THE CAHN-HILLIARD EQUATION

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Pattern formation and the Cahn-Hilliard equation

1. Steady states. There are many two component systems in which phase separation can be induced by rapidly cooling the system. Thus, if a two component system, which is spatially uniform at temperature T_1, is rapidly cooled to a second sufficiently lower temperature T_2, then the cooled system will separate into regions of higher and lower concentration. A phenomenological description of the behavior of such systems can be obtained by energy arguments. The claim would be that there exists a critical temperature T_c, such that for $T > T_c$ the free energy $F(c, T)$ of the system is a single welled function of the concentration c of one of the species, whereas for $T < T_c$ the free energy is double welled. Referring to Figure 1, a system which was spatially uniform at temperature T_1, when cooled to temperature T_2, would find it energetically preferrable to separate itself into two systems, one at concentration c_A and one at concentration c_B.

To be more specific consider now the system at temperature $T < T_c$. Assume that the *free energy* $F(c)$ per unit volume (Gibbs free energy or Landau-Ginzburg free energy) of the spatially homogeneous system has the convex/concave shape indicated in Figure 2. More precisely F is concave in the *spinodal interval* $c_A^s < c < c_B^s$ and convex elsewhere. The points c_A and c_B where the supporting tangent touches the graph are sometimes referred to as the *binodal points*. The derivative $f(c) = F'(c)$ is depicted in Figure 3.

The free energy of a spatially heterogeneous system would then be given by

$$\mathcal{F}(c) = \int_\Omega F(c(x))\,dx, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^3$ is the spatial domain in which the system is confined.

The steady states of this system are obtained as the minimizers of the free energy functional $\mathcal{F}$ under the constraint of prescribed mass

$$\frac{1}{|\Omega|}\int_\Omega c(x)\,dx = r. \tag{1.2}$$

Here the average concentration r is a given real number. This problem is easily solved with the aid of the auxiliary functional

$$\tilde{\mathcal{F}}(c) = \int_\Omega F(c(x))\,dx - \sigma\int_\Omega c(x)\,dx, \tag{1.3}$$

where σ is a Lagrange multiplier. Any critical point of $\tilde{\mathcal{F}}$ must satisfy the Euler-Lagrange equation

$$f(c) = \sigma$$

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at points of continuity of c. It follows that solutions are either constant (single phase)

$$c(x) = r, \qquad x \in \Omega,$$

or piecewise constant (two phase)

$$c(x) = \begin{cases} c_A, & x \in \Omega_A, \\ c_B, & x \in \Omega_B = \Omega \setminus \Omega_A, \end{cases}$$

where $c_A,\ c_B$ and σ are determined by

$$\begin{gathered} F(c_B) - F(c_A) = \sigma(c_B - c_A), \\ \sigma = f(c_A) = f(c_B). \end{gathered} \tag{1.4}$$

Indeed, with σ equal to the slope of the supporting tangent of the graph of F (see Figure 2), the function $\tilde{F}(c) = F(c) - \sigma c$ has the form shown in Figure 4 and any piecewise constant function with values at the minima of $\tilde{F}$ will be a global minimizer of $\tilde{\mathcal{F}}$.

The conditions (1.4) can also be phrased as *Maxwell's rule*: The coexistence concentrations c_A and c_B and the multiplier σ are determined by the condition that the shaded regions in Figure 3 have equal areas.

Applying (1.2) we find that

$$|\Omega_A| = \frac{c_B - r}{c_B - c_A}|\Omega|, \quad |\Omega_B| = \frac{r - c_A}{c_B - c_A}|\Omega|,$$

and it follows that a two phase solution is admissible only if $c_A < r < c_B$.

It is clear from the convex/concave nature of F that uniform states $c(x) = r$ are global minimizers if $r \le c_A$ and if $r \ge c_B$, they are local maxima in the spinodal region $c_A^s \le r \le c_B^s$, and they are local but not global minimizers in the remaining intervals.

Thus, if a system is prepared with an average concentration r in the spinodal region, then it will eventually end up in a final state with separated phases. But there are infinitely many such states. Are any of these two-phase solutions—in some physical sense—preferred? Experiments show that cooled systems first separate themselves rapidly into some configuration of alternately high and low concentration (spinodal decomposition). Then, slowly, the system sorts itself out, and larger and larger regions become dominated by a single phase. This points to the importance of interfacial energy.

One way to take into account the energy of the interfaces between phases is to include a gradient energy contribution in the definition of the free energy:

$$\mathcal{F}(c) = \int_\Omega \left(F(c) + \frac{1}{2}\kappa|\nabla c|^2 \right) dx. \tag{1.5}$$

Here $F(c)$ corresponds to the homogeneous free energy as above and κ is a positive coefficient. Note that the additional gradient energy term penalizes the formation of phase interfaces.

Steady states are obtained by minimizing $\mathcal{F}$ over $H^1(\Omega)$ subject to the constraint (1.2). The Euler-Lagrange equation now becomes a semilinear Neumann problem:

$$\begin{gathered} -\kappa\Delta c + f(c) = \sigma \qquad \text{in } \Omega, \\ \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega, \\ \frac{1}{|\Omega|}\int_\Omega c(x)\, dx = r, \end{gathered} \tag{1.6}$$

where again σ is a Lagrange multiplier.

This problem was studied in the one-dimensional case $\Omega = (-L, L)$ by Carr, Gurtin and Slemrod [6], who showed that for $c_A < r < c_B$

(1) when $\kappa > 0$ is small enough $\mathcal{F}$ has a unique (modulo reversal, i.e. $c(x) \to c(-x)$) global minimizer $c_\kappa(x)$;
(2) $c_\kappa(x)$ is strictly monotone;
(3) as $\kappa \to 0$, $c_\kappa(x)$ (or its reversal) approaches the single-interface solution

$$c_0(x) = \begin{cases} c_A, & -L < x < -L + l_A \\ c_B, & -L + l_A < x < L \end{cases};$$

(4) nonmonotonic solutions to (1.6) for any $\kappa > 0$ are unstable in the sense that they cannot even be local minimizers of $\mathcal{F}$.

