ERROR ESTIMATES WITH SMOOTH AND NONSMOOTH DATA FOR A FINITE ELEMENT METHOD FOR THE CAHN-HILLIARD EQUATION*

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Abstract. A finite element method for the Cahn-Hilliard equation (a semilinear parabolic equation of fourth order) is analyzed, both in a spatially semidiscrete case and in a completely discrete case based on the backward Euler method. Error bounds of optimal order over a finite time interval are obtained for solutions with smooth and nonsmooth initial data. A detailed study of the regularity of the exact solution is included. The analysis is based on local Lipschitz conditions for the nonlinearity with respect to Sobolev norms, and the existence of a Ljapunov functional for the exact and the discretized equations is essential. A result concerning the convergence of the attractor of the corresponding approximate nonlinear semigroup (upper semicontinuity with respect to the discretization parameters) is obtained as a simple application of the nonsmooth data error estimate.

1. Introduction. The Cahn-Hilliard equation

$$u_t + \Delta^2 u - \Delta\phi(u) = 0, \qquad x \in \Omega \subset \mathbf{R}^3,\ t > 0, \tag{1.1}$$

where typically $\phi(u) = u^3 - u$, together with appropriate boundary and initial conditions, is a phenomenological model for phase separation and spinodal decomposition. The boundary conditions are such that the fourth order differential operator in (1.1) can be written as the square of a second order elliptic operator. Relying on this fact we study numerical schemes for (1.1), which for the approximation of the spatial variables are based on standard Galerkin finite element methods for second order elliptic problems. We discuss spatially semidiscrete schemes as well as a completely discrete scheme based on the backward Euler method.

A semidiscrete finite element method (with numerical quadrature) of this type for the Cahn-Hilliard equation was first introduced and analyzed by Elliott, French and Milner [7]. Completely discrete schemes based on the same idea were discussed by Du and Nicolaides [5] and Du [4]. For numerical schemes based on other approximations of the fourth order elliptic operator we refer to Elliott and Zheng [8] (conforming elements in 1-D) and Elliott and French [6] (nonconforming elements in 2-D).

In these works the analysis is restricted to solutions, which are bounded uniformly in time, so that one may essentially assume that the nonlinearity ϕ satisfies a global Lipschitz condition. Due to the lack of a maximum principle this means that one has to prove (or assume) that the solution is sufficiently smooth depending on the number of space dimensions.

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The purpose of the present work is to prove error bounds that are optimal both in the order of convergence and in the regularity assumed of the initial data. In particular, we would like to allow initial data of low regularity (compared to the number of derivatives occurring in equation (1.1)). The reason for this is the existence of a Ljapunov functional for equation (1.1) and its discrete counterparts, which yields an *a priori* bound, uniform in time, for the H^1 norm of the solution and for the discrete approximations considered. The Sobolev space $H^1(\Omega)$ is therefore a natural space in which to prescribe initial data.

Moreover, error bounds for solutions with nonsmooth initial data have interesting applications in the study of the longtime behavior of discrete solutions, see Heywood and Rannacher [12], Hale, Lin and Raugel [10] and Kloeden and Lorenz [14]. As an example of this we prove a result concerning the convergence of the attractor of the corresponding approximate nonlinear semigroup. More precisely, we demonstrate that the discrete attractor is upper semicontinuous with respect to the discretization parameters.

With initial data in $H^1(\Omega)$ the solution is not bounded uniformly in time (except in the case of one space dimension). Instead we base our analysis on uniform bounds in the H^1 norm for the exact and discrete solutions and local Lipschitz conditions for the nonlinearity ϕ. These are typically of the form

$$\|\phi(u)-\phi(v)\|_X \leq C(\|u\|_{H^1}, \|v\|_{H^1})\|u-v\|_Y,$$

where $\|\cdot\|_X, \|\cdot\|_Y$ are appropriate Sobolev norms.

Nonsmooth data error estimates for finite element methods have been proved earlier by Johnson, Larsson, Thomée and Wahlbin [13], Crouzeix, Thomée and Wahlbin [3] and Crouzeix and Thomée [2] in the context of a semilinear parabolic problem of second order with globally Lipschitz continuous nonlinearity. Similar results were obtained by Helfrich [11] in an abstract framework using local Lipschitz conditions. See also Heywood and Rannacher [12] for related results in the context of the Navier-Stokes equations.

Loosely speaking our main result (Theorem 6.5) states the following: Let u_h be the spatially semidiscrete approximation using a finite element method of order r and with mesh parameter h, and let the initial approximation be chosen as the L_2 projection of the exact initial value u_0. Then for $r=2$ or 3 (piecewise linear or quadratic finite elements) we have

$$\|u_h(t)-u(t)\|_{L_2} \leq C(u_0,T)h^r t^{-\frac{r-\alpha}{4}}, \qquad 0<t\leq T,$$

for $1\leq\alpha\leq r$, provided that u_0 has α derivatives in L_2 (together with appropriate boundary conditions). An analogous result is obtained in the completely discrete case (Theorem 7.2). The restrictions $r=2$ or 3 and $\alpha\geq 1$ are probably due to our method of proof, but in the light of a counterexample in [13], [3] some restriction of this type might be expected. We have, however, not been able to adapt this counterexample to the present situation. See also Remark 2 of Section 5 below.

The outline of the paper is as follows. In Section 2 we present three initial-boundary value problems for the Cahn-Hilliard equation and put them into a common abstract framework. In Section 3 we introduce spatially semidiscrete and completely discrete finite element methods for these problems. In Section 4 we state a result concerning the regularity of the exact solution, which is needed in the subsequent error analysis. Its proof is given in an Appendix in the supplement section of this issue. In Section 5 we estimate the difference between the exact solution and the solution of a discrete linear auxiliary problem. This analysis is based on energy estimates. In Section 6 we prove error estimates for the spatially semidiscrete approximation, and in Section 7 we do the same for the fully discrete approximation. This analysis is based on semigroup techniques. Finally in Section 8 we demonstrate the existence of global attractors for the nonlinear semigroups defined by the Cahn-Hilliard equation and its approximations, and prove a result concerning the convergence of the discrete attractors.

2. The Cahn-Hilliard equation. Let Ω be a bounded domain in $\mathbf{R}^d$ for $d \leq 3$ with a sufficiently smooth boundary. We consider the finite element approximation of the following initial-boundary value problems: Find $u(x,t)$ for $x \in \Omega$, $t > 0$, such that

$$u_t - \Delta\big(-\Delta u + \phi(u)\big) = 0, \qquad x \in \Omega,\ t > 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2.2}$$

subject to one of the three sets of boundary conditions,

$$u = 0,\ -\Delta u + \phi(u) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2.3–a}$$

$$\frac{\partial u}{\partial \nu} = 0,\ \frac{\partial}{\partial \nu}\big(-\Delta u + \phi(u)\big) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2.3–b}$$

$$u(x + Le_i, t) = u(x,t), \qquad x \in \partial\Omega,\ t > 0,\ i = 1, \ldots, d. \tag{2.3–c}$$

Here ϕ is a given polynomial satisfying the structural assumptions

$$\begin{aligned} &\phi(s) = \psi'(s),\ \text{degree } \psi = 2p, \\ &\psi(s) \geq c_0|s|^{2p} - c_1,\ \psi''(s) \geq -\beta^2 \qquad \forall s \in \mathbf{R}, \end{aligned} \tag{2.4}$$

where $c_0 > 0$ and $2 \leq p < \infty$ if $d \leq 2$, $p = 2$ if $d = 3$. In the case of the Dirichlet boundary conditions (2.3–a) we make the additional assumption that $\phi(0) = 0$.

In (2.3–b) we have used the notation $\partial/\partial\nu$ for the outward normal derivative, and in (2.3–c), the case of periodic boundary conditions, we understand Ω to be a "cube" $(0,L)^d$ with e_i denoting the unit vector in the direction of the x_i-axis.

The differential equation in (2.1) is known as the Cahn-Hilliard equation. It arises in continuum models of phase separation and spinodal decomposition, cf. Cahn and Hilliard [1]. The field variable u is a scaled concentration of one species in a binary mixture and the "free energy" ψ is a double well potential. A typical example for ψ is $\psi(s) = \frac{1}{4}(s^2 - \beta^2)^2$ with $\phi(s) = s(s^2 - \beta^2)$.

In order to put these three initial-boundary value problems in a common abstract framework we introduce some notation. Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the usual norm and inner product in $L_2 = L_2(\Omega)$, and let $H^s = H^s(\Omega)$ with norms $\|\cdot\|_s$ be the usual Sobolev spaces.

For the no flux and the periodic boundary conditions (2.3–b,c) it is easy to see that a sufficiently smooth solution of (2.1), (2.2) satisfies conservation of mass

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx, \qquad t \geq 0.$$

Introducing the change of variables $\tilde{u} = u - \overline{u_0}$ and $\tilde{\psi}(\tilde{u}) = \psi(\tilde{u} + \overline{u_0})$, where $\overline{u_0}$ denotes the average of u_0, we see that the equations (2.1), (2.2), (2.3–b,c) and the structural assumptions (2.4) remain unchanged. Henceforth for the boundary conditions (2.3–b,c) we assume that the initial datum satisfies $\int_\Omega u_0(x)\,dx = 0$. For these boundary conditions we let H denote the subspace of L_2 which is orthogonal to the constants,

$H = \{v \in L_2 : (v, 1) = 0\}$, and let P be the orthogonal projection of L_2 onto H. Clearly then $Pf = f - \overline{f}$. For the Dirichlet boundary conditions (2.3–a) we let $H = L_2$ and $P = I$. We then define the linear operator $A = -\Delta$ with domain of definition

$$\begin{aligned}
&\mathcal{D}(A) = \{v \in H^2 : v = 0 \text{ on } \partial\Omega\},\\
&\mathcal{D}(A) = \{v \in H^2 \cap H : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\},\\
&\mathcal{D}(A) = \{v \in H^2 \cap H : v(x + Le_i) = v(x) \text{ for } x \in \partial\Omega,\ i = 1, \ldots, d\},
\end{aligned}$$

for the three sets of boundary conditions, respectively. Then A is a selfadjoint positive definite densely defined operator on H, and (2.1)–(2.3) may be written as an abstract initial value problem

$$\begin{aligned}
&u_t + A^2 u + AP\phi(u) = 0, \qquad t > 0,\\
&u(0) = u_0.
\end{aligned} \tag{2.5}$$

By spectral theory we may also define the spaces $\dot{H}^s = \mathcal{D}(A^{\frac{s}{2}})$ with norms $|v|_s = \|A^{\frac{s}{2}} v\|$ for real s. It is well known that, for integer $s \geq 0$, $\dot{H}^s$ is a subspace of $H^s \cap H$ characterized by certain boundary conditions, and that the norms $|\cdot|_s$ and $\|\cdot\|_s$ are equivalent on $\dot{H}^s$. This can be proved by means of the spectral theorem and trace inequalities, see Thomée [18, p. 34] for a proof in the case of the Dirichlet boundary condition. In particular, we have

$$\begin{aligned}
&\dot{H}^1 = \{v \in H^1 : v = 0 \text{ on } \partial\Omega\},\\
&\dot{H}^1 = H^1 \cap H,\\
&\dot{H}^1 = \{v \in H^1 \cap H : v(x + Le_i) = v(x) \text{ for } x \in \partial\Omega,\ i = 1, \ldots, d\},
\end{aligned}$$

for the three sets of boundary conditions, respectively, and the norm $|v|_1 = \|A^{\frac{1}{2}} v\| = \|\nabla v\|$ is equivalent to $\|v\|_1$ on $\dot{H}^1$. Apart from this we shall only need the inequality

$$\|v\|_s \leq C_s |v|_s, \qquad v \in \dot{H}^s,\ s \geq 0, \tag{2.6}$$

which follows by interpolation between the corresponding inequalities with integer s.

We also define $G : H \to \dot{H}^2$ to be the inverse of A. It is convenient to extend it to all of L_2 by $Gf = GPf$ for $f \in L_2$. Thus, $v = Gf$ if and only if $Av = Pf$, or equivalently

$$v \in \dot{H}^1, \quad (\nabla v, \nabla \chi) = (f, \chi) \qquad \forall \chi \in \dot{H}^1.$$

Clearly, G is selfadjoint positive semidefinite on L_2 and positive definite on H.

We next derive an *a priori* bound in the H^1 norm for solutions of (2.5). This bound (and its discrete counterparts) will be basic to all of our analysis below. Applying G to (2.5) we have

$$Gu_t + Au + P\phi(u) = 0,$$

and taking the inner product of this with u_t we obtain

$$(Gu_t, u_t) + \tfrac{1}{2} D_t |u|_1^2 + D_t \int_\Omega \psi(u)\, dx = 0.$$

Setting $V(u) = \frac{1}{2}|u|_1^2 + \int_\Omega \psi(u)\, dx$ ("the free energy functional") we conclude

$$\int_0^t \|G^{\frac{1}{2}} u_t\|^2\, ds + V(u(t)) = V(u_0), \qquad 0 \leq t < \infty, \tag{2.7}$$

provided that $u_0 \in \dot{H}^1$. In view of the structural assumptions (2.4) it follows that V is a Ljapunov functional for the initial value problem (2.5) (see Section 8 below for the definition of this concept). Moreover, using the Sobolev imbedding of H^1 into L_{2p} (where p is as in (2.4)) the identity (2.7) implies an *a priori* bound: If $u_0 \in \dot{H}^1$ with $\|u_0\|_1 \le R$, then

$$\|u(t)\|_1 \le C(R), \qquad 0 \le t < \infty. \tag{2.8}$$

In the sequel we shall always assume that $u_0 \in \dot{H}^1$ (at least), so that (2.8) holds. We also note that the derivative of V ("the chemical potential") is given by $w = V'(u) = Au + P\phi(u) = -Gu_t$.

Finally we let $E(t) = \exp(-tA^2)$ denote the analytic semigroup generated by $-A^2$. Much of our analysis will be based on the variations of constants formula,

$$u(t) = E(t)u_0 - \int_0^t E(t-s)AP\phi(u(s))\,ds, \tag{2.9}$$

for solutions of (2.5).

3. The finite element method. For the approximation of the Cahn-Hilliard equation we assume that we have a family $\{S_h\}_{h>0}$ of finite dimensional approximating subspaces of H^1. At the end of this section we formulate the approximation assumption upon which we shall base our error analysis. But first we formulate our discrete equations.

