THE DISCONTINUOUS GALERKIN METHOD FOR SEMILINEAR PARABOLIC PROBLEMS

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Abstract. We prove *a priori* error estimates for a space-time finite element method for semilinear parabolic problems. The finite element method has basis functions that are continuous in space and discontinuous in time, and variable spatial meshes and time steps are allowed. The effect of numerical quadrature is emphasized.

Résumé. Nous montrons des estimations d'erreur *a priori* pour une méthode des éléments finis en espace et en temps pour des problèmes paraboliques semi-linéaires. La méthode des éléments finis considerée a des fonctions de base continues en espace et discontinues en temps, et admet des maillages spatiales et des pas de temps variables. L'effet de quadrature numérique est accentué.

1. Introduction. In this paper we consider the numerical solution of the semilinear parabolic equation

$$\begin{aligned} u_t - \Delta u &= f(x,t,u), && \text{in } \Omega \times (0,t^*), \\ u &= 0, && \text{on } \partial\Omega \times (0,t^*), \\ u(\cdot,0) &= u_0, && \text{in } \Omega, \end{aligned} \tag{1.1}$$

by using a finite element method with basis functions that are continuous in space and discontinuous in time, which we refer to as the discontinuous Galerkin method. Here Ω is a bounded convex polygonal domain in $\mathbf{R}^2$, $t^* > 0$, $u_t = \partial u/\partial t$, $\Delta u = \partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2$ and f is a smooth function satisfying

$$|f(x,t,s)| + |\partial f(x,t,s)/\partial s| \le M \quad \forall x \in \Omega,\ t > 0,\ s \in \mathbf{R}. \tag{1.2}$$

The discontinuous Galerkin method for linear parabolic equations, i.e., the case of (1.1) when $f = f(x,t)$, has been analyzed by Eriksson, Johnson and Thomee [3] and Eriksson and Johnson [2]. There are two parts to the analysis in [2]: an *a posteriori* error analysis, which is used to devise a global error control for an adaptive finite element method, and an *a priori* error analysis, which guarantees convergence. The *a priori* error bound is of a specific form related to the *a posteriori* bound and is proved under conditions which allow variable spatial meshes and time steps.

The purpose of the present work is to discuss some ways of handling the nonlinear term in this context, and to prove *a priori* error bounds in the style of [2]. In particular, we want to allow the possibility that the spatial mesh will change on each time level.

To avoid some technical difficulties we assume that such refinement (un-refinement) is performed by the addition (removal) of nodes to the current mesh. In fact, it is

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known that meshes that are changed in an uncontrolled fashion may yield completely false results, see Dupont [1].

The discontinuous Galerkin method involves integrals of the right-hand side of (1.1) with respect to x and t, which we evaluate by numerical quadrature. Our main result concerns the effect of such numerical integration on the *a priori* error analysis. Eriksson and Johnson do not consider this in [2], since for linear problems this is less important.

We describe the pertinent results of [2] in Section 2. In Section 3 we formulate our numerical method and state our main result, which is proved in Section 4. We conclude in Section 5 by providing an example showing the implementation of one of our algorithms.

2. The linear case. In this section we briefly review the notation and results of Eriksson and Johnson [2], which we will use in our analysis. For the discretization of (1.1) with respect to $x = (x_1, x_2)$ we let Σ be the class of all finite element discretizations $\mathcal{S} = (h, T, S)$ satisfying the following conditions:

(1) $h \in C^1(\Omega)$ is a positive function satisfying

$$|\nabla h(x)| \le \mu \quad \forall x \in \Omega;$$

(2) $T = \{K\}$ is a partition of Ω into triangles K of diameter h_K such that

$$c_1 h_K^2 \le \text{area}(K) \quad \forall K \in T,$$

and associated with the function h through the inequalities

$$c_2 h_K \le h(x) \le h_K \quad \forall x \in \Omega,\ K \in T;$$

(3) S is the set of all functions which are continuous on $\overline{\Omega}$, linear on each $K \in T$ and vanish on $\partial\Omega$.

For a given domain Ω the positive constants c_1, c_2 and μ characterize the family Σ completely.

For the discretization with respect to t we introduce a partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n < \ldots$ of $\mathbf{R}^+$ into subintervals $I_n = (t_{n-1}, t_n)$ of lengths $k_n = t_n - t_{n-1}$, and we associate with each I_n a finite element discretization $\mathcal{S}_n = (h_n, T_n, S_n) \in \Sigma$. For $q = 0$ and 1 we define

$$\mathcal{V}_n^q = \{V : V(x,t) = \sum_{j=0}^{q} t^j \varphi_j(x),\ \varphi_j \in S_n\}.$$

The discontinuous Galerkin method for (1.1) with $f = f(x,t)$ consists in computing a function U such that $U|_{\Omega\times I_n} \in \mathcal{V}_n^q$ and

(2.1)
$$\int_{I_n} \Big((U_t, X) + (\nabla U, \nabla X) \Big)\, dt + ([U]_{n-1}, X_{n-1}^+) = \int_{I_n} (f(\cdot,t), X(t))\, dt \quad \forall X \in \mathcal{V}_n^q,$$

for $n = 1, 2, \ldots$, and $U_0^- = u_0$, where $[V]_n = V_n^+ - V_n^-$, $V_n^\pm = \lim_{s\to 0^\pm} V(t_n + s)$ (V_n^- should be considered the "nodal value" of $V \in \mathcal{V}_n^q$). Here and below $(\cdot,\cdot)$ denotes the usual inner product in $L_2 = L_2(\Omega)$ and $\|\cdot\|$ is the corresponding norm. Hence, we are computing a finite element approximation to u on space-time "slabs" $\Omega \times I_n$. By virtue of the discontinuity of U in time, we can alter the spatial mesh from one time interval to the next.

It turns out that for $q = 0$ the scheme (2.1) reduces to the following modification of the backward Euler method:

$$\frac{1}{k_n}(U_n^- - U_{n-1}^-, \chi) + (\nabla U_n^-, \nabla \chi) = \frac{1}{k_n}\int_{I_n} (f(\cdot,t),\chi)\,dt \quad \forall \chi \in S_n.$$

For $q = 1$ in the linear homogeneous case ($f \equiv 0$) the approximation agrees at the nodal points t_n with the subdiagonal third order accurate Padé difference approximation. The approximation is second order accurate in the interiors of the intervals I_n (see [3]).

