that q is even. Similarly, if $x \in \Omega_\epsilon^+$, then $B_\epsilon(x) = 0 \in \{0\} = \beta(u_\epsilon(x) - \psi(x))$, so that

$$(B_\epsilon - B)v^{q-1} \in -\Big(\beta(u_\epsilon - \psi) - \beta(u - \psi)\Big)(u_\epsilon - u)v^{q-2} \subset \mathbf{R}^- \quad \text{in } \Omega_\epsilon^+,$$

since $(\beta(s) - \beta(t))(s - t) \subset \mathbf{R}^+$. Therefore, $(B_\epsilon - B, v^{q-1})_{\Omega^+ \cup \Omega_\epsilon^+} \leq 0$ and (4.7) follows.

It now remains to bound the right side of (4.7). In order to do so, we note that in Ω^- we have $u = \psi$, so that $v = \psi - u_\epsilon$, and $\Delta u = \Delta\psi$ a. e., so that $B = f + \Delta\psi$ a. e. in Ω^-. (Note that this is where the smoothness of ψ is required; it implies that u is smooth, whence $u = \psi$ yields $\Delta u = \Delta\psi$.) Moreover, in Ω_ϵ^- we have $B_\epsilon = -\epsilon^{-1}(\psi - u_\epsilon)$ in view of (4.4).

Summing up: in $\hat{\Omega} = \Omega^- \cap \Omega_\epsilon^-$ we have $B = f + \Delta\psi$, $B_\epsilon = -\epsilon^{-1}v$, so that

$$\begin{aligned} (B_\epsilon - B, v^{q-1})_{\hat{\Omega}} &= (-\epsilon^{-1}v - (f + \Delta\psi), v^{q-1})_{\hat{\Omega}} \\ &= -\|\epsilon^{-1/q}v\|_{L_q(\hat{\Omega})}^q - (f + \Delta\psi, v^{q-1})_{\hat{\Omega}}. \end{aligned} \tag{4.10}$$

Using Hölder's and Young's inequalities

$$|(f, g)| \leq \|f\|_{L_q}\|g\|_{L_{q'}} \leq \tfrac{1}{q}\|f\|_{L_q}^q + \tfrac{1}{q'}\|g\|_{L_{q'}}^{q'},$$

we get

$$\begin{aligned} \big|(f + \Delta\psi, v^{q-1})_{\hat{\Omega}}\big| &= |(\epsilon^{1/q'}(f + \Delta\psi), (\epsilon^{-1/q}v)^{q/q'})_{\hat{\Omega}}| \\ &\leq \tfrac{1}{q}\|\epsilon^{1/q'}(f + \Delta\psi)\|_{L_q(\hat{\Omega})}^q + (1 - \tfrac{1}{q})\|\epsilon^{-1/q}v\|_{L_q(\hat{\Omega})}^q. \end{aligned} \tag{4.12}$$

Combining (4.5), (4.6), (4.7), (4.10), and (4.12), we conclude

$$\tfrac{4}{q'}\|\nabla(v^{q/2})\|_{L_2}^2 + \|\epsilon^{-1/q}v\|_{L_q(\hat{\Omega})}^q \leq \|\epsilon^{1/q'}(f + \Delta\psi)\|_{L_q(\hat{\Omega})}^q. \tag{4.13}$$

Using also Poincaré's inequality

$$\|\nabla(v^{q/2})\|_{L_2}^2 \geq c\|v^{q/2}\|_{L_2}^2 = c\|v\|_{L_q}^q,$$

we finally obtain (4.3). □

Remark 4.1. Using $q = 2$ in (4.13) we get

$$2\|\nabla(u - u_\epsilon)\|_{L_2}^2 + \|\epsilon^{-1/2}(\psi - u_\epsilon)\|_{L_2(\hat{\Omega})}^2 \leq \|\epsilon^{1/2}(f + \Delta\psi)\|_{L_2(\hat{\Omega})}^2, \tag{4.15}$$

which is (a slight improvement of) the corresponding result derived in [23] and used in [13]. More precisely, in [23] it is assumed that $-\Delta\psi \leq 0$, so that in (4.10) $(-\Delta\psi, v^{q-1})_{\hat{\Omega}} = (-\Delta\psi, (\psi - u_\epsilon)^{q-1})_{\hat{\Omega}} \leq 0$, and this term may be dropped. The right side of (4.13) then becomes $\|\epsilon^{1/q'}f\|_{L_q(\hat{\Omega})}^q$. A similar comment applies to [13], where it is assumed that $\psi = 0$ after the change of dependent variable $u \leftarrow u - \psi$. Note also that we localize the error bound to the contact set; this was not done in either [23] or [13].

Remark 4.2. Comparing with a similar analysis in [19], we note that we allow a variable penalty parameter $\epsilon = \epsilon(x)$, and we do not truncate the nonlinearities β and β_ϵ.