Consider to begin with the no flux boundary conditions (2.3–b). Recalling the usual weak formulation of the corresponding initial-boundary value problem, we state the following semidiscrete problem: Find $u_h(t), v_h(t) \in S_h$ such that

$$\begin{aligned} &(u_{h,t},\chi) + (\nabla v_h, \nabla\chi) = 0 && \forall \chi \in S_h,\ t>0,\\ &(v_h,\chi) = (\nabla u_h, \nabla\chi) + (\phi(u_h),\chi) && \forall \chi \in S_h,\ t>0,\\ &u_h(0) = u_{0h}, \end{aligned} \tag{3.1}$$

where $u_{0h} \in S_h$ is a suitable approximation of $u_0 \in \dot{H}^1$. Since we are assuming that $\overline{u_0} = 0$, it is natural to assume that $\overline{u_{0h}} = 0$, too. It is easy to see that this can be achieved, e.g., by taking u_{0h} to be the orthogonal projection of $u_0 \in \dot{H}^1$ onto S_h with respect to the L_2 inner product or with respect to the H^1 inner product. Let now

$$\dot{S}_h = \{\chi \in S_h : (\chi, 1) = 0\}.$$

It is immediate from (3.1) that $u_h(t) \in \dot{S}_h$ if $u_{0h} \in \dot{S}_h$. Therefore u_h can equivalently be obtained from the following equations: Find $u_h(t), w_h(t) \in \dot{S}_h$ such that

$$\begin{aligned} &(u_{h,t},\chi) + (\nabla w_h, \nabla\chi) = 0 && \forall \chi \in \dot{S}_h,\ t>0,\\ &(w_h,\chi) = (\nabla u_h, \nabla\chi) + (\phi(u_h),\chi) && \forall \chi \in \dot{S}_h,\ t>0,\\ &u_h(0) = u_{0h}, \end{aligned} \tag{3.2}$$

where now $u_{0h} \in \dot{S}_h$ is an approximation of $u_0 \in \dot{H}^1$. (The relation between w_h and v_h is $w_h = v_h - \overline{v_h}$.) Equivalently, we may write this as

$$\begin{aligned} &u_{h,t} + A_h^2 u_h + A_h P_h \phi(u_h) = 0, \qquad t>0,\\ &u_h(0) = u_{0h}, \end{aligned} \tag{3.3}$$

where the operator $A_h : \dot S_h \to \dot S_h$ (the "discrete Laplacian") is defined by

$$(A_h\chi, \eta) = (\nabla\chi, \nabla\eta) \qquad \forall \chi, \eta \in \dot S_h,$$

and $P_h : L_2 \to \dot S_h$ is the orthogonal projection. Clearly, A_h is selfadjoint positive definite, and we let G_h denote its inverse. As for G it is convenient to extend G_h to all of L_2 by $G_h f = G_h P_h f$ for $f \in L_2$. Thus, $v_h = G_h f$ if and only if $A_h v_h = P_h f$, or equivalently

$$v_h \in \dot S_h, \quad (\nabla v_h, \nabla\chi) = (f, \chi) \qquad \forall \chi \in \dot S_h.$$

We note that, thus defined, G_h is selfadjoint positive semidefinite on L_2 and positive definite on $\dot S_h$. We also record the facts that $\|A_h^{\frac12}\chi\| = \|\nabla\chi\| = |\chi|_1$ for all $\chi \in \dot S_h$, and that for the "discrete chemical potential" w_h in (3.2) we have $w_h = A_h u_h + P_h\phi(u_h) = -G_h u_{h,t}$.

The above refers to the no flux boundary conditions. In the case of the Dirichlet boundary conditions (2.3–a) we define instead

$$\dot S_h = \{\chi \in S_h : \chi = 0 \text{ on } \partial\Omega\},$$

and for the periodic boundary conditions (2.3–c) we set

$$\dot S_h = \{\chi \in S_h : \chi(x + Le_i) = \chi(x) \text{ on } \partial\Omega,\ i = 1, \dots, d\}.$$

Starting with (3.2) we then reiterate the above arguments and definitions. The initial value problem (3.3) is thus a common framework for our semidiscrete approximations of the three initial-boundary value problems (2.1)–(2.3).

We now derive a discrete counterpart to the *a priori* bound (2.8). In fact, $V(u) = \frac12|u|_1^2 + \int_\Omega \psi(u)\,dx$ is a Ljapunov functional for (3.3), too. To see this we argue in the same way as in the in the proof of (2.7) and obtain

$$\int_0^t \|G_h^{\frac12} u_{h,t}\|^2\,ds + V(u_h(t)) = V(u_{0h}), \qquad 0 \le t < \infty,$$

which leads to the *a priori* bound: If $u_{0h} \in \dot S_h$ with $\|u_{0h}\|_1 \le R$, then

$$\|u_h(t)\|_1 \le C(R), \qquad 0 \le t < \infty. \tag{3.4}$$

With $E_h(t) = \exp(-tA_h^2)$ we have the variations of constants formula,

$$u_h(t) = E_h(t)u_{0h} - \int_0^t E_h(t-s)A_h P_h\phi(u_h(s))\,ds, \tag{3.5}$$

for solutions of (3.3).

We next formulate a fully discrete approximation based on the backward Euler method. This means that we replace the time derivative in (3.2) or (3.3) by a backward difference quotient $\bar\partial_t U_n = (U_n - U_{n-1})/k$, where k is the timestep and U_n is the approximation to u at time $t_n = nk$, $n = 0, 1, 2, \dots$. We thus seek $U_n \in \dot S_h$ such that

$$\begin{aligned} &\bar\partial_t U_n + A_h^2 U_n + A_h P_h \phi(U_n) = 0, \qquad t_n > 0,\\ &U_0 = u_{0h}. \end{aligned} \tag{3.6}$$

Again it turns out that the functional V is a Ljapunov functional for (3.6). In fact, arguing as in the proof of (2.7) we obtain

$$(G_h\bar\partial_t U_n, \bar\partial_t U_n) + (A_h U_n, \bar\partial_t U_n) + (\phi(U_n), \bar\partial_t U_n) = 0,$$

Here

$$(A_h U_n, \bar\partial_t U_n) = \tfrac{1}{2}\bar\partial_t |U_n|_1^2 + \tfrac{1}{2}k|\bar\partial_t U_n|_1^2.$$

Recalling the condition $\psi''(s) \geq -\beta^2$ in (2.4) it follows that

$$\psi'(r)(r-s) \geq \psi(r) - \psi(s) - \tfrac{1}{2}\beta^2(r-s)^2, \tag{3.7}$$

so that

$$(\phi(U_n), \bar\partial_t U_n) \geq \bar\partial_t \int_\Omega \psi(U_n)\,dx - \tfrac{1}{2}k\beta^2\|\bar\partial_t U_n\|^2.$$

Hence

$$\|G_h^{\frac{1}{2}}\bar\partial_t U_n\|^2 + \tfrac{1}{2}k|\bar\partial_t U_n|_1^2 + \bar\partial_t V(U_n) \leq \tfrac{1}{2}k\beta^2\|\bar\partial_t U_n\|^2 \leq \tfrac{1}{8}k\beta^4\|G_h^{\frac{1}{2}}\bar\partial_t U_n\|^2 + \tfrac{1}{2}k|\bar\partial_t U_n|_1^2.$$

Thus, if $k \leq 4/\beta^4$, this shows that

$$\tfrac{1}{2}k\sum_{j=1}^n \|G_h^{\frac{1}{2}}\bar\partial_t U_n\|^2 + V(U_n) \leq V(u_{0h}), \qquad 0 \leq t_n < \infty,$$

which leads to the *a priori* bound: If $u_{0h} \in \dot S_h$ with $\|u_{0h}\|_1 \leq R$, then

$$\|U_n\|_1 \leq C(R), \qquad 0 \leq t_n < \infty. \tag{3.8}$$

This time the variation of constants formula becomes

$$U_n = E_{kh}^n u_{0h} - k\sum_{j=1}^n E_{kh}^{n-j+1} A_h P_h \phi(U_j), \tag{3.9}$$

where $E_{kh} = (I + kA_h^2)^{-1}$.

We conclude this section by formulating an approximation assumption for the spaces $\dot S_h \subset \dot H^1$, which will be the basis for our error analysis below. Let $R_h : \dot H^1 \to \dot S_h$ be the Ritz projection defined by

$$(\nabla(R_h v - v), \nabla\chi) = 0 \qquad \forall \chi \in \dot S_h.$$

We assume that, for $r = 2$ or $r = 3$,

$$|R_h v - v|_l \leq Ch^{\beta - l}|v|_\beta, \qquad -(r-2) \leq l \leq 1,\ 1 \leq \beta \leq r. \tag{3.10}$$

(Recall that $|v|_{-1} = \|G^{\frac{1}{2}}v\| = \sup_{\chi\in\dot H^1}|(v,\chi)|/|v|_1$.) From this assumption follows that

$$\|P_h v - v\| \leq Ch^r|v|_r. \tag{3.11}$$

The main examples of this situation are obtained by letting S_h be the standard piecewise linear ($r = 2$) or piecewise quadratic ($r = 3$) finite element spaces.

4. Existence and regularity of solutions. We now state a result concerning existence and regularity of solutions to the Cahn-Hilliard equation (2.5). Global existence has been proved by several authors under various assumptions of initial regularity, see, e.g., Nicolaenko, Scheurer and Temam [15], Temam [17], Elliott and Zheng [8], Zheng [21] and von Wahl [20]. Our error analysis depends on precise regularity estimates for the exact solution, most of which are not available in the literature, and we therefore develop the required results in the following theorem. Our approach is based on the techniques of [20], where global existence of solutions with initial data in H^1 was shown.

Theorem 4.1. *Let $\alpha \in [1,3]$, $\beta \in [0,4)$, $j,l = 0,1,2$, with $4j - 2l + \beta \geq \alpha$ and let $T, R \geq 0$ be arbitrary. If $u_0 \in \dot{H}^\alpha$ with $|u_0|_\alpha \leq R$, then equation (2.5) has a unique solution u which belongs to $C([0,T], \dot{H}^\alpha) \cap C^1((0,T], L_2)$. Moreover, there is a constant $C = C(T,R,\beta)$ such that*

$$\|G^l D_t^j u(t)\|_\beta \leq C t^{-j+\frac{l}{2}-\frac{\beta-\alpha}{4}}, \qquad 0 < t \leq T. \tag{4.1}$$

The estimate (4.1) means that the solution operator of the nonlinear Cahn-Hilliard equation enjoys (at least to some extent) a smoothing property analogous to that of the analytic semigroup $E(t)$:

$$|D_t^j E(t) v|_\beta \leq C_{\alpha,\beta}\, t^{-j-\frac{\beta-\alpha}{4}} |v|_\alpha, \qquad t > 0,\ 0 \leq \alpha \leq \beta. \tag{4.2}$$

The proof of Theorem 4.1 can be found in the supplement section of this issue.

5. Error analysis for a linear problem. In this section we shall discuss the following linear nonhomogeneous variant of the Cahn-Hilliard equation (2.5): Let u satisfy the initial value problem

$$\begin{aligned} u_t + A^2 u &= APf, \qquad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{5.1}$$

together with the regularity assumption that, for some $T > 0$, $\alpha \in [0,3]$, $K > 0$,

$$\|G^l D_t^j u(t)\|_\beta \leq K t^{-j+\frac{l}{2}-\frac{\beta-\alpha}{4}}, \qquad 0 < t \leq T, \tag{5.2}$$

for all $\beta \in [0,3]$, $j,l = 0,1,2$ with $4j - 2l + \beta \geq \alpha$.

We shall apply this in the following two situations: If u is the solution of (2.5) (i.e., $f = -\phi(u)$), then (5.2) holds with $K = C(T,R)$ whenever $|u_0|_\alpha \leq R$ according to Theorem 4.1. If $u(t) = E(t)u_0$ (i.e., $f = 0$), then we have $K = C|u_0|_\alpha$ according to (4.2).

We first consider a semidiscrete finite element approximation $u_h(t) \in \dot{S}_h$ given by

$$\begin{aligned} u_{h,t} + A_h^2 u_h &= A_h P_h f, \qquad t > 0, \\ u_h(0) &= P_h u_0. \end{aligned} \tag{5.3}$$

We shall estimate the difference between u_h and u under the regularity assumption (5.2). This analysis is linear in the sense that u_h depends linearly on u.

Observe that by applying G^2 to (5.1) we obtain $G^2 u_t + u = Gf$ and, similarly for (5.3), $G_h^2 u_{h,t} + u_h = G_h f$, where we have used the fact that $GP = G$, $G_h P_h = G_h$. For the difference $e = u_h - u$ we then have

$$\begin{aligned} G_h^2 e_t + e &= (G_h - G)f - (G_h^2 - G^2)u_t = (G_h - G)(Pf - Gu_t) - G_h(G_h - G)u_t \\ &= (G_h - G)Au - G_h(G_h - G)u_t = (R_h - I)u - G_h(R_h - I)Gu_t, \end{aligned}$$

where the identities $Pf - Gu_t = Au$, $R_h = G_h A$ have been employed. It follows that

$$G_h^2 e_t + e = \rho + G_h \eta, \qquad t > 0, \tag{5.4}$$

with

$$\rho = (R_h - I)u, \quad \eta = -(R_h - I)Gu_t.$$

Equation (5.4) is the basis for the estimation of e. It is convenient to first give a lemma providing estimates of ρ and η.

Lemma 5.1. *Let $r = 2$ or 3 and let u satisfy (5.1) and (5.2) for some $\alpha \in [0, r]$. Assume that $1 \le \beta \le r$, $0 \le \beta - \alpha \le 2$. Then the following bounds hold for $0 < t \le T$:*

$$t^j \|D_t^j \rho(t)\| \le CKh^\beta t^{-\frac{\beta-\alpha}{4}}, \tag{5.5}$$

$$t^j \|D_t^j \eta(t)\| \le CKh^\beta t^{-\frac{1}{2}-\frac{\beta-\alpha}{4}}, \tag{5.6}$$

$$\|\tilde{\rho}(t)\| \le CKh^\beta t^{1-\frac{\beta-\alpha}{4}}, \tag{5.7}$$

$$\|\tilde{\eta}(t)\| \le CKh^\beta t^{\frac{1}{2}-\frac{\beta-\alpha}{4}}, \tag{5.8}$$

where $\tilde{\rho}(t) = \int_0^t \rho(\tau)\, d\tau$, $\tilde{\eta}(t) = \int_0^t \eta(\tau)\, d\tau$. Moreover

$$\int_0^t \big(\tau\|\rho\|^2 + \tau^3\|\rho_t\|^2 + \tau^{-1}\|\tilde{\rho}\|^2 + \|\tilde{\eta}\|^2 + \tau^2\|\eta\|^2 + \tau^4\|\eta_t\|^2\big)\, d\tau \le CK^2 h^{2\beta} t^{2-\frac{\beta-\alpha}{2}}. \tag{5.9}$$

Proof. By (3.10) and (5.2) we have

$$t^j\|D_t^j\rho(t)\| = t^j\|(R_h - I)D_t^j u(t)\| \le Ct^j h^\beta \|D_t^j u(t)\|_\beta \le CKh^\beta t^{-\frac{\beta-\alpha}{4}},$$

which is (5.5). Similarly,

$$t^j\|D_t^j\eta(t)\| = t^j\|(R_h - I)D_t^j Gu_t(t)\| \le Ct^j h^\beta \|D_t^j Gu_t(t)\|_\beta \le CKh^\beta t^{-\frac{1}{2}-\frac{\beta-\alpha}{4}},$$

and (5.6) is proved. Using these estimates we obtain

$$\|\tilde{\rho}(t)\| \le \int_0^t \|\rho(\tau)\|\, d\tau \le CKh^\beta \int_0^t \tau^{-\frac{\beta-\alpha}{4}}\, d\tau = CKh^\beta t^{1-\frac{\beta-\alpha}{4}},$$

and

$$\|\tilde{\eta}(t)\| \le \int_0^t \|\eta(\tau)\|\, d\tau \le CKh^\beta \int_0^t \tau^{-\frac{1}{2}-\frac{\beta-\alpha}{4}}\, d\tau = CKh^\beta t^{\frac{1}{2}-\frac{\beta-\alpha}{4}},$$

provided in the latter case that $0 \le \beta - \alpha < 2$. For $\beta - \alpha = 2$ we have instead

$$\begin{aligned}\|\tilde{\eta}(t)\| &\le Ch^\beta \Big\|\int_0^t Gu_t\, d\tau\Big\|_\beta \le Ch^\beta \Big\|\int_0^t u_t\, d\tau\Big\|_{\beta-2} \\ &= Ch^\beta\|u(t) - u_0\|_\alpha \le Ch^\beta\big(\|u(t)\|_\alpha + \|u_0\|_\alpha\big) \le CKh^\beta.\end{aligned}$$

This proves (5.7) and (5.8). Finally (5.9) is an immediate consequence of the previous bounds. $\square$

Lemma 5.2. *Let $r = 2$ or 3, let u satisfy (5.1) and (5.2) for some $\alpha \in [0, r]$, and let u_h be the solution of (5.3). Assume that $0 \le r - \alpha \le 2$. Then*

$$\|u_h(t) - u(t)\|_l \le CKh^{r-l} t^{-\frac{r-\alpha}{4}}, \qquad 0 < t \le T,\ l = 0, 1. \tag{5.10}$$

Moreover, for the "chemical potential" $w = Au - Pf$ and its approximation $w_h = A_h u_h - P_h f$ we have

$$\|w_h(t) - w(t)\| \le CKh^r t^{-\frac{1}{2}-\frac{r-\alpha}{4}}, \qquad 0 < t \le T. \tag{5.11}$$

We remark that C is independent of T.