We quote the following *a priori* error bound:

Theorem 2.1. (Eriksson and Johnson [2]) *Let u be the solution of (1.1) with $f = f(x,t)$ and U that of (2.1). Suppose that μ is sufficiently small and assume that for each n one of the following assumptions hold:*

$$S_n \subset S_{n-1} \quad or \quad \big(\max_{x\in\Omega} h_n(x)\big)^2 \le \gamma k_n,$$

with γ sufficiently small and that $k_n \le c k_{n+1}$ for all n. Then there exists a constant C depending only on c_1 and c_2 such that for $q = 0, 1$ and $n = 1, 2, \ldots,$

$$\|U - u\|_{L_\infty(I_n,L_2)} \le CL_n \max_{1\le j\le n} E_j^{(q)}(u),$$

$$\|U_n^- - u(t_n)\| \le CL_n \max_{1\le j\le n} E_j^{(2q)}(u),$$

with $L_n = \frac{1}{4}\sqrt{1+\log\frac{t_n}{k_n}}$ and

$$E_j^{(p)}(u) = \|h_j^2 D^2 u\|_{L_\infty(I_j,L_2)} + \min_{s\le p+1} k_j^s \|u_t^{(s)}\|_{L_\infty(I_j,L_2)},$$

where $u_t^{(1)} = u_t$, $u_t^{(2)} = u_{tt}$, $u_t^{(3)} = \Delta u_{tt}$ and $D^m u = (\sum_{|\alpha|=m} |D_x^\alpha u|^2)^{1/2}$ (with the usual multi-index notation).

These error bounds imply that the scheme is of second order in x, of order $q+1$ uniformly in t, and of order $2q+1$ at the nodes t_n.

3. The semilinear case. In this section we discuss some ways of implementing the scheme (2.1) in the semilinear case, i.e., when $f = f(x,t,u)$. (In the sequel we sometimes write $f(u)$ instead of $f(x,t,u(x,t))$ for compactness of notation. We also use the notation $u_l = u(\cdot,t_l)$, $f(u)_l = f(\cdot,t_l,u_l)$ and $f(V)_l = f(\cdot,t_l,V_l^-)$ for $V \in \mathcal{V}_l^q$.)

In this case the right-hand side of (2.1) is an integral of $f(U)X$ over $\Omega \times I_n$, which must be evaluated by numerical quadrature if the algorithm is to be employed in a general way. This is the question that we address here. We desire that the completely discrete schemes should retain the order that the discontinuous Galerkin schemes have on linear problems. In particular, for $q = 1$, in order to retain the nominal third order accuracy of the discretization of the left-hand side of (2.1), we employ a multistep formula based on interpolation of $f(U)$ at the nodes t_n, where we expect third order accuracy for U.

For the integral with respect to x we choose the following quadrature rule. Let $\mathcal{S} = (h, T, S) \in \Sigma$ be a finite element discretization and define

$$Q_K(f) = \frac{1}{3}\text{area}(K)\sum_{j=1}^{3} f(P_{K,j}) \approx \int_K f(x)\,dx \quad \forall K \in T,$$

where $P_{K,j}$ are the vertices of the triangle K. We may then define approximations of the inner product and norm of L_2 by

$$(\chi,\psi)_S = \sum_{K\in T} Q_K(\chi\psi), \quad \|\chi\|_S = (\chi,\chi)_S^{1/2}.$$

For the approximation of the integral with respect to t we replace $f(U)$ by an interpolant with respect to t and integrate the resulting polynomials exactly. Two possibilities suggest themselves: one is to use extrapolation of an interpolant computed over previous intervals I_l, $l < n$, and the other is to use an interpolant computed over the current and previous intervals I_l, $l \le n$. The former process will yield a set of linear equations for U (semi-implicit method), whereas the latter will produce a nonlinear system (nonlinear implicit method).

We use an interpolant of order $p = 0$ when $q = 0$ and of order $p = 2$ when $q = 1$. Of course, when $p = 2$ we cannot use this kind of interpolant on the first one or two intervals, necessitating the construction of special interpolants there.

Thus our scheme is of the form: find a function U such that $U|_{\Omega\times I_n} \in \mathcal{V}_n^q$, $U_0^- = u_0$ and

$$\begin{aligned}(3.1)\quad \int_{I_n} \Big((U_t, X) + (\nabla U, \nabla X) \Big)\, dt &+ ([U]_{n-1}, X_{n-1}^+) \\ &= \int_{I_n} \langle \Pi f(U)(t), X(t)\rangle \, dt \quad \forall X \in \mathcal{V}_n^q,\ 0 < t_n \le t^*,\end{aligned}$$

where (except possibly for $n = 1$ or 2) the integrand of the right-hand side is given by
(3.2)

$$\langle \Pi f(U)(t), X(t)\rangle = \langle \Pi_{n-i}^p f(U)(t), X(t)\rangle = \sum_{l=n-i-p}^{n-i} \phi_l(t) (f(\cdot, t_l, U_l^-), X(t))_{S_l\cup S_n}.$$

Here Π_{n-i}^p is the interpolation operator for polynomials of degree $p = 2q$, computed with respect to the mesh points $\{t_l\}_{l=n-i-p}^{n-i}$ and ϕ_l are the corresponding Lagrange basis functions. Note that $i = 0$ for the nonlinear implicit scheme and $i = 1$ for the semi-implicit scheme.

In (3.2) we handle the possibility of variable meshes in the following way: we assume that all spatial meshes that occur are refinements of one common coarse mesh, and that each triangulation T_n is obtained from its precursor T_{n-1} by adding some nodes and by removing some other nodes. In this way, the union of the mesh points of two triangulations T_l and T_n form a triangulation whose finite element space is equal to $S_l \cup S_n$. In this situation we define

$$h_{l,n}(x) = \min\{h_l(x), h_n(x)\}.$$

The discrete L_2 inner product in (3.2) is thus computed over the totality of all mesh points of T_l and T_n.

When $q = 0$ the equation (3.1) becomes

$$\frac{1}{k_n}(U_n^- - U_{n-1}^-, \chi) + (\nabla U_n^-, \nabla\chi) = (f(\cdot, t_{n-i}, U_{n-i}^-), \chi)_{S_{n-i}\cup S_n} \quad \forall \chi \in S_n,\ 0 < t_n \le t^*,$$

which is a variant of the backward Euler method. In Section 5 we give details of the implementation of a scheme with $q = 1$.