Remark 4.3. Lemma 4.1 immediately generalizes to the "double obstacle problem;" cf. [23], where it is required that $\psi(x) \le u(x) \le \phi(x)$, and where the nonlinear terms are replaced by

$$\beta(u(x) - \psi(x)) - \beta(\phi(x) - u(x)), \quad \beta_\epsilon(u_\epsilon(x) - \psi(x)) - \beta_\epsilon(\phi(x) - u_\epsilon(x)).$$

The result is

$$\|u - u_\epsilon\|_{L_\infty} \le \|\epsilon(f + \Delta\psi)\|_{L_\infty(\hat\Omega_\psi)} + \|\epsilon(f + \Delta\phi)\|_{L_\infty(\hat\Omega_\phi)},$$

with the obvious definitions of the contact sets $\hat\Omega_\psi, \hat\Omega_\phi$.

To conclude this section we further point out that, in view of Theorem 3.4 and Lemma 4.1, we readily obtain an almost *a posteriori* error bound for the entire approximation procedure.

Theorem 4.2. *Let u be the solution of* (1.1) *and U_ϵ that of* (1.7)*. There are $C > 0$ and $h^* > 0$ such that if $\psi \in W^2_\infty(\Omega)$ and $h_{\max} \le h^*$, then*

$$\|u - U_\epsilon\|_{L_\infty} \le C|\log h_{\max}|\,\|h^2 R_\infty\|_{L_\infty} + \|\epsilon(f + \Delta\psi)\|_{L_\infty(\hat\Omega)}, \tag{4.18}$$

where $\hat\Omega = \{x \in \Omega : u(x) - \psi(x) = 0,\ u_\epsilon(x) - \psi(x) \le 0\}$ is the "contact set".

Remark 4.4. The error bound in Theorem 4.2 is not computable because the set $\hat\Omega$ is defined in terms of functions that are not known. To remedy this, we define a computable set

$$\Omega_\tau = \{x \in \Omega : U_\epsilon(x) - \psi(x) \le \tau\},$$

where $\tau > 0$ is a given tolerance. We claim that if

$$\|u_\epsilon - U_\epsilon\|_{L_\infty} \le \tau, \tag{4.19}$$

then $\hat\Omega \subset \Omega_\tau$. In fact, if $x \in \hat\Omega$, then

$$(U_\epsilon - \psi)(x) = (U_\epsilon - u_\epsilon)(x) + (u_\epsilon - \psi)(x) \le \tau + 0 = \tau.$$

Remark 4.5. The basis for the corresponding algorithm of Section 6 is as follows. Produce an appropriate mesh so that (4.19) holds. Then evaluate the second term on the right side of (4.18), replacing $\hat\Omega$ by Ω_τ.

Remark 4.6. Since $\beta_\epsilon(U_\epsilon - \psi) = 0$ on $\Omega \setminus \hat\Omega$, the size of ϵ in the domain $\Omega \setminus \hat\Omega$ does not affect the choice of meshsize h in $\Omega \setminus \hat\Omega$. Therefore, singularities of ψ in $\Omega \setminus \hat\Omega$ will not yield unnecessary refinement.

Remark 4.7. In light of Theorem 4.2 we may expect $\epsilon(x)$ and $h(x)$ to satisfy the *local* relation

$$\epsilon(x) \approx h^2(x), \quad \forall x \in \Omega. \tag{4.20}$$

A similar, but global, relation with ϵ constant was derived in [19] as a result of an *a priori* analysis.

5. The penalization error: rough obstacle

The proof of Lemma 4.1 does not generalize to the case when the obstacle function ψ has less than two derivatives in L_∞. This situation is analyzed in the following lemmas by means of auxiliary problems with a regularized obstacle. We stress that the auxiliary problems are only used in the analysis and thus do not alter the definitions of u_ϵ and U_ϵ. We assume the existence of $\mu \in (0, 2]$ such that

$$\psi \in C^\mu(\bar\Omega), \tag{5.1}$$

which, in case μ is not an integer, means that the derivatives of order $[\mu]$ are Hölder continuous with exponent $\mu - [\mu]$.

We start by examining the additional regularization of ψ. Let δ be a smooth function satisfying, for some constant C^*,

$$|D^k \delta(x)| \le C^* \delta(x)^{1-k}, \quad k = 0, 1, 2, \ \forall x \in \Omega. \tag{5.2}$$

This function will be used as a space dependent regularization parameter, and can thus be viewed as a distance function. Its existence is guaranteed by the following lemma.

Lemma 5.1. *For every mesh $\mathcal{T} \in \mathcal{F}$ there exists a positive function $\delta \in C^\infty(\Omega)$ satisfying*

$$ch(x)^{1-k} \le |D^k \delta(x)| \le C_k h(x)^{1-k}, \quad \forall k \ge 0, \ x \in \Omega. \tag{5.3}$$

Proof. The case $d = 2$ is proved in [21, Lemma 5.1]. The proof, which is based on a C^∞ partition of unity and the fact that any regular mesh is locally quasi-uniform, is readily adapted to any dimension d. □

We next introduce a measure of the regularity of ψ. We set, for $\nu \in (0, 2]$,

$$(\mathcal{D}^\nu_\delta \psi)(x) = \max_{y \in \mathcal{B}(x;\delta(x))} \frac{|\psi(y) - \psi(x) - V(x) \cdot (y - x)|}{\delta(x)^\nu}, \tag{5.4}$$

$$V(x) = \begin{cases} 0, & \text{if } \nu \in (0, 1], \\ \nabla\psi(x), & \text{if } \nu \in (1, 2]. \end{cases} \tag{5.5}$$

We are going to define, via convolution, a regularization ψ^δ of ψ with parameter $\delta = \delta(x)$ as in (5.2). Let $\zeta \in C^\infty(\mathbf{R}^d)$ be positive, radially symmetric, with support in $\mathcal{B}(0; 1)$ and $\int \zeta \, dz = 1$. We define

$$\varphi^\delta(y, x) = \frac{1}{\delta(x)^d} \zeta\Big(\frac{y - x}{\delta(x)}\Big), \tag{5.6}$$

$$\psi^\delta(x) = \int_{\mathbf{R}^d} \psi(y) \varphi^\delta(y, x) \, dy, \tag{5.7}$$

where ψ is extended outside Ω as a C^μ-function.