Proof. Let β be as in Lemma 5.1. We first note that by our special choice $P_h u_0$ of discrete initial value we have $G_h e(0) = 0$ where $e = u_h - u$. In order to prove the case $l = 0$ of (5.10) we start out by taking the inner product of (5.4) with e_t. Using the fact that G_h is selfadjoint positive semidefinite on L_2 we get

$$\|G_h e_t\|^2 + \tfrac{1}{2} D_t \|e\|^2 = (\rho, e_t) + (\eta, G_h e_t) \leq (\rho, e_t) + \tfrac{1}{2}\|\eta\|^2 + \tfrac{1}{2}\|G_h e_t\|^2,$$

which shows

$$\|G_h e_t\|^2 + D_t \|e\|^2 \leq 2(\rho, e_t) + \|\eta\|^2.$$

Multiplying this by t^2,

$$\begin{aligned} t^2\|G_h e_t\|^2 + D_t\left(t^2\|e\|^2\right) &\leq 2t\|e\|^2 + 2D_t\left[t^2(\rho, e)\right] - 4t(\rho, e) - 2t^2(\rho_t, e) + t^2\|\eta\|^2 \\ &\leq C\Big(D_t\left[t^2(\rho, e)\right] + t\|\rho\|^2 + t^3\|\rho_t\|^2 + t^2\|\eta\|^2 + t\|e\|^2\Big), \end{aligned}$$

and integrating with respect to t, we obtain after a simple kick-back argument

$$\int_0^t \tau^2 \|G_h e_t\|^2 \, d\tau + t^2\|e\|^2 \leq Ct^2\|\rho\|^2 + C\int_0^t \left(\tau\|\rho\|^2 + \tau^3\|\rho_t\|^2 + \tau^2\|\eta\|^2 + \tau\|e\|^2\right) d\tau.$$

Invoking the bounds for ρ and η in Lemma 5.1 we conclude that

$$\int_0^t \tau^2\|G_h e_t\|^2 \, d\tau + t^2\|e\|^2 \leq CK^2 h^{2\beta} t^{2-\frac{\beta-\alpha}{2}} + C\int_0^t \tau\|e\|^2 \, d\tau. \tag{5.12}$$

We now have to estimate $\int_0^t \tau\|e\|^2 \, d\tau$ and we therefore multiply (5.4) by e to get

$$\tfrac{1}{2} D_t\|G_h e\|^2 + \|e\|^2 = (\rho, e) + (\eta, G_h e) \leq \tfrac{1}{2}\|\rho\|^2 + \tfrac{1}{2}\|e\|^2 + \|\eta\|\,\|G_h e\|,$$

whence

$$D_t\|G_h e\|^2 + \|e\|^2 \leq \|\rho\|^2 + 2\|\eta\|\,\|G_h e\|.$$

Multiplication by t now yields

$$D_t\left(t\|G_h e\|^2\right) + t\|e\|^2 \leq t\|\rho\|^2 + t^2\|\eta\|^2 + 2\|G_h e\|^2,$$

so that, in view of (5.9),

$$\begin{aligned} t\|G_h e\|^2 + \int_0^t \tau\|e\|^2 \, d\tau &\leq \int_0^t \left(\tau\|\rho\|^2 + \tau^2\|\eta\|^2 + 2\|G_h e\|^2\right) d\tau \\ &\leq CK^2 h^{2\beta} t^{2-\frac{\beta-\alpha}{2}} + 2\int_0^t \|G_h e\|^2 \, d\tau. \end{aligned} \tag{5.13}$$

To derive an estimate of $\int_0^t \|G_h e\|^2 \, d\tau$ we integrate (5.4) with respect to t taking $G_h e(0) = 0$ into account. This yields

$$G_h^2 e + \tilde{e} = \tilde{\rho} + G_h \tilde{\eta}, \qquad t > 0,$$

where $\tilde{e}(t) = \int_0^t e \, d\tau$, etc. Multiplication by $e = D_t \tilde{e}$ gives

$$\|G_h e\|^2 + \tfrac{1}{2} D_t\|\tilde{e}\|^2 = (\tilde{\rho}, e) + (\tilde{\eta}, G_h e) \leq \|\tilde{\rho}\|\,\|e\| + \tfrac{1}{2}\|\tilde{\eta}\|^2 + \tfrac{1}{2}\|G_h e\|^2,$$

which after some simple manipulation leads to

$$\begin{aligned}\int_0^t \|G_h e\|^2\,d\tau + \|\tilde{e}\|^2 &\le \int_0^t \big(\|\tilde{\eta}\|^2 + 2\|\tilde{\rho}\|\,\|e\|\big)\,d\tau\\ &\le \int_0^t \big(\|\tilde{\eta}\|^2 + \tau^{-1}\|\tilde{\rho}\|^2 + \tfrac14\tau\|e\|^2\big)\,d\tau\\ &\le CK^2h^{2\beta}t^{2-\frac{\beta-\alpha}{2}} + \tfrac14\int_0^t \tau\|e\|^2 d\tau,\end{aligned}$$

and, upon substitution into the right-hand side of (5.13),

$$\int_0^t \tau\|e\|^2 d\tau \le CK^2h^{2\beta}t^{2-\frac{\beta-\alpha}{2}}. \tag{5.14}$$

Taken together estimates (5.12) and (5.14) yield

$$\int_0^t \tau^2\|G_h e_t\|^2\,d\tau + t^2\|e\|^2 \le CK^2h^{2\beta}t^{2-\frac{\beta-\alpha}{2}}, \tag{5.15}$$

and the case $l = 0$ of (5.10) follows.

It is now convenient to estimate the difference between w and w_h. Observe that

$$w - w_h = -Gu_t + G_h u_{h,t} = G_h(u_{h,t} - u_t) + (G_h - G)u_t = G_h e_t - \eta,$$

and hence

$$\|w - w_h\| \le \|G_h e_t\| + \|\eta\|. \tag{5.16}$$

In view of Lemma 5.1 it is therefore sufficient to bound $G_h e_t$. Differentiating (5.4) with respect to t yields

$$G_h^2 e_{tt} + e_t = \rho_t + G_h\eta_t.$$

Taking the inner product of this equation with e_t gives

$$\tfrac12 D_t\|G_h e_t\|^2 + \|e_t\|^2 = (\rho_t, e_t) + (\eta_t, G_h e_t),$$

and after multiplication by t^3,

$$\begin{aligned}\tfrac12 D_t\big(t^3\|G_h e_t\|^2\big) + t^3\|e_t\|^2 &= \tfrac32 t^2\|G_h e_t\|^2 + t^3(\rho_t, e_t) + t^3(\eta_t, G_h e_t),\\ &\le C\big(t^2\|G_h e_t\|^2 + t^3\|\rho_t\|^2 + t^4\|\eta_t\|^2\big) + \tfrac12 t^3\|e_t\|^2,\end{aligned}$$

so that

$$t^3\|G_h e_t\|^2 + \int_0^t \tau^3\|e_t\|^2\,d\tau \le C\int_0^t \big(\tau^2\|G_h e_t\|^2 + \tau^3\|\rho_t\|^2 + \tau^4\|\eta_t\|^2\big)\,d\tau. \tag{5.17}$$

Combining (5.15) and (5.9) with (5.17) we obtain

$$t^3\|G_h e_t\|^2 + \int_0^t \tau^3\|e_t\|^2\,d\tau \le CK^2h^{2\beta}t^{2-\frac{\beta-\alpha}{2}}.$$

Together with (5.16) this implies

$$\|w_h(t) - w(t)\| \le CKh^\beta t^{-\frac12-\frac{\beta-\alpha}{4}}, \tag{5.18}$$

and the desired bound (5.11) follows.

Finally we estimate the H^1 norm of e by interpolating between the known bounds for the errors in u_h and w_h. Let $e = (u_h - R_h u) + (R_h u - u) \equiv \theta + \rho$. Since, by (3.10) and (5.2),

$$\|\rho(t)\|_1 \le Ch^{r-1}\|u(t)\|_r \le Ch^{r-1}t^{-\frac{r-\alpha}{4}},$$

it is sufficient to make the following estimation:

$$\begin{aligned}
|\theta|_1^2 &= (\nabla(u_h - R_h u), \nabla(u_h - R_h u)) = (\nabla(u_h - u), \nabla(u_h - R_h u)) \\
&= (A_h u_h - Au, u_h - R_h u) = (w_h - w, u_h - R_h u) + (P_h f - Pf, u_h - R_h u) \\
&= (w_h - w, u_h - R_h u) \le \|w_h - w\|\,\|u_h - R_h u\| \\
&\le \|w_h - w\|\,\big(\|u_h - u\| + \|R_h u - u\|\big).
\end{aligned}$$

Hence, in view of (5.18), (5.15) and (5.5), we have

$$\|\theta(t)\|_1 \le Ch^{\beta}t^{-\frac{1}{4}-\frac{\beta-\alpha}{4}}, \qquad 0 < t \le T, \tag{5.19}$$

for $\alpha \le \beta \le r$. If $\alpha \le r-1$, then we can take $\beta = r-1$ and the case $l = 1$ of (5.10) follows. If $r-1 < \alpha \le r$ we argue differently. A glance at (5.4) reveals that

$$G_h^2\theta_t + \theta = -G_h^2\rho_t + G_h\eta,$$

or

$$\theta_t + A_h^2\theta = -P_h\rho_t + A_hP_h\eta.$$

An estimation of $\|\theta\|_1$ can be based on this equation via the variation of constants formula (cf. the proof of Lemma 6.8 below). We omit the details. $\square$

Remark 1. If we choose $\beta = r$ in (5.19) we obtain a bound of superconvergent order for the gradient of $\theta = u_h - R_h u$:

$$\|u_h(t) - R_h u(t)\|_1 \le Ch^r t^{-\frac{1}{4}-\frac{r-\alpha}{4}}, \qquad 0 < t \le T.$$

In the case $d \le 2$ this can be used to show an error bound of almost optimal order in the maximum norm, see Thomée [18, p. 11].

Remark 2. The restriction $r - \alpha \le 2$ occurs in (5.8); all other steps of the proof are valid under the less stringent condition $r - \alpha < 4$.

For the special case of equation (5.1) with $f = 0$, we have the following result.

Corollary 5.3. *Let $r = 2$ or 3. Then*

$$\|E_h(t)P_h v - E(t)v\| \le Ch^r t^{-\frac{r}{4}}\|v\|, \qquad t > 0,\ v \in H.$$

Proof. Lemma 5.2 shows (cf. (5.15))

$$\|E_h(t)P_h v - E(t)v\| \le Ch^2 t^{-\frac{1}{2}}\|v\|, \qquad t > 0,$$

which is the desired result when $r = 2$. For $r = 3$ we note that Lemma 5.2 also shows

$$\|E_h(t)P_h v - E(t)v\| \le Ch^3|v|_3, \qquad t > 0,$$

and the proof can be completed by Helfrich's iteration, cf. Thomée [18, pp. 39–41]. $\square$

We now turn to the fully discrete case. The backward Euler method applied to (5.3) defines $U_n \in \dot{S}_h$ by ($f_n = f(t_n)$, $u_n = u(t_n)$)

$$\begin{aligned} \bar{\partial}_t U_n + A_h^2 U_n &= A_h P_h f_n, \qquad t_n > 0, \\ U_0 &= P_h u_0. \end{aligned} \tag{5.20}$$

Analogously to (5.4) we obtain for the difference $e_n = U_n - u_n$:

$$G_h^2 \bar{\partial}_t e_n + e_n = \rho_n + G_h \zeta_n + G_h \epsilon_n, \qquad t_n > 0, \tag{5.21}$$

with

$$\rho_n = (R_h - I) u_n, \ \zeta_n = -(R_h - I) G \bar{\partial}_t u_n, \ \epsilon_n = -G[\bar{\partial}_t u_n - u_t(t_n)].$$

Equation (5.21) is the basis for the estimation of e_n. It is convenient to first give a lemma providing estimates of ρ_n, ζ_n and ϵ_n. In Lemma 5.2 we allowed $\alpha = 0$ in order to have Corollary 5.3. In the remainder of this section we assume that $\alpha \geq 1$.

Lemma 5.4. *Let $r = 2$ or 3 and let u satisfy (5.1) and (5.2) for some $\alpha \in [1, r]$, and assume that $\alpha \leq \beta \leq r$. Then the following bounds hold for $0 < t_n \leq T$:*

$$t_{n-1} \|\bar{\partial}_t \rho_n\| \leq C K h^\beta t_n^{-\frac{\beta-\alpha}{4}}, \tag{5.22}$$

$$\|\hat{\rho}_n\| \leq C K h^\beta t_n^{1-\frac{\beta-\alpha}{4}}, \tag{5.23}$$

$$\|\zeta_n\| \leq C K h^\beta t_n^{-\frac{1}{2}-\frac{\beta-\alpha}{4}}, \tag{5.24}$$

$$\|\hat{\zeta}_n\| \leq C K h^\beta t_n^{\frac{1}{2}-\frac{\beta-\alpha}{4}}, \tag{5.25}$$

$$\|\epsilon_n\| \leq C K k t_n^{-\frac{1}{2}-\frac{4-\alpha}{4}}, \tag{5.26}$$

$$\|G_h \hat{\epsilon}_n\| \leq C K \big(h^\beta t_n^{1-\frac{\beta-\alpha}{4}} + k t_n^{1-\frac{4-\alpha}{4}} \big), \tag{5.27}$$

and

$$\begin{aligned} k \sum_{j=1}^n \big(t_j \|\rho_j\|^2 + t_{j-1}^3 \|\bar{\partial}_t \rho_j\|^2 + t_j^{-1} \|\hat{\rho}_j\|^2 + t_j^2 \|\zeta_j\|^2 + \|\hat{\zeta}_j\|^2 \\ + t_j^2 \|\epsilon_j\|^2 + t_j^{-1} \|G_h \hat{\epsilon}_j\|^2 \big) \leq C K^2 \big(h^{2\beta} t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}} \big), \end{aligned} \tag{5.28}$$

where $\hat{\rho}_n = k \sum_{j=1}^n \rho_j$, $\hat{\zeta}_n = k \sum_{j=1}^n \zeta_j$, $\hat{\epsilon}_n = k \sum_{j=1}^n \epsilon_j$. Moreover, for $t_2 \leq t_n \leq T$ we have

$$\|\bar{\partial}_t \zeta_n\| \leq C K h^\beta t_n^{-\frac{3}{2}-\frac{\beta-\alpha}{4}}, \tag{5.29}$$

$$\|\bar{\partial}_t \epsilon_n\| \leq C K k t_n^{-\frac{3}{2}-\frac{4-\alpha}{4}}. \tag{5.30}$$

