SPACE-TIME DISCRETIZATION OF AN INTEGRO-DIFFERENTIAL EQUATION MODELING QUASI-STATIC FRACTIONAL ORDER VISCOELASTICITY

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ABSTRACT. We study a quasi-static model for viscoelastic materials based on a constitutive equation of fractional order. In the quasi-static case this results in a Volterra integral equation of the second kind with a weakly singular kernel in the time variable involving also partial derivatives of second order in the spatial variables. We discretize by means of a discontinuous Galerkin finite element method in time and a standard continuous Galerkin finite element method in space. To overcome the problem of the growing amount of data that has to be stored and used in each time step, we introduce sparse quadrature in the convolution integral. We prove a priori and a posteriori error estimates, which can be used as the basis for an adaptive strategy.

1. INTRODUCTION

The fractional order viscoelastic model, i.e., the linear viscoelastic model with fractional order operators in the constitutive equations, is capable of describing the behavior of many viscoelastic materials by using only a few parameters. The drawback of using fractional order operators in the constitutive equations is that they increase the mathematical complexity in the sense that the operators are nonlocal in time. This means that, when computing the fractional order derivative or integral, all function values from the previous time points need to be stored and used at each new time point. This leads to to an excessive use of memory and high computational cost. To make the fractional order models more practical to use in the analysis of complex viscoelastic structures, efficient algorithms that employ the discontinuous Galerkin method in time together with sparse quadratures have been developed in Adolfsson *et al.* (2003, 2004). Goal-oriented error estimates and adaptivity for the time integration are included in the algorithms.

It is important to be able to investigate the capability of the numerical model to produce simulations with high accuracy. For this reason estimates of the error due to discretization in both space and time, as well as adaptive strategies based on these estimations, need to be included. Our previous work emphasized the temporal discretization. Here we develop a space-time finite element formulation in the quasi-static case (i.e., inertia effects are neglected). The formulation includes error estimates and an adaptive strategy. We use a convolution integral formulation of the fractional order viscoelastic model. The convolution kernel is weakly singular and of Mittag-Leffler type. The resulting equation of motion is then a Volterrra

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integral equation of second kind with a weakly singular kernel in time and it involves partial derivatives of second order in space.

The present study is based on related results for the corresponding elastic problem in Johnson and Hansbo (1992). A priori error estimates for equations with smooth kernel were proved by Pani *et al.* (1992). Space-time discretization for fractional order viscoelasticity has also been studied by Shaw and Whiteman, see, e.g., Shaw and Whiteman (2004). The unique feature of the present work is the use of sparse quadrature.

2. Fractional order linear viscoelasticity

Let σ_{ij} and u_i denote the usual stress tensor and displacement vector and define the linear strain tensor:

$$\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

With the decompositions

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij},$$

we formulate the constitutive equations, Bagley and Torvik (1983),

$$\begin{aligned} s_{ij}(t) + \tau_1^{\alpha_1} D_t^{\alpha_1} s_{ij}(t) &= 2G_\infty e_{ij}(t) + 2G\tau_1^{\alpha_1} D_t^{\alpha_1} e_{ij}(t),\\ \sigma_{kk}(t) + \tau_2^{\alpha_2} D_t^{\alpha_2} \sigma_{kk}(t) &= 3K_\infty \epsilon_{kk}(t) + 3K\tau_2^{\alpha_2} D_t^{\alpha_2} \epsilon_{kk}(t),\end{aligned}$$

with initial conditions

$$s_{ij}(0+) = 2Ge_{ij}(0+), \quad \sigma_{kk}(0+) = 3K\epsilon_{kk}(0+),$$

meaning that the initial response follows Hooke's elastic law. Note that we have two relaxation times, $\tau_1, \tau_2 > 0$, and fractional orders of differentiation, $\alpha_1, \alpha_2 \in (0,1)$, where the fractional order derivative is defined by

$$D_t^\alpha f(t) = D_t D_t^{-(1-\alpha)} f(t) = D_t \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s)\, ds.$$

We solve for σ by means of Laplace transformation, Enelund and Olsson (1999):

$$\begin{aligned} s_{ij}(t) &= 2G\Big(e_{ij}(t) - \frac{G-G_\infty}{G}\int_0^t f_1(t-s) e_{ij}(s)\, ds\Big),\\ \sigma_{kk}(t) &= 3K\Big(\epsilon_{kk}(t) - \frac{K-K_\infty}{K}\int_0^t f_2(t-s)\epsilon_{kk}(s)\, ds\Big),\end{aligned}$$

where

$$f_i(t) = -\frac{d}{dt} E_{\alpha_i}\Big(-\Big(\frac{t}{\tau_i}\Big)^{\alpha_i}\Big)$$

and

$$E_\alpha(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(1+\alpha n)}$$

is the Mittag-Leffler function. We make the simplifying assumption (synchronous viscoelasticity):

$$\alpha = \alpha_1 = \alpha_2, \quad \tau = \tau_1 = \tau_2, \quad f = f_1 = f_2.$$

Then we may define a parameter γ, a kernel β, and the Lamé constants μ, λ,

$$\gamma = \frac{G - G_\infty}{G} = \frac{K - K_\infty}{K}, \quad \beta(t) = \gamma f(t), \quad \mu = G, \quad \lambda = K - \tfrac{2}{3}G,$$

and the constitutive equations become

$$\sigma_{ij}(t) = \Big(2\mu\epsilon_{ij}(t) + \lambda\epsilon_{kk}(t)\delta_{ij}\Big) - \int_0^t \beta(t-s)\Big(2\mu\epsilon_{ij}(s) + \lambda\epsilon_{kk}(s)\delta_{ij}\Big)\, ds.$$

