FINITE-ELEMENT METHODS FOR A STRONGLY DAMPED WAVE EQUATION*

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Abstract. Error estimates of optimal order are proved for semidiscrete and completely discrete finite element methods for a linear wave equation with strong damping, arising in viscoelastic theory. It is demonstrated that the exact solution may be interpreted in terms of an analytic semigroup, and as a result that, although the solution has essentially the spatial regularity of its initial data, it is infinitely differentiable in time for $t > 0$. The estimates for the spatially discrete method are derived by energy arguments. Rational approximation of analytic semigroups is discussed in a general setting, by means of spectral representation, and the results are used to analyze the completely discrete schemes. Both smooth (and compatible) and less smooth data are considered.

1. Introduction. In this paper we consider the model initial-boundary value problem

$$\begin{aligned} &u_{tt} + \alpha A u_t + Au = 0, \qquad \text{for } (x,t) \in \Omega \times (0,\infty), \\ &u = 0, \qquad \text{on } \partial\Omega \times (0,\infty), \\ &u(\cdot,0) = \phi, \ u_t(\cdot,0) = \psi, \qquad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^d$ with smooth boundary $\partial\Omega$, A is a self-adjoint, positive definite, linear elliptic partial differential operator of second order with smooth coefficients, and α is a fixed positive constant.

The differential equation in (1.1) is sometimes referred to as a strongly damped wave equation, and boundary value problems for it, and nonlinear versions thereof, occur in internally damped systems, e.g., in viscoelastic theory (cf. Fitzgibbon [3], Webb [15] and Massatt [10]). Our analysis below applies also, with only trivial modifications, to the equation

$$u_{tt} + (\alpha A + \beta) u_t + Au = 0,$$

i.e., in the presence of a mixture of internal and external damping.

For the purpose of numerical solution of (1.1) by the Galerkin finite element method we assume for now that we are given a family S_h of finite dimensional subspaces of $H_0^1 = H_0^1(\Omega)$ with the standard approximation property that, for some $r \geq 2$,

$$\inf_{\chi \in S_h} \big(\|v - \chi\| + h\|v - \chi\|_1 \big) \leq C h^r \|v\|_r, \qquad \text{for } v \in H^r \cap H_0^1,$$

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where $\|\cdot\|$ is the norm in $L_2 = L_2(\Omega)$ and $\|\cdot\|_s$ is that in $H^s = H^s(\Omega)$. In the technical work below this assumption will be somewhat weakened.

With $A(\cdot,\cdot)$ denoting the standard Dirichlet bilinear form corresponding to the differential operator A, we may then first pose the semidiscrete initial-boundary value problem to find $u_h(t) \in S_h$ for $t \geq 0$ such that

$$(1.2) \qquad \begin{aligned} &(u_{h,tt},\chi) + \alpha A(u_{h,t},\chi) + A(u_h,\chi) = 0, \qquad \forall \chi \in S_h,\ t > 0,\\ &u_h(0) = \phi_h,\ u_{h,t}(0) = \psi_h, \end{aligned}$$

where ϕ_h and ψ_h are approximations of ϕ and ψ in S_h, and $(\cdot,\cdot)$ denotes the inner product in L_2.

The error analysis for this method requires some information about the regularity properties of the solution of (1.1). Introducing, in addition to the norm in H^s, that in the subspace $\dot H^s = \dot H^s(\Omega)$ defined by $|v|_s = \|A^{s/2}v\|$, we show that, for any $j \geq 0$ and $s \geq 0$,

$$|u(t)|_s + t^j |D_t^{j+1}u(t)|_s \leq C\big(|\phi|_s + |\psi|_s\big), \qquad \text{for } t > 0.$$

This shows that if $\phi, \psi \in \dot H^s$, then u and u_t are also in $\dot H^s$, and further that the solution has an infinite number of time derivatives in $\dot H^s$ for $t > 0$. A mild smoothing property also with respect to the space variable is expressed by the inequality

$$|D_t^j u(t)|_s \leq Ct^{-j}\big(|\phi|_s + |\psi|_{s-\sigma}\big), \qquad \text{for } t > 0, \text{ when } 0 \leq \sigma \leq 2,$$

but further smoothing with respect to x does not hold.

The space $\dot H^s$ used above is the subspace of H^s defined essentially by the boundary conditions $A^j u = 0$ for $j < s/2$ (when s is a positive integer), and the results just described thus assume certain nontrivial boundary conditions on ϕ and ψ. A more careful analysis shows that these assumptions may be somewhat reduced. We demonstrate, for instance, that $D_t^j u(t)$ is in H^r for $t > 0$, if (and only if) $\phi \in H^r,\ \psi \in H^{r-2}$. Normally, in this case, $\|u(t)\|_r$ will be unbounded as t tends to 0, except when $\phi \in H^r \cap \dot H^{r-2},\ \psi \in \dot H^{r-2}$.

As a result of these regularity properties we will be able to show that, for the solutions of (1.2) and (1.1),

$$(1.3) \quad \|u_h(t) - u(t)\| + \min(t,1)\|u_{h,t}(t) - u_t(t)\| \leq Ch^r\big(\|\phi\|_r + |\psi|_{r-2}\big), \qquad \text{for } t \geq 0,$$

if $\phi \in H^r \cap \dot H^{\max(r-2,1)},\ \psi \in \dot H^{r-2}$, and if ϕ_h is an arbitrary approximation of ϕ of optimal order and ψ_h is more specifically chosen as the L_2-projection of ψ onto S_h. Error bounds containing negative powers of t are also deduced for higher order time derivatives under the same regularity assumptions. Note that for $r \leq 4$ the assumptions on ϕ and ψ contain no other boundary conditions than $\phi = \psi = 0$ on $\partial\Omega$. For $r = 5$ we will also be able to show an optimal order estimate for $u_h(t) - u(t)$ when t is strictly positive, under the conditions $\phi \in H^5 \cap H_0^1,\ \psi \in H^3 \cap H_0^1$.

Further we show that

$$(1.4) \qquad \|u_h(t) - u(t)\| + \|u_{h,t}(t) - u_t(t)\| \leq Ch^r\big(|\phi|_r + |\psi|_r\big), \qquad \text{for } t \geq 0,$$

if $\phi, \psi \in \dot H^r$, and both ϕ_h and ψ_h are arbitrary initial approximations of optimal order. These results are obtained by the energy method.

The error bound (1.4) shows that it is sufficient to have initial data in $\dot H^r$ in order to have optimal order convergence, for both u_h and $u_{h,t}$, uniformly for $t \geq 0$. This is in contrast to the situation for the wave equation ($\alpha = 0$ in (1.1)), where it is known that one extra derivative of initial regularity is needed, see Rauch [12]. On the other hand, (1.3) implies that the regularity of ϕ and ψ can be reduced to a limited extent, while retaining the optimal order of convergence for u_h for $t \geq 0$. But our regularity analysis shows that the initial regularity cannot be further reduced, even if we restrict our attention to strictly positive t. This differs from the situation for parabolic problems, where no initial regularity is needed to prove optimal order convergence for positive times. Thus, the behavior of the strongly damped wave equation is, in this respect, intermediate between the behaviors of hyperbolic and parabolic equations. For other investigations of finite element methods in situations with limited smoothing, see Crouzeix, Thomée and Wahlbin [2], Genis [6], and Thomée and Zhang [14].

Our regularity analysis is accomplished after introducing $w = (u, u_t)^T$ and noting that (1.1) may be written as the system

$$\begin{aligned} &w_t + Bw = 0, \qquad t > 0,\\ &w(0) = w_0 = (\phi, \psi)^T, \end{aligned}$$

where

$$B = \begin{bmatrix} 0 & -I \\ A & \alpha A \end{bmatrix}. \tag{1.5}$$

One may then show that $-B$ generates an analytic semigroup in $\dot{H}^s \times \dot{H}^{s-\sigma}$ for $0 \leq \sigma \leq 2$. This was done in special cases in [15] and [10].

The semidiscrete equation (1.2) can be written similarly as

$$\begin{aligned} &w_{h,t} + B_h w_h = 0, \qquad t > 0,\\ &w_h(0) = w_{0h} = (\phi_h, \psi_h)^T, \end{aligned} \tag{1.6}$$

where $w_h = (u_h, u_{h,t})^T \in S_h \times S_h$, and B_h is the analogue of (1.5) with A replaced by its discrete counterpart $A_h : S_h \to S_h$ defined by

$$(A_h \varphi, \chi) = A(\varphi, \chi), \qquad \forall \varphi, \chi \in S_h.$$

The formulation (1.6) may also be used as a basis for discretization in time. Thus let $r(z)$ be a rational function approximating e^{-z} to order $O(z^{p+1})$ for small z and such that $|r(z)| \leq 1$ for $\mathrm{Re}\, z \geq 0$. We may then define an approximation $W_n = (U_n, V_n)^T$ to the solution of (1.6) at time $t_n = nk$, where k is the time step, by

$$\begin{aligned} &W_n = r(kB_h)W_{n-1}, \qquad n \geq 1,\\ &W_0 = w_{0h} = (\phi_h, \psi_h)^T. \end{aligned}$$

For example, using the backward Euler method,

$$(I + kB_h)W_n = W_{n-1}, \qquad n \geq 1, \tag{1.7}$$

corresponds to $r(z) = 1/(1+z)$. After some manipulation, (1.7) may also be written as

$$\begin{aligned} &[I + (\alpha k + k^2)A_h]V_n = V_{n-1} - kA_h U_{n-1},\\ &U_n = U_{n-1} + kV_n. \end{aligned}$$

It is easy to see that, with $\bar\partial_t U_n = (U_n - U_{n-1})/k$, the discrete solution satisfies the following totally implicit three level Galerkin approximation of (1.1), namely

$$(\bar\partial_t^2 U_n, \chi) + \alpha A(\bar\partial_t U_n, \chi) + A(U_n, \chi) = 0, \qquad \forall \chi \in S_h,\ n \ge 2,$$

with the appropriate choice of U_1 from above.

Another standard example is provided by the Crank-Nicolson scheme

$$(I + \frac{k}{2} B_h) W_n = (I - \frac{k}{2} B_h) W_{n-1}, \qquad n \ge 1,$$

where $r(z) = (1 - z/2)/(1 + z/2)$. We have similarly to above

$$\begin{aligned} &[I + (\alpha \frac{k}{2} + \frac{k^2}{4}) A_h] V_n = [I - (\alpha \frac{k}{2} + \frac{k^2}{4}) A_h] V_{n-1} - k A_h U_{n-1}, \\ &U_n = U_{n-1} + \frac{k}{2}(V_n + V_{n-1}), \end{aligned}$$

and

$$(\bar\partial_t^2 U_n, \chi) + \alpha A(\tilde\partial_t U_n, \chi) + A(\hat U_n, \chi) = 0, \qquad \forall \chi \in S_h,\ n \ge 2,$$

where $\tilde\partial_t U_n = (U_n - U_{n-2})/2k$ and $\hat U_n = (U_n + 2U_{n-1} + U_{n-2})/4$.

For the general totally discrete method thus described we show that, with optimal order initial approximations, we have the smooth data error estimate

$$\|U_n - u(t_n)\| + \|V_n - u_t(t_n)\| \le C h^r \big(|\phi|_r + |\psi|_r\big) + C k^p \big(|\phi|_{2p} + |\psi|_{2p}\big),$$

for $\phi, \psi \in \dot H^s$, $s = \max(r, 2p)$. The proof depends on spectral arguments using the analyticity of the semigroup generated by $-B_h$, and a representation of initial data, used in a parabolic case by Baker, Bramble and Thomée [1]. An error estimate for U_n only is also obtained with two orders lower regularity requirement on ψ.