Lemma 5.2. *There exists $C > 0$, depending only on C^*, such that, for all $x \in \Omega$,*

$$|\psi^\delta(x) - \psi(x)| \le \delta(x)^\nu (\mathcal{D}^\nu_\delta \psi)(x), \tag{5.8}$$

$$|\Delta\psi^\delta(x)| \le C\delta(x)^{\nu-2} (\mathcal{D}^\nu_\delta \psi)(x). \tag{5.9}$$

Proof. The following properties of φ^δ are obvious from its definition:

$$\int_{\mathbf{R}^d} \varphi^\delta(y, x) \, dy = 1, \tag{5.10}$$

$$\int_{\mathbf{R}^d} \frac{\partial}{\partial x_i} \varphi^\delta(y, x) \, dy = 0, \tag{5.11}$$

$$\int_{\mathbf{R}^d} (y - x) \varphi^\delta(y, x) \, dy = 0. \tag{5.12}$$

Taking the derivative of the i^{th} component of (5.12) with respect to x_j and using (5.10) yields

$$\int_{\mathbf{R}^d} (y - x)_i \frac{\partial}{\partial x_j} \varphi^\delta(y, x) \, dy = \delta_{ij} \quad \text{(Kronecker's delta)}. \tag{5.13}$$

Taking the derivative of this with respect to x_k and using (5.11) yields

$$\int_{\mathbf{R}^d} (y-x)_i \frac{\partial^2}{\partial x_k \partial x_j} \varphi^\delta(y,x)\,dy = 0. \tag{5.14}$$

From the definition (5.7) of ψ^δ, (5.10) and (5.12), we immediately obtain

$$\begin{aligned} |\psi^\delta(x) - \psi(x)| &= \Big| \int_{\mathbf{R}^d} \big(\psi(y) - \psi(x) - V(x)\cdot(y-x)\big)\varphi^\delta(y,x)\,dy \Big| \\ &\le \delta(x)^\nu \int_{\mathbf{R}^d} \frac{|\psi(y) - \psi(x) - V(x)\cdot(y-x)|}{\delta(x)^\nu} \varphi^\delta(y,x)\,dy \\ &\le \delta(x)^\nu \mathcal{D}_\delta^\nu \psi(x), \end{aligned} \tag{5.15}$$

and (5.8) follows. In a similar way, using also (5.14), we get

$$\begin{aligned} |\Delta\psi^\delta(x)| &= \Big| \int_{\mathbf{R}^d} \big(\psi(y) - \psi(x) - V(x)\cdot(y-x)\big)\Delta_x\varphi^\delta(y,x)\,dy \Big| \\ &\le \delta(x)^\nu \int_{\mathbf{R}^d} \frac{|\psi(y) - \psi(x) - V(x)\cdot(y-x)|}{\delta(x)^\nu} |\Delta_x\varphi^\delta(y,x)|\,dy \\ &\le \delta(x)^\nu \mathcal{D}_\delta^\nu \psi(x) \int_{\mathbf{R}^d} |\Delta_x\varphi^\delta(y,x)|\,dy, \end{aligned} \tag{5.16}$$

and (5.9) follows once we have shown

$$|\Delta_x\varphi^\delta(y,x)| \le C\delta(x)^{-2} N(y,x), \quad \text{with} \int_{\mathbf{R}^d} N(y,x)\,dy \le C. \tag{5.17}$$

A tedious but straightforward calculation starting from (5.6) gives

$$|\Delta_x\varphi^\delta(y,x)| \le \delta(x)^{-2}\Big(1 + |\nabla\delta(x)| + |\nabla\delta(x)|^2 + \delta(x)|\Delta\delta(x)|\Big) N(y,x), \tag{5.18}$$

where $N(y,x)$ is a linear combination of

$$\frac{1}{\delta(x)^d}\Big(\frac{|y-x|}{\delta(x)}\Big)^j \Big| D^k\zeta\Big(\frac{y-x}{\delta(x)}\Big)\Big|, \quad j,k = 0,1,2, \tag{5.19}$$

and $|D^k\zeta(z)|$ denotes the norm of the k^{th} order gradient of ζ at z. Together with (5.2) this implies (5.17), and concludes the proof. $\square$