Note that the viscoelastic part of the model contains only three parameters:

$$0 < \gamma < 1, \quad 0 < \alpha < 1, \quad \tau > 0.$$

The kernel is weakly singular:

$$\beta(t) = -\gamma \frac{d}{dt} E_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) = \gamma\frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{-1+\alpha} E'_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) \approx Ct^{-1+\alpha}, \quad t \to 0, \tag{1}$$

and we note the properties

$$\begin{aligned} &\beta(t) \geq 0, \\ &\|\beta\|_{L_1(\mathbf{R}^+)} = \int_0^\infty \beta(t)\, dt = \gamma\Big(E_\alpha(0) - E_\alpha(\infty)\Big) = \gamma < 1. \end{aligned} \tag{2}$$

The equations of motion now become:

$$\begin{aligned} &\rho u_{i,tt} - \sigma_{ij,j} = f_i, && \text{in } \Omega, \\ &u_i = 0, && \text{on } \Gamma_{\mathrm{D}}, \\ &\sigma_{ij} n_j = g_i, && \text{on } \Gamma_{\mathrm{N}}. \end{aligned} \tag{3}$$

We consider quasi-static motion, $\rho u_{i,tt} \approx 0$, in a domain $\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$. In some of the analysis below we consider only the displacement boundary condition, $\Gamma = \Gamma_{\mathrm{D}}$.

3. Abstract formulation

We introduce the L_2 norm and scalar product:

$$\|v\| = \Big(\int_\Omega v_i v_i\, dx\Big)^{1/2}, \quad (f, v) = \int_\Omega f_i v_i\, dx,$$

and the function space:

$$V = \Big[H_0^1(\Omega)\Big]^3,$$

and a bilinear form on V:

$$a(u, v) = \int_\Omega \Big(2\mu\epsilon_{ij}(u)\epsilon_{ij}(v) + \lambda\epsilon_{ii}(u)\epsilon_{jj}(v)\Big)\, dx.$$

Recalling the constitutive equations we obtain the following weak formulation of the quasi-static equations of motion: find $u(t) \in V$ such that

$$a(u(t),v) = \int_0^t \beta(t-s)a(u(s),v)\,ds + (f(t),v) \quad \forall v \in V. \tag{4}$$

This corresponds to the strong formulation:

$$Au(t) = \int_0^t \beta(t-s)Au(s)\,ds + f(t), \tag{5}$$

where

$$(Au)_i = -(2\mu\epsilon_{ij}(u) + \lambda\epsilon_{kk}(u)\delta_{ij})_{,j}.$$

4. Regularity of solutions

In some of the analysis below we assume the regularity estimate

$$\|v\|_{H^2(\Omega)} \le C_S\|Av\| \quad \forall v \in H^2(\Omega)\cap V. \tag{6}$$

This holds if $\Gamma = \Gamma_{\mathrm{D}}$ in (3) and Ω is a convex polyhedron. In the presence of non-convex corners or mixed boundary conditions (i.e., Γ_{D} and Γ_{N} are both non-empty), this global regularity may not hold, see Remark 1 below. The presence of the constant C_S in the results below indicates where the assumption (6) is used.

Taking norms in (5) and recalling (2) we obtain

$$\begin{aligned}
\|Au(t)\| &\le \int_0^t \beta(t-s)\|Au(s)\|\,ds + \|f(t)\| \\
&\le \|\beta\|_{L_1(\mathbf{R}^+)}\|Au\|_{L_\infty(0,T;L_2)} + \|f\|_{L_\infty(0,T;L_2)} \\
&= \gamma\|Au\|_{L_\infty(0,T;L_2)} + \|f\|_{L_\infty(0,T;L_2)}.
\end{aligned}$$

Together with (6) this implies the following spatial regularity estimate for solutions of (5):

$$\|u\|_{L_\infty(0,T;H^2)} \le C_S\|Au\|_{L_\infty(0,T;L_2)} \le \frac{C_S}{1-\gamma}\|f\|_{L_\infty(0,T;L_2)}. \tag{7}$$

In order to investigate the temporal regularity we differentiate the equation (5) with respect to t:

$$\begin{aligned}
Au_t(t) &= D_t\int_0^t \beta(s)Au_t(t-s)\,ds + f_t(t) \\
&= \int_0^t \beta(t-s)Au_t(s)\,ds + \beta(t)Au(0) + f_t(t) \\
&= \int_0^t \beta(t-s)Au_t(s)\,ds + \beta(t)f(0) + f_t(t).
\end{aligned}$$

Taking norms and recalling (1) we obtain

$$\begin{aligned}\|u_t(t)\| &\leq \int_0^t \beta(t-s)\|u_t(s)\|\,ds + \beta(t)\|A^{-1}f(0)\| + \|A^{-1}f_t(t)\| \\ &\leq C\int_0^t (t-s)^{-1+\alpha}\|u_t(s)\|\,ds + Ct^{-1+\alpha}\|A^{-1}f(0)\| + \|A^{-1}f_t(t)\| \\ &\leq C\int_0^t (t-s)^{-1+\alpha}\|u_t(s)\|\,ds + C(T)t^{-1+\alpha}\|A^{-1}f\|_{W^1_\infty(0,T;L_2)},\ 0<t\leq T.\end{aligned}$$

A generalized Gronwall lemma (allowing a weak singularity $t^{-1+\alpha}$) then implies

$$\|u_t(t)\| \leq C(T)t^{-1+\alpha}\|A^{-1}f\|_{W^1_\infty(0,T;L_2)}, \quad 0<t\leq T. \tag{8}$$

The constant $C(T)$ grows exponentially with T so this inequality is only useful over short time intervals.