The single-interface solutions are therefore preferred in the sense that they represent limits, as $\kappa \to 0$, of solutions within a more general theory.

2. Dynamics. The dynamics of phase separation can be deduced from the above energy minimization principle, if one assumes that the mass flux is proportional to the gradient of the chemical potential,

$$j = -M\nabla J,$$

where $M > 0$ is the mobility. Setting, for simplicity, $M = 1$ and defining the chemical potential as the functional derivative of $\mathcal{F}$,

$$\int_\Omega J(x)v(x)\,dx = \mathcal{F}'(c)v \qquad \text{for all } v \in H^1(\Omega),$$

i.e., $J(c) = f(c)$, conservation of mass now leads to the differential equation

$$c_t = \Delta f(c) \qquad \text{in } \Omega, \tag{2.1}$$

together with a no flux boundary condition

$$\frac{\partial}{\partial n} f(c) = 0 \qquad \text{on } \partial\Omega.$$

(Note that this reduces to the usual diffusion equation when $f(c) = c$.) Here $\Delta f(c) = f'(c)\Delta c + f''(c)|\nabla c|^2$. Referring again to Figure 3 we note that $f'(c)$ is negative in the spinodal region. Equation (2.1) is therefore ill-posed—it behaves as a backwards/forwards heat equation.

Including the gradient energy term in $\mathcal{F}$ we have instead $J(c) = f(c) - \kappa\Delta c$, where we have applied the natural boundary condition from the free energy functional. This leads to the *Cahn-Hilliard equation*

$$c_t = \Delta\left(f(c) - \kappa\Delta c\right) \qquad \text{in } \Omega \tag{2.2}$$

with the boundary conditions

$$\frac{\partial}{\partial n} c = 0, \ \frac{\partial}{\partial n}\left(f(c) - \kappa\Delta c\right) = 0 \qquad \text{on } \partial\Omega. \tag{2.3}$$

Now the leading term is the fourth order elliptic operator $\kappa\Delta^2$. Thus, mathematically speaking, the inclusion of the gradient energy in $\mathcal{F}$ serves to regularize the ill-posed equation (2.1).

3. A normalization. We shall now derive a normalized version of the Cahn-Hilliard equation (2.2)-(2.3). Taking F to be a general quartic polynomial, its derivative is

$$f(c) = b_0 + b_1 c + b_2 c^2 + b_3 c^3,$$

or with $u = c - c^*$

$$f(c) = f(c^*) + f'(c^*)u + \frac{1}{2} f''(c^*)u^2 + \frac{1}{3} f'''(c^*)u^3.$$

Choosing $c^* = -b_2/3b_3$, so that $f''(c^*) = 0$, we have $f'(c^*) = b_1 - b_2^2/3b_3$, $f'''(c^*) = 3b_3$. Thus, if $b_3 > 0$ and $b_1 < b_2^2/3b_3$ (we also need $b_2 < 0$ to guarantee that the interesting stuff occurs for $c > 0$), then f can be written in the normalized form

$$f(c) = f(c^*) - \alpha u + \beta u^3$$

with $\alpha, \beta > 0$, which conforms with Figure 3. Taking, for simplicity, $\alpha = \beta = \kappa = 1$, the substitution $u = c - c^*$ thus yields

$$\begin{aligned} u_t &= \Delta\left(-\Delta u + u^3 - u\right) \qquad \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} &= 0,\ \frac{\partial}{\partial n}\left(-\Delta u + u^3 - u\right) = 0 \qquad \text{for } x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) &= u_0 \qquad \text{in } \Omega, \end{aligned} \tag{3.1}$$

where we have added an initial condition. In compact notation we also write

$$\begin{aligned} u_t &= \Delta J(u) \qquad \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} &= 0,\ \frac{\partial J(u)}{\partial n} = 0 \qquad \text{for } x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) &= u_0 \qquad \text{in } \Omega, \end{aligned} \tag{3.2}$$

where $J(u) = -\Delta u + u^3 - u$.

Remark 1. Retaining a concentration dependent mobility $M = M(u) > 0$ the derivation in Section 2 leads to

$$\begin{aligned} u_t &= \nabla \cdot (M(u)\nabla J(u)) \qquad \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} &= 0,\ \frac{\partial J(u)}{\partial n} = 0 \qquad \text{for } x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) &= u_0 \qquad \text{in } \Omega. \end{aligned} \tag{3.3}$$

Remark 2. In applications to pattern formation one might also consider periodic boundary conditions

$$u(x + Le_i, t) = u(x, t), \quad i = 1, 2, 3,$$

with L being the size of a typical pattern cell and e_i the unit vector in the direction of the x_i axis.

Remark 3. Conservation of mass. Integrating over Ω using the boundary condition we have

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega u_t\,dx = \int_\Omega \Delta J(u)\,dx = \int_{\partial\Omega} \frac{\partial J(u)}{\partial n}\,ds = 0,$$

so that

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx, \quad t > 0. \tag{3.4}$$

4. A Ljapunov functional. We shall now demonstrate that the free energy functional serves as a Ljapunov function for the Cahn-Hilliard equation. We prefer to work with the normalized equation (3.1). Thus

$$\mathcal{F}(v) = \int_\Omega \left(\frac{1}{4} v^4 - \frac{1}{2} v^2 + \frac{1}{2} |\nabla v|^2 \right) dx = \frac{1}{4} \|v\|_{L_4}^4 - \frac{1}{2} \|v\|^2 + \frac{1}{2} \|\nabla v\|^2, \tag{4.1}$$

where $\|\cdot\|$ is the norm in $L_2(\Omega)$. Since $H^1(\Omega) \subset L_4(\Omega)$ for $n \leq 3$, we see that $\mathcal{F}$ is continuous in $H^1(\Omega)$.

We must show that

(1) $\mathcal{F}$ is bounded from below on $H^1(\Omega)$;
(2) any solution $u(t)$ of (3.1) is bounded in $H^1(\Omega)$ for all time, so that $\mathcal{F}(u(t))$ is defined;
(3) $\mathcal{F}(u(t))$ is nonincreasing with time for any solution $u(t)$ of (3.1).