It remains to devise a starting procedure for the case $q = 1$, $p = 2$, i.e., to define the right-hand side $\int_{I_n} \langle \Pi f(U), X\rangle \, dt$ for $1 \le n \le 1 + i$. For the nonlinear implicit scheme ($i = 0$) we simply take

$$\langle \Pi f(U), X\rangle = \langle \Pi_1^1 f(U), X\rangle \quad \text{on } I_1,$$

i.e., we use a linear interpolant in (3.2). For the semi-implicit method ($i = 1$) we similarly define

$$\langle \Pi f(U), X\rangle = \langle \Pi_1^1 f(U), X\rangle \quad \text{on } I_2.$$

For $n = 1$ we use a prediction-correction procedure in order to obtain the correct accuracy. For the predicted value of U_1^- we define $\tilde U|_{\Omega\times I_1} \in \mathcal{V}_1^q$ such that $\tilde U_0^- = u_0$ and

$$\int_{I_1} \Big((\tilde U_t, X) + (\nabla \tilde U, \nabla X) \Big)\, dt + ([\tilde U]_0, X_0^+) = \int_{I_1} \langle \Pi_0^0 f(\tilde U), X\rangle \, dt \quad \forall X \in \mathcal{V}_1^q,$$

that is,

$$\int_{I_1} \Big((\tilde U_t, X) + (\nabla \tilde U, \nabla X) \Big)\, dt + ([\tilde U]_0, X_0^+) = \int_{I_1} (f(\cdot, 0, u_0), X(t))_{S_0 \cup S_1} \, dt \quad \forall X \in \mathcal{V}_1^q.$$

Then we compute $U|_{\Omega\times I_1} \in \mathcal{V}_1^q$ such that $U_0^- = u_0$ and

$$\int_{I_1} \Big((U_t, X) + (\nabla U, \nabla X) \Big)\, dt + ([U]_0, X_0^+) = \int_{I_1} \langle \Pi_1^1 f(\tilde U), X\rangle \, dt \quad \forall X \in \mathcal{V}_1^q.$$

For the scheme described above we have the following *a priori* error bound. We need all the assumptions of this and the previous sections and, in addition, we assume that, for some $c_3, c_4 > 0$,

$$c_3 \le \frac{k_n}{k_{n-1}} \le c_4, \quad 0 < t_n \le t^*. \tag{3.3}$$

It is also convenient to define

$$m = \max\{i + p - 1, 0\} = \begin{cases} 0, & \text{if } q = 0, \\ 1 + i, & \text{if } q = 1, \end{cases} \tag{3.4}$$

to distinguish between the general case of (3.2) ($t_n \ge t_{m+1}$) and the special initialization procedure ($0 < t_n \le t_m$) needed when $q = 1$.

Theorem 3.1. *Let u be the solution of (1.1) and U be that of (3.1). Then there are constants k and C depending on M, t^* and c_l, $l = 1, \dots, 4$, such that, if $k_n \le k$ for $0 < t_n \le t^*$, then*

$$\|U - u\|_{L_\infty(I_n, L_2)} \le C L_n \max_{1\le j\le n} E_j^{(q)}(u) + C \max_{1\le j \le n} F_j^{(2q)}(u), \quad 0 < t_n \le t^*,$$
$$\|U_n^- - u(t_n)\| \le C L_n \max_{1\le j\le n} E_j^{(2q)}(u) + C \max_{1\le j \le n} F_j^{(2q)}(u), \quad 0 < t_n \le t^*.$$

For the semi-implicit schemes ($i = 1$) the condition $k_n \le k$ is not needed. Here L_n and $E_j^{(p)}(u)$ are as in Theorem 2.1 and

$$F_j^{(p)}(u) = \sum_{l=j-i-p}^{j-i} \big(\|h_{l,j}^2 D_x^2 u_l\| + \|h_{l,j}^2 D_x^2 f(u)_l\| + \|h_{l,j}^2 D_x f(u)_l\| \big)$$
$$+ \min_{s \le p+1} \big(k_j^s \|D_t^s f(u)\|_{L_\infty(I_j, L_2)} \big), \quad t_{m+1} \le t_j \le t^*,$$

and for $q=1$ we have, in addition, if $i=0$,

$$F_1^{(2)}(u)=\sum_{l=0}^{1}\big(\|h_{l,1}^2 D_x^2 u_l\|+\|h_{l,1}^2 D_x^2 f(u)_l\|+\|h_{l,1}^2 D_x f(u)_l\|\big) \\ +\min_{s\le 2}\big\{k_1^{s+1}\|D_t^s f(u)\|_{L_\infty(I_1,L_2)}\big\},$$

and, if $i=1$,

$$F_1^{(2)}(u)=\sum_{l=0}^{1}\big(\|h_{l,1}^2 D_x^2 u_l\|+\|h_{l,1}^2 D_x^2 f(u)_l\|+\|h_{l,1}^2 D_x f(u)_l\|\big) \\ +\min_{s\le 2}\big\{k_1^{s+1}\|D_t^s f(u)\|_{L_\infty(I_1,L_2)}\big\}+\min_{s\le 1}\big\{k_1^{s+2}\|D_t^s f(u)\|_{L_\infty(I_1,L_2)}\big\},$$

and

$$F_2^{(2)}(u)=\sum_{l=0}^{1}\big(\|h_{l,1}^2 D_x^2 u_l\|+\|h_{l,1}^2 D_x^2 f(u)_l\|+\|h_{l,1}^2 D_x f(u)_l\|\big) \\ +\max_{1\le l\le 2}\min_{s\le 2}\big\{k_l^{s+1}\|D_t^s f(u)\|_{L_\infty(I_l,L_2)}\big\}.$$

Remark. If one would settle for second order convergence (when $q=1$), then $\Pi f(U)$ in (3.2) could be taken to be a linear interpolant, which could even be computed at interior points of the interval I_{n-1} or I_n, with corresponding simplifications of the analysis and implementation.

4. Proof of Theorem 3.1. For the proof of Theorem 3.1 we shall compare the solution U of (3.1) with the solution V of the discrete linear problem (2.1) with $f(x,t)$ replaced by $f(x,t,u(x,t))$. That is, $V|_{\Omega\times I_n}\in\mathcal{V}_n^q$, $0<t_n\le t^*$ is defined by $V_0^-=u_0$ and

$$\int_{I_n}\Big((V_t,X)+(\nabla V,\nabla X)\Big)\,dt+([V]_{n-1},X_{n-1}^+)=\int_{I_n}(f(u),X)\,dt\quad \forall X\in\mathcal{V}_n^q. \tag{4.1}$$

Theorem 2.1 then shows that

$$\|V_n^- - u_n\|\le CL_n\max_{1\le j\le n}E_j^{(2q)}(u),\quad 0<t_n\le t^*, \tag{4.2}$$

and the proof of Theorem 3.1 will be accomplished once we have shown that for $\theta=U-V$ we have

$$\|\theta\|_{L_\infty(I_n,L_2)}\le CL_n\max_{1\le j\le n}E_j^{(2q)}(u)+C\max_{1\le j\le n}F_j^{(2q)}(u),\quad 0<t_n\le t^*. \tag{4.3}$$