For methods with $|r(\infty)| < 1$ it is possible to show a nonsmooth data error estimate for the semidiscrete solution of the form

$$\|U_n - u_h(t_n)\| + \|V_n - u_{h,t}(t_n)\| \le C k^p t_n^{-p} \big(\|\phi_h\| + \|\psi_h\|\big), \qquad \text{for } t_n > 0. \tag{1.8}$$

In spite of this, since the smoothing properties of (1.1) with respect to x are limited as described above, we are only able to use (1.8) to obtain optimal order error estimates for the completely discrete scheme in situations when such estimates are available for the semidiscrete problem. In particular, combination of (1.8) and (1.3) yields that if $\phi \in H^r \cap \dot H^{\max(r-2,1)}$, $\psi \in \dot H^{r-2}$, and with ϕ_h an optimal order approximation to ϕ, and ψ_h chosen as the L_2-projection of ψ, then we have, for t_n positive,

$$\|U_n - u(t_n)\| + \|V_n - u_t(t_n)\| \le C(t_n)(h^r + k^p)\big(\|\phi\|_r + |\psi|_{r-2}\big).$$

For $r = 5$ an optimal order result for $t_n > 0$ is obtained when $\phi \in H^5 \cap H_0^1$, $\psi \in H^3 \cap H_0^1$.

Error estimates in the case of a smooth solution have been given earlier in Lin, Thomée and Wahlbin [9] for a semidiscrete version of the equation

$$u_{tt} + A(t) u_t + B(t) u = f(t), \qquad \text{for } x \in \Omega,\ t > 0, \tag{1.9}$$

where $A(t)$ and $B(t)$ are second order differential operators, with $A(t)$ self-adjoint positive definite elliptic. Our equation is a special case of (1.9), with the additional feature that $B(t) = A(t) = A$. This causes the problem to be associated with a contractive analytic semigroup and also permits a precise analysis by the spectral method.

The plan of the paper is as follows. In Section 2 we present the initial-boundary value problem to be studied, introduce a more general class of finite element methods than the one described above for the discretization with respect to the spatial variable, and discuss existence and regularity of the exact solution of (1.1).

In Section 3 we estimate the error in the spatially semidiscrete approximation. In Section 4 we discuss, in a general setting, approximation of an analytic semigroup by a rational function of its generator. In Section 5 we combine results from Sections 3 and 4 and obtain error bounds for a class of completely discrete schemes.

In the following we let C and c denote various positive constants, not necessarily the same at each occurrence, that are independent of the functions and discretization parameters. In particular, they may depend on α.

2. The strongly damped wave equation. We will thus consider the initial-boundary value problem (1.1). We assume that A is a second order elliptic operator given by

$$Au = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u,$$

where the coefficients are smooth, $a_0(x) \geq 0$, and $\{a_{ij}\}$ is uniformly positive definite. We may then think of A as a self-adjoint, positive definite, unbounded linear operator in $L_2 = L_2(\Omega)$ with dense domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and (1.1) can be written as an evolution problem in L_2:

$$\begin{aligned} &u_{tt} + \alpha A u_t + Au = 0, \qquad t > 0, \\ &u(0) = \phi, \ \ u_t(0) = \psi. \end{aligned} \tag{2.1}$$

In a standard way we may write (2.1) as a 2×2 system,

$$\begin{aligned} &w_t + Bw = 0, \qquad t > 0, \\ &w(0) = w_0, \end{aligned} \tag{2.2}$$

where $w = (u, u_t)^T$, $w_0 = (\phi, \psi)^T$ and

$$B = \begin{bmatrix} 0 & -I \\ A & \alpha A \end{bmatrix}.$$

We will show below that $-B$ generates an analytic semigroup in the Hilbert space $L_2 \times L_2$, so that (2.2) has a unique solution $w(t) = \exp(-tB)w_0$ for any $w_0 \in L_2 \times L_2$.

We next introduce a class of finite element methods for discretization with respect to the spatial variable that is more general than the standard Galerkin finite element method described in the Introduction.

The inverse of A is a self-adjoint, positive definite, compact, linear operator on L_2, which we denote by T. For the approximation of the solution of (1.1) we assume that we are given a family $\{S_h\}_{h>0}$ of finite-dimensional subspaces of L_2 and corresponding linear operators $T_h : L_2 \to S_h$ with the following properties:

(i) T_h is self-adjoint, positive semi-definite on L_2, and positive definite on S_h;

(ii) for some integer $r \geq 2$ we have

$$\|(T_h - T)f\| \leq C\,h^r\,\|f\|_{r-2}, \qquad \text{for } f \in H^{r-2}. \tag{2.3}$$

Here and in the following $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product in L_2, or $L_2\times L_2$, unless otherwise stated, and $\|\cdot\|_s$ is the norm in $H^s=H^s(\Omega)$.

We note that (ii) implies that

$$\begin{aligned}\inf_{\chi\in S_h}\|\chi-v\|&\le\|T_hAv-v\|=\|(T_h-T)Av\|\\&\le Ch^r\|Av\|_{r-2}\le Ch^r\|v\|_r,\qquad\text{for } v\in H^r\cap H_0^1.\end{aligned}$$

In other words, for the orthogonal projection $P_h: L_2\to S_h$ we have

$$\|P_hv-v\|\le Ch^r\|v\|_r,\qquad\text{for } v\in H^r\cap H_0^1. \tag{2.4}$$

Since T_h is invertible on S_h, we may define $A_h=T_h^{-1}: S_h\to S_h$, and a semidiscrete approximation $u_h(t)\in S_h$ by

$$\begin{aligned}&u_{h,tt}+\alpha A_hu_{h,t}+A_hu_h=0,\qquad t>0,\\&u_h(0)=\phi_h,\ u_{h,t}(0)=\psi_h,\end{aligned} \tag{2.5}$$

where $\phi_h,\psi_h\in S_h$ are suitable approximations of ϕ,ψ. We can also write this as

$$\begin{aligned}&w_{h,t}+B_hw_h=0,\qquad t>0,\\&w_h(0)=w_{0h},\end{aligned} \tag{2.6}$$

where $w_h(t)=(u_h(t),u_{h,t}(t))^T\in S_h\times S_h$, $w_{0h}=(\phi_h,\psi_h)^T$ and

$$B_h=\begin{bmatrix}0&-I\\A_h&\alpha A_h\end{bmatrix}.$$

An example of the situation described is provided by the standard Galerkin finite element method (1.2), but our assumptions also allow isoparametric elements as well as more general finite element methods, where the S_h do not need to satisfy the boundary condition in (1.1), see Thomée [13, Chapter 2].

Some of our analysis will take place in the framework of the spaces $\dot H^s=\dot H^s(\Omega)$ defined for real s by $\dot H^s=\mathcal{D}(A^{s/2})$ with norms

$$|v|_s=\|A^{s/2}v\|=\Big(\sum_{j=1}^\infty\lambda_j^s|(v,\omega_j)|^2\Big)^{1/2}, \tag{2.7}$$

where $\{\lambda_j\}_{j=1}^\infty$ are the eigenvalues and $\{\omega_j\}_{j=1}^\infty$ is the corresponding normalized eigensystem of A. It is well-known (cf. Grisvard [7] or Fujiwara [5]) that, for $s\ge0$, $\frac12s-\frac14$ not an integer, $\dot H^s$ is the subspace of H^s characterized by the boundary conditions $A^jv=0$ on $\partial\Omega$ for $j<\frac12s-\frac14$, and that the norm of $\dot H^s$ is equivalent to that of H^s. In particular, we have $\dot H^1=H_0^1$, $\dot H^2=H^2\cap H_0^1$ with equivalent norms.

Lemma 2.1. *The operators $-B$ and $-B_h$ generate analytic semigroups* $\exp(-tB)$ *in $L_2\times L_2$ and* $\exp(-tB_h)$ *in $S_h\times S_h$ (supplied with the norm of $L_2\times L_2$), respectively. Moreover,* $\exp(-tB)$ *is an analytic semigroup in $\dot H^s\times\dot H^{s-\sigma}$ for $s\in\mathbf{R}$, $0\le\sigma\le2$.*

Proof. The desired result follows from the following lemma, see, e.g., Pazy [11, Theorem 2.5.2].

Lemma 2.2. *Let $s \in \mathbf{R}$, $0 \le \sigma \le 2$. Then there is an angle $\theta_0 \in (0, \pi/2)$ and a positive number C such that the sector*

$$\Sigma = \{z \in \mathbf{C} : \theta_0 \le |\arg z| \le \pi\} \cup \{0\}$$

is contained in the resolvent sets of B and B_h for $h > 0$, and

$$\|(zI - B)^{-1}\| \le \frac{C}{|z|}, \qquad \text{for } z \in \Sigma,\ z \ne 0, \tag{2.8}$$

$$\|(zI - B_h)^{-1}\| \le \frac{C}{|z|}, \qquad \text{for } z \in \Sigma,\ z \ne 0,\ h > 0, \tag{2.9}$$

where in (2.8) and (2.9) $\|\cdot\|$ denotes the norm of bounded linear operators in $\dot H^s \times \dot H^{s-\sigma}$ and $S_h \times S_h$, respectively.

Proof. Let $c_0 > 0$ be a lower bound for the eigenvalues of A and A_h. (It follows from (2.3) that there is an upper bound for $\|T_h\|$, and hence a positive lower bound for the spectrum of A_h, independently of h.) We prove the statements about B; the argument depends only on c_0 and therefore applies equally well to B_h.

We want to estimate the solution of the equation

$$(zI - B)w = f,$$

for any $f \in \dot H^s \times \dot H^{s-\sigma}$. With $f = \sum_j \hat f_j \omega_j$ the eigenfunction expansion of f, where $\hat f_j \in \mathbf{C}^2$, it follows that $w = \sum_j \hat w_j \omega_j$ with $\hat w_j = (zI - B(\lambda_j))^{-1} \hat f_j$, where

$$B(\lambda) = \begin{bmatrix} 0 & -1 \\ \lambda & \alpha\lambda \end{bmatrix}, \tag{2.10}$$

$$(zI - B(\lambda))^{-1} = (z^2 - \alpha\lambda z + \lambda)^{-1} \begin{bmatrix} z - \alpha\lambda & -1 \\ \lambda & z \end{bmatrix}.$$

In view of Parseval's relation and (2.7) we must show that

$$\begin{aligned} &\left| \begin{bmatrix} \lambda^s & 0 \\ 0 & \lambda^{s-\sigma} \end{bmatrix} (zI - B(\lambda))^{-1} \begin{bmatrix} \lambda^s & 0 \\ 0 & \lambda^{s-\sigma} \end{bmatrix}^{-1} \right| \\ &= \left| (z^2 - \alpha\lambda z + \lambda)^{-1} \begin{bmatrix} z - \alpha\lambda & -\lambda^{\sigma/2} \\ \lambda^{1-\sigma/2} & z \end{bmatrix} \right| \le \frac{C}{|z|}, \qquad z \in \Sigma,\ \lambda \ge c_0, \end{aligned} \tag{2.11}$$

for a suitable sector Σ, where $|\cdot|$ on the left denotes the norm of the 2×2 matrix.