5. Spatial approximation

We introduce a standard finite element space $V_h \subset V$ consisting of continuous piecewise linear functions on a triangulation $\{K\}$ of Ω. We define the mesh function h as the piecewise constant function given by $h(x) = h_K$ for $x \in K$, and we define $h_{\max} = \max h_K$.

The spatially semidiscrete finite element problem is: find $u_h(t) \in V_h$ such that

$$a(u_h(t), v_h) = \int_0^t \beta(t-s)a(u_h(s), v_h)\,ds + (f(t), v_h) \quad \forall v_h \in V_h. \tag{9}$$

Defining $A_h : V_h \to V_h$ by

$$(A_h w_h, v_h) = a(w_h, v_h) \quad \forall w_h, v_h \in V_h, \tag{10}$$

and the orthogonal projection $P_h : L_2(\Omega) \to V_h$ by

$$(P_h g, v_h) = (g, v_h) \quad \forall g \in L_2(\Omega), v_h \in V_h, \tag{11}$$

we may write the equation in strong form as

$$A_h u_h(t) = \int_0^t \beta(t-s)A_h u_h(s)\,ds + P_h f(t). \tag{12}$$

We begin by proving an a priori error estimate in the energy norm, defined by $\|v\|_V = \sqrt{a(v,v)}$. We use the Ritz projection $R_h : V \to V_h$ defined by

$$a(R_h v - v, v_h) = 0, \quad \forall v_h \in V_h. \tag{13}$$

This means that $R_h v$ is the finite element solution of the stationary elastic problem who's exact solution is v. We recall error estimates for R_h. In the energy norm we have

$$\|R_h v - v\|_V \leq C\|hD^2 v\| \leq Ch_{\max}\|v\|_{H^2}. \tag{14}$$

By the standard duality argument we also have

$$\|R_h v - v\| \leq CC_S h_{\max}^2\|v\|_{H^2}. \tag{15}$$

Theorem 1. *Let u and u_h denote the solutions of* (4) *and* (9)*, respectively, and let $e(t) = u_h(t) - u(t)$ denote the error. Then*

$$\|e\|_{L_\infty(0,T;V)} \le \frac{1+\gamma}{1-\gamma} \|R_h u - u\|_{L_\infty(0,T;V)} \tag{16}$$

$$\le C \frac{1+\gamma}{1-\gamma} \|hD^2 u\|_{L_\infty(0,T;L_2)} \tag{17}$$

$$\le C C_S \frac{1+\gamma}{(1-\gamma)^2} h_{\max} \|f\|_{L_\infty(0,T;L_2)}. \tag{18}$$

Proof. Since $V_h \subset V$ we may take $v = v_h \in V_h$ in (4) and subtract it from (9) to get

$$a(e(t), v_h) - \int_0^t \beta(t-s) a(e(s), v_h)\, ds = 0 \quad \forall v_h \in V_h. \tag{19}$$

Hence, with $\rho(t) = R_h u(t) - u(t)$,

$$\begin{aligned}
\|e(t)\|_V^2 &= a(e(t), e(t)) - \int_0^t \beta(t-s) a(e(s), e(t))\, ds + \int_0^t \beta(t-s) a(e(s), e(t))\, ds \\
&= a(e(t), \rho(t)) - \int_0^t \beta(t-s) a(e(s), \rho(t))\, ds + \int_0^t \beta(t-s) a(e(s), e(t))\, ds \\
&\le \Big(1 + \|\beta\|_{L_1}\Big) \|e\|_{L_\infty(0,T;V)} \|\rho\|_{L_\infty(0,T;V)} + \|\beta\|_{L_1} \|e\|^2_{L_\infty(0,T;V)} \\
&\le (1+\gamma) \|e\|_{L_\infty(0,T;V)} \|\rho\|_{L_\infty(0,T;V)} + \gamma \|e\|^2_{L_\infty(0,T;V)},
\end{aligned}$$

which proves (16). Combining this with (14) we then obtain (17) and finally (7) proves (18). ☐

Remark 1. The error estimate (17) means

$$\|e\|_{L_\infty(0,T;V)} \le C \frac{1+\gamma}{1-\gamma} \sup_{t\in[0,T]} \Big(\sum_K \big(h_K \|D^2 u\|_{L_2(K)} \big)^2 \Big)^{1/2}.$$

The regularity estimate (6) is only used to prove (18). So if (6) does not hold and u is not globally in H^2, then we may replace (17) by

$$\|e\|_{L_\infty(0,T;V)} \le C \frac{1+\gamma}{1-\gamma} \sup_{t\in[0,T]} \Big(\sum_K \big(\min\{ \|Du\|_{L_2(K)}, h_K \|D^2 u\|_{L_2(K)} \} \big)^2 \Big)^{1/2}.$$

Here we may use $\|Du\|_{L_2(K)}$ for elements K where $u \notin H^2(K)$.

The next result is an a priori error estimate in the L_2-norm.