Proof. To begin with (5.5) implies

$$\|\bar{\partial}_t \rho_n\| \leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\rho_t\| \, d\tau \leq C K h^\beta \frac{1}{k} \int_{t_{n-1}}^{t_n} \tau^{-1-\frac{\beta-\alpha}{4}} \, d\tau \leq C K h^\beta t_n^{-1-\frac{\beta-\alpha}{4}},$$

for $n \geq 2$, since $t_n \leq 2 t_{n-1}$. This proves (5.22). The bound (5.23) is proved in the same way as (5.7). Next we note that $\hat{\zeta}_n = (R_h - I) G (u_n - u_0) = \tilde{\eta}_n$, and hence (5.25) is the same as (5.8). For the proof of (5.24) we have by (3.10) and (5.2)

$$\begin{aligned} \|\zeta_n\| &= \|(R_h - I) \frac{1}{k} \int_{t_{n-1}}^{t_n} G u_t \, d\tau\| \leq C h^\beta \frac{1}{k} \int_{t_{n-1}}^{t_n} \|G u_t\|_\beta \, d\tau \\ &\leq C K h^\beta \frac{1}{k} \int_{t_{n-1}}^{t_n} \tau^{-\frac{1}{2}-\frac{\beta-\alpha}{4}} \, d\tau \leq C K h^\beta t_{n-1}^{-\frac{1}{2}-\frac{\beta-\alpha}{4}} \leq C K h^\beta t_n^{-\frac{1}{2}-\frac{\beta-\alpha}{4}}, \end{aligned}$$

for $n \geq 2$. For $n = 1$ we have instead $\|\zeta_1\| = \frac{1}{k}\|\hat{\zeta}_1\| \leq CKh^\beta t_1^{-\frac{1}{2}-\frac{\beta-\alpha}{4}}$ by (5.25). In order to prove (5.26) we use Taylor's formula to get

$$\begin{aligned}\|\epsilon_n\| &= \|\frac{1}{k}\int_{t_{n-1}}^{t_n}(\tau - t_{n-1})Gu_{tt}\,d\tau\| \leq \int_{t_{n-1}}^{t_n}\|Gu_{tt}\|\,d\tau \\ &\leq CK\int_{t_{n-1}}^{t_n}\tau^{-\frac{6-\alpha}{4}}\,d\tau \leq CKkt_{n-1}^{-\frac{6-\alpha}{4}} \leq CKkt_n^{-\frac{1}{2}-\frac{4-\alpha}{4}},\end{aligned}$$

for $n \geq 2$, and for $n = 1$

$$\|\epsilon_1\| \leq \frac{1}{k}\int_0^k \tau\|Gu_{tt}\|\,d\tau \leq CK\frac{1}{k}\int_0^k \tau^{1-\frac{6-\alpha}{4}}\,d\tau \leq CKk^{1-\frac{6-\alpha}{4}} = CKkt_1^{-\frac{1}{2}-\frac{4-\alpha}{4}}.$$

In a similar way we get

$$\|\bar{\partial}_t\zeta_n\| \leq Ch^\beta\frac{1}{k}\int_{t_{n-2}}^{t_n}\|Gu_{tt}\|_\beta\,d\tau \leq CKh^\beta t_n^{-\frac{3}{2}-\frac{\beta-\alpha}{4}},$$

which is (5.29), and

$$\|\bar{\partial}_t\epsilon_n\| \leq \int_{t_{n-2}}^{t_n}\|Gu_{ttt}\|\,d\tau \leq CKkt_n^{-\frac{3}{2}-\frac{4-\alpha}{4}},$$

where we have used the fact that $\|Gu_{ttt}(t)\| \sim \|u_{tt}(t)\|_2 \sim CKt^{-\frac{3}{2}-\frac{4-\alpha}{4}}$, cf. the end of the proof of Theorem 4.1. This proves (5.30). For the estimation of $G_h\hat{\epsilon}_n$ we write

$$\|G_h\hat{\epsilon}_n\| \leq \|G\hat{\epsilon}_n\| + \|(G_h - G)\hat{\epsilon}_n\|.$$

Here

$$\begin{aligned}\|G\hat{\epsilon}_n\| &= \|k\sum_{j=1}^n G\epsilon_j\| = \|k\sum_{j=1}^n\frac{1}{k}\int_{t_{j-1}}^{t_j}(\tau - t_{j-1})G^2u_{tt}\,d\tau\| \\ &\leq k\int_0^{t_n}\|G^2u_{tt}\|\,d\tau \leq CKk\int_0^{t_n}\tau^{-\frac{4-\alpha}{4}}\,d\tau \leq CKkt_n^{1-\frac{4-\alpha}{4}},\end{aligned}$$

since $\alpha \geq 1$, and

$$\begin{aligned}\|(G_h - G)\hat{\epsilon}_n\| &= \|k\sum_{j=1}^n(R_h - I)G\epsilon_j\| \leq Ch^\beta\|k\sum_{j=1}^n G\epsilon_j\|_\beta \\ &\leq Ch^\beta\Big(\|k\sum_{j=1}^n G^2\bar{\partial}_t u_j\|_\beta + \|k\sum_{j=1}^n G^2u_t(t_j)\|_\beta\Big) \\ &\leq Ch^\beta\Big(\int_0^{t_n}\|G^2u_t\|_\beta\,d\tau + k\sum_{j=1}^n\|G^2u_t(t_j)\|_\beta\Big) \\ &\leq CKh^\beta\Big(\int_0^{t_n}\tau^{-\frac{\beta-\alpha}{4}}\,d\tau + k\sum_{j=1}^n t_j^{-\frac{\beta-\alpha}{4}}\Big) \leq CKh^\beta t_n^{1-\frac{\beta-\alpha}{4}},\end{aligned}$$

and (5.27) is proved. Finally (5.28) is an immediate consequence of the previous bounds. □

Lemma 5.5. *Let $r = 2$ or 3, let u satisfy (5.1) and (5.2) for some $\alpha \in [1, r]$, and let U_n be the solution of (5.20). Then*

$$\|U_n - u(t_n)\|_l \le CK\big(h^{r-l} t_n^{-\frac{r-\alpha}{4}} + k t_n^{-\frac{4+l-\alpha}{4}}\big), \qquad 0 < t_n \le T,\ l = 0, 1. \tag{5.31}$$

Moreover, for the "chemical potential" $w_n = Au_n - Pf_n$ and its approximation $W_n = A_h U_n - P_h f_n$ we have

$$\|W_n - w(t_n)\| \le CK\big(h^r t_n^{-\frac{1}{2}-\frac{r-\alpha}{4}} + k t_n^{-\frac{1}{2}-\frac{4-\alpha}{4}}\big), \qquad 0 < t_n \le T. \tag{5.32}$$

Proof. Let β be as in Lemma 5.4. In order to prove the case $l = 0$ of (5.31) we start out by taking the inner product of (5.21) with $\bar\partial_t e_n$. Using the fact that G_h is selfadjoint positive semidefinite on L_2 we get

$$\begin{aligned}\|G_h \bar\partial_t e_n\|^2 + (e_n, \bar\partial_t e_n) &= (\rho_n, \bar\partial_t e_n) + (\zeta_n, G_h \bar\partial_t e_n) + (\epsilon_n, G_h \bar\partial_t e_n)\\ &\le (\rho_n, \bar\partial_t e_n) + \|\zeta_n\|^2 + \|\epsilon_n\|^2 + \tfrac{1}{2}\|G_h \bar\partial_t e_n\|^2,\end{aligned}$$

which shows

$$\|G_h \bar\partial_t e_n\|^2 + 2(e_n, \bar\partial_t e_n) \le 2(\rho_n, \bar\partial_t e_n) + 2\|\zeta_n\|^2 + 2\|\epsilon_n\|^2.$$

Using the identity

$$\bar\partial_t(a_n b_n) = (\bar\partial_t a_n) b_n + a_{n-1}(\bar\partial_t b_n) \tag{5.33–a}$$
$$\qquad\qquad = (\bar\partial_t a_n) b_n + a_n(\bar\partial_t b_n) - k(\bar\partial_t a_n)(\bar\partial_t b_n), \tag{5.33–b}$$

we obtain

$$\begin{aligned}\|G_h \bar\partial_t e_n\|^2 + \bar\partial_t \|e_n\|^2 + k\|\bar\partial_t e_n\|^2 &\le 2\bar\partial_t(\rho_n, e_n) - 2(\bar\partial_t \rho_n, e_n) + 2k(\bar\partial_t \rho_n, \bar\partial_t e_n)\\ &\quad + 2\|\zeta_n\|^2 + 2\|\epsilon_n\|^2 \le 2\bar\partial_t(\rho_n, e_n) - 2(\bar\partial_t \rho_n, e_n)\\ &\quad + k\|\bar\partial_t \rho_n\|^2 + k\|\bar\partial_t e_n\|^2 + 2\|\zeta_n\|^2 + 2\|\epsilon_n\|^2.\end{aligned}$$

Cancelling the term $k\|\bar\partial_t e_n\|^2$, multiplying by t_{n-1}^2 and using (5.33–a) yields

$$\begin{aligned}t_{n-1}^2\|G_h \bar\partial_t e_n\|^2 &+ \bar\partial_t\big(t_n^2\|e_n\|^2\big) \le 2t_{n-\frac{1}{2}}\|e_n\|^2 + 2\bar\partial_t\left[t_n^2(\rho_n, e_n)\right]\\ &- 4t_{n-\frac{1}{2}}(\rho_n, e_n) - 2t_{n-1}^2(\bar\partial_t \rho_n, e_n) + t_{n-1}^3\|\bar\partial_t \rho_n\|^2 + 2t_{n-1}^2\|\zeta_n\|^2 + 2t_{n-1}^2\|\epsilon_n\|^2\\ &\le C\big(\bar\partial_t\left[t_n^2(\rho_n, e_n)\right] + t_n\|\rho_n\|^2 + t_{n-1}^3\|\bar\partial_t \rho_n\|^2 + t_n^2\|\zeta_n\|^2 + t_n^2\|\epsilon_n\|^2 + t_n\|e_n\|^2\big).\end{aligned}$$

Multiplying by k and summing with respect to n, we obtain after a simple kick-back argument

$$\begin{aligned}k\sum_{j=1}^n t_{j-1}^2\|G_h \bar\partial_t e_j\|^2 + t_n^2\|e_n\|^2 &\le Ct_n^2\|\rho_n\|^2\\ &\quad + Ck\sum_{j=1}^n\big(t_j\|\rho_j\|^2 + t_{j-1}^3\|\bar\partial_t \rho_j\|^2 + t_j^2\|\zeta_j\|^2 + t_j^2\|\epsilon_j\|^2 + t_j\|e_j\|^2\big).\end{aligned}$$

Invoking the bounds for ρ_n, ζ_n and ϵ_n in Lemmas 5.1 and 5.4 we conclude that

$$k\sum_{j=1}^n t_{j-1}^2\|G_h \bar\partial_t e_j\|^2 + t_n^2\|e_n\|^2 \le CK^2\left(h^{2\beta} t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}} + k\sum_{j=1}^n t_j\|e_j\|^2\right). \tag{5.34}$$

We now have to estimate $k\sum_{j=1}^n t_j\|e_j\|^2$ and we therefore multiply (5.21) by e_n to get

$$\begin{aligned}(G_h^2\bar\partial_t e_n, e_n) + \|e_n\|^2 &= (\rho_n, e_n) + (\zeta_n, G_h e_n) + (\epsilon_n, G_h e_n)\\ &\le \tfrac12\|\rho_n\|^2 + \tfrac12\|e_n\|^2 + (\|\zeta_n\| + \|\epsilon_n\|)\,\|G_h e_n\|,\end{aligned}$$

whence, by (5.33–b),

$$\bar\partial_t\|G_h e_n\|^2 + k\|G_h\bar\partial_t e_n\|^2 + \|e_n\|^2 \le \|\rho_n\|^2 + 2\,(\|\zeta_n\| + \|\epsilon_n\|)\,\|G_h e_n\|.$$

Multiplication by t_n and using (5.33–a) now yields

$$\begin{aligned}&\bar\partial_t\big(t_{n+1}\|G_h e_n\|^2\big) + kt_n\|G_h\bar\partial_t e_n\|^2 + t_n\|e_n\|^2\\ &\quad\le \|G_h e_n\|^2 + t_n\|\rho_n\|^2 + 2t_n\,(\|\zeta_n\| + \|\epsilon_n\|)\,\|G_h e_n\|\\ &\quad\le t_n\|\rho_n\|^2 + 2t_n^2\|\zeta_n\|^2 + 2t_n^2\|\epsilon_n\|^2 + 2\|G_h e_n\|^2,\end{aligned}$$

so that, since $G_h e_0 = 0$ and in view of (5.28),

(5.35)
$$\begin{aligned}k\sum_{j=1}^n t_j\|e_j\|^2 &\le Ck\sum_{j=1}^n\big(t_j\|\rho_j\|^2 + t_j^2\|\zeta_j\|^2 + t_j^2\|\epsilon_j\|^2 + \|G_h e_j\|^2\big)\\ &\le CK^2\big(h^{2\beta}t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}}\big) + 2k\sum_{j=1}^n\|G_h e_j\|^2.\end{aligned}$$

To derive an estimate of $k\sum_{j=1}^n\|G_h e_j\|^2$ we sum (5.21) with respect to n, taking $G_h e_0 = 0$ into account, which yields

$$G_h^2 e_n + \hat e_n = \hat\rho_n + G_h\hat\zeta_n + G_h\hat\epsilon_n, \qquad t_n > 0,$$

where $\hat e_n = k\sum_{j=1}^n e_j$, $\hat e_0 = 0$, etc. Multiplication by $e_n = \bar\partial_t\hat e_n$ gives

$$\begin{aligned}\|G_h e_n\|^2 + \tfrac12\bar\partial_t\|\hat e_n\|^2 + \tfrac12 k\|\bar\partial_t\hat e_n\|^2 &= (\hat\rho_n, e_n) + (\hat\zeta_n, G_h e_n) + (G_h\hat\epsilon_n, e_n)\\ &\le \tfrac12\|\hat\zeta_n\|^2 + \tfrac12\|G_h e_n\|^2 + (\|\hat\rho_n\| + \|G_h\hat\epsilon_n\|)\,\|e_n\|,\end{aligned}$$

which after some simple manipulation leads to

(5.36)
$$\begin{aligned}k\sum_{j=1}^n\|G_h e_j\|^2 + \|\hat e_n\|^2 &\le k\sum_{j=1}^n\big(\|\hat\zeta_j\|^2 + 2\,(\|\hat\rho_j\| + \|G_h\hat\epsilon_j\|)\,\|e_j\|\big)\\ &\le k\sum_{j=1}^n\big(\|\hat\zeta_j\|^2 + 2t_j^{-1}\|\hat\rho_j\|^2 + 2t_j^{-1}\|G_h\hat\epsilon_j\|^2 + \tfrac14 t_j\|e_j\|^2\big)\\ &\le CK^2\big(h^{2\beta}t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}}\big) + \tfrac14 k\sum_{j=1}^n t_j\|e_j\|^2,\end{aligned}$$

and, upon substitution into the right-hand side of (5.35),

$$k\sum_{j=1}^n t_j\|e_j\|^2 \le CK^2\big(h^{2\beta}t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}}\big). \tag{5.37}$$

Estimates (5.34) and (5.37) now yield

$$k\sum_{j=1}^{n} t_{j-1}^2\|G_h\bar\partial_t e_j\|^2 + t_n^2\|e_n\|^2 \le CK^2\big(h^{2\beta}t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}}\big), \tag{5.38}$$

and the case $l=0$ of (5.31) follows.