We first locate the zeros of $z^2 - \alpha\lambda z + \lambda$ (i.e., the eigenvalues of $B(\lambda)$) for $\lambda \ge c_0$. They are

$$\tau_\pm(\lambda) = \frac{\alpha\lambda}{2} \pm \sqrt{\left(\frac{\alpha\lambda}{2}\right)^2 - \lambda} \tag{2.12}$$

and it is easily seen that they lie in the set

$$\{z \in \mathbf{C} : |z - \alpha^{-1}| = \alpha^{-1},\ \operatorname{Re} z \ge \alpha c_0/2\} \cup \{z \in \mathbf{C} : \operatorname{Re} z > \alpha^{-1},\ \operatorname{Im} z = 0\}.$$

Now let $\theta_0 = \arctan(2/(\alpha^2 c_0))$. This defines a sector Σ of the desired form and a constant $c_1 > 0$, such that

$$|z - \tau_\pm(\lambda)| \ge c_1 |z|, \qquad z \in \Sigma,\ \lambda \ge c_0.$$

Since, in addition,

$$|z-\alpha\lambda| \le c_2|z-\tau_+(\lambda)|, \qquad z \in \Sigma,\ \lambda \ge c_0,$$

we hence find that

$$\begin{aligned}\left|\frac{z-\alpha\lambda}{z^2-\alpha\lambda z+\lambda}\right| &= \left|\frac{z-\alpha\lambda}{(z-\tau_+(\lambda))(z-\tau_-(\lambda))}\right| \\ &\le \frac{c_2}{|z-\tau_-(\lambda)|} \le \frac{c_2}{c_1|z|}, \qquad z \in \Sigma,\ \lambda \ge c_0,\end{aligned}$$

and similarly for the other entries on the left-hand side of (2.11). This shows (2.11) and the proof is complete.

The special case $0 \le \sigma \le s < 2$ of Lemma 2.1 was shown in [10] in a more general context. The case $s = \sigma = 2$ was obtained in [15] by a simple argument.

We shall now turn to a discussion of the regularity of the solution of the initial boundary value problem (1.1). The first result, which refers to the scale of spaces $\dot{H}^s$, follows directly from the analyticity of the semigroup as described in Lemma 2.1.

Lemma 2.3. *For any integer $j \ge 0$ and real numbers s and σ with $0 \le \sigma \le 2$ there is a positive number C such that for the solution u of (2.1) we have*

$$|D_t^j u(t)|_s + |D_t^{j+1} u(t)|_{s-\sigma} \le Ct^{-j}\big(|\phi|_s + |\psi|_{s-\sigma}\big), \qquad \textit{for } t > 0.$$

Proof. Since $\exp(-tB)$ is an analytic semigroup in $\dot{H}^s \times \dot{H}^{s-\sigma}$, we have, in the corresponding norm, for $w(t) = \exp(-tB)w_0$,

$$|D_t^j w(t)|_{s\times(s-\sigma)} \le Ct^{-j}|w_0|_{s\times(s-\sigma)}, \qquad t > 0,$$

which is the desired result.

This result can also be easily derived directly from the eigenfunction expansion of the solution of (1.1), namely

$$u(t) = \sum_{i=1}^{\infty} \hat{u}_i(t)\omega_i,$$

where, with $\tau_\pm(\lambda)$ defined in (2.12),

$$\hat{u}_i(t) = \frac{\tau_+(\lambda_i)e^{-t\tau_-(\lambda_i)} - \tau_-(\lambda_i)e^{-t\tau_+(\lambda_i)}}{\tau_+(\lambda_i)-\tau_-(\lambda_i)}\hat{\phi}_i + \frac{e^{-t\tau_-(\lambda_i)} - e^{-t\tau_+(\lambda_i)}}{\tau_+(\lambda_i)-\tau_-(\lambda_i)}\hat{\psi}_i, \tag{2.13}$$

with the obvious modification if $\lambda_i = 4/\alpha^2$, i.e., when $\tau_+(\lambda_i) = \tau_-(\lambda_i)$. This representation may be obtained directly from (1.1) by the standard separation of variables argument.

We shall now use this representation to demonstrate that, in contrast to the situation in the case of a parabolic equation, there is no more smoothing in x with respect to the scale of spaces $\dot{H}^s$ than that exhibited in Lemma 2.3. In fact, as $\lambda \to \infty$ we have by (2.12) that $\tau_-(\lambda) \sim 1/\alpha$, $\tau_+(\lambda) \sim \alpha\lambda$, and hence, for any $t > 0$,

$$\hat{u}_i(t) \sim e^{-t/\alpha}(\hat{\phi}_i + \frac{1}{\alpha\lambda_i}\hat{\psi}_i), \qquad \text{as } i \to \infty. \tag{2.14}$$

Thus the least conditions required on ϕ and ψ in order to guarantee that $u(t)$ belongs to $\dot{H}^s$ are $\phi \in \dot{H}^s$ and $\psi \in \dot{H}^{s-2}$.

It is clear from (2.13) that

$$|D_t^j \hat{u}_i(t)| \le C(t)\big(|\hat{\phi}_i| + |\hat{\psi}_i|\big), \qquad \text{for } i \ge 1,\ t > 0, \tag{2.15}$$

and hence $u(t) = \sum_{i=1}^{\infty} \hat{u}_i(t)\omega_i$ is well defined in L_2 for $\phi, \psi \in L_2$ together with its time derivatives for $t > 0$. We shall think of this as a generalized solution of (1.1); when the $\hat{u}_i(t)$ are such that u and u_t are in $\mathcal{D}(A) = \dot{H}^2$, this is a genuine solution.

A natural question to ask is now what regularity the solution possesses in the situation that ϕ and ψ are smooth without satisfying the higher order boundary conditions of the $\dot{H}^s$ spaces. It will turn out that if $\phi \in H^s$ and $\psi \in H^{s-2}$, then the generalized solution and its time derivatives belong to H^s for $t > 0$. This is expressed in the following lemma, which shows, in particular, that if ϕ and ψ are smooth and vanish on $\partial\Omega$, then they are the initial values of a smooth genuine solution.

Lemma 2.4. *Let $s \ge 2$, $\sigma, \tau \ge 0$, and assume that $\phi \in H^s \cap \dot{H}^\sigma$ and $\psi \in H^{s-2} \cap \dot{H}^\tau$. Then $D_t^j u(t) \in H^s$ for $t > 0$, and, with $[y]_+ = \max(y, 0)$, we have*

$$\|D_t^j u(t)\|_s \le C\big(\|\phi\|_s + \|\psi\|_{s-2} + t^{-[(s-\sigma)/2+j-1]_+}|\phi|_\sigma + t^{-[(s-\tau)/2+j-1]_+}|\psi|_\tau\big).$$

Proof. Since $\omega_i \in \dot{H}^s$ for any $s \ge 0$, the smoothness properties of $u(t)$ depend only on the behavior of $\hat{u}_i(t)$ for large i. We write (2.13) in the form

$$\hat{u}_i(t) = [a_i^-(t) - a_i^+(t)]\hat{\phi}_i + [b_i^-(t) - b_i^+(t)]\hat{\psi}_i,$$

where $+$ corresponds to $\exp(-t\tau_+(\lambda_i))$ and correspondingly for $-$. We set, with $\kappa = 8/\alpha^2$,

$$u_\pm^\phi(t) = \sum_{\lambda_i > \kappa} a_i^\pm \hat{\phi}_i \omega_i, \quad u_\pm^\psi(t) = \sum_{\lambda_i > \kappa} b_i^\pm \hat{\psi}_i \omega_i.$$

Clearly then, by (2.15), we have for any $s \ge 0$, with $C = C_s$,

$$\begin{aligned} &|D_t^j(u(t) - [u_-^\phi(t) - u_+^\phi(t)] - [u_-^\psi(t) - u_+^\psi(t)])|_s \\ &\qquad = |\sum_{\lambda_i \le \kappa} D_t^j \hat{u}_i(t)\omega_i|_s = \Big(\sum_{\lambda_i \le \kappa} \lambda_i^s |D_t^j \hat{u}_i(t)|^2\Big)^{1/2} \le C\big(\|\phi\| + \|\psi\|\big). \end{aligned} \tag{2.16}$$

We shall also show

$$\|D_t^j u_-^\phi(t)\|_s \le C\|\phi\|_s, \tag{2.17}$$

$$\|D_t^j u_-^\psi(t)\|_s \le C\|\psi\|_{s-2}, \tag{2.18}$$

$$|D_t^j u_+^\phi(t)|_s \le Ct^{-[(s-\sigma)/2+j-1]_+}|\phi|_\sigma, \tag{2.19}$$

$$|D_t^j u_+^\psi(t)|_s \le Ct^{-[(s-\tau)/2+j-1]_+}|\psi|_\tau, \tag{2.20}$$

which estimates together then will complete the proof.

We start with (2.19) and note that

$$D_t^j u_+^\phi(t) = \sum_{\lambda_i > \kappa} (-1)^j \tau_+(\lambda_i)^j a_i^+(t)\hat{\phi}_i\omega_i.$$

Here

$$\begin{aligned} \lambda_i^{s/2}|\tau_+(\lambda_i)^j a_i^+(t)| &= \lambda_i^{s/2}\tau_+(\lambda_i)^j \tau_-(\lambda_i)[\tau_+(\lambda_i) - \tau_-(\lambda_i)]^{-1}e^{-t\tau_+(\lambda_i)} \\ &\le C\lambda_i^{s/2+j-1}e^{-ct\lambda_i} \le Ct^{-[(s-\sigma)/2+j-1]_+}\lambda_i^{\sigma/2}, \end{aligned}$$

so that

$$|D_t^j u_+^\phi(t)|_s \le Ct^{-[(s-\sigma)/2+j-1]_+}\Big(\sum_{\lambda_i>\kappa}\lambda_i^\sigma|\hat\phi_i|^2\Big)^{1/2} \le Ct^{-[(s-\sigma)/2+j-1]_+}|\phi|_\sigma.$$

The proof of (2.20) is analogous.

We now turn to the proof of (2.17) and (2.18) and note first that

$$D_t^j u_-^\phi(t) = \sum_{\lambda_i>\kappa}(-1)^j\tau_-(\lambda_i)^j a_i^-(t)\hat\phi_i\omega_i.$$

We shall show below that, for any $j \ge 0$ and $N > 0$, there exist a polynomial $P_{j,N}(\cdot,\cdot)$ and a constant $C = C_{j,N}$ such that

$$|\tau_-(\lambda_i)^j a_i^-(t) - e^{-t/\alpha}P_{j,N}(t,\lambda_i^{-1})| \le C\lambda_i^{-N}, \qquad \text{for } t>0,\ \lambda_i \ge \kappa. \tag{2.21}$$

Assuming this for a moment we may define, with $T = A^{-1}$ as above,

$$\tilde u_-^{\phi,j}(t) = e^{-t/\alpha}P_{j,N}(t,T)\phi = e^{-t/\alpha}\sum_{i=1}^\infty P_{j,N}(t,\lambda_i^{-1})\hat\phi_i\omega_i.$$

Clearly then, if $N \ge s/2$,

$$|D_t^j u_-^\phi(t) - \tilde u_-^{\phi,j}(t)|_s \le C\Big(\sum_{i=1}^\infty \lambda_i^s\lambda_i^{-2N}|\hat\phi_i|^2\Big)^{1/2} \le C\|\phi\|. \tag{2.22}$$

Further, since $T^l\phi \in H^{s+2l} \subset H^s$ for $l \ge 0$, we have

$$\|\tilde u_-^{\phi,j}(t)\|_s \le C\|\phi\|_s.$$

Together these estimates show (2.17).