Let u^δ and u^δ_ϵ denote the solutions of (1.1) and (1.4), respectively, with ψ replaced by ψ^δ and with Dirichlet boundary data $\max(0,\psi^\delta)$; note that the convolution may cause $\psi^\delta > 0$ on $\partial\Omega$. The following lemma provides a localized error estimate of the effect of penalization. To state it, we need the enlarged contact set

$$\tilde{\Omega}_0 = \bigcup_{i=1}^4 \Xi_i \tag{5.20}$$

defined by

$$\begin{aligned} \Xi_1 &:= \{x\in\Omega : u^\delta(x) - \psi^\delta(x) = 0\}, \quad \Xi_2 := \{x\in\Omega : u(x) - \psi(x) = 0\}, \\ \Xi_3 &:= \{x\in\Omega : u^\delta_\epsilon(x) - \psi^\delta(x) \le 0\}, \quad \Xi_4 := \{x\in\Omega : u_\epsilon(x) - \psi(x) \le 0\}. \end{aligned}$$

Lemma 5.3. *Let $\nu \in (0,2]$ and let δ satisfy (5.2). Let u and u_ϵ be the solutions of (1.1) and (1.4), respectively. There exist C and δ^* such that if (5.1) is valid and $\|\delta\|_{L_\infty} \le \delta^*$, then*

$$\|u - u_\epsilon\|_{L_\infty} \le \|\epsilon f\|_{L_\infty(\tilde{\Omega}_0)} + C\|\epsilon\delta^{\nu-2}\mathcal{D}_\delta^\nu\psi\|_{L_\infty(\tilde{\Omega}_0)} + 2\|\delta^\nu\mathcal{D}_\delta^\nu\psi\|_{L_\infty(\tilde{\Omega}_0)}. \tag{5.21}$$

Proof. We have

$$\|u - u_\epsilon\|_{L_\infty} \le \|u - u^\delta\|_{L_\infty} + \|u^\delta - u^\delta_\epsilon\|_{L_\infty} + \|u^\delta_\epsilon - u_\epsilon\|_{L_\infty}. \tag{5.22}$$

Lemma 4.1 gives

$$\|u^\delta - u^\delta_\epsilon\|_{L_\infty} \le \|\epsilon(f + \Delta\psi^\delta)\|_{L_\infty(\hat\Omega_{\psi^\delta})} \le \|\epsilon f\|_{L_\infty(\tilde\Omega_0)} + \|\epsilon\Delta\psi^\delta\|_{L_\infty(\tilde\Omega_0)}, \tag{5.23}$$

because $\hat\Omega_{\psi^\delta} = \{x \in \Omega : u^\delta(x) - \psi^\delta(x) = 0,\ u^\delta_\epsilon(x) - \psi^\delta(x) \le 0\}$ is contained in $\tilde\Omega_0$; note that $u^\delta - u^\delta_\epsilon = 0$ on $\partial\Omega$, whence the proof of Lemma 4.1 is still valid. In view of (5.9) we readily get

$$\|\epsilon\Delta\psi^\delta\|_{L_\infty(\tilde\Omega_0)} \le C\|\epsilon\delta^{\nu-2}\mathcal{D}^\nu_\delta\psi\|_{L_\infty(\tilde\Omega_0)}. \tag{5.24}$$

We now assert that

$$\|u - u^\delta\|_{L_\infty} \le \|\psi - \psi^\delta\|_{L_\infty(\Xi_1\cup\Xi_2)}, \tag{5.25}$$

$$\|u_\epsilon - u^\delta_\epsilon\|_{L_\infty} \le \|\psi - \psi^\delta\|_{L_\infty(\Xi_3\cup\Xi_4)}, \tag{5.26}$$

which combined with (5.8) yield (5.21). We resort to the maximum principle to prove (5.25) and (5.26). In order to prove (5.25) we define

$$\sigma = \|\psi - \psi^\delta\|_{L_\infty(\Xi_1\cup\Xi_2)},$$

set $w = \sigma + u$, and show that $w \ge u^\delta$. Suppose by contradiction that the open set $\mathcal{G} = \{x \in \Omega : w(x) < u^\delta(x)\}$ is nonempty. Since $\sigma \ge \psi - \psi^\delta$ in Ξ_1, we deduce that $\mathcal{G} \cap \Xi_1$ is empty, because

$$u^\delta - \psi^\delta > w - \psi^\delta = \sigma + u - \psi^\delta \ge \sigma + \psi - \psi^\delta \ge 0 \quad \text{in } \mathcal{G} \cap \Xi_1$$

which is inconsistent with the definition of Ξ_1. In addition, $u^\delta - \psi^\delta > 0$ outside Ξ_1, so that $\beta(u^\delta - \psi^\delta) = \{0\}$ in $\mathcal{G}$.

Also the set $\mathcal{G} \cap \partial\Omega$ is empty, because in $\mathcal{G} \cap \partial\Omega$ we have

$$0 > u + \sigma - u^\delta = \sigma - \max\{0, \psi^\delta\},$$

which implies $u^\delta = \psi^\delta$ and hence $\mathcal{G} \cap \partial\Omega \subset \Xi_1$. But then, since $\sigma \ge \psi^\delta - \psi$, we have $0 > \sigma - \psi^\delta \ge -\psi \ge 0$ in $\mathcal{G} \cap \partial\Omega$, which is a contradiction.