Observing that

$$\mathcal{F}(v) = \frac{1}{2} \|\nabla v\|^2 + \frac{1}{4} \int_\Omega \left(v^2 - 1 \right)^2 dx - \frac{1}{4} |\Omega|, \tag{4.2}$$

we see that $\mathcal{F}$ is bounded from below. Next, multiplying the differential equation in (3.1) by $J(u) = -\Delta u + u^3 - u$ and integrating by parts over Ω, we have

$$(u_t, J(u)) + \|\nabla J(u)\|^2 = 0.$$

Since $\frac{d}{dt} \mathcal{F}(u) = (u_t, J(u))$, this implies

$$\frac{d}{dt} \mathcal{F}(u) \leq 0,$$

so that $\mathcal{F}(u(t))$ is nonincreasing and

$$\mathcal{F}(u(t)) \leq \mathcal{F}(u_0), \quad t \geq 0. \tag{4.3}$$

Using (4.2) this implies

$$\frac{1}{2} \|\nabla u(t)\|^2 + \frac{1}{4} \int_\Omega \left(u(t)^2 - 1 \right)^2 dx \leq \frac{1}{2} \|\nabla u_0\|^2 + \frac{1}{4} \int_\Omega \left(u_0^2 - 1 \right)^2 dx,$$

so that H^1 boundedness follows, viz.

$$\|\nabla u(t)\| \leq \|\nabla u_0\| + \frac{1}{\sqrt{2}} \left(\int_\Omega \left(u_0^2 - 1 \right)^2 dx \right)^{1/2}, \quad t \geq 0. \tag{4.4}$$

Remark. The above argument works also in the case (3.3) of a concentration dependent mobility.

5. The Viscous Cahn-Hilliard equation. Other regularizations are possible. In some systems, such as polymer-polymer systems, viscosity can be important. Novick-Cohen and Pego derive the following Viscous *Cahn-Hilliard Equation* in an attempt to incorporate both viscous and gradient energy effects:

$$\begin{gathered} u_t = \Delta(f(u) + \nu u_t - \kappa \Delta u) \qquad \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial n}(f(u) + \nu u_t - \kappa \Delta u) = 0,\ \frac{\partial u}{\partial n} = 0 \qquad \text{for } x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0 \qquad \text{for } x \in \Omega, \end{gathered} \tag{5.1}$$

where ν is a positive viscosity coefficient and $f(u) = \beta u^3 - \alpha u$. Without gradient energy effects ($\kappa = 0$) (5.1) reduces to the *Viscous Diffusion Equation*:

$$
\begin{aligned}
u_t &= \Delta(f(u) + \nu u_t) && \text{for } x \in \Omega,\ t > 0, \\
\frac{\partial}{\partial n}(f(u) + \nu u_t) &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) &= u_0 && \text{for } x \in \Omega.
\end{aligned}
\tag{5.2}
$$

Now note that with $J = f(u) + \nu u_t - \kappa\Delta u$ we have $u_t = \Delta J$ and $-\nu\Delta J + J = f(u) - \kappa\Delta u$, so that, setting $v = J - \frac{\kappa}{\nu}u$, (5.1) is equivalent to

$$
\begin{aligned}
\nu u_t - \kappa\Delta u &= v - f(u) + \tfrac{\kappa}{\nu}u && \text{for } x \in \Omega,\ t > 0, \\
-\nu\Delta v + v &= f(u) - \tfrac{\kappa}{\nu}u && \text{for } x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) &= u_0 && \text{for } x \in \Omega.
\end{aligned}
\tag{5.3}
$$

Similarly, (5.2) becomes

$$
\begin{aligned}
\nu u_t &= v - f(u) && \text{for } x \in \Omega,\ t > 0, \\
-\nu\Delta v + v &= f(u) && \text{for } x \in \Omega,\ t > 0, \\
\frac{\partial v}{\partial n} &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) &= u_0 && \text{for } x \in \Omega.
\end{aligned}
\tag{5.4}
$$

Or, in terms of the solution operator $T = (-\nu\Delta + I)^{-1}$ of the Neumann problem, these problems can be written as

$$
\begin{aligned}
\nu u_t - \kappa\Delta u &= (T - I)\,(f(u) - \tfrac{\kappa}{\nu}u) && \text{for } x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial n} &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) &= u_0 && \text{for } x \in \Omega,
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
\nu u_t &= (T - I)\,f(u) && \text{for } x \in \Omega,\ t > 0, \\
u(\cdot, 0) &= u_0 && \text{for } x \in \Omega.
\end{aligned}
\tag{5.6}
$$

Thus, (5.1) can be viewed as a semilinear heat equation with a nonlocal nonlinearity and (5.2) is a nonlocal ordinary differential equation.

Existence of solutions

6. Introduction. In this chapter we shall be concerned with proving existence of solutions to the Cahn-Hilliard equation (3.1). Replacing the nonlinear boundary condition by an equivalent linear one, we have

$$
\begin{aligned}
u_t + \Delta^2 u &= \Delta f(u) && \text{for } x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial n} = 0,\ \frac{\partial \Delta u}{\partial n} &= 0 && \text{for } x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) &= u_0 && \text{in } \Omega.
\end{aligned}
\tag{6.1}
$$

We shall assume that Ω is a bounded smooth domain in $\mathbf{R}^n$, $n \le 3$. We recall the *a priori* bound (4.4): For any possible solution u of (6.1) with $\|u_0\|_{H^1} \le \rho$ we have

$$\|u(t)\|_{H^1} \le C(\rho), \qquad 0 \le t < \infty. \tag{6.2}$$

Note that the H^1 norm is a rather weak norm in the context of our fourth order equation—we need four derivatives to make sense of the equation (or two derivatives for the weak form of the equation)—and also in the context of applying the Sobolev inequality—the maximum norm cannot be bounded in terms of the H^1 norm in two or three dimensions.