To show (2.21) we first note that $0 \le \tau_-(\lambda) - 1/\alpha \le C\lambda^{-1}$ for $\lambda \ge \kappa$ and that hence by Taylor expansion of $\exp(-t(\tau_-(\lambda)-1/\alpha))$,

$$|e^{-t\tau_-(\lambda)} - e^{-t/\alpha}\sum_{j=0}^{N-1}\frac{(-t(\tau_-(\lambda)-1/\alpha))^j}{j!}| \le C(t/\lambda)^N e^{-t/\alpha} \le C\lambda^{-N}, \quad \lambda \ge \kappa.$$

Thus by expanding $\tau_-(\lambda)$ in powers of λ^{-1} we find for a suitable polynomial $P_N(\cdot,\cdot)$,

$$|e^{-t\tau_-(\lambda)} - e^{-t/\alpha}P_N(t,\lambda^{-1})| \le C\lambda^{-N}, \qquad \lambda \ge \kappa.$$

Expanding similarly $\tau_-(\lambda)^j\tau_+(\lambda)[\tau_+(\lambda)-\tau_-(\lambda)]^{-1}$ in powers of λ^{-1} completes the proof of (2.21). We note, in particular, for later use that $P_{0,N}(t,0) = P_N(t,0) \equiv 1$.

To show (2.18) we need instead of (2.21) the existence of polynomials $Q_{j,N}$ such that

$$|\tau_-(\lambda_i)^j b_i^-(t) - e^{-t/\alpha}\lambda_i^{-1}Q_{j,N}(t,\lambda_i^{-1})| \le C\lambda_i^{-N}, \qquad \text{for } t>0,\ \lambda_i \ge \kappa, \tag{2.23}$$

which is demonstrated in the analogous way. The additional factor λ_i^{-1} in front of the polynomial in (2.23) comes from the fact that the coefficient $[\tau_+(\lambda_i)-\tau_-(\lambda_i)]^{-1}$ of $\exp(-t\tau_-(\lambda_i))$ in $b_i^-(t)$ behaves like $(\alpha\lambda_i)^{-1}$ (cf. (2.14)). In this case we define

$$\tilde u_-^{\psi,j}(t) = e^{-t/\alpha}TQ_{j,N}(t,T)\psi = e^{-t/\alpha}\sum_{i=1}^\infty \lambda_i^{-1}Q_{j,N}(t,\lambda_i^{-1})\hat\psi_i\omega_i.$$

and find, similarly to above, since $T^l\psi \in H^{s-2+2l} \subset H^s$ for $l \geq 1$, that

$$\|\tilde{u}_-^{\psi,j}(t)\|_s \leq C\|\psi\|_{s-2}.$$

The proof of Lemma 2.4 is now complete.

The results of Lemma 2.4 are also sharp. To indicate a proof of this, let us first show that if $\psi = 0$ and $u(t)$ is in H^s for some positive t, then $\phi \in H^s$. To see this we write

$$\tilde{u}_-^{\phi,0} = u_+^\phi + u - [u - (u_-^\phi - u_+^\phi)] - [u_-^\phi - \tilde{u}_-^{\phi,0}].$$

Since $u_\pm^\psi(t) = 0$, (2.16) shows that $u(t) - (u_-^\phi(t) - u_+^\phi(t))$ is in $\dot{H}^s$. By (2.22) the same is true for $u_-^\phi(t) - \tilde{u}_-^{\phi,0}(t)$. But by (2.19) we have $u_+^\phi(t) \in \dot{H}^s$, and hence $\tilde{u}_-^{\phi,0}(t) \in H^s$. This function is of the form

$$\tilde{u}_-^{\phi,0}(t) = e^{-t/\alpha}\sum_{l=0}^{M} a_l(t)T^l\phi, \qquad \text{with } a_0(t) = P_{0,N}(t,0) \equiv 1, \tag{2.24}$$

where the $a_l(t)$ are polynomials in t. Hence, since $T^l\phi$ is smoother than ϕ for any $l > 0$, we conclude easily that $\phi \in H^s$.

We next show that if $\psi = 0$ and $u(t)$ is bounded in H^s, uniformly for $t \geq 0$, then $\phi \in \dot{H}^{s-2}$. For this we first note that by (2.24) $\tilde{u}_-^{\phi,0}(t)$ is bounded in H^s uniformly for $t \geq 0$, so that we now know that this holds for $u_+^\phi(t)$. But since this function belongs to $\dot{H}^s$, we have by equivalence of norms that $u_+^\phi(t)$ is bounded in $\dot{H}^s$ so that, in particular, for $t = 0$,

$$\sum_{\lambda_i > \kappa} \lambda_i^s a_i^+(0)^2|\hat{\phi}_i|^2 < \infty.$$

Since

$$a_i^+(0) = \tau_-(\lambda_i)[\tau_+(\lambda_i) - \tau_-(\lambda_i)]^{-1} \sim \frac{1}{\alpha^2\lambda_i}, \qquad \text{as } i \to \infty,$$

we finally conclude that

$$|\phi|_{s-2}^2 = \sum_{i=1}^{\infty} \lambda_i^{s-2}|\hat{\phi}_i|^2 < \infty.$$

A similar discussion shows that, if $\phi = 0$ and $u(t) \in H^s$ for some positive t, then $\psi \in H^{s-2}$ and, if in addition $u(t)$ is bounded in H^s for small t, then $\psi \in \dot{H}^{s-2}$.

We end this section with the following stability result for the semidiscrete version of our model problem.

Lemma 2.5. *For any integer $j \geq 0$ we have for the solution of (2.5)*

$$\|D_t^j u_h(t)\| + \|D_t^{j+1}u_h(t)\| \leq Ct^{-j}\left(\|\phi_h\| + \|\psi_h\|\right), \qquad \text{for } t > 0.$$

Proof. This follows at once by the analyticity of $\exp(-tB_h)$ in $S_h \times S_h$.

3. Semidiscretization with respect to space. In this section we estimate the error between the solution of (1.1) and its semidiscrete approximation defined by (2.5).

The following is our main result. In addition to giving a uniform error bound for u_h for all positive times, it shows that, as a result of the regularity of u in time, the time derivatives of the error may be bounded, with error bounds which contain negative powers of t for small t, but which are uniform for large t.

Theorem 3.1. *Assume that* $\phi \in H^r \cap \dot{H}^{\max(r-2,1)}$, $\psi \in \dot{H}^{r-2}$, *and let*

$$\|\phi_h - \phi\| \le Ch^r \|\phi\|_r, \quad \psi_h = P_h \psi.$$

Then we have, with $t_- = \min(t,1)$,

$$\|D_t^j (u_h(t) - u(t))\| \le Ch^r t_-^{-j} \big(\|\phi\|_r + |\psi|_{r-2}\big), \qquad \text{for } t > 0\ . \tag{3.1}$$

Proof. We begin by showing the result for $t \le 1$. We note that without restricting the generality we may assume that $\phi_h = P_h \phi$, since, by the stability result of Lemma 2.5 and (2.4), and since $\phi \in H^r \cap H_0^1$, the additional error caused by the general choice of ϕ_h is bounded by

$$Ct^{-j} \|\phi_h - P_h \phi\| \le Ch^r t^{-j} \|\phi\|_r.$$

Applying the operator $T = A^{-1}$ to the differential equation in (2.1) and integrating with respect to t, we have

$$Tu_t + \alpha u + \int_0^t u \, d\tau = T\psi + \alpha\phi. \tag{3.2}$$

Performing the analogous operations to (2.5) and subtracting (3.2) from the resulting equation, we obtain for the error $e = u_h - u$,

$$\begin{aligned} &T_h e_t + \alpha e + \int_0^t e \, d\tau = \eta \equiv -(T_h - T)u_t + (T_h - T)\psi + \alpha(P_h - I)\phi, \ t > 0, \\ &T_h e(0) = 0. \end{aligned} \tag{3.3}$$

Here we have used the fact that $T_h P_h = T_h$, which follows from

$$(T_h P_h f, g) = (P_h f, T_h g) = (f, T_h g) = (T_h f, g), \qquad \forall f, g \in L_2.$$

The core of the proof of (3.1) for small t is the following lemma, which is deduced by an energy argument.

Lemma 3.2. *For each* $t^* > 0$, $\beta \ge 0$ *and* $\epsilon > 0$ *there is* $C = C(\beta t^*, \epsilon)$ *such that, if* e *satisfies*

$$\begin{aligned} &T_h e_t + \alpha e + \beta \int_0^t e(\tau) \, d\tau = \eta, \qquad 0 < t < t^*, \\ &T_h e(0) = 0, \end{aligned} \tag{3.4}$$

then

$$\|e(t)\| \le C \sup_{0<\tau<t} \|\eta(\tau)\| + \epsilon \sup_{0<\tau<t} \big(\tau \|\eta_t(\tau)\|\big), \qquad 0 < t < t^*. \tag{3.5}$$

Proof. Writing $\tilde{e}(t) = \int_0^t e(\tau)\, d\tau$ and taking the inner product of (3.4) with e_t, we have

$$(T_h e_t, e_t) + \frac{\alpha}{2} \frac{d}{dt} \|e\|^2 = -\beta(\tilde{e}, e_t) + (\eta, e_t).$$

Multiplication by t leads to

$$\begin{aligned} t(T_h e_t, e_t) + \frac{\alpha}{2} \frac{d}{dt} \big(t\|e\|^2\big) &= \frac{\alpha}{2} \|e\|^2 - \beta t(\tilde{e}, e_t) + t(\eta, e_t) \\ = \frac{\alpha}{2} \|e\|^2 + \beta \frac{d}{dt} \big(\frac{1}{2} \|\tilde{e}\|^2 &- t(\tilde{e}, e)\big) + \beta t \|e\|^2 + \frac{d}{dt} \big(t(\eta, e)\big) - (\eta, e) - t(\eta_t, e), \end{aligned}$$

and since T_h is positive semi-definite, we obtain by simple estimates

$$\frac{d}{dt}(t\|e\|^2) \le C\,(1+\beta t^*)\|e\|^2 + C\frac{d}{dt}\big(\frac{\beta}{2}\|\tilde{e}\|^2 - \beta t(\tilde{e},e) + t(\eta,e)\big) + C\|\eta\|^2 + \frac{\epsilon^2}{2}t^2\|\eta_t\|^2$$

for $0 < t < t^*$, and after integration with respect to t,

$$\begin{aligned} t\|e(t)\|^2 &\le C(1+\beta t^*)\int_0^t \|e\|^2\,d\tau + C\big(\frac{\beta}{2}\|\tilde{e}\|^2 - \beta t(\tilde{e},e) + t(\eta,e)\big) \\ &\quad + C\int_0^t \|\eta\|^2\,d\tau + \frac{\epsilon^2}{2}\int_0^t \tau^2\|\eta_t\|^2\,d\tau \\ &\le C(1+\beta t^* + \beta^2 t^{*2})\int_0^t \|e\|^2\,d\tau + C\int_0^t \|\eta\|^2\,d\tau + \frac{\epsilon^2}{2}\int_0^t \tau^2\|\eta_t\|^2\,d\tau \\ &\quad + Ct\|\eta(t)\|^2 + \frac{1}{2}t\|e(t)\|^2, \qquad 0 < t < t^*, \end{aligned}$$

where we have used the fact that $\tilde{e}(0) = 0$ and $\|\tilde{e}(t)\|^2 \le t^*\int_0^t \|e\|^2\,d\tau$ for $0 < t < t^*$. Hence, absorbing βt^* into the constant,

$$\begin{aligned} t\|e(t)\|^2 &\le C\int_0^t \|e\|^2\,d\tau + C\int_0^t \|\eta\|^2\,d\tau \\ &\quad + \epsilon^2\int_0^t \tau^2\|\eta_t\|^2\,d\tau + Ct\|\eta(t)\|^2, \qquad 0 < t < t^*. \end{aligned} \tag{3.6}$$

In order to estimate $\int_0^t \|e\|^2\,d\tau$ we take the inner product of (3.4) with e to obtain

$$(T_h e_t, e) + \alpha\|e\|^2 + \beta(\tilde{e},e) = (\eta,e).$$

Since T_h is self-adjoint, this leads to

$$\frac{d}{dt}(T_h e,e) + 2\alpha\|e\|^2 + \beta\frac{d}{dt}\|\tilde{e}\|^2 \le \frac{1}{\alpha}\|\eta\|^2 + \alpha\|e\|^2,$$

and, upon integration,

$$(T_h e,e) + \alpha\int_0^t \|e\|^2\,d\tau + \beta\|\tilde{e}\|^2 \le C\int_0^t \|\eta\|^2\,d\tau,$$

since $T_h e(0) = 0,\ \tilde{e}(0) = 0$. Together with (3.6) this shows

$$t\|e(t)\|^2 \le C\int_0^t \|\eta\|^2\,d\tau + \epsilon^2\int_0^t \tau^2\|\eta_t\|^2\,d\tau + Ct\|\eta(t)\|^2, \qquad 0 < t < t^*, \tag{3.7}$$

from which (3.5) follows. The proof of the lemma is now complete.