It is now convenient to estimate the difference between $w(t_n)$ and W_n. Observe that

$$w(t_n) - W_n = -Gu_t(t_n) + G_h\bar\partial_t U_n = G_h\bar\partial_t e_n - \zeta_n - \epsilon_n.$$

The last two terms ζ_n and ϵ_n are estimated as desired by (5.24) and (5.26). In order to estimate $G_h\bar\partial_t e_n$, we form the backward difference of (5.21):

$$G_h^2\bar\partial_t^2 e_n + \bar\partial_t e_n = \bar\partial_t\rho_n + G_h\bar\partial_t\zeta_n + G_h\bar\partial_t\epsilon_n, \qquad n\ge 2.$$

Taking the inner product with $\bar\partial_t e_n$ and using (5.33–b) we get

$$\begin{aligned}\tfrac12\bar\partial_t\|G_h\bar\partial_t e_n\|^2 + \tfrac12 k\|G_h\bar\partial_t e_n\|^2 + \|\bar\partial_t e_n\|^2 \\ = (\bar\partial_t\rho_n, \bar\partial_t e_n) + (\bar\partial_t\zeta_n, G_h\bar\partial_t e_n) + (\bar\partial_t\epsilon_n, G_h\bar\partial_t e_n).\end{aligned}$$

Hence, by a simple kick-back argument,

$$\bar\partial_t\|G_h\bar\partial_t e_n\|^2 \le \|\bar\partial_t\rho_n\|^2 + 2(\|\bar\partial_t\zeta_n\| + \|\bar\partial_t\epsilon_n\|)\|G_h\bar\partial_t e_n\|.$$

Multiplying by t_{n-2}^3, using (5.33) and the fact that $\bar\partial_t(t_{n-1}^3)\le 3t_{n-1}^2$, now yields

$$\begin{aligned}\bar\partial_t\big(t_{n-1}^3\|G_h\bar\partial_t e_n\|^2\big) &\le 3t_{n-1}^2\|G_h\bar\partial_t e_n\|^2 + t_{n-2}^3\|\bar\partial_t\rho_n\|^2 \\ &\quad + 2t_{n-2}^3(\|\bar\partial_t\zeta_n\| + \|\bar\partial_t\epsilon_n\|)\|G_h\bar\partial_t e_n\| \\ &\le C\big(t_{n-1}^2\|G_h\bar\partial_t e_n\|^2 + t_{n-1}^3\|\bar\partial_t\rho_n\|^2 + t_{n-2}^4\|\bar\partial_t\zeta_n\|^2 + t_{n-2}^4\|\bar\partial_t\epsilon_n\|^2\big).\end{aligned}$$

In a standard way we conclude

$$t_{n-1}^3\|G_h\bar\partial_t e_n\|^2 \le Ck\sum_{j=2}^{n}\big(t_{j-1}^2\|G_h\bar\partial_t e_j\|^2 + t_{j-1}^3\|\bar\partial_t\rho_j\|^2 + t_{j-2}^4\|\bar\partial_t\zeta_j\|^2 + t_{j-2}^4\|\bar\partial_t\epsilon_j\|^2\big),$$

and hence, by (5.38) and Lemma 5.5,

$$t_{n-1}^3\|G_h\bar\partial_t e_n\|^2 \le CK^2\big(h^{2\beta}t_n^{2-\frac{\beta-\alpha}{2}} + k^2 t_n^{2-\frac{4-\alpha}{2}}\big),$$

for $n\ge 2$. For $n=1$ we recall that $G_h e_0 = 0$, so that in view of (5.36) and (5.37),

$$\|G_h\bar\partial_t e_1\| = k^{-1}\|G_h e_1\| \le CK\big(h^\beta t_1^{-\frac12-\frac{\beta-\alpha}{4}} + kt_1^{-\frac12-\frac{4-\alpha}{4}}\big).$$

Taken together these estimates prove (5.32).

Finally the estimate of the H^1 norm of e_n is proved by interpolation between the known bounds for the errors in U_n and W_n just as in the proof of Lemma 5.2. □

6. Error bounds for the semidiscretization in space. In this section we shall estimate the difference between the solution u of the nonlinear Cahn-Hilliard equation (2.5) and its semidiscrete approximation u_h defined in (3.3). We begin by settling the question of existence, uniqueness and stability for u_h. Recall the *a priori* bound

$$\|u_h(t)\|_1 \le C(\|u_{0h}\|_1), \qquad 0\le t<\infty, \tag{6.1}$$

that we obtained in (3.4). Since (3.3) is a finite dimensional system of ordinary differential equations with differentiable nonlinearity, this bound immediately gives global existence:

Lemma 6.1. *The initial value problem (3.3) has a unique solution, which exists for all time.*

In our error analysis we shall use the following bounds for the nonlinearity $\phi(u)$.

Lemma 6.2. *Let* $\|v\|_1, \|w\|_1 \le R$. *Then*

$$\|\phi'(v)z\| \le C(R)\,\|z\|_1, \tag{A.1}$$

$$\|\phi(v)-\phi(w)\| \le C(R)\,\|v-w\|_1, \tag{A.2}$$

$$\|\phi(v)\| \le C(R), \tag{A.3}$$

$$\|G_h^{\frac{1}{2}}[\phi'(v)z]\| \le C(R)\,\|z\|, \tag{6.5}$$

$$\|G_h^{\frac{1}{2}}[\phi(v)-\phi(w)]\| \le C(R)\,\|v-w\|, \tag{6.6}$$

$$\|G_h^{\frac{1}{2}}([\phi'(v)-\phi'(w)]z)\| \le C(R)\,\|v-w\|\,\|z\|_1. \tag{6.7}$$

Proof. We only demonstrate (6.5) and (6.6); the remaining bounds are proved in a similar way. First note that, by Hölder's and Sobolev's inequalities ($d \le 3$),

$$\|G_h^{\frac{1}{2}} f\| = \sup_{\chi\in \dot{S}_h} \frac{|(f,\chi)|}{|\chi|_1} \le C\|f\|_{L_{6/5}}. \tag{6.8}$$

Since by assumption (2.4) ϕ is a cubic polynomial if $d = 3$, we thus have

$$\begin{aligned}\|G_h^{\frac{1}{2}}[\phi'(v)z]\| &\le C\|\phi'(v)z\|_{L_{6/5}} \le C\|\phi'(v)\|_{L_3}\|z\|_{L_2}\\ &\le C\,(1+\|v\|_{L_6}^2)\|z\| \le C\,(1+\|v\|_1^2)\|z\|,\end{aligned}$$

which is (6.5), and (6.6) readily follows. The modification needed when $d \le 2$ and ϕ has arbitrary degree is obvious. $\square$

Remark. The local Lipschitz condition (6.6) was used by Thomée and Wahlbin [19] in the error analysis of finite element methods for semilinear parabolic problems of second order.

We also need the following well known generalization of Gronwall's lemma. We include a proof for the sake of completeness.

Lemma 6.3. *Let the function* $\varphi(t,\tau) \ge 0$ *be continuous for* $0 \le \tau < t \le T$. *If*

$$\varphi(t,\tau) \le A\,(t-\tau)^{-1+\alpha} + B\int_\tau^t (t-s)^{-1+\beta}\varphi(s,\tau)\,ds, \qquad 0\le\tau<t\le T,$$

for some constants $A, B \ge 0$, $\alpha, \beta > 0$, *then there is a constant* $C = C(B,T,\alpha,\beta)$ *such that*

$$\varphi(t,\tau) \le CA\,(t-\tau)^{-1+\alpha}, \qquad 0\le\tau<t\le T.$$

Proof. Iterating the given inequality $N-1$ times, using the identity

$$\int_\tau^t (t-s)^{-1+\alpha}(s-\tau)^{-1+\beta}\,ds = C(\alpha,\beta)\,(t-\tau)^{-1+\alpha+\beta}, \qquad \alpha,\beta>0, \tag{6.9}$$

and estimating $(t-\tau)^\beta$ by T^β, we obtain

$$\varphi(t,\tau) \le C_1A\,(t-\tau)^{-1+\alpha} + C_2\int_\tau^t (t-s)^{-1+N\beta}\varphi(s,\tau)\,ds, \qquad 0\le\tau<t\le T,$$

where $C_1 = C_1(B, T, \alpha, \beta, N)$, $C_2 = C_2(B, \beta, N)$. We now choose the smallest N such that $-1 + N\beta \geq 0$, and estimate $(t-s)^{-1+N\beta}$ by $T^{-1+N\beta}$. If $-1+\alpha \geq 0$ we obtain the desired conclusion by the standard version of Gronwall's lemma. Otherwise we set $\psi(t,\tau) = (t-\tau)^{1-\alpha}\varphi(t,\tau)$ to obtain

$$\psi(t,\tau) \leq C_1 A + C_3 \int_\tau^t (s-\tau)^{-1+\alpha}\psi(s,\tau)\, ds, \qquad 0 \leq \tau < t \leq T,$$

and the standard version of Gronwall's lemma yields $\psi(t,\tau) \leq CA$ for $0 \leq \tau < t \leq T$, which is the desired result. □

We now turn to the stability of u_h with respect to perturbations of the initial value.

Lemma 6.4. *Let $u_h^{(i)}$, $i = 1, 2$, be two solutions of (3.3) with initial values $u_{0h}^{(i)}$ and satisfying $\|u_h^{(i)}(t)\|_1 \leq R$ for $0 \leq t \leq T$, $i = 1, 2$. Then for $j = -1, 0, 1$, $j \leq l = 0, 1$ we have*

$$|u_h^{(1)}(t) - u_h^{(2)}(t)|_l \leq C(R,T) t^{-\frac{l-j}{4}} |u_{0h}^{(1)} - u_{0h}^{(2)}|_j, \qquad 0 < t \leq T.$$

Proof. The proof is more or less the same as that of Theorem 6.5 and we omit it. □

We are now ready to formulate our main result.

Theorem 6.5. *Let $r = 2$ or 3 and assume that for some $\alpha \in [1, r]$ we have $u_0 \in \dot{H}^\alpha$ with*

$$\|u_0\|_\alpha \leq R_1; \qquad \|u(t)\|_1 + \|u_h(t)\|_1 \leq R_2, \qquad 0 \leq t \leq T, \tag{6.10}$$

where u and u_h are the solutions of (2.5) and (3.3), respectively. Then

$$\|u_h(t) - u(t)\| \leq C\|u_{0h} - P_h u_0\| + Ch^r t^{-\frac{r-\alpha}{4}}, \tag{6.11}$$

$$\|u_h(t) - u(t)\|_1 \leq C t^{-\frac{1-j}{4}} |u_{0h} - P_h u_0|_j + Ch^{r-1} t^{-\frac{r-\alpha}{4}}, \qquad j = 0, 1, \tag{6.12}$$

for $0 < t \leq T$, where $C = C(R_1, R_2, T)$.

Proof. It follows from Lemma 6.4 that we may assume that $u_{0h} = P_h u_0$; otherwise the additional errors in (6.11) and (6.12) caused by such a perturbation of the discrete initial value are bounded by

$$C\|u_{0h} - P_h u_0\| \quad \text{and} \quad Ct^{-\frac{1-j}{4}} |u_{0h} - P_h u_0|_j, \qquad j = -1, 0, 1,\ 0 < t \leq T,$$

respectively. Assuming thus that $u_{0h} = P_h u_0$ we shall compare u_h to the auxiliary function $\tilde{u}_h(t) \in \dot{S}_h$ defined by

$$\begin{aligned} \tilde{u}_{h,t} + A_h^2 \tilde{u}_h &= -A_h P_h \phi(u), \qquad t > 0, \\ \tilde{u}_h(0) &= P_h u_0. \end{aligned} \tag{6.13}$$

Setting $e = u_h - u$ and $\tilde{e} = \tilde{u}_h - u$, we know from Lemma 5.2 and (6.10) that

$$\|\tilde{e}(t)\|_l \leq C(R_1, T)\, h^{r-l} t^{-\frac{r-\alpha}{4}}, \qquad 0 < t \leq T,\ l = 0, 1. \tag{6.14}$$

By Duhamel's principle (3.5) we have

$$\begin{aligned} e(t) = \tilde{e}(t) + (u_h(t) - \tilde{u}_h(t)) &= \tilde{e}(t) - \int_0^t E_h(t-\tau) A_h P_h[\phi(u_h(\tau)) - \phi(u(\tau))]\, d\tau \\ &= \tilde{e}(t) - \int_0^t A_h^{\frac{3}{2}} E_h(t-\tau) G_h^{\frac{1}{2}}[\phi(u_h(\tau)) - \phi(u(\tau))]\, d\tau. \end{aligned}$$

By (6.14), the Lipschitz condition (6.6) and (6.10) we obtain

$$\|e(t)\| \le C(R_1, T)\, h^r t^{-\frac{r-\alpha}{4}} + C(R_2) \int_0^t (t-\tau)^{-\frac{3}{4}} \|e(\tau)\|\, d\tau,$$

and Gronwall's lemma 6.3 shows

$$\|e(t)\| \le C(R_1, R_2, T)\, h^r t^{-\frac{r-\alpha}{4}}, \qquad 0 < t < T.$$

For the proof of (6.12) we use (A.2) instead:

$$\begin{aligned} \|e(t)\|_1 &\le \|\tilde{e}(t)\|_1 + C \int_0^t \|A_h^{\frac{3}{2}} E_h(t-\tau) P_h[\phi(u_h(\tau)) - \phi(u(\tau))]\|\, d\tau \\ &\le C(R_1, T)\, h^{r-1} t^{-\frac{r-\alpha}{4}} + C(R_2) \int_0^t (t-\tau)^{-\frac{3}{4}} \|e(\tau)\|_1\, d\tau, \end{aligned}$$

and (6.12) follows by Gronwall's lemma. $\square$

In order to apply the above result we must verify assumption (6.10). In view of (2.8) and (6.1) we find that (6.10) holds, for example, if it can be proved that $\|u_{0h}\|_1 \le C\|u_0\|_1$ independently of h. Clearly this holds if $u_{0h} = R_h u_0$. Another possibility is to choose $u_{0h} = P_h u_0$, provided that we have the inverse inequality

$$\|\chi\|_1 \le C h^{-1} \|\chi\| \qquad \forall \chi \in \dot{S}_h. \tag{6.15}$$

It is easy to see that (6.15) and (3.10) imply $\|P_h u_0\|_1 \le C\|u_0\|_1$.

In view of (3.10) and (3.11) we have

$$|R_h u_0 - P_h u_0|_j \le |R_h u_0 - u_0|_j + |u_0 - P_h u_0|_j \le C h^{\beta - j} |u_0|_\beta,$$

for $1 \le \beta \le r$ and $j = 0$ if $r = 2$, $j = -1, 0$ if $r = 3$. (The negative norm bound for the error in P_h follows from (3.11) by a well known duality argument.) The following corollaries are now evident.

Corollary 6.6. (Smooth data) *Let $r = 2$ or 3 and assume that $u_0 \in \dot{H}^r$ with $|u_0|_r \le R$.*

(1) *If $u_{0h} = R_h u_0$, then*

$$\|u_h(t) - u(t)\| \le C(R, T) h^r, \qquad 0 \le t \le T;$$

(2) *if $u_{0h} = P_h u_0$ and (6.15) holds, then*

$$\|u_h(t) - u(t)\|_l \le C(R, T) h^{r-l}, \qquad 0 \le t \le T,\ l = 0, 1.$$

Corollary 6.7. (Data in H^1) *Let $r = 2$ or 3 and assume that $u_0 \in \dot{H}^1$ with $|u_0|_1 \le R$.*

(1) *If $u_{0h} = R_h u_0$, then*

$$\|u_h(t) - u(t)\|_1 \le C(R,T) h^{r-1} t^{-\frac{r-1}{4}}, \qquad 0 < t \le T;$$

(2) *if $u_{0h} = P_h u_0$ and (6.15) holds, then*

$$\|u_h(t) - u(t)\|_l \le C(R,T) h^{r-l} t^{-\frac{r-1}{4}}, \qquad 0 < t \le T,\ l = 0, 1.$$

The estimation of the error in the semidiscrete "chemical potential" $w_h = A_h u_h + P_h \phi(u_h)$ is more technical. We shall only present a result for the case of nonsmooth data: $u_0 \in \dot{H}^1$. In the proof of this we shall need the following bound for $u_{h,t}$.