Nevertheless we shall see that the *a priori* bound (6.2) will lead to global existence. We shall describe two kinds of existence results:

(1) Global existence for smooth initial data, essentially $u_0 \in H^4(\Omega)$, by a standard technique.
(2) Global existence for non-smooth initial data, $u_0 \in H^1$, by a method of von Wahl.

Note also that (6.2) was proved by the energy method. We shall now abandon energy methods and work with semigroup techniques.

7. An analytic semigroup. Let X be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ and let A be an unbounded, closed and densely defined linear operator in X with domain $D(A)$. We assume that A is self-adjoint and positive definite and has a compact inverse. Then we have at our disposal all the power of the spectral theorem. In particular, A generates an analytic semigroup $\{e^{-tA}\}_{t\ge 0}$. In fact, e^{-tA} is the solution operator of the linear homogeneous evolution problem

$$\begin{aligned} u_t + Au &= 0, \qquad t > 0,\\ u(0) &= v, \end{aligned}$$

and we have

$$u(t) = e^{-tA}v = \sum_{j=1}^{\infty} e^{-t\lambda_j}\hat{v}_j\varphi_j,$$

where $\{\lambda_j, \varphi_j\}_{j=1}^{\infty}$ are the eigenvalues and normalized eigenvectors of A and $\hat{v}_j = (v, \varphi_j)$.

We define, also by spectral theory, the fractional powers of A and we set for $\alpha \in \mathbf{R}$

$$X_\alpha = D(A^\alpha),$$

$$\|v\|_\alpha = \|A^\alpha v\| = \left(\sum_{j=1}^{\infty} \lambda_j^{2\alpha}|\hat{v}_j|^2\right)^{1/2}. \tag{7.1}$$

It follows that $\{X_\alpha\}$ is a scale of Hilbert spaces and in particular we have

$$D(A) = X_1 \subset X_\beta \subset X_\alpha \subset X_0 = X$$

with continuous and compact imbeddings for $0 < \alpha < \beta < 1$.

We easily find that for $0 \le \alpha \le \beta$

$$\|e^{-tA}v\|_\beta \le C_{\alpha,\beta}t^{-(\beta-\alpha)}e^{-ct}\|v\|_\alpha, \qquad t > 0, \tag{7.2}$$

where $C_{\alpha,\beta}$ and c are positive numbers, or in terms of the operator norm

$$\|e^{-tA}\|_{\alpha,\beta} \le C_{\alpha,\beta}t^{-(\beta-\alpha)}e^{-ct}, \qquad t > 0. \tag{7.3}$$

We also have the *moment inequality* or *interpolation inequality*

$$\|u\|_\alpha \le C\|u\|_\beta^\theta\|u\|_\gamma^{1-\theta}, \qquad \alpha = \theta\beta + (1-\theta)\gamma,\ 0 \le \theta \le 1. \tag{7.4}$$

Note that $\theta = \frac{\alpha-\gamma}{\beta-\gamma}$, $1-\theta = \frac{\beta-\alpha}{\beta-\gamma}$. See Pazy [18], Theorem 2.6.10.

8. An abstract evolution equation. Assume that for some $\alpha \in [0,1)$ $M : X_\alpha \to X$ is a nonlinear operator that satisfies a *local Lipschitz condition*: If $\|u\|_\alpha, \|v\|_\alpha \le \rho$, then

$$\|M(u) - M(v)\| \le g(\rho)\|u - v\|_\alpha. \tag{8.1}$$

Consider the problem

$$\begin{aligned} u_t + Au &= M(u), \qquad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{8.2}$$

A solution of (8.2) on the interval $[0,T]$ is a function $u \in C([0,T], X_\alpha) \cap C^1((0,T), X)$ with $u(t) \in D(A)$ for $0 < t \le T$ and which satisfies (8.2).

By the variation of constants formula any solution of (8.2) satisfies the integral equation

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} M(u(\tau))\, d\tau, \qquad t \ge 0. \tag{8.3}$$

The contrary is also true: If $u \in C([0,T], X_\alpha)$ is a solution of (8.3), then it is also a solution of (8.2) on the interval $[0,T]$. See, e.g., Pazy [18], the proof of Theorem 6.3.1.

We now have the following standard *local existence theorem.*

Theorem 8.1. *For any $\rho \ge 0$ there is a $T = T(\rho) > 0$ such that (8.2) has a unique solution on the interval $[0,T]$ for all $u_0 \in X_\alpha$ with $\|u_0\|_\alpha \le \rho$.*

Proof. Let $\|u_0\|_\alpha \le \rho$. We shall apply Banach's fixed point theorem to the mapping G defined by

$$G(v)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} M(v(\tau))\, d\tau.$$

Let $Y = C([0,T], X_\alpha)$ and

$$\|u\|_Y = \max_{0 \le t \le T} \|u(t)\|_\alpha.$$

and set

$$\mathcal{B} = \{u \in Y : \|u\|_Y \le R\}.$$

The numbers T and R are to be determined so that the mapping G is a contraction of $\mathcal{B}$ into itself.

If $v \in \mathcal{B}$, then by (8.1)

$$\|M(v(\tau))\| \le \|M(0)\| + \|M(v(\tau)) - M(0)\| \le \|M(0)\| + g(R)R = K(R),$$

so that, in view of (7.3),

$$\begin{aligned} \|G(v)(t)\|_\alpha &\le \|e^{-tA}\|_{\alpha,\alpha}\|u_0\|_\alpha + \int_0^t \|e^{-(t-\tau)A}\|_{0,\alpha}\|M(v(\tau))\|\, d\tau \\ &\le C_{\alpha,\alpha}\rho + C_{0,\alpha}K(R)\int_0^t (t-\tau)^{-\alpha} d\tau \\ &= C_{\alpha,\alpha}\rho + C_{0,\alpha}K(R)\frac{T^{1-\alpha}}{1-\alpha}, \end{aligned}$$

for $0 \le t \le T$. We set $R = C_{\alpha,\alpha}\rho + 1$ and choose T such that $C_{0,\alpha}K(R)T^{1-\alpha}/(1-\alpha) \le \frac{1}{2}$. Then $\|G(v)\|_Y < R$ so that G maps $\mathcal{B}$ into $\mathcal{B}$.