We now show (3.1) for $j = 0$. We have by Lemma 3.2, using also (3.3), (2.3), and (2.4),

$$\begin{aligned} \|e(t)\| &\le C\sup_{0<\tau<t}\big(\|\eta(\tau)\| + \tau\|\eta_t(\tau)\|\big) \\ &\le Ch^r\big(\|\phi\|_r + |\psi|_{r-2} + \sup_{0<\tau<t}\big(\|u_t(\tau)\|_{r-2} + \tau\|u_{tt}(\tau)\|_{r-2}\big)\big),\ 0 < t \le 1. \end{aligned}$$

Here, by Lemma 2.3 (or Lemma 2.4),

$$\|u_t(\tau)\|_{r-2} + \tau\|u_{tt}(\tau)\|_{r-2} \le C\big(|\phi|_{r-2} + |\psi|_{r-2}\big),$$

which completes the proof for $j = 0$.

To show the result for general j we proceed by induction to demonstrate that

$$t^j\|D_t^j e(t)\| \le C \sup_{0<\tau<t}\Big(\sum_{l=0}^{j+1} \tau^l\|D_t^l \eta(\tau)\|\Big), \qquad \text{for } 0 < t \le 1. \tag{3.8}$$

Assuming this for a moment we have now

$$t^j\|D_t^j e(t)\| \le Ch^r\Big\{\|\phi\|_r + |\psi|_{r-2} + \sup_{0<\tau<t}\Big(\sum_{l=0}^{j+1}\tau^l\|D_t^{l+1}u(\tau)\|_{r-2}\Big)\Big\},\ 0 < t \le 1,$$

from which the desired estimate again follows by Lemma 2.3 (or Lemma 2.4).

For the proof of (3.8) we differentiate (3.3) j times and multiply by t^j to obtain

$$T_h(t^j e^{(j)})_t + \alpha t^j e^{(j)} = -t^j e^{(j-1)} + t^j\eta^{(j)} + jt^{j-1}T_h e^{(j)},$$

where $e^{(j)} = D_t^j e$ and η is as above. Now, by (3.3),

$$T_h e^{(j)} = -\alpha e^{(j-1)} - \tilde e^{(j-1)} + \eta^{(j-1)},$$

so that

$$T_h(t^j e^{(j)})_t + \alpha t^j e^{(j)} = -t^j e^{(j-1)} - jt^{j-1}\tilde e^{(j-1)} - \alpha j t^{j-1}e^{(j-1)} + t^j\eta^{(j)} + jt^{j-1}\eta^{(j-1)}.$$

Hence, by Lemma 3.2 (with $\beta = 0$) and using $t \le 1$, we have

$$\begin{aligned}\sup_{0<\tau<t}\big(\tau^j\|e^{(j)}\|\big) &\le C\sup_{0<\tau<t}\big(\tau^{j-1}\|e^{(j-1)}\| + \tau^{j-1}\|\tilde e^{(j-1)}\|\big)\\ &\quad + C\sup_{0<\tau<t}\big(\tau^{j+1}\|\eta^{(j+1)}\| + \tau^j\|\eta^{(j)}\| + \tau^{j-1}\|\eta^{(j-1)}\|\big)\\ &\quad + 2\epsilon\sup_{0<\tau<t}\big(\tau^j\|e^{(j)}\|\big).\end{aligned}$$

Here $\tilde e^{(j-1)} = e^{(j-2)}$ for $j \ge 2$, and $\|\tilde e^{(j-1)}(\tau)\| \le \tau\sup_{0<\sigma<\tau}\|e(\sigma)\|$ for $j = 1$. Hence, choosing $\epsilon < 1/2$, (3.8) follows by induction. This completes the proof of Theorem 3.1 for $t \le 1$.

We now turn to $t > 1$. This time we make the special choice $\phi_h = R_h\phi$, where R_h denotes the "elliptic projection" defined by $R_h = T_hA$, which as above shows the general result by stability, since $\phi \in H^r \cap H_0^1$, and

$$\|R_hv - v\| = \|(T_h - T)Av\| \le Ch^r\|Av\|_{r-2} \le Ch^r\|v\|_r, \qquad v \in H^r \cap H_0^1. \tag{3.9}$$

We split the error as

$$e = u_h - u = (u_h - R_hu) + (R_hu - u) \equiv \theta + \rho,$$

and note that, by our assumptions, $\theta \in S_h$ and

$$\theta(0) = 0, \quad \theta_t(0) = P_h\psi - R_h\psi = -P_h\rho_t(0). \tag{3.10}$$

By (3.9) and Lemma 2.4 we have

$$\begin{aligned}\|D_t^j\rho(t)\| &= \|(R_h - I)D_t^j u(t)\| \le Ch^r\|D_t^j u(t)\|_r \\ &\le Ch^r t^{-j}\big(\|\phi\|_r + |\phi|_{r-2} + |\psi|_{r-2}\big), \qquad \text{for } t>0,\end{aligned} \tag{3.11}$$

and it remains to estimate $D_t^j\theta$.

From (2.1), (2.5) and the definition of R_h we obtain

$$\theta_{tt} + \alpha A_h\theta_t + A_h\theta = -P_h(R_h - I)u_{tt} = -P_h\rho_{tt}.$$

Integrating with respect to t and using also (3.10) we find that θ satisfies

$$\begin{aligned}&\theta_t + \alpha A_h\theta + A_h\tilde\theta = -P_h\rho_t, \qquad t>0,\\ &\theta(0) = 0,\end{aligned} \tag{3.12}$$

where $\tilde\theta(t) = \int_0^t \theta(\tau)\,d\tau$. We will show by induction that, with $|\theta|_{1,h} = \|A_h^{\frac12}\theta\|$ a discrete analogue of the H_0^1 norm, we have

$$\|D_t^j\theta(t)\|^2 + \int_1^t |D_t^j\theta|_{1,h}^2\,d\tau \le Ch^{2r}\big(\|\phi\|_r + |\psi|_{r-2}\big)^2, \qquad \text{for } t\ge 1. \tag{3.13}$$

Together with (3.11) this will complete the proof.

We begin with a lemma which implies (3.13) for $j=0$.

Lemma 3.3. *The solution of (3.12) satisfies, with $e=\theta+\rho$,*

$$\|\theta(t)\|^2 + \alpha\int_1^t |\theta|_{1,h}^2\,d\tau \le C\sup_{0<\tau<1}\big(\|e(\tau)\|^2 + \|\rho(\tau)\|^2\big) + C\int_1^t \|\rho_t\|^2\,d\tau, \qquad t\ge 1.$$

Proof. Taking the inner product of (3.12) by θ, and using the facts that A_h is self-adjoint, positive definite and that $T_h = A_h^{-1}$ is bounded independently of h, we get

$$\frac12\frac{d}{dt}\|\theta\|^2 + \alpha|\theta|_{1,h}^2 + \frac12\frac{d}{dt}|\tilde\theta|_{1,h}^2 \le \|\rho_t\|\|\theta\| \le C\|\rho_t\|^2 + \frac{\alpha}{2}|\theta|_{1,h}^2.$$

Integrating with respect to t we obtain by simple estimates

$$\|\theta(t)\|^2 + \alpha\int_1^t |\theta|_{1,h}^2\,d\tau + |\tilde\theta(t)|_{1,h}^2 \le C\big(\|\theta(1)\|^2 + |\tilde\theta(1)|_{1,h}^2\big) + C\int_1^t \|\rho_t\|^2\,d\tau.$$

In order to estimate $|\tilde\theta(1)|_{1,h}$ we now take the inner product of (3.12) by $\tilde\theta$ and find

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}|\tilde\theta|_{1,h}^2 + |\tilde\theta|_{1,h}^2 &= -(\theta_t,\tilde\theta) - (\rho_t,\tilde\theta) = -(e_t,\tilde\theta)\\ &= -\frac{d}{dt}(e,\tilde\theta) + (e,\theta) \le -\frac{d}{dt}(e,\tilde\theta) + C\big(\|e\|^2 + \|\rho\|^2\big).\end{aligned}$$

Hence

$$|\tilde\theta(1)|_{1,h}^2 \le C\|e(1)\|\,\|\tilde\theta(1)\| + C\int_0^1 \big(\|e\|^2 + \|\rho\|^2\big)\,d\tau,$$

which yields

$$|\tilde\theta(1)|_{1,h}^2 \le C\sup_{0<\tau<1}\big(\|e(\tau)\|^2 + \|\rho(\tau)\|^2\big).$$

Since $\|\theta(1)\|$ is obviously estimated by the same quantity, this completes the proof of the lemma.

To show (3.13) for $j=0$ we note that $e(t)$ and $\rho(t)$ have already been bounded as desired for $t \le 1$ and that by Lemma 2.4,

$$\int_1^t \|\rho_t\|^2\, d\tau \le Ch^{2r} \int_1^t \|u_t\|_r^2\, d\tau \le Ch^{2r} \int_1^t \tau^{-2}\, d\tau \, \big(\|\phi\|_r + |\psi|_{r-2}\big)^2.$$

There remains only the induction step in the proof of (3.13). For this purpose we differentiate (3.12) j times with respect to t and multiply by $\theta^{(j)} = D_t^j \theta$ to obtain for $j \ge 1$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta^{(j)}\|^2 + \alpha|\theta^{(j)}|_{1,h}^2 &= -(A_h\theta^{(j-1)}, \theta^{(j)}) - (\rho^{(j+1)}, \theta^{(j)}) \\ &\le C\big(|\theta^{(j-1)}|_{1,h}^2 + \|\rho^{(j+1)}\|^2\big) + \frac{\alpha}{2}|\theta^{(j)}|_{1,h}^2. \end{aligned}$$

Hence, assuming (3.13) for $j-1$, we have by the estimates for $t \le 1$ and by (3.11) that

$$\begin{aligned} \|\theta^{(j)}(t)\|^2 + \int_1^t |\theta^{(j)}|_{1,h}^2\, d\tau &\le \|\theta^{(j)}(1)\|^2 + C\int_1^t \big(|\theta^{(j-1)}|_{1,h}^2 + \|\rho^{(j+1)}\|^2\big)\, d\tau \\ &\le Ch^{2r}\big(\|\phi\|_r + |\psi|_{r-2}\big)^2, \qquad \text{for } t \ge 1. \end{aligned}$$

The proof of Theorem 3.1 is now complete.