In order to prove this we shall show below that

$$\|\theta_n^-\|\le CL_n\max_{1\le j\le n}E_j^{(2q)}(u)+C\max_{1\le j\le n}F_j^{(2q)}(u),\quad 0<t_n\le t^*. \tag{4.4}$$

It is convenient to begin by demonstrating that (4.3) follows from (4.4). From (3.1) and (4.1) it follows that $\theta|_{\Omega\times I_n}\in\mathcal{V}_n^q$, $0<t_n\le t^*$, satisfies $\theta_0^-=0$ and

$$\int_{I_n}\Big((\theta_t,X)+(\nabla\theta,\nabla X)\Big)\,dt+([\theta]_{n-1},X_{n-1}^+) \\ =\int_{I_n}\Big(\langle\Pi f(U),X\rangle-(f(u),X)\Big)\,dt\quad\forall X\in\mathcal{V}_n^q. \tag{4.5}$$

We split the integrand on the right-hand side into three terms,

$$\begin{aligned}\langle \Pi f(U), X\rangle - (f(u), X) &= \langle \Pi[f(U) - f(u)], X\rangle \\ &\quad + \big(\langle \Pi f(u), X\rangle - (\Pi f(u), X)\big) \\ &\quad + ([\Pi - I]f(u), X).\end{aligned} \tag{4.6}$$

These are estimated in the following three lemmas. The first term requires a uniform Lipschitz condition on the nonlinearity f, which holds in view of assumption (1.2).

Lemma 4.1. *Let $U|_{\Omega\times I_l} \in \mathcal{V}_l^q$ for $j-i-p \le l \le j-i$ and $X \in \mathcal{V}_j^q$. Then*

$$|\langle \Pi_{j-i}^p[f(U)-f(u)](t), X(t)\rangle| \le CM \sum_{l=j-i-p}^{j-i} \big(\|U_l^- - u_l\| + \|h_{l,j}^2 D_x^2 u_l\|\big)\|X(t)\|, \quad t \in I_j.$$

Proof. Using the uniform boundedness of the Lagrangian basis functions,

$$|\phi_l(t)| \le C, \quad t \in I_j, \tag{4.7}$$

under the assumption (3.3), and employing the Lipschitz condition for f, we have

$$\begin{aligned}|\langle \Pi_{j-i}^p[f(U) - f(u)](t), X(t)\rangle| &= \left|\sum_{l=j-i-p}^{j-i} \phi_l(t)\,([f(U)_l - f(u)_l], X(t))_{S_l\cup S_j}\right| \\ &\le CM\left(\sum_{l=j-i-p}^{j-i} \|U_l^- - u_l\|_{S_l\cup S_j}\right)\|X(t)\|_{S_l\cup S_j}.\end{aligned}$$

But $\|\cdot\|_{S_l\cup S_j}$ and $\|\cdot\|$ are uniformly equivalent norms on $S_l \cup S_j$, and $X(t) \in S_j \subset S_l \cup S_j$, so that $\|X(t)\|_{S_l\cup S_j} \le C\|X(t)\|$. Similarly, with $J_{S_l\cup S_j} : C(\overline{\Omega}) \to S_l \cup S_j$ the Lagrange interpolation operator, we have

$$\begin{aligned}\|U_l^- - u_l\|_{S_l\cup S_j} &= \|J_{S_l\cup S_j}(U_l^- - u_l)\|_{S_l\cup S_j} \le C\|U_l^- - J_{S_l\cup S_j} u_l\| \\ &\le C\big(\|U_l^- - u_l\| + \|u_l - J_{S_l\cup S_j} u_l\|\big),\end{aligned}$$

since $U_l^- \in S_l \subset S_l \cup S_j$. In view of a well-known error bound for $J_{S_l\cup S_j}$, this proves the lemma. $\square$

The second term on the right of (4.6) involves the error in spatial quadrature.

Lemma 4.2. *Let $X \in \mathcal{V}_j^q$. Then*

$$\begin{aligned}&\big|\langle [\Pi_{j-i}^p f(u)](t), X(t)\rangle - \big([\Pi_{j-i}^p f(u)](t), X(t)\big)\big| \\ &\qquad \le C \sum_{l=j-i-p}^{j-i} \big(\|h_{l,j}^2 D_x^2 f(u)_l\| + \|h_{l,j}^2 D_x f(u)_l\|\big)\|X(t)\|_{H^1}, \quad t \in I_j.\end{aligned}$$

Proof. Using (4.7) we have

$$\begin{aligned}\big|\langle [\Pi_{j-i}^p f(u)](t), X(t)\rangle - ([\Pi_{j-i}^p f(u)](t), X(t))\big| &= \left|\sum_{l=j-i-p}^{j-i} \phi_l(t)\,\epsilon_{l,j}(f(u)_l, X(t))\right| \\ &\le C \sum_{l=j-i-p}^{j-i} |\epsilon_{l,j}(f(u)_l, X(t))|,\end{aligned}$$

where $\epsilon_{l,j}(\cdot,\cdot) = (\cdot,\cdot)_{S_l\cup S_j} - (\cdot,\cdot)$ is the quadrature error, for which we have

$$|\epsilon_{l,j}(f,\chi)| \le C\big(\|h_{l,j}^2 D_x^2 f\| + \|h_{l,j}^2 D_x f\|\big)\|\chi\|_{H^1}, \quad f \in H^2(\Omega),\ \chi \in S_l \cup S_j.$$

This follows by a simple modification of the proof of Lemma 2.3 of [4] or Lemma 1 on p. 170 of [5]. Since $X(t) \in S_j \subset S_l \cup S_j$ this proves the lemma. □

The third term in (4.6) involves the error in interpolation with respect to t. The proof of the following lemma is well-known and we omit it. The assumption (3.3) is used here again.

Lemma 4.3. *Let $m = \max\{i+p-1, 0\}$. We have*

$$\|(\Pi_{j-i}^p - I)f(u)\|_{L_\infty(I_j,L_2)} \le C \max_{j-m\le l\le j}\ \min_{s\le p+1} \big\{k_l^s \|D_t^s f(u)\|_{L_\infty(I_l,L_2)}\big\}.$$

We can now estimate $\|\theta\|_{L_\infty(I_n,L_2)}$ by applying a simple energy argument to (4.5). We formulate this as a lemma.