By the variational inequalities defining u^δ and u, we thus have

$$-\Delta u^\delta = f \le -\Delta u = -\Delta w \quad \text{in } \mathcal{G}.$$

Hence $-\Delta(w - u^\delta) \ge 0$ in $\mathcal{G}$, $w - u^\delta = 0$ on $\partial\mathcal{G}$, so that $w - u^\delta \ge 0$ in $\mathcal{G}$, which is a contradiction. This proves $w \ge u^\delta$, i.e., $u^\delta - u \le \sigma$. The inequality $u - u^\delta \le \sigma$ is proved by interchanging the roles of ψ and ψ^δ, setting $w = \sigma + u^\delta$, and making use of Ξ_2.

The proof of (5.26) is accomplished in a similar way: we define $\sigma = \|\psi - \psi^\delta\|_{L_\infty(\Xi_3\cup\Xi_4)}$, $w = \sigma + u_\epsilon$ and show that $w \ge u^\delta_\epsilon$. Suppose by contradiction that the open set $\mathcal{G} = \{x \in \Omega : w(x) < u^\delta_\epsilon(x)\}$ is nonempty. Since $\sigma \ge \psi - \psi^\delta$ in Ξ_3 we have in $\mathcal{G} \cap \Xi_3$

$$u^\delta_\epsilon - \psi^\delta \ge w - \psi^\delta = \sigma + u_\epsilon - \psi^\delta \ge u_\epsilon - \psi,$$

whence, exploiting the monotonicity of β_ϵ,

$$\beta_\epsilon(u^\delta_\epsilon - \psi^\delta) \ge \beta_\epsilon(u_\epsilon - \psi).$$

On the other hand, we have $u^\delta_\epsilon - \psi^\delta > 0$ outside Ξ_3 and consequently

$$0 = \beta_\epsilon(u^\delta_\epsilon - \psi^\delta) \ge \beta_\epsilon(u_\epsilon - \psi).$$

Thus $\beta_\epsilon(u_\epsilon^\delta - \psi^\delta) \geq \beta_\epsilon(u_\epsilon - \psi)$ in $\mathcal{G}$ and therefore, from the equations defining u_ϵ^δ and u_ϵ we infer that

$$-\Delta u_\epsilon^\delta = f - \beta_\epsilon(u_\epsilon^\delta - \psi^\delta) \leq f - \beta_\epsilon(u_\epsilon - \psi) = -\Delta u_\epsilon = -\Delta w \quad \text{in } \mathcal{G}.$$

We check that $\mathcal{G} \cap \partial\Omega$ is empty in the same way as above. This implies that $-\Delta(w - u_\epsilon^\delta) \geq 0$ in $\mathcal{G}$, and since $w - u_\epsilon^\delta = 0$ on $\partial\mathcal{G}$, that $w - u_\epsilon^\delta \geq 0$ in $\mathcal{G}$, which is a contradiction. This proves $w \geq u_\epsilon^\delta$, i.e., $u_\epsilon^\delta - u_\epsilon \leq \sigma$. The inequality $ue - u_\epsilon^\delta \leq \sigma$ is proved by interchanging the roles of ψ and ψ^δ, setting $w = \sigma + u_\epsilon^\delta$, and making use of Ξ_4. ☐

We see from (5.21) that a natural relation between δ and ϵ is

$$\epsilon(x) \approx \delta(x)^2, \quad \forall x \in \tilde{\Omega}_0. \tag{5.35}$$

Coupling Theorem 3.4 with Lemma 5.3, we readily obtain an almost *a posteriori* error estimate for the whole approximation process.

Theorem 5.4. *Let $\nu \in (0, 2]$ and let δ satisfy* (5.2)*. Let u and u_ϵ be the solutions of* (1.1) *and* (1.4)*, respectively. There exist C, $h^* > 0$, and $\delta^* > 0$ such that if* (5.1) *is valid, $h_{\max} \leq h^*$, and $\|\delta\|_{L_\infty} \leq \delta^*$, then (with $\tilde{\Omega}_0$ defined in* (5.20)*)*

$$\begin{aligned} \|u - U_\epsilon\|_{L_\infty} &\leq C|\log h_{\max}|\, \|h^2 R_\infty\|_{L_\infty} \\ &\quad + C\Big(\|\epsilon f\|_{L_\infty(\tilde{\Omega}_0)} + \|\epsilon\delta^{\nu-2}\mathcal{D}_\delta^\nu \psi\|_{L_\infty(\tilde{\Omega}_0)} + \|\delta^\nu \mathcal{D}_\delta^\nu \psi\|_{L_\infty(\tilde{\Omega}_0)}\Big). \end{aligned} \tag{5.36}$$

Remark 5.1. As in Theorem 4.2 for smooth obstacles, the error bound (5.36) is not computable because the enlarged contact set $\tilde{\Omega}_0$ is defined in terms of functions that are not known. We now show that $\tilde{\Omega}_0$ may be replaced by a slightly larger, but computable, set Ω_τ. For a given tolerance $\tau > 0$ let

$$\Omega_\tau = \{x \in \Omega : U_\epsilon(x) - \psi(x) \leq 3\tau\}.$$

We recall from (5.20) that $\tilde{\Omega}_0 = \bigcup_{i=1}^4 \Xi_i$ and suppose $x \in \Xi_1$ so that $(u^\delta - \psi^\delta)(x) = 0$. From now on we assume that the full right side of (5.36) is less than or equal to τ. From Lemma 5.3 and its proof it follows that the quantities $\|u^\delta - u\|_{L_\infty}$ and $\|\psi - \psi^\delta\|_{L_\infty}$ are each less than or equal to τ. Then

$$U_\epsilon - \psi = (U_\epsilon - u) + (u - u^\delta) + (u^\delta - \psi^\delta) + (\psi^\delta - \psi) \leq 3\tau;$$

thus $\Xi_1 \subset \Omega_\tau$. The remaining inclusions $\Xi_i \subset \Omega_\tau$ are obtained in the same way. Thus $\tilde{\Omega}_0 \subset \Omega_\tau$ if the right side of (5.36) is less than or equal to τ.