We note that the assumptions of Theorem 3.1 are exactly those which are required in Lemma 2.4 to guarantee that $u(t)$ is bounded in H^r, uniformly for $t \ge 0$, and weaker than what is needed for $u(t)$ to be bounded in $\dot H^r$ by Lemma 2.3. In particular, for $r \le 4$, none of the higher order boundary conditions of the $\dot H^s$ spaces are required, but only $\phi = 0$, and, if $r = 3, 4$ also $\psi = 0$, on $\partial\Omega$.

For $r = 5$ the boundary conditions $A\phi = A\psi = 0$ on $\partial\Omega$ are also needed since $\phi, \psi \in \dot H^3$ is assumed in that case. By a slightly more careful analysis we are able to omit these extra artificial boundary conditions, at the expense of a weak singularity for t small. We shall show that in this case, when $\phi \in H^5 \cap H_0^1$ and $\psi \in H^3 \cap H_0^1$ we have for any $\epsilon \in (0, 1/2)$

$$\|e(t)\| \le C_\epsilon h^5 t^{-(1+\epsilon)/4}\big(\|\phi\|_5 + \|\psi\|_3\big), \qquad \text{for } 0 < t \le t^*. \tag{3.14}$$

To see this we note from (3.7) of the proof of Lemma 3.2 that

$$\|e(t)\|^2 \le Ct^{-1}\int_0^t \big(\|\eta\|^2 + \tau^2\|\eta_t\|^2\big)\, d\tau.$$

(The last term on the right in (3.7) may easily be estimated by the first two terms.) Hence, by the definition of η, cf. (3.3), we find

$$\|e(t)\|^2 \le Ch^{10}\Big(\|\phi\|_5^2 + \|\psi\|_3^2 + t^{-1}\int_0^t \big(\|u_t\|_3^2 + \tau^2\|u_{tt}\|_3^2\big)\, d\tau\Big). \tag{3.15}$$

But under our assumptions on ϕ and ψ, we have $\phi, \psi \in \dot H^{(5-\epsilon)/2}$ and hence, by Lemma 2.4,

$$\begin{aligned} \|u_t(\tau)\|_3 + \tau\|u_{tt}(\tau)\|_3 &\le C_\epsilon\big(\|\phi\|_3 + \|\psi\|_1 + \tau^{-(1+\epsilon)/4}\big(|\phi|_{(5-\epsilon)/2} + |\psi|_{(5-\epsilon)/2}\big)\big) \\ &\le C\tau^{-(1+\epsilon)/4}\big(\|\phi\|_5 + \|\psi\|_3\big), \end{aligned}$$

which, inserted into (3.15) gives (3.14).

In view of the system formulation (2.2) of our model problem (1.1) and the estimates of Lemma 2.3, it appears natural to attempt to derive an error estimate for $u_{h,t}$, which is uniform down to $t = 0$. In the following theorem such an estimate is derived which requires $\phi, \psi \in \dot{H}^r$, thus with higher regularity for ψ and (if $r > 2$) more boundary conditions on ϕ than in Theorem 3.1. Comparing with Lemmas 2.3 and 2.4 these extra assumptions are not surprising.

Theorem 3.4. *Let ϕ, $\psi \in \dot{H}^r$ and assume that*

$$\|\phi_h - \phi\| \le Ch^r |\phi|_r, \quad \|\psi_h - \psi\| \le Ch^r |\psi|_r.$$

Then there is a constant C such that

$$\|u_h(t) - u(t)\| + \|u_{h,t}(t) - u_t(t)\| \le Ch^r \big(|\phi|_r + |\psi|_r\big), \qquad \textit{for } t \ge 0.$$

Proof. We again note that, without restricting generality, we may assume that $\phi_h = P_h\phi$, $\psi_h = P_h\psi$. In fact, by Lemma 2.5, the additional error caused by the general choice may be bounded by

$$C\big(\|\phi_h - P_h\phi\| + \|\psi_h - P_h\psi\|\big) \le Ch^r \big(|\phi|_r + |\psi|_r\big).$$

Theorem 3.1 already bounds $\|e\|$ in the desired way. In order to estimate $\|e_t\|$ we differentiate (3.3) once to obtain

$$T_h e_{tt} + \alpha e_t = -(T_h - T)u_{tt} - e, \qquad t > 0,$$

and note also that $T_h e_t(0) = 0$. Hence by Lemma 3.2 (with $\beta = 0$, so that C is independent of the length of the time interval)

$$\|e_t(t)\| \le C \sup_{0<\tau<t} \big(\|(T_h - T)u_{tt}(\tau)\| + \tau\|(T_h - T)u_{ttt}(\tau)\| + \|e(\tau)\| + \tau\|e_t(\tau)\|\big).$$

By Theorem 3.1 we have

$$\|e(\tau)\| + \tau\|e_t(\tau)\| \le Ch^r \big(|\phi|_r + |\psi|_{r-2}\big).$$

Further, using the differential equation, and Lemma 2.3,

$$\begin{aligned}\|(T_h - T)u_{tt}(\tau)\| &\le Ch^r \|u_{tt}(\tau)\|_{r-2} \le Ch^r \big(\|u_t(\tau)\|_r + \|u(\tau)\|_r\big) \\ &\le Ch^r \big(|\phi|_r + |\psi|_r\big),\end{aligned}$$

and similarly

$$\tau\|(T_h - T)u_{ttt}(\tau)\| \le Ch^r \tau\big(\|u_{tt}(\tau)\|_r + \|u_t(\tau)\|_r\big) \le Ch^r \big(|\phi|_r + |\psi|_r\big),$$

which completes the proof.

We finally remark that the higher order time derivatives of the error can also be estimated under the same regularity assumptions as in Theorem 3.4, but with negative powers of t occurring in the bounds.

4. Rational approximation of analytic semigroups. In this section we discuss the approximation of an analytic semigroup by a rational function of its generator. In the following section we shall combine this with our previous discussion of the spatially semidiscrete problem (2.5) and obtain error bounds for a class of completely discrete schemes for the strongly damped wave equation (1.1).

The technique that we employ here is similar that that of Le Roux [8]. The results of [8], however, do not apply directly to the non-parabolic problem (1.1). For earlier related work see also Fujita and Mizutani [4].

Let A be a densely defined linear operator in a Banach space X, with norm $\|\cdot\|$, such that the resolvent set $\rho(A)$ of A satisfies, for some $\theta_0 \in (0, \frac{\pi}{2})$,

$$\rho(A) \supset \Sigma = \{z \in \mathbf{C} : \theta_0 \le |\arg z| \le \pi\} \cup \{0\}, \tag{4.1}$$

and such that

$$\|(zI - A)^{-1}\| \le \frac{M}{|z|}, \qquad \text{for } z \in \Sigma,\ z \ne 0. \tag{4.2}$$

Then $-A$ is the infinitesimal generator of an analytic semigroup $\{\exp(-tA)\}_{t\ge 0}$, see Pazy [11, Theorem 2.5.2 and Section 2.6].

Let $r(z)$ be a rational function approximating e^{-z} to order $p+1 \ge 2$, i.e.,

$$r(z) = e^{-z} + O(z^{p+1}), \qquad \text{as } z \to 0. \tag{4.3}$$

We also assume that $r(z)$ satisfies the stability requirement

$$|r(z)| \le 1, \qquad \text{for } \operatorname{Re} z \ge 0. \tag{4.4}$$

Under the latter assumption we can write

$$r(z) = \gamma + \sum_{l,m} \gamma_{l,m}(\alpha_m - z)^{-l},$$

where the sum is finite, $\gamma = r(\infty)$ and the poles α_m lie in the left half-plane. We define for $k > 0$

$$r(kA) = \gamma I + \sum_{l,m} \gamma_{l,m}(\alpha_m I - kA)^{-l}.$$

This is a bounded linear operator on X, since $\alpha_m/k \in \Sigma$. We want to show that the iterated operator $r(kA)^n$ approximates $\exp(-t_n A)$ for $t_n = nk > 0$. We first formulate a simple consequence of (4.3) and (4.4).

Lemma 4.1. *Assume that (4.3) and (4.4) hold. Then for any $\theta_0 \in (0, \frac{\pi}{2})$ and $R > 0$ there are positive numbers C, c such that*

$$|r(z)^n - e^{-nz}| \le Cn|z|^{p+1}e^{-cn|z|}, \qquad \text{for } |z| \le R,\ |\arg z| \le \theta_0,\ n \ge 1.$$

Proof. We first note that we can find $C > 0$ such that

$$|r(z) - e^{-z}| \le C|z|^{p+1}, \qquad \text{for } |z| \le R,\ \operatorname{Re} z > 0. \tag{4.5}$$

By (4.3) this holds for some small R, and in view of (4.4) then also for arbitrary $R > 0$. We next observe that

$$|e^{-z}| \le e^{-c|z|}, \qquad \text{for } |\arg z| \le \theta_0, \tag{4.6}$$

if $c \le \cos\theta_0$. We may then select $c > 0$ such that

$$|r(z)| \le e^{-c|z|}, \qquad \text{for } |z| \le R, \ |\arg z| \le \theta_0. \tag{4.7}$$

This follows from (4.5) and (4.6) for some small R, and then for arbitrary R by noting that, by (4.4) and the maximum principle, we have $|r(z)| < 1$ for $\operatorname{Re} z > 0$. Hence

$$\begin{aligned} |r(z)^n - e^{-nz}| &= |(r(z) - e^{-z}) \sum_{j=0}^{n-1} r(z)^j e^{-(n-1-j)z}| \\ &\le C|z|^{p+1} n e^{-(n-1)c|z|} \le Cn|z|^{p+1} e^{-cn|z|}, \end{aligned}$$

and the lemma is proved.

We can now show that if $v \in \mathcal{D}(A^p)$, then

$$\|[r(kA)^n - \exp(nkA)]v\| \le Ck^p\|A^p v\|.$$

This is a consequence of the following theorem.

Theorem 4.2. *Under the hypotheses (4.1)–(4.4) on A and $r(z)$, there is a constant C such that*

$$\|[r(kA)^n - \exp(-nkA)]A^{-j}\| \le C\,k^j, \qquad \text{for } 1 \le j \le p, \ n \ge 1, \ 0 < k < 1.$$

The constant is the same for all A satisfying (4.1) and (4.2).

Proof. Let θ_0 be as in (4.1). Since the rational function $r(kz)^n z^{-j}$ vanishes at infinity, the operator $r(kA)^n A^{-j}$ may be represented by Cauchy's integral formula, cf. [11, p. 69]. Using also the contour integral representation of $\exp(-t_n A)$, we thus have

$$[r(kA)^n - \exp(-nkA)]A^{-j} = \frac{1}{2\pi i} \int_\Gamma F_n(kz) z^{-j} (zI - A)^{-1}\, dz,$$

where $F_n(z) = r(z)^n - e^{-nz}$ and $\Gamma = \Gamma_+ \cup \Gamma_-$, with $\Gamma_\pm = \{\rho e^{\pm i\theta_0} : 0 \le \rho < \infty\}$, is the boundary of Σ traversed in the positive sense.