If $u, v \in \mathcal{B}$, then we have in a similar manner

$$\begin{aligned} \|G(u)(t) - G(v)(t)\| &\le \int_0^t \|e^{-(t-\tau)A}\|_{0,\alpha}\|M(u(\tau)) - M(v(\tau))\|\, d\tau \\ &\le C_{0,\alpha}g(R)\frac{T^{1-\alpha}}{1-\alpha}\, \|u - v\|_Y \end{aligned}$$

for $0 \le t \le T$. Requiring also that $C_{0,\alpha}g(R)T^{1-\alpha}/(1-\alpha) \le \frac{1}{2}$, we have a contraction.

Corollary 8.2. *For each $u_0 \in X_\alpha$ there is a $T^* = T^*(u_0) \in (0,\infty]$ such that (8.2) has a unique solution on the interval $[0,T]$ for any $T < T^*$. Moreover, if $T^* < \infty$, then*

$$\lim_{t \uparrow T^*} \|u(t)\|_\alpha = \infty. \tag{8.4}$$

Proof. By Theorem 8.1 we have a solution on some (possibly short) interval $[0,T_1]$. Since $u(T_1) \in X_\alpha$, the solution can be continued to a second (possibly shorter) interval $[T_1,T_2]$ and so on. Let $T^* = \sup\{T : \text{ (8.2) has a solution on } [0,T]\}$ be the maximal interval of existence. If $T^* < \infty$ and (8.4) fails, then there is a number R and a sequence $\{t_i\}$ with $\lim_i t_i = T^*$ and $\|u(t_i)\|_\alpha \le R$. It now follows from Theorem 8.1 that the solution can be continued beyond T^*. This contradicts the maximality of T^*.

Theorem 8.1 (or its corollary) immediately leads to the following *global existence theorem.*

Theorem 8.3. *If for some $u_0 \in X_\alpha$ and some T and R we have an* a priori *bound*

$$\|u(t)\|_\alpha \le R, \qquad 0 \le t \le T,$$

for a possible solution of (8.2), then a unique solution exists on the interval $[0,T]$.

9. Application to the Cahn-Hilliard equation. Smooth initial data. In order to formulate the Cahn-Hilliard equation (6.1) in the abstract setting of Section 8 we set $X = L_2(\Omega)$ and define

$$A_1 u = -\Delta u + u, \quad D(A_1) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\}.$$

Then A_1 satisfies all the assumptions of Section 7. In particular, the spectral theorem yields fractional powers of A_1 and we define

$$A = A_1^2 = \Delta^2 - 2\Delta + I, \quad M(u) = \Delta f(u) - 2\Delta u + u.$$

It turns out that

$$\begin{aligned}
X_1 &= D(A^1) = D(A_1^2) = \{u \in H^4(\Omega) : \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}\Big|_{\partial\Omega} = 0\},\\
X_{3/4} &= D(A^{3/4}) = D(A_1^{3/2}) = \{u \in H^3(\Omega) : \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\},\\
X_{1/2} &= D(A^{1/2}) = D(A_1) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\},\\
X_{1/4} &= D(A^{1/4}) = D(A_1^{1/2}) = H^1(\Omega).
\end{aligned}$$

We also have equivalence of norms

$$c\|u\|_{m/4} \le \|u\|_{H^m} \le C\|u\|_{m/4}, \qquad u \in X_{m/4},\ m = 1,2,3,4. \tag{9.1}$$

In addition, we shall need the following Sobolev type inequalities for $0 \le \alpha \le 1$

$$\begin{aligned}
\|u\|_{W_p^m} &\le C\|u\|_\alpha, \qquad m - \frac{n}{p} < 4\alpha - \frac{n}{2},\ 2 \le p \le \infty,\\
\|u\|_{C^\nu} &\le C\|u\|_\alpha, \qquad 0 \le \nu < 4\alpha - \frac{n}{2}.
\end{aligned} \tag{9.2}$$

See Theorem 8.4.3 in Pazy [18].

The following result shows that $M(u)$ satisfies the local Lipschitz condition (8.1) with $\alpha = \frac{1}{2}$.

Lemma 9.1. *For $u, v \in X_{1/2}$ we have*

$$\|\Delta(f(u)-f(v))\| \le C\left(1+\|u\|_{1/2}^2+\|v\|_{1/2}^2\right)\|u-v\|_{1/2}.$$

Proof. Since $f(u)-f(v)=\int_0^1 f'(su+(1-s)v)\,ds\,(u-v)=w(u-v)$, Hölder's inequality implies

$$\begin{aligned}\|\Delta(f(u)-f(v))\| &= \|w\Delta(u-v)+2\nabla w\cdot\nabla(u-v)+\Delta w(u-v)\|\\ &\le \|w\|_{L_\infty}\|\Delta(u-v)\|+2\|\nabla w\|_{L_3}\|\nabla(u-v)\|_{L_6}+\|\Delta w\|\,\|u-v\|_{L_\infty}.\end{aligned}$$

Writing for simplicity $u_s = su+(1-s)v$, we have here

$$\|w\|_{L_\infty} = \|\int_0^1 f'(u_s)\,ds\|_{L_\infty} = \|\int_0^1 (6u_s^2-1)\,ds\|_{L_\infty} \le C\left(1+\int_0^1 \|u_s\|_{L_\infty}^2 ds\right).$$