Theorem 2. *Let u and u_h denote the solutions of* (4) *and* (9)*, respectively, and let $e(t) = u_h(t) - u(t)$ denote the error. Then*

$$\|e\|_{L_\infty(0,T;L_2)} \le \frac{1+\gamma}{1-\gamma}\|R_h u - u\|_{L_\infty(0,T;L_2)} \tag{20}$$

$$\le CC_S\frac{1+\gamma}{1-\gamma}h_{\max}^2\|u\|_{L_\infty(0,T;H^2)} \tag{21}$$

$$\le CC_S^2\frac{1+\gamma}{(1-\gamma)^2}h_{\max}^2\|f\|_{L_\infty(0,T;L_2)}. \tag{22}$$

Proof. We use a duality argument based on the stationary adjoint elastic problem with arbitrary data g: find $\psi \in V$ such that

$$a(w,\psi) = (w,g), \quad \forall w \in V. \tag{23}$$

Its strong form is $A\psi = g$. We write $\eta = \psi - R_h\psi$, $\rho(t) = R_h u(t) - u(t)$, and use (13) and (23) to get

$$a(e,\eta) = a(\rho,\psi) = (\rho, A\psi) = (\rho, g), \quad a(e,\psi) = (e, A\psi) = (e,g).$$

Taking $w = e(t)$ in (23) and using (19), then leads to

$$\begin{aligned}
(e(t),g) &= a(e(t),\psi) - \int_0^t \beta(t-s)a(e(s),\psi)\,ds + \int_0^t \beta(t-s)a(e(s),\psi)\,ds \\
&= a(e(t),\eta) - \int_0^t \beta(t-s)a(e(s),\eta)\,ds + \int_0^t \beta(t-s)a(e(s),\psi)\,ds \\
&= (\rho(t),g) - \int_0^t \beta(t-s)(\rho(s),g)\,ds + \int_0^t \beta(t-s)(e(s),g)\,ds \\
&\le \Big(1 + \|\beta\|_{L_1}\Big)\|\rho\|_{L_\infty(0,T;L_2)}\|g\| + \|\beta\|_{L_1}\|e\|_{L_\infty(0,T;L_2)}\|g\| \\
&\le (1+\gamma)\|\rho\|_{L_\infty(0,T;L_2)}\|g\| + \gamma\|e\|_{L_\infty(0,T;L_2)}\|g\|.
\end{aligned}$$

With $g = e(t)$ we conclude (20). Combination with (15) and (7) then proves (21) and (22). ☐

We next turn to a posteriori error estimates. We introduce the residual:

$$\langle \mathcal{R}(t), v\rangle = a(u_h(t),v) - \int_0^t \beta(t-s)a(u_h(s),v)\,ds - (f(t),v) \quad \forall v \in V, \tag{24}$$

and note that (19) means

$$\langle \mathcal{R}(t), v_h\rangle = 0 \quad \forall v_h \in V_h. \tag{25}$$

Combining (4) and (24) we obtain an equation for the error: $e(t) \in V$ satisfies

$$a(e(t),v) - \int_0^t \beta(t-s)a(e(s),v)\,ds = \langle \mathcal{R}(t), v\rangle \quad \forall v \in V. \tag{26}$$

We prove

Theorem 3. *Let u and u_h denote the solutions of* (4) *and* (9), *respectively, and let $e(t) = u_h(t) - u(t)$ denote the error. Then*

$$\|e\|_{L_\infty(0,T;V)} \leq \frac{C}{1-\gamma}\|hR\|_{L_\infty(0,T;L_2)},$$

where the computational residual is divided into three parts:

$$R = R_1 + R_2 + R_3,$$

which are defined piecewise with respect to the mesh $\{K\}$, i.e., for $x \in K$ they are defined by

$$\begin{aligned} R_1(x,t) &= -\nabla \cdot \sigma(u_h)(x,t) + \int_0^t \beta(t-s)\nabla \cdot \sigma(u_h)(x,s)\,ds - f(x,t), \\ R_2(x,t) &= \tfrac{1}{2}h^{-1/2}|K|^{-1/2}\|[\sigma(u_h)(\cdot,t)\cdot n]\|_{L_2(\partial K)}, \\ R_3(x,t) &= \tfrac{1}{2}h^{-1/2}|K|^{-1/2}\int_0^t \beta(t-s)\|[\sigma(u_h)(\cdot,s)\cdot n]\|_{L_2(\partial K)}\,ds. \end{aligned}$$

Here

$$\sigma(u_h) = 2\mu\epsilon(u_h) + \lambda\nabla \cdot u_h I.$$

and $[\sigma(u_h)\cdot n]$ is the jump in the normal stress over the element edge ∂K.

Proof. Since $\mathcal{R}(t)$ satisfies the orthogonality (25), we may use the results of Johnson and Hansbo (1992) to prove the following estimate of the residual:

$$\langle \mathcal{R}(t), v\rangle \leq C\|hR\|_{L_\infty(0,T;L_2)}\|v\|_V \quad \forall v \in V.$$

We then use $v = e(t)$ in (26):

$$\begin{aligned} \|e(t)\|_V^2 &= \langle \mathcal{R}(t), e(t)\rangle + \int_0^t \beta(t-s)a(e(s),e(t))\,ds \\ &\leq C\|hR\|_{L_\infty(0,T;L_2)}\|e\|_{L_\infty(0,T;V)} + \|\beta\|_{L_1}\|e\|^2_{L_\infty(0,T;V)} \\ &\leq C\|hR\|_{L_\infty(0,T;L_2)}\|e\|_{L_\infty(0,T;V)} + \gamma\|e\|^2_{L_\infty(0,T;V)}, \end{aligned}$$

which leads to

$$\|e(t)\|_V \leq C\|hR\|_{L_\infty(0,T;L_2)} + \gamma\|e\|_{L_\infty(0,T;V)},$$

and the desired result follows. □

We also have an estimate in the L_2-norm.