Using (4.2) we get

$$\begin{aligned} \|\int_{\Gamma_+} F_n(kz) z^{-j} (zI - A)^{-1}\, dz\| &\le M \int_0^\infty |F_n(k\rho e^{i\theta_0})| \frac{d\rho}{\rho^{j+1}} \\ &\le Mk^j \int_0^\infty |F_n(\rho e^{i\theta_0})| \frac{d\rho}{\rho^{j+1}}. \end{aligned}$$

Here, by Lemma 4.1, we have

$$\int_0^1 |F_n(\rho e^{i\theta_0})| \frac{d\rho}{\rho^{j+1}} \le Cn \int_0^1 \rho^{p-j} e^{-cn\rho}\, d\rho \le C,$$

and, in view of (4.4),

$$\int_1^\infty |F_n(\rho e^{i\theta_0})| \frac{d\rho}{\rho^{j+1}} \le 2 \int_1^\infty \frac{d\rho}{\rho^{j+1}} = C.$$

The integral along Γ_- is estimated in the same way. This completes the proof.

We next show that under the additional assumption that $|r(\infty)| < 1$, we have, for any "nonsmooth" initial data $v \in X$,

$$\|[r(kA)^n - \exp(nkA)]v\| \le Ck^p t_n^{-p}\|v\|.$$

We will need the following result about the behavior of $r(z)^n$ for large z.

Lemma 4.3. *Let the rational function r be such that(4.4) holds and assume that $|r(\infty)| < 1$. Then for any $R > 0$ and $\theta_0 \in (0, \frac{\pi}{2})$ there are positive numbers C, c such that, with $\gamma = r(\infty)$,*

$$|r(z)^n - \gamma^n| \le C|z|^{-1}e^{-cn}, \qquad \text{for } |z| \ge R, \ |\arg z| \le \theta_0, \ n \ge 1.$$

Proof. By the maximum principle we have $|r(z)| < 1$ for $\operatorname{Re} z > 0$. Using also the fact that the rational function $r(z) - \gamma$ is zero at infinity, we obtain

$$|r(z) - \gamma| \le C|z|^{-1},$$
$$|r(z)| \le e^{-c},$$

for $|z| \ge R, \ |\arg z| \le \theta_0$. Hence

$$|r(z)^n - \gamma^n| = |(r(z) - \gamma)\sum_{j=0}^{n-1} r(z)^j\gamma^{n-1-j}| \le C|z|^{-1}ne^{-(n-1)c} \le C|z|^{-1}e^{-cn}.$$

We are now ready to state and prove the following result.

Theorem 4.4. *In addition to the assumptions of Theorem 4.2 we assume that $|r(\infty)| < 1$. Then there is a constant C such that*

$$\|r(kA)^n - \exp(-nkA)\| \le Ck^p t_n^{-p}, \qquad \text{for } k > 0, \ t_n = nk > 0.$$

The constant is the same for all A satisfying (4.1) and (4.2).

Proof. Let $F_n(z) = r(z)^n - e^{-nz}$, $\gamma = r(\infty)$ and define $G_n(z) = F_n(z) - \gamma^n z/(1+z)$. Since $|\gamma| < 1$ and $kA(I+kA)^{-1}$ is bounded, we have

$$\|\gamma^n kA(I+kA)^{-1}\| \le C|\gamma|^n \le \frac{C}{n^p} = Ck^p t_n^{-p},$$

and it remains to estimate $G_n(kA)$. The rational function $r(kz)^n - \gamma^n kz/(1+kz)$ is zero at infinity, so that

$$G_n(kA) = \frac{1}{2\pi i}\int_\Gamma G_n(kz)(zI - A)^{-1}\,dz.$$

In view of (4.2) we have

$$\|\int_{\Gamma_+} G_n(kz)(zI - A)^{-1}\,dz\| \le M\int_0^\infty |G_n(k\rho e^{i\theta_0})|\frac{d\rho}{\rho} = M\int_0^\infty |G_n(\rho e^{i\theta_0})|\frac{d\rho}{\rho}.$$

By Lemma 4.1 and since $|z/(1+z)| \le |z|$ for $\operatorname{Re} z > 0$, we have

$$\begin{aligned}\int_0^1 |G_n(\rho e^{i\theta_0})|\frac{d\rho}{\rho} &\le \int_0^1 |F_n(\rho e^{i\theta_0})|\frac{d\rho}{\rho} + |\gamma|^n\int_0^1 d\rho \\ &\le Cn\int_0^\infty \rho^p e^{-cn\rho}\,d\rho + |\gamma|^n \\ &\le \frac{C}{n^p}\int_0^\infty \rho^p e^{-\rho}\,d\rho + |\gamma|^n \le \frac{C}{n^p}.\end{aligned}$$

Next we write $G_n(z) = (r(z)^n - \gamma^n) + \gamma^n/(1+z) - e^{-nz}$ and use Lemma 4.3 to get

$$\int_1^\infty |G_n(\rho e^{i\theta_0})|\frac{d\rho}{\rho} \le Ce^{-nc}\int_1^\infty \frac{d\rho}{\rho^2} + |\gamma|^n\int_1^\infty \frac{d\rho}{\rho^2} + \int_1^\infty e^{-cn\rho}\frac{d\rho}{\rho} \le \frac{C}{n^p}.$$

The proof is now complete.

We remark that assumption (4.4) is A-stability in the sense of Dahlquist. Clearly, this can be replaced in the above discussion by $A(\theta_0)$-stability, i.e., $|r(z)| \le 1$ needs to hold only in the sector $|\arg z| \le \theta_0$.

5. Completely discrete schemes. As indicated in the introduction we shall study a class of completely discrete schemes for (1.1), where the approximate solution $W_n = (U_n, V_n)^T \in S_h \times S_h$ at discrete time $t_n = nk$ is defined by the recursion

$$\begin{aligned} W_n &= r(kB_h)W_{n-1}, \qquad n \geq 1, \\ W_0 &= w_{0h} = (\phi_h, \psi_h)^T, \end{aligned} \tag{5.1}$$

or, equivalently, $W_n = r(kB_h)^n w_{0h}$. Here $k > 0$ is a time step and $r(z)$ is a rational function approximating e^{-z}, and

$$B_h = \begin{bmatrix} 0 & -I \\ A_h & \alpha A_h \end{bmatrix}$$

is the discrete analogue of the operator B introduced in Section 2. Lemma 2.2 shows that B_h satisfies the hypotheses of Section 4 independently of h.

We shall first show the following smooth data result.

Theorem 5.1. *Assume that $r(z)$ satisfies conditions (4.3) and (4.4). Let $\phi, \psi \in \dot{H}^s$, $s = \max(r, 2p)$, and assume that*

$$\|\phi_h - \phi\| \leq Ch^r|\phi|_r, \quad \|\psi_h - \psi\| \leq Ch^r|\psi|_r.$$

Then for the solutions of (1.1) and of (5.1) we have

$$\begin{aligned} \|U_n - u(t_n)\| + \|V_n - u_t(t_n)\| &\leq Ch^r\left(|\phi|_r + |\psi|_r\right) \\ &\quad + Ck^p\left(|\phi|_{2p} + |\psi|_{2p}\right), \qquad \text{for } t_n \geq 0. \end{aligned}$$

Equivalently the result of the theorem can be stated as

$$\|W_n - w(t_n)\| \leq Ch^r|w_0|_r + Ck^p|w_0|_{2p}, \qquad \text{for } t_n \geq 0, \tag{5.2}$$

where we now let $|\cdot|_s$ denote also the norm in $\dot{H}^s \times \dot{H}^s$. In order to prove the latter bound we recall that Theorem 3.4 shows

$$\|w_h(t) - w(t)\| \leq Ch^r|w_0|_r, \qquad \text{for } t \geq 0, \tag{5.3}$$

Applying Theorem 4.2 with A replaced by B_h and $X = S_h \times S_h$, we have

$$\|W_n - w_h(t_n)\| \leq Ck^p\|B_h^p w_{0h}\|, \qquad \text{for } t_n \geq 0.$$

The latter estimate is unsatisfactory in that it places a discrete norm on the discrete initial data. By means of a technique used for a parabolic problem in Baker, Bramble and Thomée [1] (see also Thomée [13, Chapter 7]) we shall obtain an estimate in terms of norms of the continuous initial values as stated in (5.2).

We begin by showing that the discrete solution operator $r(kB_h)^n$ is stable in $L_2 \times L_2$.

Lemma 5.2. *There is a constant C such that*

$$\|r(kB_h)^n\| \leq C, \qquad \text{for } n \geq 0,\ 0 < k < 1.$$

Proof. In the notation of the proof of Lemma 2.2 we must show the uniform boundedness of the family of 2×2 matrices $\{r(kB(\lambda))^n : n \geq 1,\ 0 < k < 1,\ \lambda \geq c_0\}$, where $B(\lambda)$ is defined in (2.10). By Schur's lemma $B(\lambda)$ is orthogonally equivalent to

$$S(\lambda) = \begin{bmatrix} \tau_-(\lambda) & \beta(\lambda) \\ 0 & \tau_+(\lambda) \end{bmatrix},$$

where $|\beta(\lambda)|$ is bounded by the sum of the moduli of the elements of $B(\lambda)$, so that $|\beta(\lambda)| \leq C\lambda$ for $\lambda \geq c_0$. An elementary calculation shows that for an arbitrary rational function f we have

$$f(S(\lambda)) = \begin{bmatrix} f(\tau_-(\lambda)) & \beta(\lambda)\,\dfrac{f(\tau_+(\lambda)) - f(\tau_-(\lambda))}{\tau_+(\lambda) - \tau_-(\lambda)} \\ 0 & f(\tau_+(\lambda)) \end{bmatrix},$$

when $\tau_+(\lambda) \neq \tau_-(\lambda)$. In particular, when $f(z) = r(kz)^n$, we find by (4.4) that the diagonal elements are bounded by 1 in modulus, and hence it now suffices to show

(5.4)
$$|\beta_{k,n}(\lambda)| = |\beta(\lambda)\,\frac{r(k\tau_+(\lambda))^n - r(k\tau_-(\lambda))^n}{\tau_+(\lambda) - \tau_-(\lambda)}| \leq C, \qquad \lambda \geq c_0,\ n \geq 1,\ 0 < k < 1.$$

To see this, we first consider λ large, $\lambda \geq 8/\alpha^2$, say. Then

$$|\tau_+(\lambda) - \tau_-(\lambda)| \geq \frac{\alpha}{\sqrt{2}}\lambda \geq c|\beta(\lambda)|, \qquad \text{with } c > 0,$$

and since $|r(z)| \leq 1$, (5.4) follows. For the remaining case, $c_0 \leq \lambda \leq 8/\alpha^2$, we have $c_1 \leq |\tau_\pm(\lambda)| \leq c_2$ for some $c_1,\ c_2 > 0$, so that by (4.7)

$$|r(k\tau_\pm(\lambda))^n| \leq e^{-cnk}, \qquad \text{with } c > 0.$$

Hence, since now $\beta(\lambda)$ is bounded, we conclude that

$$|\beta_{k,n}(\lambda)| \leq Ck|\sum_{j=0}^{n-1} r(k\tau_-(\lambda))^{n-1-j} r(k\tau_+(\lambda))^j| \leq Cnke^{-c(n-1)k} \leq C,$$

showing the uniform boundedness when $\tau_+(\lambda) \neq \tau_-(\lambda)$, i.e., when $\lambda \neq 4/\alpha^2$, and thus, by continuity, also for $\lambda = 4/\alpha^2$. The proof is now complete.