Lemma 4.4. *Let $U \in \mathcal{V}_l^q$ for $j-i-p \le l \le j-i$ and assume that $\theta \in \mathcal{V}_j^q$ satisfies*

$$\int_{I_j} \Big((\theta_t, X) + (\nabla\theta, \nabla X)\Big)\,dt + ([\theta]_{j-1}, X_{j-1}^+) = \int_{I_j} \Big(\langle \Pi_{j-i}^p f(U), X\rangle - (f(u), X)\Big)\,dt \quad \forall X \in \mathcal{V}_j^q. \tag{4.8}$$

Then with $m = \max\{i+p-1, 0\}$ we have

$$\begin{aligned}
\|\theta\|_{L_\infty(I_j,L_2)} \le{}& C\|\theta_{j-1}^-\| + CMk_j \sum_{l=j-i-p}^{j-i} \big(\|U_l^- - u_l\| + \|h_{l,j}^2 D_x^2 u_l\|\big)\\
&+ C\sqrt{k_j} \sum_{l=j-i-p}^{j-i} \big(\|h_{l,j}^2 D_x^2 f(u)_l\| + \|h_{l,j}^2 D_x f(u)_l\|\big)\\
&+ Ck_j \max_{j-m\le l\le j}\ \min_{s\le p+1} \big(k_l^s \|D_t^s f(u)\|_{L_\infty(I_l,L_2)}\big).
\end{aligned}$$

Proof. Since $\theta(t)$ is a polynomial in t of degree 0 or 1, we have

$$\|\theta\|_{L_\infty(I_j,L_2)} \le \max\{\|\theta_j^-\|, \|\theta_{j-1}^+\|\},$$

so it suffices to estimate the two terms on the right. Taking $X = \theta$ in (4.8), we have

$$\int_{I_j} \Big((\theta_t, \theta) + (\nabla\theta, \nabla\theta)\Big)\,dt + ([\theta]_{j-1}, \theta_{j-1}^+) = \int_{I_j} \Big(\langle \Pi_{j-i}^p f(U), \theta\rangle - (f(u), \theta)\Big)\,dt,$$

whence,

$$\|\theta_j^-\|^2 + \|\theta_{j-1}^+\|^2 + \|\theta\|_{L_2(I_j,H^1)}^2 \le C\|\theta_{j-1}^-\|^2 + C\int_{I_j} \Big(\langle \Pi_{j-i}^p f(U), \theta\rangle - (f(u), \theta)\Big)\,dt.$$

In view of Lemmas 4.1, 4.2 and 4.3, we have here

$$\begin{aligned}
\int_{I_j} \Big(\langle \Pi_{j-i}^p f(U), X\rangle - (f(u), \theta)\Big)\,dt ={}& \int_{I_j} \langle \Pi_{j-i}^p [f(U) - f(u)], \theta\rangle\,dt\\
&+ \int_{I_j} \Big(\langle \Pi_{j-i}^p f(u), \theta\rangle - (\Pi_{j-i}^p f(u), \theta)\Big)\,dt + \int_{I_j} ([\Pi_{j-i}^p - I] f(u), \theta)\,dt\\
\le{}& Ck_j R_1 \|\theta\|_{L_\infty(I_j,L_2)} + C\sqrt{k_j} R_2 \|\theta\|_{L_2(I_j,H^1)} + Ck_j R_3 \|\theta\|_{L_\infty(I_j,L_2)}\\
\le{}& C\epsilon^{-1} k_j^2 (R_1^2 + R_3^2) + C\epsilon^{-1} k_j R_2^2 + \epsilon\|\theta\|_{L_\infty(I_j,L_2)}^2 + \epsilon\|\theta\|_{L_2(I_j,H^1)}^2,
\end{aligned}$$

where $\epsilon > 0$ and

$$R_1 = M \sum_{l=j-i-p}^{j-i} \big(\|U_l^- - u_l\| + \|h_{l,j}^2 D_x^2 u_l\|\big),$$

$$R_2 = \sum_{l=j-i-p}^{j-i} \big(\|h_{l,j}^2 D_x^2 f(u)_l\| + \|h_{l,j}^2 D_x f(u)_l\|\big),$$

$$R_3 = \max_{j-m\le l\le j} \min_{s\le p+1} \big(k_l^s \|D_t^s f(u)\|_{L_\infty(I_l,L_2)}\big).$$

Hence, choosing ϵ small enough we obtain the desired result. $\square$

We are now in a position to finish the proof that (4.4) implies (4.3). We have to distinguish between the general case and the startup terms. In the general case, i.e., when $n \ge m+1$, where m is defined in (3.4), we have $\Pi = \Pi_{n-i}^{2q}$ in (4.5) and Lemma 4.4, (4.2) and (4.4) show

$$\begin{aligned}\|\theta\|_{L_\infty(I_n,L_2)} &\le C \max_{1\le j\le n}\Bigg(\|\theta_j^-\| + \|V_j^- - u_j\| + \sum_{l=j-i-2q}^{j-i} \big(\|h_{l,j}^2 D_x^2 u_l\| \\ &\qquad + \|h_{l,j}^2 D_x^2 f(u)_l\| + \|h_{l,j}^2 D_x f(u)_l\|\big) \\ &\qquad + \min_{s\le 2q+1} \big(k_j^s \|D_t^s f(u)\|_{L_\infty(I_j,L_2)}\big)\Bigg) \\ &\le C L_n \max_{1\le j\le n} E_j^{(2q)}(u) + C \max_{1\le j\le n} F_j^{(2q)}(u), \quad t_{m+1} \le t_n \le t^*.\end{aligned}$$

This is the desired result in the general case.

If $q = 1$ we also have to consider the startup terms. If $q = 1$, $i = 0$, and $n = 1$, then we have $\Pi = \Pi_1^1$ in (4.5) and Lemma 4.4 shows

$$\begin{aligned}\|\theta\|_{L_\infty(I_1,L_2)} &\le C\Bigg(\|\theta_1^-\| + \|V_1^- - u_1\| + \sum_{l=0}^{1}\big(\|h_{l,1}^2 D_x^2 u_l\| \\ &\qquad + \|h_{l,1}^2 D_x^2 f(u)_l\| + \|h_{l,1}^2 D_x f(u)_l\|\big)\Bigg) \\ &\qquad + C \min_{s\le 2}\big\{k_1^{s+1}\|D_t^s f(u)\|_{L_\infty(I_1,L_2)}\big\} \\ &\le C L_1 E_1^{(2)}(u) + C F_1^{(2)}(u).\end{aligned} \tag{4.9}$$