Since, by Sobolev's inequality (recall that $n \le 3$) and (9.1),

$$\|u_s\|_{L_\infty} \le C\|u_s\|_{H^2} \le C\|u_s\|_{1/2},$$

this shows

$$\|w\|_{L_\infty} \le C\left(1+\|u\|_{1/2}^2+\|v\|_{1/2}^2\right).$$

Similarly,

$$\begin{aligned}\|\nabla w\|_{L_3} &= \|\int_0^1 f''(u_s)\nabla u_s\,ds\|_{L_3} = \|6\int_0^1 u_s\nabla u_s\,ds\|_{L_3}\\ &\le C\int_0^1 \|u_s\|_{L_6}\|\nabla u_s\|_{L_6} ds \le C\left(\|u\|_{1/2}^2+\|v\|_{1/2}^2\right),\end{aligned}$$

because

$$\|u_s\|_{L_6}\|\nabla u_s\|_{L_6} \le C\|u_s\|_{H^2}^2 \le C\|u_s\|_{1/2}^2.$$

By the same token

$$\begin{aligned}\|\Delta w\| &= \|\int_0^1 (f''(u_s)\Delta u_s + f'''(u_s)|\nabla u_s|^2)\,ds\| = \|6\int_0^1 (u_s\Delta u_s+|\nabla u_s|^2)\,ds\|\\ &\le C\int_0^1 (\|u_s\|_{L_\infty}\|\Delta u_s\|+\|\nabla u_s\|_{L_4}^2)\,ds \le C\left(\|u\|_{1/2}^2+\|v\|_{1/2}^2\right).\end{aligned}$$

Further, we have

$$\begin{aligned}\|\Delta(u-v)\| &\le C\|u-v\|_{H^2} \le C\|u-v\|_{1/2},\\ \|\nabla(u-v)\|_{L_6} &\le C\|u-v\|_{H^2} \le C\|u-v\|_{1/2},\\ \|u-v\|_{L_\infty} &\le C\|u-v\|_{H^2} \le C\|u-v\|_{1/2}.\end{aligned}$$

This completes the proof.

We thus have local existence for initial data $u_0 \in X_{1/2}$ and we would have global existence provided we could prove an *a priori* bound in $X_{1/2}$. But so far we have only a bound in the weaker norm of $H^1(\Omega) = X_{1/4}$, see (6.2). The solution can be estimated *a priori* in H^2 by the energy method, see e.g. [12], but we shall stick to the integral formulation (8.3) of our problem. Our aim is to obtain an *a priori* bound in X_α for some $\alpha \ge \frac{1}{2}$. We shall use our H^1 bound together with an appropriate growth condition on the nonlinear term. We begin with the bound for M.

Lemma 9.2. *For $u \in X_{3/4}$ we have*

$$\|\Delta f(u)\| \le C\left(1+\|u\|_{1/4}^2\right)\|u\|_{3/4}.$$

Proof. We have

$$\begin{aligned}
\|\Delta f(u)\| &= \|f'(u)\Delta u + f''(u)|\nabla u|^2\| \\
&\le \|f'(u)\|_{L_3}\|\Delta u\|_{L_6} + \|f''(u)\|_{L_6}\|\nabla u\|_{L_6}^2 \\
&\le C\left(1+\|u\|_{L_6}^2\right)\|\Delta u\|_{L_6} + C\|u\|_{L_6}\|\nabla u\|_{L_6}^2 \\
&\le C\left(1+\|u\|_{H^1}^2\right)\|u\|_{H^3} + C\|u\|_{H^1}\|\nabla u\|_{H^2}^2 \\
&\le C\left(1+\|u\|_{1/4}^2\right)\|u\|_{3/4} + C\|u\|_{1/4}\|u\|_{1/2}^2 \\
&\le C\left(1+\|u\|_{1/4}^2\right)\|u\|_{3/4},
\end{aligned}$$

since by the moment inequality $\|u\|_{1/2} \le C\|u\|_{1/4}^{1/2}\|u\|_{3/4}^{1/2}$. The lemma is proved.

Thus, M satisfies for $F = X_{1/4}$, $\alpha = \beta = \frac{3}{4}$, $\gamma = 1$ the hypotheses of the following theorem.

Theorem 9.3. *Let $0 \le \alpha < 1$, $1 \le \gamma < \frac{1}{\alpha}$, $\beta = \gamma\alpha$ and let F be a Banach space with continuous imbeddings*

$$X_\alpha \subset F \subset X \tag{9.3}$$

and assume that $M : X_\alpha \to X$ is such that $\|u\|_F \le \rho$ implies

$$\|M(u)\| \le g(\rho)\left(1+\|u\|_\alpha^\gamma\right). \tag{9.4}$$

If $u \in C([0,T], X_\alpha)$ is a solution of (8.3) with $u_0 \in X_\beta$ and

$$\|u(t)\|_F \le R, \qquad 0 \le t \le T,$$

for some R and T, then

$$\|u(t)\|_\beta \le C(R,T)\left(1+\|u_0\|_\beta\right), \qquad 0 \le t \le T.$$

Proof. Note that $0 \le \alpha \le \beta < 1$. By the moment inequality we have

$$\|u\|_\alpha \le C\|u\|^{1-\alpha/\beta}\|u\|_\beta^{\alpha/\beta},$$

so that, since $F \subset X$,

$$\|u\|_\alpha^\gamma \le C\|u\|^{\gamma-1}\|u\|_\beta \le C\|u\|_F^{\gamma-1}\|u\|_\beta.$$

Hence

$$\begin{aligned}
\|u(t)\|_\beta &\le \|e^{-tA}\|_{\beta,\beta}\|u_0\|_\beta + \int_0^t \|e^{-(t-\tau)A}\|_{0,\beta}\|M(u(\tau))\|\,d\tau \\
&\le C_{\beta,\beta}\|u_0\|_\beta + C_{0,\beta}g(R)\int_0^t (t-\tau)^{-\beta}\left(1+CR^{\gamma-1}\|u(\tau)\|_\beta\right)d\tau \\
&\le C(R,T)\left(1+\|u_0\|_\beta\right) + C(R)\int_0^t (t-\tau)^{-\beta}\|u(\tau)\|_\beta\,d\tau
\end{aligned}$$

and the result follows from the generalized Grönwall lemma.

In view of the bound (6.2) we may thus apply Theorem 8.3 with $\alpha = \frac{3}{4}$. The result is global existence for smooth initial data, i.e., for $u_0 \in X_\alpha$, $\alpha \ge \frac{3}{4}$.