Theorem 4. *Let u and u_h denote the solutions of* (4) *and* (9), *respectively, and let $e(t) = u_h(t) - u(t)$ denote the error. Then, with R as in Theorem 3,*

$$\|e\|_{L_\infty(0,T;L_2)} \leq \frac{CC_S}{1-\gamma}\|h^2R\|_{L_\infty(0,T;L_2)}.$$

Proof. Since $\mathcal{R}(t)$ satisfies the orthogonality (25), we may use the results of Johnson and Hansbo (1992) to prove the following estimate of the residual:

$$\langle \mathcal{R}(t), v\rangle \leq C\|h^2R\|_{L_\infty(0,T;L_2)}\|v\|_{H^2} \quad \forall v \in V \cap H^2(\Omega).$$

We then use $w = e(t)$ in (23) and $v = \psi$ in (26):

$$\begin{aligned}(e(t), g) = a(e(t), \psi) &= \langle \mathcal{R}(t), \psi \rangle + \int_0^t \beta(t-s) a(e(s), \psi)\, ds \\ &= \langle \mathcal{R}(t), \psi - R_h \psi \rangle + \int_0^t \beta(t-s)(e(s), g)\, ds \\ &\le C \|h^2 R\|_{L_\infty(0,T;L_2)} \|\psi\|_{L_\infty(0,T;H^2)} + \|\beta\|_{L_1} \|e\|_{L_\infty(0,T;L_2)} \|g\| \\ &\le C C_S \|h^2 R\|_{L_\infty(0,T;L_2)} \|g\| + \gamma \|e\|_{L_\infty(0,T;L_2)} \|g\|.\end{aligned}$$

Here we also applied the regularity estimate (6) to ψ, recalling that $A\psi = g$. With $g = e(t)$ we obtain the desired result. □

6. Temporal discretization – discontinuous Galerkin

We introduce a temporal mesh, $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$, with intervals $I_n = (t_{n-1}, t_n)$ and steps $k_n = t_n - t_{n-1}$, and discrete function space:

$$\mathcal{W}_D = \Big\{ w : w(t) = w_n \text{ for } t \in I_n,\ w_n \in V_h,\ n = 1, \ldots, N \Big\}.$$

The completely discrete finite element problem is: find $U \in \mathcal{W}_D$, such that for $n = 1, \ldots, N$

$$\int_{I_n} \Big(a(U(t), v(t)) - \int_0^t \beta(t-s) a(U(s), v(t))\, ds - (f(t), v(t)) \Big)\, dt = 0 \quad \forall v \in \mathcal{W}_D. \tag{27}$$

Writing $U_n = U|_{I_n} \in V_h$, $v|_{I_n} = \chi \in V_h$, and recalling A_h and P_h from (10) and (11), we note that this is a time-stepping method, where in each step we solve the equation, cf. (12),

$$A_h U_n - q_n(A_h U) - P_h \bar{f}_n = 0,$$

with

$$\begin{aligned}\bar{f}_n &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(t)\, dt, \\ q_n(A_h U) &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_0^t \beta(t-s) A_h U(s)\, ds\, dt \\ &= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \sum_{j=1}^n \int_{t_{j-1}}^{t_j \wedge t} \beta(t-s) A_h U_j\, ds\, dt \\ &= \sum_{j=1}^n k_j \omega_{nj} A_h U_j, \\ \omega_{nj} &= \frac{1}{k_n k_j} \int_{t_{n-1}}^{t_n} \int_{t_{j-1}}^{t_j \wedge t} \beta(t-s)\, ds\, dt, \quad t_j \wedge t = \min(t_j, t).\end{aligned}$$

Thus, in each step, we have to solve

$$(I - k_n\omega_{nn})A_hU_n = \sum_{j=1}^{n-1} k_j\omega_{nj}A_hU_j + P_h\bar{f}_n,$$

where according to (1), for k_n small,

$$\begin{aligned} k_n\omega_{nn} &= \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \beta(t-s)\,ds\,dt \\ &\approx \frac{\gamma}{(1+\alpha)\Gamma(1+\alpha)}\Big(\frac{k_n}{\tau}\Big)^{\alpha} < 1. \end{aligned}$$

Therefore the equation is solvable. Note that the right-hand side of the above equation is a convolution sum, which requires that the whole history is stored and which must be re-computed in each time step. This leads to an excessive use of memory and high computational cost. This can be alleviated by means of sparse quadrature as shown in Section 8 below.

7. Error estimates

We begin by proving an a priori error estimate.

Theorem 5. *Let u and U denote the solutions of* (4) *and* (27), *respectively, and let $e(t) = U(t) - u(t)$ denote the error. Then, with the piecewise constant function $\bar{u}$ defined by $\bar{u}(t) = \frac{1}{k_n}\int_{I_n} u(s)\,ds$ for $t \in I_n$,*

$$\|e\|_{L_\infty(0,T;L_2)} \leq \frac{1+\gamma}{1-\gamma}\|R_hu - u\|_{L_\infty(0,T;L_2)} + \frac{1}{1-\gamma}\|\bar{u} - u\|_{L_\infty(0,T;L_2)} \tag{28}$$

$$\leq CC_S\frac{1+\gamma}{1-\gamma}h_{\max}^2\|u\|_{L_\infty(0,T;H^2)} + \frac{1}{1-\gamma}\max_{1\leq n\leq N}\|u_t\|_{L_1(I_n;L_2)}. \tag{29}$$

Remark 2. The last term can estimated as follows

$$\max_{1\leq n\leq N}\|u_t\|_{L_1(I_n;L_2)} \leq \max_{1\leq n\leq N}\min\Big(\|u_t\|_{L_1(I_n;L_2)}, k_n\|u_t\|_{L_\infty(I_n;L_2)}\Big).$$