Lemma 6.8. *Let $\|u_{0h}\|_1 \le R$. Then*

$$\|u_{h,t}(t)\|_l \le C(R,T) t^{-1-\frac{l-1}{4}}, \qquad 0 < t \le T,\ l = 0, 1. \tag{6.16}$$

Proof. Let $z_h = u_{h,t}$. Then by differentiation of (3.3) we have

$$z_{h,t} + A_h^2 z_h = -A_h P_h \phi'(u_h) z_h,$$

and after multiplication by t

$$\begin{aligned} D_t(t z_h) + A_h^2 (t z_h) &= z_h - t A_h P_h \phi'(u_h) z_h \\ &= -A_h^2 u_h - A_h P_h \phi(u_h) - t A_h P_h \phi'(u_h) z_h, \end{aligned} \tag{6.17}$$

where we have used (3.3) in the last step. Hence

$$t z_h(t) = -\int_0^t E_h(t-s)\big(A_h^2 u_h(s) + A_h P_h \phi(u_h(s)) + s A_h P_h \phi'(u_h(s)) z_h(s)\big)\, ds,$$

so that, by the boundedness of $\|u_h\|_1$, (A.3) and (6.5),

$$\begin{aligned} t\|z_h(t)\| &\le C \int_0^t (t-s)^{-\frac{3}{4}} \big(\|u_h(s)\|_1 + \|G_h^{\frac{1}{2}} \phi(u_h(s))\| + s\|G_h^{\frac{1}{2}}[\phi'(u_h(s)) z_h(s)]\|\big)\, ds \\ &\le C(R) t^{\frac{1}{4}} + C(R) \int_0^t (t-s)^{-\frac{3}{4}} s \|z_h(s)\|\, ds. \end{aligned}$$

Now Gronwall's lemma 6.3 yields $t\|z_h(t)\| \le C(R,T) t^{\frac{1}{4}}$ for $0 < t \le T$, which proves the case $l = 0$ of the lemma. The proof for the case $l = 1$ can be based on the first identity in (6.17). We proceed in the same way using the known bound for $\|z_h\|$ and the bound (A.1) for $\|\phi'(u_h) z_h\|$. $\square$

Theorem 6.9. *Let $r = 2$ or 3 and let $u_0 \in \dot{H}^1$ with $|u_0|_1 \le R$, and $u_{0h} = P_h u_0$. Then for the "chemical potential" $w = Au + P\phi(u)$ and its approximation $w_h = A_h u_h + P_h \phi(u_h)$ we have*

$$\|w_h(t) - w(t)\| \le C(R,T) h^r t^{-\frac{1}{2}-\frac{r-1}{4}}, \qquad 0 < t \le T.$$

Proof. Again we use the auxiliary function $\tilde{u}_h$ defined in (6.13). Let $e = u_h - u$, $z_h = u_h - \tilde{u}_h$, and $\tilde{w}_h = A_h\tilde{u}_h + P_h\phi(u) = -G_h\tilde{u}_{h,t}$. Since $w_h - w = (w_h - \tilde{w}_h) + (\tilde{w}_h - w)$, where by Lemma 5.2

$$\|\tilde{w}_h(t) - w(t)\| \le Ch^r t^{-\frac{1}{2}-\frac{r-1}{4}} = Ch^r t^{-\frac{r+1}{4}}, \qquad 0 < t \le T, \tag{6.18}$$

it remains to estimate $w_h - \tilde{w}_h = -G_h(u_{h,t} - \tilde{u}_{h,t}) = -G_h z_{h,t}$. The function z_h satisfies

$$z_{h,t} + A_h^2 z_h = A_h P_h f, \quad f = \phi(u) - \phi(u_h). \tag{6.19}$$

Differentiating this equation we get

$$z_{h,tt} + A_h^2 z_{h,t} = A_h P_h f_t.$$

Multiplying by t and using (6.19) yields

$$D_t(tz_{h,t}) + A_h^2(tz_{h,t}) = z_{h,t} + tA_hP_hf_t = -A_h^2 z_h + A_hP_hf + tA_hP_hf_t.$$

Hence, by Duhamel's principle,

$$\begin{aligned} tG_h z_{h,t}(t) = &-\int_0^t A_h E_h(t-s) z_h(s)\, ds + \int_0^t A_h^{\frac{1}{2}} E_h(t-s) G_h^{\frac{1}{2}} f(s)\, ds \\ &+ \int_0^t E_h(t-s) P_h f_t(s) s\, ds \equiv I_1 + I_2 + I_3. \end{aligned}$$

Here by the error bounds in Lemma 5.2 and Theorem 6.5 we have

$$\|I_1\| \le C\int_0^t (t-s)^{-\frac{1}{2}} \|z_h(s)\|\, ds \le Ch^r \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{r-1}{4}}\, ds \le Ch^r t^{1-\frac{r+1}{4}}.$$

Similarly by (6.6)

$$\|I_2\| \le C\int_0^t (t-s)^{-\frac{1}{4}} \|e(s)\|\, ds \le Ch^r t^{1-\frac{r}{4}}.$$

For I_3 we write $f_t = -\big[\phi'(u_h) - \phi'(u)\big] u_{h,t} - \phi'(u) e_t$, so that

$$\begin{aligned} I_3 = &-\int_0^t A_h^{\frac{1}{2}} E_h(t-s) G_h^{\frac{1}{2}} \big([\phi'(u_h(s)) - \phi'(u(s))] u_{h,t}(s) \big) s\, ds \\ &- \int_0^t \big[E_h(t-s)P_h - E(t-s)\big]\, \big[\phi'(u(s)) e_t(s)\big] s\, ds \\ &- \int_0^t E(t-s) \big[\phi'(u(s)) e_t(s)\big] s\, ds \equiv I_4 + I_5 + I_6. \end{aligned}$$

Here, by (6.7), Theorem 6.5 and Lemma 6.8, we have

$$\|I_4\| \le C\int_0^t (t-s)^{-\frac{1}{4}} \|u_{h,t}(s)\|_1 \|e(s)\|\, s\, ds \le Ch^r t^{1-\frac{r}{4}}.$$

Further, by Corollary 5.3, (A.1) and the bounds for u_t and $u_{h,t}$ in Theorem 4.1 and Lemma 6.8, we obtain

$$\|I_5\| \le Ch^r \int_0^t (t-s)^{-\frac{r}{4}} \|e_t(s)\|_1\, s\, ds \le Ch^r t^{1-\frac{r}{4}}.$$

For I_6 we argue as follows. Let $\chi \in L_2$ be arbitrary. Then

$$\begin{aligned}(E(t-s)[\phi'(u(s))e_t(s)],\chi) &= (Ge_t(s), AP[\phi'(u(s))E(t-s)\chi])\\ &\leq \|Ge_t(s)\|\,\|AP[\phi'(u(s))E(t-s)\chi]\|.\end{aligned}$$

By a careful exploitation of Sobolev's inequality ($d \leq 3$) and the moment inequality

$$|v|_\beta \leq C|v|_\delta^{1-\theta}|v|_\gamma^\theta \qquad \beta = (1-\theta)\delta + \theta\gamma,\ \theta \in [0,1], \tag{6.20}$$

we may show

$$\|AP[\phi'(u)v]\| \leq C\left(1+|u|_1|u|_\gamma\right)|v|_2,$$

where $\frac{5}{2} < \gamma < 3$ (cf. the proof of Lemma A.1 in the supplement). Hence, by the regularity estimates for $u(t)$ and $E(t)$, we have

$$\|A[\phi'(u(s))E(t-s)\chi]\| \leq C\left(1+|u(s)|_1|u(s)|_\gamma\right)|E(t-s)\chi|_2 \leq C\,(t-s)^{-\frac{1}{2}}s^{-\frac{\gamma-1}{4}}\|\chi\|,$$

and, since χ is arbitrary, we conclude that

$$\|E(t-s)[\phi'(u(s))e_t(s)]\| \leq C\,(t-s)^{-\frac{1}{2}}s^{-\frac{\sigma}{4}}\|Ge_t(s)\|,$$

where $\sigma = \gamma - 1 \in (\frac{3}{2}, 2)$. Therefore

$$\|I_6\| \leq C\int_0^t (t-s)^{-\frac{1}{2}}s^{1-\frac{\sigma}{4}}\|Ge_t(s)\|\,ds.$$

Here $Ge_t = -(G_h - G)u_{h,t} - (\tilde{w}_h - w) + G_h z_{h,t}$, where by (3.10) and Lemma 6.8 with $l = r-2$ (hence $l = 0$ or 1),

$$\|(G_h - G)u_{h,t}(t)\| \leq Ch^r\|Gu_{h,t}(t)\| \leq Ch^r\|u_{h,t}(t)\|_{r-2} \leq Ch^r t^{-\frac{r+1}{4}}.$$

Taking this together with (6.18) and the above bounds for $I_j,\ j = 1,\ldots,6$, we now have

$$tG_h z_{h,t}(t) \leq Ch^r t^{1-\frac{r+1}{4}} + C\int_0^t (t-s)^{-\frac{1}{2}}s^{1-\frac{\sigma}{4}}\|G_h z_{h,t}(s)\|\,ds, \qquad 0 < t \leq T,$$

or, with $\varphi(t) = tG_h z_{h,t}(t)$,

$$\varphi(t) \leq Ch^r t^{1-\frac{r+1}{4}} + C\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{\sigma}{4}}\varphi(s)\,ds, \qquad 0 < t \leq T.$$

Iterating this inequality once, recalling that $\sigma < 2$, we obtain (cf. the proof of the Gronwall lemma 6.3)

$$\varphi(t) \leq Ch^r t^{1-\frac{r+1}{4}} + C\int_0^t s^{-\frac{\sigma}{2}}\varphi(s)\,ds, \qquad 0 < t \leq T,$$

and since $1 - (r+1)/4 \geq 0$ the standard Gronwall lemma shows $\varphi(t) \leq Ch^r t^{1-\frac{r+1}{4}}$ for $0 < t \leq T$, which implies the desired bound for $G_h z_{h,t}$. $\square$

7. Error bounds for the completely discrete scheme. The purpose of this section is to estimate the difference between the solution u of the Cahn-Hilliard equation (2.5) and its completely discrete approximation U_n defined in (3.6). The argument is completely parallel to that of the previous section and we only present an outline indicating the modifications needed. We first recall that, if $k \le 4/\beta^4$, then we have the *a priori* bound (3.8). Using this bound we conclude that (3.6) has a unique solution U_n for all t_n if k is small.

In the proof of our main result we need a discrete version of the Gronwall lemma 6.3:

Lemma 7.1. *Let $0 \le \varphi_n \le R$ for $0 \le t_n \le T$. If*

$$\varphi_n \le A_1 t_n^{-1+\alpha_1} + A_2 t_n^{-1+\alpha_2} + B\,k \sum_{j=1}^{n} t_{n-j+1}^{-1+\beta}\,\varphi_j, \qquad 0 < t_n \le T,$$

for some constants $A_1, A_2, B \ge 0$, $\alpha_1, \alpha_2, \beta > 0$, then there are constants $k_0 = k_0(R, B, \beta)$ and $C = C(B, T, \alpha_1, \alpha_2, \beta)$ such that, for $k \le k_0$,

$$\varphi_n \le C\left(A_1 t_n^{-1+\alpha_1} + A_2 t_n^{-1+\alpha_2}\right), \qquad 0 < t_n \le T.$$

Proof. The proof is completely analogous to the proof of Lemma 6.3. Iterating the given inequality, using

$$k \sum_{j=1}^{n} t_{n-j+1}^{-1+\alpha}\, t_j^{-1+\beta} \le C(\alpha,\beta) t_n^{-1+\alpha+\beta}, \qquad \alpha, \beta > 0,$$

which follows by comparison with the integral in (6.9), we get

$$\begin{aligned}
\varphi_n &\le C_1\left(A_1 t_n^{-1+\alpha_1} + A_2 t_n^{-1+\alpha_2}\right) + C_2\,k \sum_{j=1}^{n} t_{n-j+1}^{-1+N\beta}\,\varphi_j \\
&\le C_1\left(A_1 t_n^{-1+\alpha_1} + A_2 t_n^{-1+\alpha_2}\right) + C_2\,k \sum_{j=1}^{n-1} t_{n-j+1}^{-1+N\beta}\,\varphi_j + C_2\,k^{N\beta}\varphi_n, \qquad 0 < t_n \le T,
\end{aligned}$$

where C_1 and C_2 are the same as before and $-1 + N\beta \ge 0$. If $C_2 k_0^{N\beta} R < 1$ and $k \le k_0$, then we may cancel the last term on the right and the proof is completed by means of the standard discrete Gronwall lemma. In this connection, if $\alpha_1 \ge \alpha_2$, say, and $-1 + \alpha_2 < 0$, we first set $\psi_n = t_n^{1-\alpha_2}\varphi_n$ to get

$$\psi_n \le C_3\left(A_1 t_n^{\alpha_1-\alpha_2} + A_2\right) + C_4\,k \sum_{j=1}^{n-1} t_j^{-1+\alpha_2}\,\psi_j, \qquad 0 < t_n \le T,$$

which leads to $\psi_n \le C\left(A_1 t_n^{\alpha_1-\alpha_2} + A_2\right)$ for $0 < t_n \le T$. $\square$

We can now state our main result. For simplicity of presentation we assume that $u_{0h} = P_h u_0$. The modifications needed for other choices of discrete initial data are exactly the same as in the previous section.