Theorem 9.4. *For any* $u_0 \in \{u \in H^4(\Omega) : \frac{\partial u}{\partial n}\big|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}|_{\partial\Omega} = 0\}$ *the Cahn-Hilliard equation (6.1) has a unique solution for all time.*

10. Non-smooth initial data. We shall now describe a method of von Wahl [22] which can be used to prove existence of solutions to (8.2) with non-smooth initial values. We first replace the Lipschitz condition (8.1) by a more general one. Let $0 \le \alpha \le \beta < 1$. Assume that the nonlinear operator $M : X_\beta \to X$ satisfies the Lipschitz condition: If $u, v \in X_\beta$ with $\|u\|_\alpha, \|v\|_\alpha \le \rho$, then

$$\|M(u) - M(v)\| \le g(\rho)\big(\|u - v\|_\beta + (1 + \|u\|_\beta + \|v\|_\beta)\,\|u - v\|_\alpha\big). \tag{10.1}$$

Under this condition we have local existence.

Theorem 10.1. *For any* $\rho \ge 0$ *there is a* $T = T(\rho) > 0$ *such that (8.2) has a unique solution on the interval* $[0, T]$ *for all* $u_0 \in X_\alpha$ *with* $\|u_0\|_\alpha \le \rho$.

Proof. Let $\|u_0\|_\alpha \le \rho$ and set $\delta = \beta - \alpha$. Note that $0 \le \delta < 1$. We shall apply Banach's fixed point theorem to the mapping G defined by

$$G(v)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} M(v(\tau))\, d\tau.$$

We let Y be the Banach space

$$Y = C([0, T], X_\alpha) \cap C((0, T], X_\beta)$$

with norm

$$\|u\|_Y = \max\{\sup_{0\le t\le T} \|u(t)\|_\alpha,\ \sup_{0<t\le T} (t^\delta \|u(t)\|_\beta)\}$$

and set

$$\mathcal{B} = \{u \in Y : \|u\|_Y \le R\}.$$

The numbers T and R are to be determined so that the mapping G is a contraction of $\mathcal{B}$ into itself.

If $v \in \mathcal{B}$, then by (10.1)

$$\begin{aligned}\|M(v(\tau))\| &\le \|M(0)\| + \|M(v(\tau)) - M(0)\|\\ &\le \|M(0)\| + g(R)(R + R\tau^{-\delta} + R^2\tau^{-\delta}) \le K(R)(1 + \tau^{-\delta}),\end{aligned}$$

Using the formula

$$\int_0^t (t - \tau)^{-\alpha}\tau^{-\beta} d\tau = c_{\alpha,\beta}\, t^{1-\alpha-\beta}, \qquad 0 \le \alpha, \beta < 1,$$

we have for $0 \le t \le T,\ \alpha \le \gamma \le \beta$

$$\begin{aligned}\|G(v)(t)\|_\gamma &\le \|e^{-tA}\|_{\alpha,\gamma}\|u_0\|_\alpha + \int_0^t \|e^{-(t-\tau)A}\|_{0,\gamma}\|M(v(\tau))\|\, d\tau\\ &\le C_{\alpha,\gamma}\rho + C_{0,\gamma}K(R)\int_0^t (t - \tau)^{-\gamma}(1 + \tau^{-\delta})\, d\tau\\ &= C_{\alpha,\gamma}\rho + C_{0,\gamma}K(R)c_{\gamma,\delta}(t^{1-\gamma} + t^{1-\gamma-\delta})\\ &= t^{-(\gamma-\alpha)}\big(C_{\alpha,\gamma}\rho + C_{0,\gamma}K(R)c_{\gamma,\delta}(t^{1-\alpha} + t^{1-\beta})\big)\\ &\le t^{-(\gamma-\alpha)}\big(C_1\rho + C_2(R)(T^{1-\alpha} + T^{1-\beta})\big).\end{aligned}$$

We set $R = C_1\rho + 1$ and choose T such that $C_2(R)(T^{1-\alpha} + T^{1-\beta}) \le \frac{1}{2}$. Then application of the above with $\gamma = \alpha$ and $\gamma = \beta$ shows that $\|G(v)\|_Y < R$, so that G maps $\mathcal{B}$ into $\mathcal{B}$.

If $u, v \in \mathcal{B}$, then we have in a similar manner

$$
\begin{aligned}
&\|G(u)(t) - G(v)(t)\|_\gamma \le \int_0^t \|e^{-(t-\tau)A}\|_{0,\gamma} \|M(u(\tau)) - M(v(\tau))\| \, d\tau \\
&\le C_{0,\gamma} g(R) \int_0^t (t-\tau)^{-\gamma} \left(\|u(\tau) - v(\tau)\|_\beta + (1 + \|u(\tau)\|_\beta + \|v(\tau)\|_\beta) \|u(\tau) - v(\tau)\|_\alpha \right) d\tau \\
&\le C(R) \int_0^t (t-\tau)^{-\gamma}(1 + \tau^{-\delta}) \, d\tau \, \|u - v\|_Y \le t^{-(\gamma-\alpha)} C_3(R)(T^{1-\alpha} + T^{1-\beta}) \|u - v\|_Y
\end{aligned}
$$

for $0 \le t \le T$ and $\alpha \le \gamma \le \beta$. Requiring also that $C_3(R)(T^{1-\alpha} + T^{1-\beta}) \le \frac{1}{2}$, we have a contraction.

Corollary 10.2. *For each $u_0 \in X_\alpha$ there is a $T^* = T^*(u_0) \in (0, \infty]$ such that (8.2) has a unique solution on the interval $[0, T]$ for any $T < T^*$. Moreover, if $T^* < \infty$, then*

$$
\lim_{t \uparrow T^*} \|u(t)\|_\alpha = \infty. \tag{10.2}
$$

Proof. See the proof of Corollary 8.2.

Theorem 10.1 (or its corollary) immediately leads to the following *global existence theorem.*

Theorem 10.3. *If for some $u_0 \in X_\alpha$ and some T and R we have an* a priori *bound*

$$
\|u(t)\|_\alpha \le R, \qquad 0 \le t \le T,
$$

for a possible solution of (8.2), then a unique solution exists on the interval $[0, T]$.