Proof of Theorem 5.1. In view of (5.3) we only have to estimate

$$W_n - w_h(t_n) = [r(kB_h)^n - \exp(-t_n B_h)]w_{0h}.$$

By stability of the discrete solution operator $r(kB_h)^n$ in $L_2 \times L_2$ and (2.4) there is no restriction of generality in assuming that $w_{0h} = P_h w_0$, where we extend the notation P_h to mean also the orthogonal projection of $L_2 \times L_2$ onto $S_h \times S_h$. Thus, we shall bound $F_n P_h w_0$, where $F_n = F_n(kB_h) = r(kB_h)^n - \exp(-nkB_h)$.

We set

$$w_0^k = \sum_{k\lambda_j \leq 1} \hat{w}_{0,j}\omega_j,$$

where $\hat{w}_{0,j} \in \mathbf{C}^2$ are the Fourier coefficients of w_0 with respect to the eigensystem of A (see the notation of the proof of Lemma 2.2). It is easy to verify that

$$\|w_0 - w_0^k\| \leq Ck^p|w_0|_{2p}, \tag{5.5}$$

$$|w_0^k|_{2p} \leq |w_0|_{2p}, \tag{5.6}$$

$$|w_0^k|_{r+2j} \leq Ck^{-j}|w_0|_r, \qquad \text{for } 0 \leq j \leq p. \tag{5.7}$$

Using the identity

$$P_h = \sum_{j=0}^{p-1} B_h^{-j}(P_h B^{-1} - B_h^{-1}P_h)B^{j+1} + B_h^{-p}P_h B^p,$$

we have

$$F_n P_h w_0 = \sum_{j=0}^{p-1} F_n B_h^{-j}(P_h B^{-1} - B_h^{-1} P_h) B^{j+1} w_0^k \\ + F_n B_h^{-p} P_h B^p w_0^k + F_n P_h (w_0 - w_0^k).$$

In order to bound the terms on the right we combine Theorem 4.2 (for $1 \le j \le p$) and Lemma 5.2 (for $j = 0$) to obtain

$$\|F_n B_h^{-j}\| \le C k^j, \qquad \text{for } 0 \le j \le p.$$

Further, we have

$$B^{-1} = \begin{bmatrix} \alpha I & T \\ -I & 0 \end{bmatrix},$$

and similarly for B_h^{-1}. Therefore

$$P_h B^{-1} - B_h^{-1} P_h = \begin{bmatrix} 0 & P_h T - T_h \\ 0 & 0 \end{bmatrix},$$

which, in view of (2.3), implies, for $f = (f_1, f_2)^T$,

$$\|(P_h B^{-1} - B_h^{-1} P_h) f\| = \|(P_h T - T_h) f_2\| \le C h^r \|f\|_{r-2}.$$

Hence, using also (5.7), and the fact that B is a second order partial differential operator, we find

$$\begin{aligned} \|F_n B_h^{-j}(P_h B^{-1} - B_h^{-1} P_h) B^{j+1} w_0^k\| &\le C k^j \|(P_h B^{-1} - B_h^{-1} P_h) B^{j+1} w_0^k\| \\ &\le C k^j h^r \|B^{j+1} w_0^k\|_{r-2} \le C h^r k^j |w_0^k|_{r+2j} \\ &\le C h^r |w_0|_r, \qquad \text{for } 0 \le j \le p-1. \end{aligned}$$

Similarly, using (5.6), we have

$$\|F_n B_h^{-p} P_h B^p w_0^k\| \le C k^p \|B^p w_0^k\| \le C k^p |w_0^k|_{2p} \le C k^p |w_0|_{2p}.$$

Finally, by stability and (5.5), we conclude

$$\|F_n P_h (w_0 - w_0^k)\| \le C \|w_0 - w_0^k\| \le C k^p |w_0|_{2p}.$$

Together with (5.3) these estimates show (5.2) and the theorem is proved.

Similarly to the situation for the semidiscrete problem it is possible to obtain an estimate for the error in the function values u alone under slightly reduced regularity demands on ψ. More precisely we have the following.

Theorem 5.3. *Assume that $r(z)$ satisfies (4.3) and (4.4). Let $\phi \in \dot H^s$, $\psi \in \dot H^{s-2}$ with $s = \max(r, 2p)$, and assume that*

$$\|\phi_h - \phi\| \le C h^r |\phi|_r, \quad \psi_h = P_h \psi.$$

Then for the solutions of (1.1) and of (5.1) we have

$$\|U_n - u(t_n)\| \le C h^r \big(|\phi|_r + |\psi|_{r-2}\big) \\ + C k^p \big(|\phi|_{2p} + |\psi|_{2p-2}\big), \qquad \textit{for } t_n \ge 0. \tag{5.8}$$

Proof. The proof will parallel that of Theorem 5.1 and we shall merely give the essential changes. The error in the semidiscrete solution is now taken care of by Theorem 3.1. By the same argument as in the proof of Theorem 5.1 we may assume without loss of generality that $\phi_h = P_h\phi$. Letting as before $W_n - w_h(t_n) = F_nP_hw_0$, we shall estimate

$$M_hF_nP_hw_0 = M_hF_nM_h^{-1}M_hP_hw_0 \equiv \tilde{F}_nM_hP_hw_0, \qquad \text{with } M_h = \begin{bmatrix} I & 0 \\ 0 & T_h \end{bmatrix},$$

in $L_2 \times L_2$, thus in effect estimating the second component of the error in a discrete negative norm. One easily checks that

$$\tilde{F}_n = r(k\tilde{B}_h)^n - \exp(-t_n\tilde{B}_h),$$

where

$$\tilde{B}_h = \tilde{B}(A_h) = M_hB_hM_h^{-1} = \begin{bmatrix} 0 & -A_h \\ I & \alpha A_h \end{bmatrix}.$$

Further, since $\tilde{B}(\lambda)$ has the same eigenvalues as $B(\lambda)$, the estimate for the resolvent in Lemma 2.2 holds for $\tilde{B}_h$, so that the results of Section 4 apply. Also, the stability statement of Lemma 5.2 remains valid for $\tilde{B}_h$ by the same argument as for B_h.

We now write

$$\tilde{F}_nM_hP_hw_0 = \tilde{F}_nP_hMw_0 + \tilde{F}_n\begin{bmatrix} 0 \\ (T_h - P_hT)\psi \end{bmatrix}, \qquad \text{with } M = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}. \tag{5.9}$$

By the stability in $L_2 \times L_2$, the latter term is bounded by

$$C\|(T_h - P_hT)\psi\| \le Ch^r|\psi|_{r-2}. \tag{5.10}$$

For the first term on the right of (5.9) we write as in the proof of Theorem 5.1,

$$\begin{aligned} \tilde{F}_nP_hMw_0 = &\sum_{j=0}^{p-1}\tilde{F}_n\tilde{B}_h^{-j}(P_h\tilde{B}^{-1} - \tilde{B}_h^{-1}P_h)\tilde{B}^{j+1}Mw_0^k \\ &+ \tilde{F}_n\tilde{B}_h^{-p}P_h\tilde{B}^pMw_0^k + \tilde{F}_nP_hM(w_0 - w_0^k), \end{aligned} \tag{5.11}$$

where now

$$\tilde{B} = MBM^{-1} = \begin{bmatrix} 0 & -A \\ I & \alpha A \end{bmatrix}.$$

For the terms in the sum we have by Theorem 4.2 and Lemma 5.2, and since

$$P_h\tilde{B}^{-1} - \tilde{B}_h^{-1}P_h = \begin{bmatrix} 0 & 0 \\ T_h - P_hT & 0 \end{bmatrix},$$

and by (5.7), that

$$\begin{aligned} \|\tilde{F}_n\tilde{B}_h^{-j}(P_h\tilde{B}^{-1} - \tilde{B}_h^{-1}P_h)\tilde{B}^{j+1}Mw_0^k\| &\le Ck^j\|(P_h\tilde{B}^{-1} - \tilde{B}_h^{-1}P_h)\tilde{B}^{j+1}Mw_0^k\| \\ &\le Ck^jh^r\|\tilde{B}^{j+1}Mw_0^k\|_{r-2} \\ &\le Ch^rk^j|Mw_0^k|_{r+2j} \\ &\le Ch^r\big(|\phi|_r + |\psi|_{r-2}\big). \end{aligned}$$

Finally, as in the proof of Theorem 5.1, the last two terms on the right of (5.11) are easily bounded by

$$Ck^p\big(|\phi|_{2p} + |\psi|_{2p-2}\big).$$

This completes the proof of the theorem.

We note that in the proof of Theorem 5.3 we have, in fact, estimated also the discrete negative norm

$$|u_{h,t}(t_n) - V_n|_{-2,h} = \|T_h(u_{h,t}(t_n) - V_n)\|.$$

This is somewhat analogous to the case $\sigma = 2$ in Lemma 2.3. A corresponding estimate for the semidiscrete solution may be easily obtained directly by solving (3.3) for $T_h e_t$, and we may thus bound $|V_n - u_t(t_n)|_{-2,h}$ by the right-hand side of (5.8).

We also remark that the assumption $\psi_h = P_h\psi$ in Theorem 5.3 may be weakened to the condition

$$|\psi_h - \psi|_{-2,h} \le Ch^r|\psi|_{r-2}.$$

For, the error in $M_h W_n$ caused by such a perturbation is bounded by

$$\|M_h\, r(kB_h)^n \begin{bmatrix} 0 \\ \psi_h - \psi \end{bmatrix}\| = \|r(k\tilde{B}_h)^n M_h \begin{bmatrix} 0 \\ \psi_h - \psi \end{bmatrix}\| \le C|\psi_h - \psi|_{-2,h}.$$

This also applies to the error in the time derivative just mentioned. We shall not insist on the details.

We conclude by an observation relating to initial data of low regularity. Recall that for time discretization schemes with $|r(\infty)| < 1$, Theorem 4.4 yields the nonsmooth data error estimate

$$\|U_n - u_h(t_n)\| + \|V_n - u_{h,t}(t_n)\| \le Ck^p t_n^{-p}\big(\|\phi_h\| + \|\psi_h\|\big), \qquad \text{for } t_n > 0.$$

Unfortunately, no nonsmooth data estimate for the error in the spatially semidiscrete solution is available that matches this exactly. However, in view of Theorem 3.1, we may state the following result in which the regularity assumptions are weaker than in Theorem 5.1.

Theorem 5.4. *Assume that $r(z)$ satisfies (4.3) and (4.4), and in addition that $|r(\infty)| < 1$. Let $\phi \in H^r \cap \dot{H}^{\max(r-2,1)}$, $\psi \in \dot{H}^{r-2}$, and assume that*

$$\|\phi_h - \phi\| \le Ch^r\|\phi\|_r, \quad \psi_h = P_h\psi.$$

Then for the solutions of (1.1) and of (5.1) we have

$$\|U_n - u(t_n)\| + \|V_n - u_t(t_n)\| \le C(t_n)(h^r + k^p)\big(\|\phi\|_r + |\psi|_{r-2}\big), \qquad \text{for } t_n > 0.$$

In particular, for $r \le 4$, we thus have an optimal order error estimate for positive time as long as $\phi \in H^r \cap H_0^1$ and $\psi \in H^{r-2} \cap H_0^1$ if $r = 3, 4$ ($\psi \in L_2$ if $r = 2$). For $r = 5$ the corresponding error estimate follows for the function values by use of (3.14).

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