Hence this term converges as $O(k_n)$ except near $t = 0$, where according to (8)

$$\|u_t\|_{L_1(I_1;L_2)} \approx C\int_0^{k_1} t^{-1+\alpha}\,dt = Ck_1^{\alpha}.$$

Proof. We imitate the proof of Theorem 2. Taking $w = e(t)$ in (23) and using (27), leads to

$$\begin{aligned}
&\int_{I_n} (e(t), g)\,dt \\
&\quad = \int_{I_n} \Big(a(e(t),\psi) - \int_0^t \beta(t-s) a(e(s),\psi)\,ds + \int_0^t \beta(t-s) a(e(s),\psi)\,ds \Big)\,dt \\
&\quad = \int_{I_n} \Big(a(e(t),\eta) - \int_0^t \beta(t-s) a(e(s),\eta)\,ds + \int_0^t \beta(t-s) a(e(s),\psi)\,ds \Big)\,dt \\
&\quad = \int_{I_n} \Big((\rho(t),g) - \int_0^t \beta(t-s) (\rho(s),g)\,ds + \int_0^t \beta(t-s) (e(s),g)\,ds \Big)\,dt \\
&\quad \le k_n \Big(1 + \|\beta\|_{L_1}\Big) \|\rho\|_{L_\infty(0,T;L_2)} \|g\| + k_n \|\beta\|_{L_1} \|e\|_{L_\infty(0,T;L_2)} \|g\| \\
&\quad \le k_n (1+\gamma) \|\rho\|_{L_\infty(0,T;L_2)} \|g\| + k_n \gamma \|e\|_{L_\infty(0,T;L_2)} \|g\|.
\end{aligned}$$

With $\bar{e}_n = \frac{1}{k_n} \int_{I_n} e(t)\,dt$ this implies

$$(\bar{e}_n, g) \le \Big((1+\gamma) \|\rho\|_{L_\infty(0,T;L_2)} + \gamma \|e\|_{L_\infty(0,T;L_2)} \Big) \|g\|,$$

and hence, using $g = \bar{e}_n$,

$$\|\bar{e}_n\| \le (1+\gamma) \|\rho\|_{L_\infty(0,T;L_2)} + \gamma \|e\|_{L_\infty(0,T;L_2)}.$$

To complete the proof we note that, since $U(t) = U_n$ we have $U - \bar{u} = \bar{e}_n$, so that

$$\begin{aligned}
\|e\|_{L_\infty(0,T;L_2)} &\le \|U - \bar{u}\|_{L_\infty(0,T;L_2)} + \|\bar{u} - u\|_{L_\infty(0,T;L_2)} \\
&= \max_{1 \le n \le N} \|\bar{e}_n\| + \|\bar{u} - u\|_{L_\infty(0,T;L_2)} \\
&\le (1+\gamma) \|\rho\|_{L_\infty(0,T;L_2)} + \gamma \|e\|_{L_\infty(0,T;L_2)} + \|\bar{u} - u\|_{L_\infty(0,T;L_2)}.
\end{aligned}$$

This proves (28). Then (29) follows by (15) and the mean value theorem. ☐

We finally prove an a posteriori error estimate.

Theorem 6. *Let u and U denote the solutions of* (4) *and* (27)*, respectively, and let $e(t) = U(t) - u(t)$ denote the error. Then*

$$\|e\|_{L_\infty(0,T;L_2)} \le \frac{1}{1-\gamma} \Big(C C_S \|h^2 R\|_{L_\infty(0,T;L_2)} + C \|R_4\|_{L_\infty(0,T;L_2)} \Big),$$

where the spatial residual $R = R_1 + R_2 + R_3$ is as in Theorem 3 and the new temporal residual R_4 is defined in each mesh simplex K by

$$R_4(x,t) = (A_h)U(x,t) - \int_0^t \beta(t-s)(A_h U)(x,s)\,ds - (P_h f)(x,t).$$

Proof. The proof is based on a time-dependent adjoint problem: find $\phi(t) \in V$ such that

$$a(w,\phi(t)) - \int_t^T \beta(s-t) a(w,\phi(s))\,ds = (w, g(t)) \quad \forall w \in V. \tag{30}$$

Taking $w(t) \in V$ and integrating leads to

$$\int_0^T \Big(a(w(t), \phi(t)) - \int_0^t \beta(t-s) a(w(s), \phi(t))\, ds \Big)\, dt = \int_0^T (w(t), g(t))\, dt \qquad \forall w \in L_\infty(0,T;V).$$

Summing (27) from 1 to N gives

$$\int_0^T \Big(a(U(t), v(t)) - \int_0^t \beta(t-s) a(U(s), v(t))\, ds - (f(t), v(t)) \Big)\, dt = 0 \quad \forall v \in \mathcal{W}_D,$$

and integrating (4) from 0 to $T = t_N$ gives

$$\int_0^T \Big(a(u(t), v(t)) - \int_0^t \beta(t-s) a(u(s), v(t))\, ds - (f(t), v(t)) \Big)\, dt = 0 \qquad \forall v \in L_\infty(0,T;V).$$

Subtracting them we get

$$\int_0^T \Big(a(e(t), v(t)) - \int_0^t \beta(t-s) a(e(s), v(t))\, ds - (f(t), v(t)) \Big)\, dt = 0 \qquad \forall v \in \mathcal{W}_D. \tag{31}$$