Theorem 7.2. *Let $r = 2$ or 3 and assume that for some $\alpha \in [1, r]$ we have $u_0 \in \dot{H}^\alpha$ and $u_{0h} = P_h u_0$ with*

$$\|u_0\|_\alpha \le R_1; \qquad \|u(t_n)\|_1 + \|U_n\|_1 \le R_2, \qquad 0 \le t_n \le T,$$

where u and U_n are the solutions of (2.5) and (3.6), respectively. Then there are $k_0 = k_0(R_2)$ and $C = C(R_1, R_2, T)$ such that, for $k \le k_0$,

$$\|U_n - u(t_n)\|_l \le C\left(h^{r-l} t_n^{-\frac{r-\alpha}{4}} + k t_n^{-\frac{4+l-\alpha}{4}}\right), \qquad 0 < t_n \le T,\ l = 0, 1.$$

Proof. We define $\tilde{U}_n \in \dot{S}_h$ by

$$\begin{aligned} \bar{\partial}_t \tilde{U}_n + A_h^2 \tilde{U}_n &= -A_h P_h \phi(u(t_n)), \qquad t_n > 0, \\ \tilde{U}_0 &= P_h u_0. \end{aligned}$$

With $e_n = [U_n - \tilde{U}_n] + [\tilde{U}_n - u(t_n)] \equiv Z_n + \tilde{e}_n$ we know from Lemma 5.5 that

$$\|\tilde{e}_n\|_l \le C(R_1, T)\left(h^{r-l} t_n^{-\frac{r-\alpha}{4}} + k t_n^{-\frac{4+l-\alpha}{4}}\right), \qquad 0 < t_n \le T,\ l = 0, 1. \tag{7.1}$$

We first demonstrate the case $l = 0$. By the variation of constants formula (3.9) we have

$$e_n = \tilde{e}_n - k \sum_{j=1}^n E_{kh}^{n-j+1} A_h P_h \big[\phi(U_n) - \phi(u(t_n))\big].$$

Using the fact that

$$\|A_h^{\frac{\beta}{2}} E_{kh}^n v\| \le C_\beta t_n^{-\frac{\beta}{4}} \|v\|, \qquad t_n > 0,\ \beta \ge 0,$$

and the Lipschitz condition (6.6) we obtain

$$\|e_n\| \le C(R_1, T)\left(h^r t_n^{-\frac{r-\alpha}{4}} + k t_n^{-\frac{4-\alpha}{4}}\right) + C(R_2)\, k \sum_{j=1}^n t_{n-j+1}^{-\frac{3}{4}} \|e_j\|, \qquad 0 < t_n \le T,$$

and the desired bound follows by the discrete Gronwall lemma 7.1. Similarly for the case $l = 1$ we use the discrete analogue of the proof of (6.12). However this does not work when $\alpha = 1$ due to the strength of the singularity of the term $k t_n^{-\frac{4+l-\alpha}{4}}$ in (7.1). Instead we argue as follows when $\alpha = 1$: From the equation for Z_n and (5.33) it follows that

$$\bar{\partial}_t(t_n Z_n) + A_h^2(t_n Z_n) = Z_{n-1} - t_n A_h P_h[\phi(U_n) - \phi(u(t_n))], \qquad t_n > 0.$$

Using $Z_0 = 0$ and (A.2) we obtain by the variation of constants formula

$$t_n \|Z_n\|_1 \le k \sum_{j=2}^n \|A_h^{\frac{1}{2}} E_{kh}^{n-j+1}\| \, \|Z_{j-1}\| + C(R_2)\, k \sum_{j=1}^n \|A_h^{\frac{3}{2}} E_{kh}^{n-j+1}\| \, t_j \|e_j\|_1.$$

By a modification of the first part of this proof we have here (with $\alpha = 1$)

$$\|Z_j\| \le C(R_1, R_2, T)\left(h^{r-1} t_j^{-\frac{r-2}{4}} + k t_j^{-\frac{3}{4}}\right), \qquad 0 < t_j \le T.$$

Together with (7.1) this shows

$$t_n \|e_n\|_1 \le C(R_1, R_2, T)\left(h^{r-1} t_n^{1-\frac{r-1}{4}} + k\right) + C(R_2)\, k \sum_{j=1}^n t_{n-j+1}^{-\frac{3}{4}} t_j \|e_j\|_1, \quad 0 < t_n \le T,$$

and the desired result follows. □

8. Stability of attractors. Let $u(t)=\mathcal{T}(t)(u_0)$ denote the solution of the Cahn-Hilliard equation (2.5). Then $\mathcal{T}(t)$ is a nonlinear semigroup in $\dot H^1$. Similarly (3.3) and (3.6) define nonlinear semigroups $\mathcal{T}_h(t)$ and $\mathcal{T}_{hk}^n$ in $\dot S_h\subset\dot H^1$ by $u_h(t)=\mathcal{T}_h(t)(u_{0h})$ and $U_n=\mathcal{T}_{hk}^n(u_{0h})$. We show below that $\mathcal{T}(t)$ has a *global attractor* $\mathcal{A}$. This means that $\mathcal{A}\subset\dot H^1$ is a *maximal compact invariant set* which *attracts every bounded subset* of $\dot H^1$. See Hale [9] for the definitions of these terms. We also show that $\mathcal{T}_h(t)$ and $\mathcal{T}_{hk}^n$ have global attractors $\mathcal{A}_h$ and $\mathcal{A}_{hk}$, respectively, in $\dot S_h\subset\dot H^1$.

We may think of $\mathcal{T}_h(t)$ and $\mathcal{T}_{hk}^n$ as perturbations of $\mathcal{T}(t)$, and the purpose of this section is to use our error bounds for solutions with initial data in $\dot H^1$ to prove a stability property of the perturbed attractors $\mathcal{A}_h$ and $\mathcal{A}_{hk}$.

In fact, applying Theorems 6.5 and 7.2 with $u_0=u_{0h}\in\dot S_h\subset\dot H^1$ we immediately obtain

Corollary 8.1. *Let $r=2$ or 3, $R>0$ and let $J\subset(0,\infty)$ be a compact interval. Then, for small k, we have*

$$\delta_h(R,J)\equiv\sup_{\substack{v\in\dot S_h\\ \|v\|_1\le R}}\sup_{t\in J}\|\mathcal{T}_h(t)(v)-\mathcal{T}(t)(v)\|_1\le C(R,J)h^{r-1},$$
$$\delta_{hk}(R,J)\equiv\sup_{\substack{v\in\dot S_h\\ \|v\|_1\le R}}\sup_{t_n\in J}\|\mathcal{T}_{hk}^n(v)-\mathcal{T}(t_n)(v)\|_1\le C(R,J)(h^{r-1}+k).$$

(Note that the constant blows up as J approaches 0 or ∞.) Since $\delta_h(R,J)\to0$, $\delta_{hk}(R,J)\to0$ as $h,k\to0$ for any R,J, it follows that

$$d(\mathcal{A}_h,\mathcal{A})\to0,\quad d(\mathcal{A}_{hk},\mathcal{A})\to0\qquad\text{as } h,k\to0, \tag{8.1}$$

see Temam [17, Theorem I.1.2]. Here $d(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_1$ is the (unsymmetric) semidistance between the sets A,B. Thus (8.1) means that for any $\epsilon>0$ there is h such that $\mathcal{A}_h$ lies in an ϵ-neighborhood of $\mathcal{A}$, or, in the terminology of [9], that $\mathcal{A}_h$ is upper semicontinuous at $h=0$.

The idea of the proof of (8.1) is to compare a discrete trajectory $u_h(t)$ on time intervals $[NT,(N+1)T]$, $N=1,2,\dots$, to exact trajectories $u(t)$ with $u(NT)=u_h(NT)$ using Corollary 8.1. The length T of these intervals is determined as the time it takes for $\mathcal{A}$ to attract any initial value into a $\frac12\epsilon$-neighborhood of itself. Everything being uniform on bounded sets of $\dot H^1$, one may conclude that $u_h(t)$ belongs to an ϵ-neighborhood of $\mathcal{A}$ for $h\le h_0=h_0(\epsilon)$, $t\ge T=T(\epsilon)$.

The same idea of using a nonsmooth data error bound to obtain a result about the longtime behavior of discrete solutions can be found in Heywood and Rannacher [12]. See also Hale, Lin and Raugel [10] and Kloeden and Lorenz [14] for related results on the upper semicontinuity of attractors.

We conclude this section by demonstrating the existence of the attractors $\mathcal{A}$, $\mathcal{A}_h$ and $\mathcal{A}_{hk}$. This follows easily from a general result about *asymptotically smooth gradient systems*, see Hale [9, Theorem 3.8.5]. We verify the assumptions of this theorem.

First we note that $\mathcal{T}(t)$ is a C^1-semigroup in $\dot H^1$. This means that for fixed t the mapping $u_0\mapsto\mathcal{T}(t)(u_0)$ is Fréchet differentiable, which is easily proved using the techniques of the proof of Theorem 4.1 in the supplement. Next we note that the smoothing property of $\mathcal{T}(t)$ obtained in Theorem 4.1 implies that $\mathcal{T}(t)$ is completely continuous. This implies that $\mathcal{T}(t)$ is asymptotically smooth and that all positive orbits $\gamma^+(v)=\{\mathcal{T}(t)v : t\ge0\}$ are precompact (see [9, Corollary 3.2.2, Lemma 3.2.1]). We also note that $\mathcal{T}(t)$ is a gradient system, i.e., it is a C^1-semigroup with the additional properties:

(1) each bounded positive orbit is precompact;

(2) there is a Ljapunov functional for $\mathcal{T}(t)$, i.e., there is a continuous mapping $V : \dot{H}^1 \to \mathbf{R}$ such that
(3) V is bounded below;
(4) $V(u) \to \infty$ as $|u|_1 \to \infty$;
(5) $t \mapsto V(\mathcal{T}(t)v)$ is nondecreasing;
(6) if v is such that $V(\mathcal{T}(t)v) = V(v)$ for all t, then v is an equilibrium point of $\mathcal{T}(t)$.

We have already verified (1). Moreover, we saw in Section 2 that $V(v) = \frac{1}{2}|v|_1^2 + \int_\Omega \psi(v)\,dx$ is a Ljapunov functional for $\mathcal{T}(t)$.

Finally we need to check that the set of equilibrium points $\mathcal{E}$ of $\mathcal{T}(t)$ is bounded in $\dot{H}^1$. To see this let $v \in \mathcal{E}$. Then $Av + P\phi(v) = 0$, so that $|v|_1^2 + (\phi(v), v) = 0$. Using (3.7) and (2.4) we get

$$|v|_1^2 + c_0\|v\|_{L_{2p}}^{2p} \le C + \tfrac{1}{2}\beta^2\|v\|^2 \le C\,(1 + \|v\|_{L_{2p}}^2),$$

which shows that $|v|_1 \le C$.

We are now in a position to apply [9, Theorem 3.8.5]. We conclude that $\mathcal{T}(t)$ has a global attractor $\mathcal{A}$. Moreover, the attractor is connected and equal to the unstable manifold of the set $\mathcal{E}$. Similar arguments apply to $\mathcal{T}_h(t)$ and $\mathcal{T}_{hk}^n$. In this case the complete continuity is automatic by finite dimensionality.

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SUPPLEMENT TO ERROR ESTIMATES WITH SMOOTH AND NONSMOOTH DATA FOR A FINITE ELEMENT METHOD FOR THE CAHN-HILLIARD EQUATION*

Charles M. Elliott and Stig Larsson

Appendix. In this appendix we present the proof of Theorem 4.1. The proof is based on estimating the right-hand side of the variation of constants formula (2.9) for solutions of (2.5), using the analyticity (4.2) of $E(t)$ together with certain bounds for the nonlinearity $\phi(u)$, and the *a priori* bound (2.8) for the H^1 norm of u. We begin with the required bounds for $\phi(u)$.

Lemma A.1. *Assume that $u, v \in \dot{H}^1$ with $|u|_1, |v|_1 \le R$. Then there is a constant $C = C(R)$ such that, under the appropriate regularity assumptions for w and z, we have*

$$|\phi'(u)w|_0 \le C\,|w|_1, \tag{A.1}$$

$$|\phi(u) - \phi(v)|_0 \le C\,|u - v|_1, \tag{A.2}$$

$$|\phi(u)|_0 \le C, \tag{A.3}$$

$$|AP(\phi'(u)w)|_0 \le C\,\big(|w|_3 + |u|_3|w|_1\big), \tag{A.4}$$

$$|AP(\phi(u) - \phi(v))|_0 \le C\,\big(|u - v|_3 + (|u|_3 + |v|_3)|u - v|_1\big), \tag{A.5}$$

$$|AP\phi(u)|_0 \le C\,|u|_3, \tag{A.6}$$

$$|\phi''(u)wz|_0 \le C\,|w|_1|z|_1, \tag{A.7}$$

$$|AP(\phi''(u)wz)|_0 \le C\,\big(|w|_1|z|_3 + |w|_3|z|_1 + |u|_3|w|_1|z|_1\big), \tag{A.8}$$

$$\begin{aligned} |AP\big([\phi'(u) - \phi'(v)]\,w\big)|_0 \le C\,\big(&|w|_1|u - v|_3 + |w|_3|u - v|_1 \\ &+ (|u|_3 + |v|_3)|w|_1|u - v|_1\big). \end{aligned} \tag{A.9}$$

Proof. We only demonstrate (A.8) from which (A.9) readily follows. The bound (A.4) is proved in a similar way, and (A.5), (A.6) then follow. The same is true of (A.1)–(A.3) and (A.7). We present the proof for the case $d = 3$; the case $d \le 2$ is analogous.

For the proof of (A.8) we first note that

$$\|\Delta(fg)\|_{L_2} \le \|\Delta f\|_{L_6}\|g\|_{L_3} + 2\|\nabla f\|_{L_6}\|\nabla g\|_{L_3} + \|f\|_{L_6}\|\Delta g\|_{L_3}.$$

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With the intention of applying this with $f = \phi''(u)$ and $g = wz$, we derive by Sobolev's inequality

$$\begin{aligned}
\|wz\|_{L_3} &\le \|w\|_{L_6}\|z\|_{L_6} \le C\|w\|_1\|z\|_1, \\
\|\nabla(wz)\|_{L_3} &\le \|\nabla w\|_{L_6}\|z\|_{L_6} + \|w\|_{L_6}\|\nabla z\|_{L_6} \le C\big(\|w\|_2\|z\|_1 + \|w\|_1\|z\|_2\big), \\
\|\Delta(wz)\|_{L_3} &\le \|\Delta w\|_{L_6}\|z\|_{L_6} + 2\|\nabla w\|_{L_6}\|\nabla z\|_{L_6} + \|w\|_{L_6}\|\Delta z\|_{L_6} \\
&\le C\big(\|w\|_3\|z\|_1 + \|w\|_2\|z\|_2 + \|w\|_1\|z\|_3\big),
\end{aligned}$$

and, using also the assumption (2.4) that ϕ is a cubic polynomial,

$$\begin{aligned}
\|\phi''(u)\|_{L_6} &\le C\left(1 + \|u\|_{L_6}\right) \le C\left(1 + \|u\|_1\right), \\
\|\nabla\phi''(u)\|_{L_6} &\le \|\phi'''(u)\|_{L_\infty}\|\nabla u\|_{L_6} \le C\|u\|_2, \\
\|\Delta\phi''(u)\|_{L_6} &\le \|\phi'''(u)\|_{L_\infty}\|\Delta u\|_{L_6} \le C\|u\|_3.
\end{aligned}$$

Taking these bounds together, we obtain

$$\|\Delta\big(\phi''(u)wz\big)\| \le C\left(1 + \|u\|_1\right)\big(\|w\|_1\|z\|_3 + \|w\|_3\|z\|_1 + \|u\|_3\|w\|_1\|z\|_1\big),$$

where we have used the fact that

$$\|u\|_2\|v\|_2 \le C\left(\|u\|_1\|u\|_3\|v\|_1\|v\|_3\right)^{1/2} \le C\left(\|u\|_1\|v\|_3 + \|u\|_3\|v\|_1\right).$$

In view of the equivalence of the norms $\|\cdot\|_s$ and $|\cdot|_s$ on $\dot{H}^s$, this proves (A.8). ☐

Remark. If we replace the norm $|\cdot|_3$ by $|\cdot|_\gamma$ for some $\gamma \in (3,4)$ in the above result, then we may replace our assumption that ϕ is a cubic polynomial ($d \le 3$) by a more general polynomial bound: $|\phi^{(j)}(s)| \le C\,(1+|s|^{q-j})$ for $q \in [3,5)$, $j = 1,\dots,4$, cf. von Wahl [20]. Our regularity analysis then works in the same way, but we have chosen the present setup for the ease of presentation.

Before embarking on the proof of Theorem 4.1, it is convenient to formulate three technical lemmas concerning the following situation: Let τ, γ, K, T be fixed with $0 \le \tau < T$, $\gamma \in [0,1]$, $K \ge 0$, and assume that

$$w(t) = E(t-\tau)w(\tau) + \int_\tau^t E(t-s)F(s)\,ds, \qquad \tau \le t \le T, \tag{A.10}$$

$$|w(\tau)|_\gamma \le K. \tag{A.11}$$

Below we will encounter this situation with w replaced by u, u_t and u_{tt}.