If $q = 1$, $i = 1$, then we first apply Lemma 4.4 with θ replaced by $\tilde\theta = \tilde U - V$ and with $\Pi = \Pi_0^0$. This gives

$$\begin{aligned}\|\tilde\theta\|_{L_\infty(I_1,L_2)} &\le C\Bigg(\|h_{0,1}^2 D_x^2 u_0\| + \|h_{0,1}^2 D_x^2 f(u)(t_0)\| + \|h_{0,1}^2 D_x f(u)(t_0)\| \\ &\qquad + \min_{s\le 1}\big\{k_1^{s+1}\|D_t^s f(u)\|_{L_\infty(I_1,L_2)}\big\}\Bigg).\end{aligned}$$

Next we apply the lemma with $\theta = U - V$ and with $\Pi_1^1 f(\tilde U) - f(u)$ on the right-hand side. This gives

$$\begin{aligned}\|\theta\|_{L_\infty(I_1,L_2)} &\le C\Bigg(k_1\|\tilde\theta_1^-\| + \|V_1^- - u_1\| + \sum_{l=0}^{1}\big(\|h_{l,1}^2 D_x^2 u_l\| \\ &\qquad + \|h_{l,1}^2 D_x^2 f(u)_l\| + \|h_{l,1}^2 D_x f(u)_l\|\big)\Bigg) \\ &\qquad + C \min_{s\le 2}\big\{k_1^{s+1}\|D_t^s f(u)\|_{L_\infty(I_1,L_2)}\big\}.\end{aligned}$$

Using the above bound for $\tilde\theta = \tilde U - V$ we conclude

$$\|\theta\|_{L_\infty(I_1,L_2)} \le CL_1E_1^{(2)}(u) + CF_1^{(2)}(u).$$

We then continue to the interval I_2 and use Lemma 4.4 with $\theta = U - V,\ \Pi = \Pi_1^1$:

$$\begin{aligned}
\|\theta\|_{L_\infty(I_2,L_2)} \le C\biggl(&\|\theta_1^-\| + \|V_1^- - u_1\| + \sum_{l=0}^{1}\bigl(\|h_{l,1}^2 D_x^2 u_l\| \\
&+ \|h_{l,1}^2 D_x^2 f(u)_l\| + \|h_{l,1}^2 D_x f(u)_l\|\bigr)\biggr) \\
&+ C \max_{1\le l\le 2}\min_{s\le 2}\bigl\{k_l^{s+1}\|D_t^s f(u)\|_{L_\infty(I_l,L_2)}\bigr\} \\
\le{}& CL_1E_1^{(2)}(u) + CF_2^{(2)}(u).
\end{aligned}$$

Assumption (3.3) was also employed here. This completes the proof that (4.4) implies (4.3).

We now proceed to prove (4.4). Following [2] we represent $\|\theta_n^-\|$ by duality. For this purpose we define $\mathcal{V}^q = \{V : V \in \mathcal{V}_n^q,\ 0 < t_n \le t^*\}$. We then note that, by summation of equation (4.5), we have

$$B(\theta, X) = \sum_{j=1}^{n}\int_{I_j}\langle \Pi f(U), X\rangle\, dt - \int_0^{t_n}(f(u), X)\, dt \quad \forall X \in \mathcal{V}^q, \tag{4.10}$$

where

$$\begin{aligned}
B(V,W) &= \sum_{j=1}^{n}\int_{I_j}\Bigl((V_t, W) + (\nabla V, \nabla W)\Bigr)\, dt + \sum_{j=1}^{n-1}([V]_j, W_j^+) + (V_0^+, W_0^+) \\
&= \sum_{j=1}^{n}\int_{I_j}\Bigl(-(V, W_t) + (\nabla V, \nabla W)\Bigr)\, dt - \sum_{j=1}^{n-1}(V_j^-, [W]_j) + (V_n^-, W_n^-).
\end{aligned}$$

Next we consider the discrete analog of the "backward problem"

$$-z_t - \Delta z = 0, \quad 0 < t < t; \quad z(t_n) = \theta_n^-.$$

In view of the second form of $B(\cdot,\cdot)$ above, it is clear that the corresponding discrete problem consists in finding $Z \in \mathcal{V}^q$ such that

$$B(X, Z) = (X_n^-, \theta_n^-) \quad \forall X \in \mathcal{V}^q. \tag{4.11}$$

The following stability bound is proved in [3]:

$$\|Z\|_{L_\infty([0,t_n],L_2)} + \|Z\|_{L_2([0,t_n],H^1)} \le C\|\theta_n^-\|. \tag{4.12}$$

Taking $X = \theta$ in (4.11) and using (4.10), we obtain

$$\|\theta_n^-\|^2 = B(\theta, Z) = \Bigl(\sum_{j=1}^{m} + \sum_{j=m+1}^{n}\Bigr)\int_{I_j}\Bigl(\langle \Pi f(U), Z\rangle - (f(u), Z)\Bigr)\, dt,$$

where m is as in (3.4) and the first sum is empty if $m=0$. Splitting the integrand on the right as in (4.6) and using Lemmas 4.1, 4.2 and 4.3, we obtain

$$
\begin{aligned}
&\left|\sum_{j=m+1}^{n}\int_{I_j}\Big(\langle \Pi f(U),Z\rangle-(f(u),Z)\Big)\,dt\right| \\
&\le C\sum_{j=m+1}^{n}k_j\sum_{l=j-i-2q}^{j-i}\Big(\|\theta_l^-\|+\|V_l^--u_l\|+\|h_{l,j}^2D_x^2u_l\|\Big)\|Z\|_{L_\infty([0,t_n],L_2)} \\
&\quad+C\sqrt{t_n}\max_{1\le j\le n}\sum_{l=j-i-2q}^{j-i}\Big(\|h_{l,j}^2D_x^2f(u)_l\|+\|h_{l,j}^2D_xf(u)_l\|\Big)\|Z\|_{L_2([0,t_n],H^1)} \\
&\quad+Ct_n\max_{1\le j\le n}\min_{s\le 2q+1}\big(k_j^s\|D_t^sf(u)\|_{L_\infty(I_j,L_2)}\big)\|Z\|_{L_\infty([0,t_n],L_2)} \\
&\le C\bigg((1-i)k_n\|\theta_n^-\|+\sum_{j=1}^{n-1}k_j\|\theta_j^-\|+L_n\max_{1\le j\le n}E_j^{(2q)}(u)+\max_{1\le j\le n}F_j^{(2q)}(u))\bigg)\|\theta_n^-\|.
\end{aligned}
$$

Here we have also used (4.12), (4.2) and the fact that $\theta_0^-=0$. The factor $1-i$ in the first term on the right means that this term vanishes for the lagged schemes ($i=1$).