Remark 5.2. An important consequence of the *localized* estimate (5.36) is that if the obstacle ψ were rough outside the enlarged contact set $\tilde{\Omega}_0$ it would not lead to unnecessary mesh refinement in $\Omega \setminus \tilde{\Omega}_0$. On the other hand, the size of ϵ in $\Omega \setminus \tilde{\Omega}_0$ does not affect the choice of meshsize h in $\Omega \setminus \tilde{\Omega}_0$ because $\beta_\epsilon(U_\epsilon - \psi) = 0$. This is consistent with both Remark 4.6 for smooth obstacles and Example 6.5 below.

6. Numerical experiments

In this section we discuss the results of preliminary computations in 1D using adaptive algorithms with h and ϵ refinement. We examine three distinct examples: one has a smooth obstacle and the other two have rough obstacles with jump discontinuities in the first derivative. In the first rough obstacle example the discontinuity occurs inside the contact set and we observe the algorithm setting ϵ to a very small number and refining the mesh heavily near this point. In the second rough obstacle example the discontinuity is outside the contact set. The algorithm sets ϵ to a small value near this point but does not refine the mesh heavily there, since the solution does not touch the rough obstacle and hence is smooth. The small value of ϵ outside the contact set does not degrade the performance of the iteration scheme, since β_ϵ is zero in that region. Our preliminary conclusions are that the *a posteriori* estimates provide useful upper bounds on the true errors and the possibility to vary ϵ provides more accuracy in the rough obstacle case.

6.1. **Implementation.** Let $\tau > 0$ be a given tolerance, and let Ω_τ be the computable set of either Remark 4.4 or Remark 5.1. We use the *a posteriori* estimate

$$\|u - U_\epsilon\|_{L_\infty(\Omega)} \leq C_1 \max_{K\in\mathcal{T}} h_K^2 \|R_\infty\|_{L_\infty(K)} + C_2\|\epsilon(f + \Delta\psi)\|_{L_\infty(\Omega_\tau)} \tag{6.1}$$

for both smooth obstacles (Theorem 4.2) and rough obstacles (Theorem 5.4), in the latter case with the second term on the right side replaced by

$$C_2 \left(\|\epsilon f\|_{L_\infty(\Omega_\tau)} + \|\epsilon \mathcal{D}_h^2 \psi\|_{L_\infty(\Omega_\tau)}\right).$$

This keeps both algorithms almost identical and reflects the local nature of the *a posteriori* estimates (4.18) and (5.36). The term $\mathcal{D}_h^2\psi$ is defined by a difference at nodes on the mesh.

We use continuous piecewise linear and quadratic finite element functions in our approximations. In order to linearize we use Newton's method and thus solve the following problem on each iteration: Find $U_\epsilon^{j+1} \in V_h$ such that

$$(\nabla U_\epsilon^{j+1}, \nabla\chi) + (\beta_\epsilon(U_\epsilon^j - \psi) + (\beta_\epsilon'(U_\epsilon^j - \psi)(U_\epsilon^{j+1} - U_\epsilon^j), \chi)_h = (f,\chi)_h,$$

for all $\chi \in V_h$. The inner products with subscript h are Gauss quadrature approximations of the corresponding integrals.

6.2. **Adaptive Algorithm.** We now describe the full adaptive algorithm for the smooth obstacle example.

<u>Choose</u> the functions ϵ_0, U_ϵ^{-1}, and U_ϵ^0; define a mesh $\mathcal{T}_0$ (we started with a uniform mesh with N element domains); set index $j = 0$; and define tolerances τ (for adaptivity) and Tol (for the nonlinear iteration).
<u>Do Until</u>
$C_1 \max_{K\in\mathcal{T}_j} h_K^2 \|R_\infty\|_{L_\infty(K)} + C_2\|\epsilon(f+\Delta\psi)\|_{L_\infty(\Omega_\tau)} \leq \tau$ and $\|U_\epsilon^j - U_\epsilon^{j-1}\|_{L_\infty(\Omega)} \leq$ Tol:

- Compute U_ϵ^{j+1} on mesh $\mathcal{T}_j$ with function ϵ_j and using function U_ϵ^j to linearize.
- Define new mesh $\mathcal{T}_{j+1}$ by refining any elements, K, in the mesh $\mathcal{T}_j$ where $h_K^2\|R_\infty(U_\epsilon^{j+1}, \epsilon_j)\|_{L_\infty(K)} > \tau/2$.
- Define $\Omega_{j+1} = \{x \in \Omega : U_\epsilon^{j+1} - \psi \leq \tau\}$.
- Choose $\epsilon_{j+1} = \tau/(2\max\{1, |f + \Delta\psi|\})$ on the set Ω_{j+1}.
- Increment j.