Lemma A.2. *Let w satisfy (A.10) and (A.11). (a) If, in addition,*

$$|F(t)|_0 \le C|w(t)|_3 + K\,(t-\tau)^{-(3-\gamma)/4}, \qquad \tau < t < T, \tag{A.12}$$

then

$$|w(t)|_\rho \le C(T,\rho)K\,(t-\tau)^{-(\rho-\gamma)/4}, \qquad \rho \in [\gamma, 4),\ \tau < t < T. \tag{A.13}$$

(b) If

$$|GF(t)|_0 \le C|w(t)|_1 + K\,(t-\tau)^{-(2-\gamma)/4}, \qquad \tau < t < T,$$

then

$$|w(t)|_1 \le C(T)K\,(t-\tau)^{-(1-\gamma)/4}, \qquad \tau < t < T.$$

Proof. Using (A.12) together with (A.11) and (4.2), we obtain

$$
\begin{aligned}
|w(t)|_3 &\le |E(t-\tau)w(\tau)|_3 + \int_\tau^t |E(t-s)F(s)|_3\,ds \\
&\le C\,(t-\tau)^{-(3-\gamma)/4}|w(\tau)|_\gamma + C\int_\tau^t (t-s)^{-3/4}|F(s)|_0\,ds \\
&\le CK\,(t-\tau)^{-(3-\gamma)/4} + C\int_\tau^t (t-s)^{-3/4}|w(s)|_3\,ds \\
&\quad + CK\int_\tau^t (t-s)^{-3/4}(s-\tau)^{-(3-\gamma)/4}\,ds \\
&\le CK\,(t-\tau)^{-(3-\gamma)/4} + C\int_\tau^t (t-s)^{-3/4}|w(s)|_3\,ds, \qquad \tau < t < T,
\end{aligned}
$$

and the Gronwall Lemma 6.3 yields $|w(t)|_3 \le CK\,(t-\tau)^{-(3-\gamma)/4}$ for $\tau < t < T$. Substituting this back into (A.12) and repeating the above argument, we arrive at (A.13). Part (b) is proved in a similar way, using

$$
|E(t-s)F(s)|_1 = |E(t-s)GF(s)|_3 \le C\,(t-s)^{-3/4}|GF(s)|_0. \quad \square
$$

Lemma A.3. *Let w satisfy (A.10) and (A.11), and assume that, in addition,*

$$
|w(t)|_\rho \le K\,(t-\tau)^{-(\rho-\gamma)/4},\ |F(t)|_0 \le K\,(t-\tau)^{-(3-\gamma)/4},\ |GF(t)|_0 \le K\,(t-\tau)^{-(1-\gamma)/4},
$$

for $\rho \in [\gamma, 4)$, $\tau < t < T$. Then, for $\rho \in [1,3]$, $\epsilon \in [0, 4-\rho)$, $\tau < s < t < T$,

$$
|w(t)-w(s)|_\rho \le C(T,\epsilon)K\,(t-s)^{\epsilon/4}(t-\tau)^{-(\rho+\epsilon-\gamma)/4}.
$$

Proof. A simple calculation using (A.10) shows

$$
w(t)-w(s) = \big(E(t-s)-I\big)w(s) + \int_s^t E(t-\sigma)F(\sigma)\,d\sigma \equiv I_1 + I_2.
$$

Using our assumption, we have for the first term on the right

$$
|I_1|_\rho \le C\,(t-s)^{\epsilon/4}|w(s)|_{\rho+\epsilon} \le CK\,(t-s)^{\epsilon/4}(s-\tau)^{-(\rho+\epsilon-\gamma)/4},
$$

see, e.g., Pazy [16, Theorem 2.6.13]. For the second term we get

$$
|I_2|_3 \le C\int_s^t (t-\sigma)^{-3/4}|F(\sigma)|_0\,d\sigma \le CK\,(t-s)^{1/4}(s-\tau)^{-(3-\gamma)/4},
$$

and

$$
|I_2|_1 \le C\int_s^t (t-\sigma)^{-3/4}|GF(\sigma)|_0\,d\sigma \le CK\,(t-s)^{1/4}(s-\tau)^{-(1-\gamma)/4},
$$

and the desired result follows by interpolation using the moment inequality (6.20). $\square$

Lemma A.4. *Let w satisfy (A.10) and (A.11). Assume that, in addition, for some $\epsilon \in (0,1)$ the following bounds hold:*

$$|F(t)|_0 \leq K\,(t-\tau)^{-(3-\gamma)/4}, \ |F(t)-F(s)|_0 \leq K\,(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon-\gamma)/4},$$

for $\tau < s < t < T$. Then $w \in C^1((\tau,T),L_2)$ with $w(t) \in \dot{H}^4$, $w_t + A^2 w = F$, and

$$|w_t(t)|_0 \leq C(T)K\,(t-\tau)^{-1+\gamma/4}, \qquad \tau < t < T. \tag{A.14}$$

Moreover,

$$w_t(t) = E(t-\tau_1)w_t(\tau_1) + \int_{\tau_1}^t E(t-s)F'(s)\,ds, \qquad \tau < \tau_1 < t < T. \tag{A.15}$$

Proof. The first claims follow from a standard regularity result for linear nonhomogeneous evolution equations, see, e.g., Pazy [16, Corollary 4.3.3]. We only need to verify (A.14) and (A.15). Differentiating (A.10), we obtain

$$w_t(t) = D_t E(t-\tau)w(\tau) + F(t) + \int_\tau^t D_t E(t-s)F(s)\,ds, \tag{A.16}$$

or, using $D_t E(t-s) = -A^2 E(t-s) = -D_s E(t-s)$,

$$w_t(t) = -A^2 E(t-\tau)w(\tau) + E(t-\tau)F(t) + \int_\tau^t A^2 E(t-s)\big(F(t)-F(s)\big)\,ds.$$

Hence,

$$\begin{aligned}
|w_t(t)|_0 &\leq C\,(t-\tau)^{-1+\gamma/4}|w(\tau)|_\gamma + C|F(t)|_0 + C\int_\tau^t (t-s)^{-1}|F(t)-F(s)|_0\,ds \\
&\leq CK\,(t-\tau)^{-1+\gamma/4} + CK\,(t-\tau)^{-(3-\gamma)/4} \\
&\quad + CK\int_\tau^t (t-s)^{-1+\epsilon/4}(s-\tau)^{-(3+\epsilon-\gamma)/4}\,ds \\
&\leq CK\,(t-\tau)^{-1+\gamma/4}, \qquad \tau < t < T.
\end{aligned}$$

Finally, (A.15) now follows essentially by integration by parts in (A.16). □

Proof of Theorem 4.1. We first apply a standard argument based on the local Lipschitz condition (A.2) to obtain local existence: For any $R_1 \geq 0$ there is $T_1 > 0$ such that equation (2.9) has a unique solution $u \in C([0,T_1],\dot{H}^1)$, whenever $u_0 \in \dot{H}^1$ with $|u_0|_1 \leq R_1$. Using the *a priori* bound (2.8), we may then conclude that the solution exists for all time.

It remains to show that u is a solution of (2.5) and that it has the regularity claimed in Theorem 4.1. For simplicity of exposition we present the proof for the special case $\alpha = 1$ only.

From (A.6) and (2.8) it follows that

$$|AP\phi(u(t))|_0 \leq C(R)|u(t)|_3,$$

and we apply part (a) of Lemma A.2 with $\tau = 0$, $\gamma = 1$ and $K = R$, to obtain

$$|u(t)|_\beta \leq C(T,R,\beta)t^{-(\beta-1)/4}, \qquad 0 < t < T,\ \beta \in [1,4). \tag{A.17}$$

In view of the inequality (2.6), this proves the special case $j = l = 0$ of (4.1). (In the sequel we shall not indicate the dependence on T, R, β of various constants.) Substituted into (A.6) and (A.3), the estimate (A.17) also implies that

$$|AP\phi(u(t))|_0 \le Ct^{-1/2}, \ |\phi(u(t))|_0 \le C,$$

and Lemma A.3 yields

$$\text{(A.18)} \qquad |u(t) - u(s)|_\rho \le C\,(t-s)^{\epsilon/4}(t-\tau)^{-(\rho+\epsilon-1)/4},$$

for $\rho \in [1,3]$, $\epsilon \in [0, 4-\rho)$, $0 < s < t < T$. Substituting this and (A.17) into (A.5), we now have

$$\begin{aligned} |AP[\phi(u(t)) - \phi(u(s))]|_0 &\le C\left(|u(t) - u(s)|_3 + (|u(t)|_3 + |u(s)|_3)|u(t) - u(s)|_1\right) \\ &\le C\,(t-s)^{\epsilon/4}s^{-(2+\epsilon)/4}, \qquad 0 < s < t < T, \end{aligned}$$

and Lemma A.4 shows that u is a solution of (2.5) and that

$$\text{(A.19)} \qquad |u_t(t)|_0 \le Ct^{-3/4}, \qquad 0 < t < T.$$

Moreover, we have

$$\text{(A.20)} \quad u_t(t) = E(t-\tau)u_t(\tau) - \int_\tau^t E(t-s)AP\big[\phi'(u(s))u_t(s)\big]\,ds, \qquad 0 < \tau < t < T.$$

In view of (A.1), we have here $|\phi'(u(t))u_t(t)|_0 \le C|u_t(t)|_1$ and, by (A.19), we may take $\gamma = 0$, $K = C\tau^{-3/4}$, and apply Lemma A.2 (b). Hence,

$$|u_t(t)|_1 \le C\tau^{-3/4}(t-\tau)^{-1/4}, \qquad 0 < \tau < t < T.$$

Using this result together with (A.17) in (A.4), we get

$$|AP[\phi'(u(t))u_t(t)]|_0 \le C\left(|u_t(t)|_3 + |u(t)|_3|u_t(t)|_1\right) \le C|u_t(t)|_3 + C\tau^{-3/4}(t-\tau)^{-3/4},$$

and Lemma A.2 (a) yields

$$\text{(A.21)} \qquad |u_t(t)|_\beta \le C\tau^{-3/4}(t-\tau)^{-\beta/4}, \qquad 0 < \tau < t < T, \ \beta \in [0,4),$$

or, with $\tau = t/2$,

$$\text{(A.22)} \qquad |u_t(t)|_\beta \le Ct^{-(3+\beta)/4}, \qquad 0 < t < T, \ \beta \in [0,4),$$

which implies the special case $j = 1$, $l = 0$ of (4.1). The bound (A.21) also implies that

$$|AP[\phi'(u(t))u_t(t)]|_0 \le C\tau^{-3/4}(t-\tau)^{-3/4}, \ |\phi'(u(t))u_t(t)|_0 \le C\tau^{-3/4}(t-\tau)^{-1/4},$$

so that, by Lemma A.3,

$$\text{(A.23)} \qquad |u_t(t) - u_t(s)|_\rho \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(\rho+\epsilon)/4},$$

for $0 < \tau < s < t < T$, $\rho \in [1,3]$, $\epsilon \in [0, 4-\rho)$. Writing next

$$\begin{aligned} AP\big[\phi'(u(t))u_t(t) - \phi'(u(s))u_t(s)\big] &= AP\big(\phi'(u(t))\big[u_t(t) - u_t(s)\big]\big) \\ &\quad + AP\big(\big[\phi'(u(t)) - \phi'(u(s))\big]u_t(s)\big) \equiv I_1 + I_2, \end{aligned}$$

we have, by (A.4), (A.17) and (A.23),

$$|I_1|_0 \le C\left(|u_t(t)-u_t(s)|_3 + |u(t)|_3|u_t(t)-u_t(s)|_1\right) \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon)/4}.$$

Similarly, by (A.9) and (A.18),

$$\begin{aligned}|I_2|_0 \le C\Big(&\left[|u_t(s)|_3 + (|u(t)|_3+|u(s)|_3)\,|u_t(s)|_1\right]|u(t)-u(s)|_1 \\ &+|u_t(s)|_1|u(t)-u(s)|_3\Big) \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon)/4}.\end{aligned}$$

We conclude that

$$|AP[\phi'(u(t))u_t(t)-\phi'(u(s))u_t(s)]|_0 \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon)/4},$$

for $0<\tau<s<t<T$. Hence, by Lemma A.4, $|u_{tt}(t)|_0 \le C\tau^{-3/4}(t-\tau)^{-1} \le Ct^{-7/4}$, and further estimates of u_{tt} can be based on

$$u_{tt}(t) = E(t-\tau)u_{tt}(\tau) - \int_\tau^t E(t-s)AP\left[\phi'(u(s))u_{tt}(s)+\phi''(u(s))u_t(s)^2\right]ds.$$

Using (A.1) and (A.7), we first get

$$\begin{aligned}|\phi'(u(t))u_{tt}(t)+\phi''(u(t))u_t(t)^2|_0 &\le C\left(|u_{tt}(t)|_1+|u_t(t)|_1^2\right)\\ &\le C|u_{tt}(t)|_1 + C\tau^{-7/4}(t-\tau)^{-1/2},\end{aligned}$$

and Lemma A.2 (b) may be applied with $\gamma=0$, $K=C\tau^{-7/4}$ to yield

$$|u_{tt}(t)|_1 \le C\tau^{-7/4}(t-\tau)^{-1/4}.$$

Hence, by (A.4) and (A.8),

$$\begin{aligned}&|AP[\phi'(u(t))u_{tt}(t)+\phi''(u(t))u_t(t)^2]|_0\\ &\quad\le C\Big(|u_{tt}(t)|_3+|u|_3|u_{tt}(t)|_1+(|u_t(t)|_3+|u(t)|_3|u_t(t)|_1)|u_t(t)|_1\Big)\\ &\quad\le C|u_{tt}(t)|_3 + C\tau^{-7/4}(t-\tau)^{-3/4},\end{aligned}$$

and Lemma A.2 (a) yields $|u_{tt}(t)|_\beta \le C\tau^{-7/4}(t-\tau)^{-\beta/4}$, $\beta\in[0,4)$, which proves the special case $j=2,\ l=0$ of (4.1).

We now turn to the cases $l=1,2$. From (A.22) we have

$$|Gu_t(t)|_4 = |u_t(t)|_2 \le Ct^{-5/4},$$

and, using equation (2.5) and (A.3), (A.17),

$$|Gu_t(t)|_0 \le |Au(t)|_0 + |\phi(u(t))|_0 \le |u(t)|_2 + C \le Ct^{-1/4}.$$

Interpolating between these results by means of the moment inequality (6.20), we obtain

$$|Gu_t(t)|_\beta \le Ct^{-(\beta+1)/4}, \qquad \beta\in[0,4],$$

which is the desired result when $j = l = 1$. In a similar way we find

$$
\begin{aligned}
&|G^2 u_t(t)|_4 = |u_t(t)|_0 \le C t^{-3/4}, \\
&|G^2 u_t(t)|_1 \le |u(t)|_1 + |G\phi(u(t))|_1 \le C,
\end{aligned}
$$

since $|G\phi(u(t)))|_1 \le C|\phi(u(t))|_0 \le C$, and interpolation yields the desired result when $j = 1,\ l = 2,\ \beta \in [1, 4]$. Next

$$
\begin{aligned}
&|G u_{tt}(t)|_4 = |u_{tt}(t)|_2 \le C t^{-9/4}, \\
&|G u_{tt}(t)|_0 \le |A u_t(t)|_0 + |\phi'(u(t)) u_t(t)|_0 \le |u_t(t)|_2 + C|u_t(t)|_1 \le C t^{-5/4},
\end{aligned}
$$

which covers the case $j = 2,\ l = 1,\ \beta \in [0, 4]$. Finally,

$$
\begin{aligned}
&|G^2 u_{tt}(t)|_4 = |u_{tt}(t)|_0 \le C t^{-7/4}, \\
&|G^2 u_{tt}(t)|_0 \le |u_t(t)|_0 + |G\big(\phi'(u(t)) u_t(t)\big)|_0 \le C|u_t(t)|_0 \le C t^{-3/4},
\end{aligned}
$$

since, by Sobolev's inequality,

$$
|Gf|_0 = \sup_{\chi \in \dot{H}^2} \frac{|(f, \chi)|}{|\chi|_2} \le C\|f\|_{L_1}, \quad \|\phi'(u) u_t\|_{L_1} \le |\phi'(u)|_0 |u_t|_0 \le C|u_t|_0.
$$

This proves the remaining case $j = l = 2,\ \beta \in [0, 4]$ of (4.1). $\square$

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