If $m=1$, i.e., $q=1,\ i=0$, we have also the term

$$
\begin{aligned}
&\left|\int_{I_1}\Big(\langle \Pi_1^1 f(U),Z\rangle-(f(u),Z)\Big)\,dt\right| \\
&\quad\le Ck_1\bigg(\|\theta_1^-\|+\|V_1^--u_1\|+\sum_{l=0}^{1}\|h_{l,1}^2D_x^2u_l\|\bigg)\|Z\|_{L_\infty(I_1,L_2)} \\
&\quad+C\sqrt{k_1}\sum_{l=0}^{1}\Big(\|h_{l,1}^2D_x^2f(u)_l\|+\|h_{l,1}^2D_xf(u)_l\|\Big)\|Z\|_{L_2(I_1,H^1)} \\
&\quad+C\min_{s\le 2}\big(k_1^{s+1}\|D_t^sf(u)\|_{L_\infty(I_1,L_2)}\big)\|Z\|_{L_\infty(I_1,L_2)} \\
&\quad\le\Big(CL_1E_1^{(2)}(u)+CF_1^{(2)}(u)\Big)\|\theta_n^-\|,
\end{aligned}
$$

where we have used the estimate of $\|\theta_1^-\|$ from (4.9).

If $m=2$, i.e., $q=1,\ i=1$, we have the terms

$$
\begin{aligned}
&\left|\int_{I_1}\Big(\langle \Pi_1^1 f(\tilde U),Z\rangle-(f(u),Z)\Big)\,dt\right|+\left|\int_{I_2}\Big(\langle \Pi_1^1 f(U),Z\rangle-(f(u),Z)\Big)\,dt\right| \\
&\quad\le Ck_1\bigg(\|\tilde\theta_1^-\|+\|V_1^--u_1\|+\sum_{l=0}^{1}\|h_{l,1}^2D_x^2u_l\|\bigg)\|Z\|_{L_\infty(I_1,L_2)} \\
&\quad+Ck_2\bigg(\|\theta_1^-\|+\|V_1^--u_1\|+\sum_{l=0}^{1}\|h_{l,1}^2D_x^2u_l\|\bigg)\|Z\|_{L_\infty(I_2,L_2)} \\
&\quad+C\sqrt{t_2}\max_{1\le j\le 2}\sum_{l=0}^{1}\Big(\|h_{l,j}^2D_x^2f(u)_l\|+\|h_{l,j}^2D_xf(u)_l\|\Big)\|Z\|_{L_2([0,t_2],H^1)} \\
&\quad+C\max_{1\le l\le 2}\min_{s\le 2}\big(k_l^{s+1}\|D_t^sf(u)\|_{L_\infty(I_l,L_2)}\big)\|Z\|_{L_\infty([0,t_2],L_2)} \\
&\quad\le\Big(CL_1E_1^{(2)}(u)+C\max_{1\le j\le 2}F_j^{(2)}(u)\Big)\|\theta_n^-\|,
\end{aligned}
$$

where we have used the already proven bounds for $\|\tilde{\theta}_1^-\|$ and $\|\theta_1^-\|$.

Altogether we now have

$$\|\theta_n^-\| \le (1-i)Ck_n\|\theta_n^-\| + C\sum_{j=1}^{n-1} k_j\|\theta_j^-\| + CL_n \max_{1\le j\le n} E_j^{(2q)}(u) + C\max_{1\le j\le n} F_j^{(2q)}(u),$$

and (4.4) follows by Gronwall's Lemma. If $i=0$ we first have to eliminate the first term on the right by taking k_n small. This completes the proof of Theorem 3.1.

5. Example. We conclude this paper by providing an example showing the implementation of one of our algorithms in the case $q=1$. We solve the equation $u_t - \Delta u = 10(u-u^3)$ with homogeneous Dirichlet boundary conditions and $\Omega = (0,1)\times(0,1)$. For the spatial discretization we use the piecewise linear finite element method computed on the fixed mesh shown in Figure 1 with mesh spacing $h=1/m$. We number the nodes as indicated so that there are $m=1/h$ nodes on a side and m^2 nodes in total.

FIGURE 1. Mesh for the sample computation.

Associated with this mesh are the finite element space S, the mass matrix A, $A_{ij} = (\varphi_i,\varphi_j)$ and stiffness matrix B, $B_{ij} = (\nabla\varphi_i,\nabla\varphi_j)$, where $\{\varphi_i\}_{i=1}^{m^2}$ is the standard computational basis for S. We will also need the lumped mass matrix $\bar{A}$, $\bar{A}_{ij} = (\varphi_i,\varphi_j)_S$, see [4] or [5].

We derive the equations for the semi-implicit $q=1$, $i=1$ scheme, which in some sense is the most complicated of our algorithms. We also have experience computing with $q=0$, $i=0,1$ and $q=1$, $i=0,1$ in one and two dimensions using fixed spatial meshes, and with $q=1$, $i=0$ in the fully adaptive two dimensional code TRANSI.

Throughout the startup procedure and the rest of the steps the same discrete system results from the left-hand side of (3.1). We let $U\in\mathcal{V}_n^q$ be given by

$$U(t) = U_{n-1}^+\frac{t-t_n}{-k_n} + U_n^-\frac{t-t_{n-1}}{k_n},$$

where U_{n-1}^+ and U_n^- now denote vectors of values at the spatial nodes, that is, $U_n^\pm = ((U_n^\pm)_1, (U_n^\pm)_2, \dots, (U_n^\pm)_{m^2})^T$, corresponding to the basis of S. (3.1) yields the pair of equations

$$
\begin{aligned}
A\frac{U_n^- - U_{n-1}^+}{k_n}\int_{I_n}\frac{t-t_n}{-k_n}\,dt + BU_{n-1}^+\int_{I_n}\frac{(t-t_n)^2}{k_n^2}\,dt \\
+ BU_n^-\int_{I_n}\frac{(t-t_n)(t-t_{n-1})}{-k_n^2}\,dt + AU_{n-1}^+ = AU_{n-1}^- + \bar{A}\int_{I_n}\Pi f(U)\frac{t-t_n}{-k_n}\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
A\frac{U_n^- - U_{n-1}^+}{k_n}\int_{I_n}\frac{t-t_{n-1}}{k_n}\,dt + BU_{n-1}^+\int_{I_n}\frac{(t-t_n)(t-t_{n-1})}{-k_n^2}\,dt \\
+ BU_n^-\int_{I_n}\frac{(t-t_n)^2}{k_n^2}\,dt = \bar{A}\int_{I_n}\Pi f(U)\frac{t-t_{n-1}}{k_n}\,dt,
\end{aligned}
$$