We define the residual by

$$\begin{aligned} &\int_0^T \langle \mathcal{R}(t), v(t) \rangle\, dt \\ &\qquad = \int_0^T \Big(a(U(t), v(t)) - \int_0^t \beta(t-s) a(U(s), v(t))\, ds - (f(t), v(t)) \Big)\, dt \\ &\qquad\qquad \forall v \in L_\infty(0,T;V). \end{aligned}$$

We get the error equation

$$\int_0^T \Big(a(e(t), v(t)) - \int_0^t \beta(t-s) a(e(s), v(t))\, ds \Big)\, dt = \int_0^T \langle \mathcal{R}(t), v(t) \rangle\, dt \qquad \forall v \in L_\infty(0,T;V).$$

We now take $w = e$ in the adjoint equation and $v = \phi$ in the error equation to get, using $v = P_h \bar{\phi} \in \mathcal{W}_D$ in (31),

$$\begin{aligned} \int_0^T (e(t), g(t))\, dt &= \int_0^T \langle \mathcal{R}(t), \phi(t) \rangle\, dt \\ &= \int_0^T \langle \mathcal{R}(t), \phi(t) - P_h \bar{\phi}(t) \rangle\, dt \\ &= \int_0^T \langle \mathcal{R}(t), \phi(t) - P_h \phi(t) \rangle\, dt + \int_0^T \langle \mathcal{R}(t), P_h(\phi(t) - \bar{\phi}(t)) \rangle\, dt. \end{aligned}$$

Here the first term on the right is of the same type as the spatial residual in Theorem 3 and the second term is the new temporal residual:

$$
\begin{aligned}
\int_0^T (e(t),g(t))\,dt &= \int_0^T \langle \mathcal{R}(t),\phi(t)-P_h\phi(t)\rangle\,dt + \int_0^T (R_4(t),(\phi(t)-\bar\phi(t)))\,dt\\
&\le C\|h^2R\|_{L_\infty(0,T;L_2)}\|\phi\|_{L_1(0,T;H^2)} + 2\|R_4\|_{L_\infty(0,T;L_2)}\|\phi\|_{L_1(0,T;L_2)}\\
&\le \frac{CC_S}{1-\gamma}\|h^2R\|_{L_\infty(0,T;L_2)}\|g\|_{L_1(0,T;L_2)} + \frac{2}{1-\gamma}\|R_4\|_{L_\infty(0,T;L_2)}\|A^{-1}g\|_{L_1(0,T;L_2)}\\
&\le \Big(\frac{CC_S}{1-\gamma}\|h^2R\|_{L_\infty(0,T;L_2)} + \frac{C}{1-\gamma}\|R_4\|_{L_\infty(0,T;L_2)}\Big)\|g\|_{L_1(0,T;L_2)}.
\end{aligned}
$$

Here we used the fact that the adjoint equation is formally the same as (5) so that we have the regularity estimates

$$
\begin{aligned}
&\|\phi\|_{L_1(0,T;H^2)} \le \frac{C_S}{1-\gamma}\|g\|_{L_1(0,T;L_2)},\\
&\|\phi\|_{L_1(0,T;L_2)} \le \frac{1}{1-\gamma}\|A^{-1}g\|_{L_1(0,T;L_2)} \le \frac{C}{1-\gamma}\|g\|_{L_1(0,T;L_2)},
\end{aligned}
$$

which are proved in the same way as (7). We finish the proof by using that

$$
\|e\|_{L_\infty(0,T;L_2)} = \sup_g \frac{\int_0^T (e(t),g(t))\,dt}{\|g\|_{L_1(0,T;L_2)}}.
$$

□

8. Sparse quadrature

We briefly describe how the convolution integral can be computed by a sparse quadrature rule introduced by Sloan and Thomée (1986) and analyzed for pure temporal discretization of viscoelasticity in Adolfsson *et al.* (2004).

We introduce sparse time levels $0 = t_{M_0} < t_{M_1} < t_{M_2} < \cdots < t_{M_L}$ and replace the kernel $\beta(t-s)$ by a piecewise linear interpolant:

$$
\tilde\beta(t,s) = \begin{cases} \beta(t-t_{M_{l-1}})\phi_{1,l}(s) + \beta(t-t_{M_l})\phi_{2,l}(s), & s\in[t_{M_{l-1}},t_{M_l}],\\ & \quad l=1,\dots,L,\\ \beta(t-s), & s\in[t_{M_L},t], \end{cases}
$$

where

$$
\phi_{1,l}(s) = \frac{t_{M_l}-s}{K_l},\quad \phi_{2,l}(s) = \frac{s-t_{M_{l-1}}}{K_l},\quad K_l = t_{M_l}-t_{M_{l-1}},
$$

are piecewise linear basis functions. Due to the singularity of $\beta(t-s)$ at $s=t$ we take L to be the largest integer such that $t-t_{M_L}\ge\tau$, where τ is the relaxation time in Section 2. We showed in Adolfsson *et al.* (2004) that the quadrature error

$$
E_Q(t) = \int_0^t (\tilde\beta(t,s)-\beta(t-s))U(s)\,ds
$$

is bounded by

$$\|E_Q\|_{L_\infty(0,T;L_2)} \le \sum_{j=1}^{M_L} k_j e_{nj} \|U_j\|, \tag{32}$$

where

$$e_{nj} = \frac{1}{8}\|\beta''\|_{L_\infty(I_{nl})} K_l^2, \quad j = M_{l-1}, \dots, M_l,$$

and

$$I_{nl} = [t_{n-1} - t_{M_l}, t_n - t_{M_{l-1}}].$$

Hence, we may choose the large steps $K \approx \sqrt{k}$ and save on storage and computational work.