11. Application to the Cahn-Hilliard equation. Non-mooth initial data. We must show that $M(u) = \Delta f(u) - 2\Delta u + u$ satisfies a Lipschitz condition of the form (10.1).

Lemma 11.1. *Let $\frac{3}{4} < \beta < 1$. For $u, v \in X_{1/4}$ we have*

$$
\begin{aligned}
\|\Delta(f(u) - f(v))\| \le C &\left(1 + \|u\|_{1/4}^2 + \|v\|_{1/4}^2\right) \|u - v\|_\beta \\
&+ C \left(\|u\|_{1/4} + \|v\|_{1/4}\right) \left(\|u\|_\beta + \|v\|_\beta\right) \|u - v\|_{1/4}.
\end{aligned}
$$

Proof. Take $\beta \in (\frac{3}{4}, 1)$. Since $f(u) - f(v) = \int_0^1 f'(su + (1-s)v) \, ds \, (u - v) = w(u - v)$, Hölder's inequality implies

$$
\begin{aligned}
\|\Delta(f(u) - f(v))\| &= \|w\Delta(u - v) + 2\nabla w \cdot \nabla(u - v) + \Delta w(u - v)\| \\
&\le \|w\|_{L_3} \|\Delta(u - v)\|_{L_6} + 2\|\nabla w\|_{L_\infty} \|\nabla(u - v)\| + \|\Delta w\|_{L_3} \|u - v\|_{L_6}.
\end{aligned}
$$

Writing for simplicity $u_s = su + (1-s)v$, we have here

$$
\|w\|_{L_3} = \| \int_0^1 f'(u_s) \, ds \|_{L_3} = \| \int_0^1 (6u_s^2 - 1) \, ds \|_{L_3} \le C \left(1 + \int_0^1 \|u_s\|_{L_6}^2 ds\right).
$$

Since, by Sobolev's inequality (recall that $n \le 3$) and (9.1),

$$
\|u_s\|_{L_6} \le C\|u_s\|_{H^1} \le C\|u_s\|_{1/4},
$$

this shows

$$\|w\|_{L_3} \le C\left(1+\|u\|_{1/4}^2+\|v\|_{1/4}^2\right).$$

Similarly,

$$\begin{aligned}\|\nabla w\|_{L_\infty} &= \|\int_0^1 f''(u_s)\nabla u_s\,ds\|_{L_\infty} = \|6\int_0^1 u_s\nabla u_s\,ds\|_{L_\infty}\\ &\le C\int_0^1 \|u_s\|_{L_\infty}\|\nabla u_s\|_{L_\infty}ds.\end{aligned}$$

Using (9.2) and (7.4) we have for $4\sigma > \frac{n}{2}$

$$\|u_s\|_{L_\infty}\|\nabla u_s\|_{L_\infty} \le C\|u_s\|_\sigma\|u_s\|_{\sigma+1/4} \le C\|u_s\|_{1/4}^{1-\theta}\|u_s\|_\beta^\theta\cdot\|u_s\|_{1/4}^{1-\gamma}\|u_s\|_\beta^\gamma,$$

where $\theta = \frac{4\sigma-1}{4\beta-1}$ and $\gamma = \frac{4\sigma}{4\beta-1}$. Choosing $\sigma = \frac{\beta}{2}$ (which is possible since $\beta > \frac{3}{4}$ and $n \le 3$) so that $\theta+\gamma = 1$, we thus have

$$\|\nabla w\|_{L_\infty} \le C\left(\|u\|_{1/4}+\|v\|_{1/4}\right)\left(\|u\|_\beta+\|v\|_\beta\right).$$

By the same token

$$\begin{aligned}\|\Delta w\|_{L_3} &= \|\int_0^1 (f''(u_s)\Delta u_s \ + f'''(u_s)|\nabla u_s|^2)\,ds\|_{L_3} = \|6\int_0^1 (u_s\Delta u_s + |\nabla u_s|^2)\,ds\|_{L_3}\\ &\le C\int_0^1 (\|u_s\|_{L_6}\|\Delta u_s\|_{L_6} + \|\nabla u_s\|_{L_6}^2)\,ds.\end{aligned}$$

Now

$$\|u_s\|_{L_6}\|\Delta u_s\|_{L_6} \le C\|u_s\|_{1/4}\|u_s\|_{3/4}$$

and

$$\|\nabla u_s\|_{L_6}^2 \le C\|u_s\|_{1/2} \le C\|u_s\|_{1/4}\|u_s\|_{3/4}$$

so that

$$\|\Delta w\|_{L_3} \le C\left(\|u\|_{1/4}+\|v\|_{1/4}\right)\left(\|u\|_\beta+\|v\|_\beta\right).$$

Finally, we have

$$\begin{aligned}\|\Delta(u-v)\|_{L_6} &\le C\|u-v\|_{H^3} \le C\|u-v\|_{3/4},\\ \|\nabla(u-v)\| &\le C\|u-v\|_{H^1} \le C\|u-v\|_{1/4},\\ \|u-v\|_{L_6} &\le C\|u-v\|_{H^1} \le C\|u-v\|_{1/4}.\end{aligned}$$

This completes the proof.

In view of the *a priori* bound (6.2), we can now apply Theorem 10.3 with $\alpha = \frac{1}{4}$ and some $\beta \in (\frac{3}{4}, 1)$.

Theorem 11.2. *For any $u_0 \in H^1(\Omega)$ the Cahn-Hilliard equation (6.1) has a unique solution for all time.*

Remark. The above results can be obtained under more general conditions on the function f. In the proof of Lemma 11.1 we actually only need the fact that

$$|f^{(j)}(u)| \le C\left(1+|u|^{3-j}\right), \qquad j = 1,2,3.$$

In von Wahl [22] f is allowed to grow like $|u|^{5-\epsilon}$ for $n=3$ and like an arbitrary polynomial for $n=2$. There is no growth restriction for $n=1$. The argument is essentially the same as the one given here. In the proof of the *a priori* bound (6.2) we need

$$F(u) = \int_0^u f(s)\,ds \ge -c, \qquad u \in \mathbf{R}.$$

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