where $f(U) = (f((U)_1), f((U)_2), \ldots, f((U)_{m^2}))^T$. This is equivalent to

$$
\begin{bmatrix} \frac{1}{2}A + \frac{1}{3}k_nB & \frac{1}{2}A + \frac{1}{6}k_nB \\ -\frac{1}{2}A + \frac{1}{6}k_nB & \frac{1}{2}A + \frac{1}{3}k_nB \end{bmatrix}
\begin{bmatrix} U_{n-1}^+ \\ U_n^- \end{bmatrix}
= \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}
\begin{bmatrix} U_{n-2}^+ \\ U_{n-1}^- \end{bmatrix}
+ \begin{bmatrix} \bar{A}\int_{I_n}\Pi f(U)\frac{t-t_n}{-k_n}\,dt \\ \bar{A}\int_{I_n}\Pi f(U)\frac{t-t_{n-1}}{k_n}\,dt \end{bmatrix},
$$

for $n = 1, 2, \ldots$, with $U_0^+ := 0$. We let

$$
\mathbf{U}_n := \begin{bmatrix} U_{n-1}^+ \\ U_n^- \end{bmatrix}, \ \mathbf{B}_n := \begin{bmatrix} \frac{1}{2}A + \frac{1}{3}k_nB & \frac{1}{2}A + \frac{1}{6}k_nB \\ -\frac{1}{2}A + \frac{1}{6}k_nB & \frac{1}{2}A + \frac{1}{3}k_nB \end{bmatrix}, \ \mathbf{A} := \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}.
$$

Then, in order of application, the equations are

$$
\mathbf{B}_1\tilde{\mathbf{U}}_1 = \mathbf{A}\mathbf{U}_0 + \begin{bmatrix} 0 & \frac{1}{2}k_1\bar{A} \\ 0 & \frac{1}{2}k_1\bar{A} \end{bmatrix}\begin{bmatrix} 0 \\ f(U_0^-) \end{bmatrix},
$$

$$
\mathbf{B}_1\mathbf{U}_1 = \mathbf{A}\mathbf{U}_0 + \begin{bmatrix} \frac{1}{3}k_1\bar{A} & \frac{1}{6}k_1\bar{A} \\ \frac{1}{6}k_1\bar{A} & \frac{1}{3}k_1\bar{A} \end{bmatrix}\begin{bmatrix} f(U_0^-) \\ f(\tilde{U}_1^-) \end{bmatrix},
$$

$$
\mathbf{B}_2\mathbf{U}_2 = \mathbf{A}\mathbf{U}_1 + \begin{bmatrix} -\frac{k_2^2}{6k_1}\bar{A} & \frac{k_2}{k_1}\left(\frac{k_2}{6} + \frac{k_1}{2}\right)\bar{A} \\ -\frac{k_2^2}{3k_1}\bar{A} & \frac{k_2}{k_1}\left(\frac{k_2}{3} + \frac{k_1}{2}\right)\bar{A} \end{bmatrix}\begin{bmatrix} f(U_0^-) \\ f(U_1^-) \end{bmatrix},
$$

and finally, in general,

$$
\begin{aligned}
\mathbf{B}_n\mathbf{U}_n = \mathbf{A}\mathbf{U}_{n-1}
+ \Bigg[& \begin{matrix} \frac{k_n^2}{k_{n-2}(k_{n-2}+k_{n-1})}\left(\frac{k_n}{12} + \frac{k_{n-1}}{6}\right)\bar{A} & -\frac{k_n^2}{k_{n-2}k_{n-1}}\left(\frac{k_n}{12} + \frac{k_{n-1}}{6} + \frac{k_{n-2}}{6}\right)\bar{A} \\ \frac{k_n^2}{k_{n-2}(k_{n-2}+k_{n-1})}\left(\frac{k_n}{4} + \frac{k_{n-1}}{3}\right)\bar{A} & -\frac{k_n^2}{k_{n-2}k_{n-1}}\left(\frac{k_n}{4} + \frac{k_{n-1}}{3} + \frac{k_{n-2}}{3}\right)\bar{A} \end{matrix} \\
& \begin{matrix} \frac{k_n}{k_{n-1}(k_{n-2}+k_{n-1})}\left(\frac{k_n^2}{12} + \frac{k_nk_{n-1}}{3} + \frac{k_nk_{n-2}}{6} + \frac{k_{n-1}^2}{2} + \frac{k_{n-1}k_{n-2}}{2}\right)\bar{A} \\ \frac{k_n}{k_{n-1}(k_{n-2}+k_{n-1})}\left(\frac{k_n^2}{4} + \frac{2k_nk_{n-1}}{3} + \frac{k_nk_{n-2}}{3} + \frac{k_{n-1}^2}{2} + \frac{k_{n-1}k_{n-2}}{2}\right)\bar{A} \end{matrix} \Bigg]
\begin{bmatrix} f(U_{n-3}^-) \\ f(U_{n-2}^-) \\ f(U_{n-1}^-) \end{bmatrix}.
\end{aligned}
$$

Implementing these in code is now straightforward. There is one point that should be discussed, namely how to solve the associated systems of equations. In particular, the $q = 1$ formulas yield nonsymmetric linear systems which can be expensive to solve. (This difficulty is common in systems arising from implicit high-order Runge-Kutta schemes, which are related to these formulas.) Both B and A have a "principal" tridiagonal band as well as two other bands located $O(1/h)$ above and below the diagonal. Thus, for the $q=0$ scheme (backward Euler), using just this band structure without further rearrangement yields an operations count of $O(1/h^4)$ for solving the system once. In our case $\mathbf{B}_n$ has a block 2×2 band structure. Solving this system directly however leads to an operations count of $O(1/h^6)$ (as if the matrix were full) because of fill-in. Rearranging the matrix can alleviate this load. For example, it is straightforward to rearrange the matrix to achieve an operations count of $O(1/h^4)$ once again. First permute the rows of $\mathbf{B}_n$ into the order $1,\ m^2+1,\ 2,\ m^2+2,\ \ldots,\ m^2,\ 2m^2$ and then do the same by columns. The resulting matrix has a band of width 7 down the diagonal and bands of width 5 located $O(1/h)$ above and below the diagonal.

The following computation was made on a 35×35 mesh with initial data equal to 1 inside the disc $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2 \leq \frac{1}{16}$ and equal to 0 outside. The solution eventually becomes all 0 at a fairly slow rate. See Figure 2.

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FIGURE 2. The discontinuous Galerkin $q = 1$, $i = 0$ approximation for the equation $u_t - \Delta u = 10(u - u^3)$.

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