Theorem 7. *Let u be the solution of* (4) *and U the solution of* (27) *with β replaced by $\tilde\beta$, and let $e(t) = U(t) - u(t)$ denote the error. Then*

$$\|e\|_{L_\infty(0,T;L_2)} \le \frac{1}{1-\gamma}\Big(CC_S\|h^2\tilde R\|_{L_\infty(0,T;L_2)} + C\|\tilde R_4\|_{L_\infty(0,T;L_2)} + \|E_Q\|_{L_\infty(0,T;L_2)}\Big),$$

where the spatial residual $\tilde R = \tilde R_1 + \tilde R_2 + \tilde R_3$ and the temporal residual $\tilde R_4$ are as in Theorem 3 and Theorem 6 but with β replaced by $\tilde\beta$ and E_Q is bounded as in (32).

Proof. We modify the proof of Theorem 6:

$$\begin{aligned}
\int_0^T (e(t), g(t))\,dt &= \int_0^T \langle \mathcal{R}(t), \phi(t)\rangle\,dt \\
&= \int_0^T \Big(a(U(t),\phi(t)) - \int_0^t \beta(t-s) a(U(s),\phi(t))\,ds - (f(t),\phi(t))\Big)\,dt \\
&= \int_0^T \Big(a(U(t),\phi(t)) - \int_0^t \tilde\beta(t,s) a(U(s),\phi(t))\,ds - (f(t),\phi(t)) \\
&\quad + \int_0^t (\tilde\beta(t,s) - \beta(t-s)) a(U(s),\phi(t))\,ds\Big)\,dt \\
&= \int_0^T \langle \tilde{\mathcal{R}}(t), \phi(t)\rangle\,dt + \int_0^T (E_Q(t), A\phi(t))\,dt.
\end{aligned}$$

The first term is handled as in Theorem 6. The additional term is estimated as

$$\begin{aligned}
\int_0^T (E_Q(t), A\phi(t))\,dt &\le \|E_Q\|_{L_\infty(0,T;L_2)} \|A\phi\|_{L_1(0,T;L_2)} \\
&\le \|E_Q\|_{L_\infty(0,T;L_2)} \frac{1}{1-\gamma} \|g\|_{L_1(0,T;L_2)}.
\end{aligned}$$

□

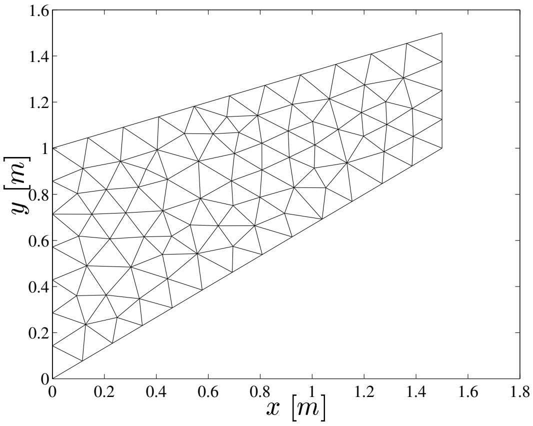

FIG. 1. Undeformed mesh.

Note that use of $\tilde{\beta}$ in the residuals $\tilde{R}_i$ means that they can be computed in a sparse way. The a posteriori error estimate of Theorem 7 can be used as a basis for an adaptive algorithm as described in Johnson and Hansbo (1992) and Adolfsson *et al.* (2004).

9. Numerical experiment

We illustrate the theory by a numerical experiment: Cooke's membrane in two dimensions, see Figure 1. We use the boundary conditions: $u = (0, 0)$ at $x = 0$, $g = (0, -1)$ at $x = 1.5$, and $g = (0, 0)$ on the remaining boundaries, cf. (3). We use the model parameters: $\gamma = 0.5$, $\tau = 0.5$, $\alpha = 0.5$. The deformed mesh at $t/\tau = 20$ is displayed in Figure 2 with the displacement magnified by the factor 10^5. The time evolution of the node displacement at the point (1.5,1.5) is shown in Figure 3.

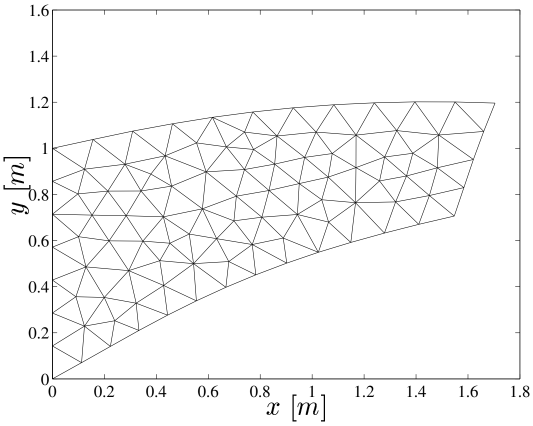

FIG. 2. Deformed mesh.

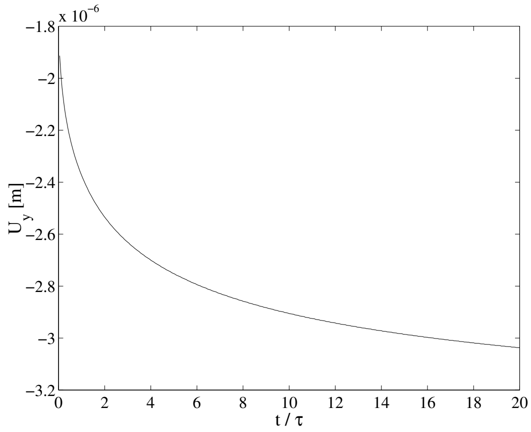

FIG. 3. Time evolution.

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