<u>End Do Until Loop.</u>

Note that the mesh and ϵ refinement all happen at the same time. Moreover, there is only one Newton step for each pass through the loop.

Note that, in view of the discussions after Theorems 4.2 and 5.4, once the algorithm has converged, except for the possibility that the constants were chosen incorrectly, we expect that the estimate (6.1) with U_ϵ defined as the current iterate is an upper bound for the error.

N	Estimated Error	$\|u - U_\epsilon\|_{L_\infty(\Omega)}$	Rate
12	3.0(-2)	2.6(-2)	—
24	3.2(-3)	2.6(-3)	3.3
71	8.7(-5)	7.6(-5)	3.3
154	5.3(-6)	5.0(-6)	3.5

TABLE 1. Example 6.3: Smooth obstacle.

6.3. **Example: Smooth obstacle.** We take $\Omega = (-1, 1)$, $\psi(x) = 1 - 4x^2$, $f(x) = -8$, and

$$u(x) = \begin{cases} 4x^2 - 16bx - (1 + 16b), & \text{if } x > b, \\ \psi(x), & \text{if } -b \le x \le b, \\ 4x^2 + 16bx - (1 + 16b), & \text{if } x < -b, \end{cases}$$

where $b = 1 - \sqrt{6}/4 \cong 0.3876275\ldots$. Note that b is an irrational number. This ensures that the typical mesh does not have any nodes at the free boundary.

Since quadratics usually give L_∞ a priori estimates with an N^{-3} convergence rate assuming the function being approximated has three derivatives this example is interesting since the true solution has only W^2_∞ regularity with jumps in the second derivative at the free boundary.

Table 1 shows the results of a series of test runs on this problem, where we explore the rate of convergence obtained as a result of the mesh refinement and variable epsilon. We find that $C_1 = 1/50$ is an upper bound for the first constant and that $C_2 = 1$ is sufficient for the second. (Other estimates of the constant in front of the $\|u - u_\epsilon\|_{L_\infty}$ term have shown that the 6 may be replaced by a 1 with the addition of a very small term on the right side of the inequality).

Examining the table we note that the estimated error was above the true error as was generally the case and that the rate was very close to 3 as one would hope for quadratics.

6.4. **Example: Rough obstacle.** We first consider the specific case where the roughness is inside the contact set and thus the true solution, u, is also rough. Let $\Omega = (-1, 1)$, $f(x) = -8$,

$$\tilde{\psi}(x) = \begin{cases} 3 - 4x, & \text{if } x > 0, \\ 3 + 4x, & \text{if } x \le 0, \end{cases} \qquad \tilde{u}(x) = \begin{cases} 4(x-1)^2, & \text{if } x > 1/2, \\ \psi(x), & \text{if } -1/2 \le x \le 1/2, \\ 4(x+1)^2, & \text{if } x < -1/2, \end{cases}$$

and then define $\psi(x) = \tilde{\psi}(x - \delta)$ as well as $u(x) = \tilde{u}(x - \delta)$, where $\delta = \sqrt{2}/20 \cong .0707106\ldots$, so that the point of discontinuity is not likely to occur at a mesh point.

Below we list some results from the experiments with this example. We use continuous piecewise linear approximation functions in this case with $C_1 = C_2 = 2$ in the *a posteriori* error estimate.

Again we note that the mesh and ϵ refinement helps us recover a convergence rate of N^{-2} and that the estimator is an upper bound. The variable ϵ is particularly useful in this example. Near the jump discontinuity it becomes very small to compensate for the large $\mathcal{D}^2_h\psi$ term in the a posteriori estimate.

N	Estimated Error	$\|u - U_\epsilon\|_{L_\infty(\Omega)}$	Rate
74	9.5(-2)	1.2(-2)	—
96	3.3(-2)	5.0(-3)	3.3
168	8.2(-3)	1.3(-3)	2.4
243	4.5(-3)	6.3(-4)	2.0

TABLE 2. Example 6.4: Rough obstacle.

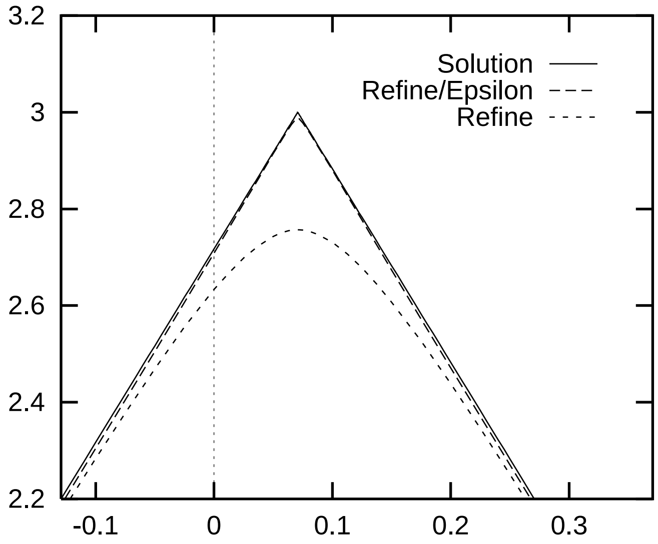

FIGURE 1. Example 6.4: Comparison of the true solution and the refined approximations with variable and constant ϵ.

To demonstrate this we show the results of two computations, one with and one without allowing ϵ to vary in x. In the "Refine" case shown in Figure 6.4 the mesh has 78 elements, the L_∞ error is .243, and $\epsilon = .003$. In the "Refine/Epsilon" graph in Figure 6.4 the mesh has 74 elements, the L_∞ error is .012, the mesh is shown in Figure 6.4, and the function ϵ is shown in Figure 6.4. The (apparent) payoff for the variable ϵ is seen in Figure 6.4, which shows how much closer and sharper the variable ϵ approximation is to the true solution. Note that the results displayed in Figure 6.4 are consistent with the solution function involved. On $[-1, -0.5]$ and $[0.5, 1]$ the size of the mesh elements is small, since u is quadratic, while it is piecewise linear on $(-0.5, 0.5)$. The refinement is heavy at δ, where u' has its only jump.

6.5. **Example: Rough obstacle.** We finally turn to the case, where the rough part of the obstacle is outside the contact set. Again, we take $\Omega = (-1, 1)$, and let $f(x) = -9/2$,

$$\tilde{\psi}(x) = \begin{cases} -3 + 3x, & \text{if } x > 0, \\ -3 - 3x, & \text{if } x \le 0, \end{cases} \qquad \tilde{u}(x) = \begin{cases} \psi(x), & \text{if } x > 2/3, \\ 9x^2/4 - 2, & \text{if } -2/3 \le x \le 2/3, \\ \psi(x) & \text{if } x < -2/3, \end{cases}$$

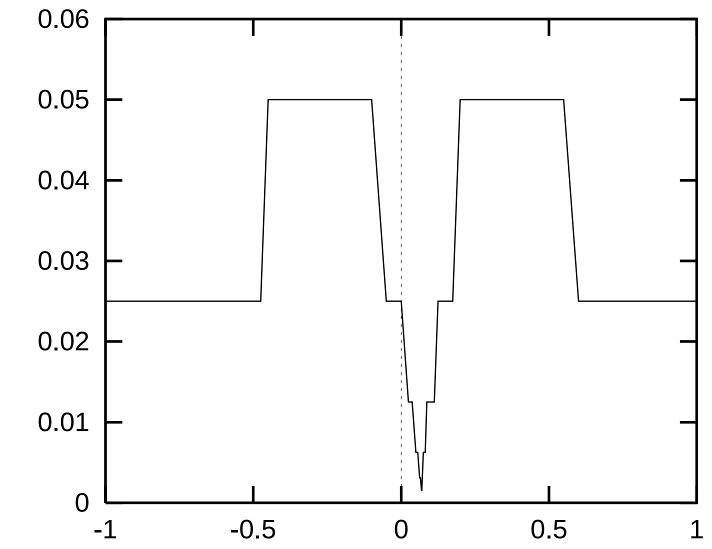

FIGURE 2. Example 6.4: The mesh function h versus x.

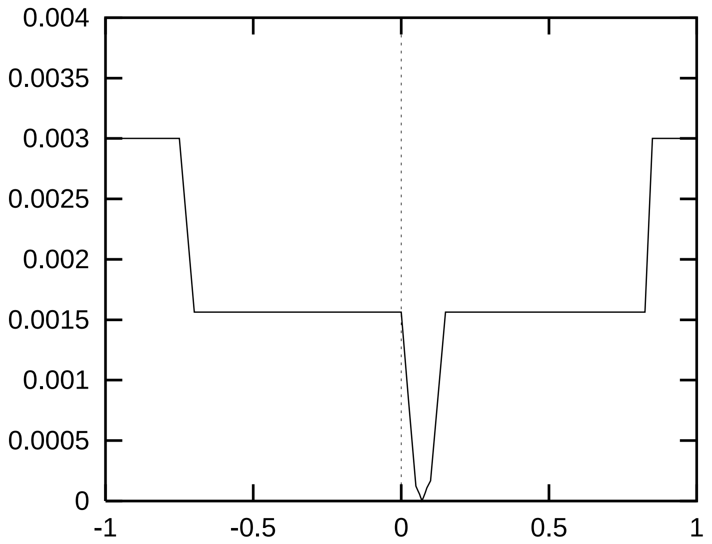

FIGURE 3. Example 6.4: The penalty function ϵ versus x.

and define $\psi(x) = \tilde{\psi}(x - \delta)$ as well as $u(x) = \tilde{u}(x - \delta)$, where $\delta = \sqrt{2}/20 \cong .0707106\ldots$, so that the point of discontinuity is not likely to occur at a mesh point. We use continuous piecewise linear approximation functions, and take the constants in the estimators to each be 2.

Since the jump discontinuity of ψ' is not in the contact set, and so u is smooth, one would expect that the algorithm does not refine such a singularity of ψ and this is what we observe in the experiments. In Table 3 we display some results for

N	Estimated Error	$\|u - U_\epsilon\|_{L_\infty(\Omega)}$	Rate
38	9.5(-2)	1.2(-2)	—
68	2.4(-2)	3.1(-3)	2.3
136	5.8(-3)	7.5(-4)	2.0
275	1.4(-3)	1.9(-4)	2.0

TABLE 3. Example 6.5: Rough obstacle with jump discontinuity in ψ' outside the contact set.

this problem: we see that the estimator bounds the true error and the convergence rate is proportional to N^{-2}.

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