SPECTRAL ASYMPTOTICS OF PSEUDODIFFERENTIAL OPERATORS

K. Kh. Boimatov

1. We consider the symmetric operator

(1) display

in the space $L_2(R_n)$ with domain of definition $D(P_0) = C_0^\infty(R_n)$.

We assume that $P(x,s) \in C^\infty(R_n \times R_n)$ is a real symbol. Let the following conditions hold:

a) $(1+|s|)^a \le MP(x,s), \quad M, a > 0$,

b) $|D_x^\alpha D_s^\beta P(x,s)| \le M_{\alpha\beta} P^{1-\theta|\beta|}(x,s)$ for all α, β with some $\theta > 0$,

c) $P(x,s) \le MP(y,s)$ for all $x, y, s \in R_n$.

Under these assumptions the closure P of P_0 is (cf. [1, § 4]) a self-adjoint operator which is semi-bounded from below. In this note we study the asymptotic behavior of the spectral function $e(\lambda, x, y)$ of the operator P. The results obtained are applied to the study of the asymptotic functions for the distribution of eigenvalues of pseudodifferential operators defined in domains with finite measure. The state of the art of research on the spectral asymptotics of differential and pseudodifferential operators is presented in the works [2,3].

We use the notation

(A) display

Let $l(\lambda)$ be a positive non-decreasing function such that $l(2\lambda) = O(1)l(\lambda)$ $(\lambda \to +\infty)$ and for some $\eta \in R_n$ the inequality

(B) display

holds.

Theorem 1. *There exist numbers $N, \epsilon, \lambda_0 > 0, \sigma \in (0,1)$ such that for $\lambda \ge \lambda_0$ we have the bound*

(2) *display*

1991 *Mathematics Subject Classification.* 35S05, 35P20.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. We give a *sketch of the proof.* For simplicity we assume that $a > 2n$, $8\rho n < 1$ (for the general case see no. 3 below). Let $\theta_i, \psi_{ij}, \varphi_{ij}, x_{ij}$ be as in 1,2, § 4 of [1]. We put

(3) display

where $G_{ij}(z) = \exp(-zP_{ij})$, and P_{ij} is the pseudodifferential operator in $L_2(R_n)$ with "constant" symbol $P(x_{ij}, s)$.

Lemma 1. (cf. Theorem 4.1 of [1]) *There exists a number $\kappa \in (0,1)$ such that, with arbitrary $\sigma \in (\kappa, 1)$, for some $c > 0$ the integral representation*

(4) *display*

holds in the region

(C) *display*

where the integral is taken over a rectilinear segment, and the operator function $\Phi(z)$ satisfies the bound $|\Phi(z)| = O(|\,Re\, z|^{-\kappa})$ in the region $W_{\sigma,c}$.

From the conditions $a > 2n$, $8\rho n < 1$ is not hard to deduce (see [1, § 4]) that $|(-\Delta)^n G_0(z)| \leq M|\operatorname{Re} z|^{-1/2}$. Hence and from (4) follows the estimate

(D) display

By a well-known criterion we conclude that the operators $G_0(z), G(z) = \exp(-zP)$,

(E) display

$z \in W_{\sigma,c}$, are integral operators.

For an integral operator T with kernel $T(x,y)$ we put

(F) display

It is obvious that $a(T_2)a(T_1^*) \geq |(T_1T_2)(x,y)|$.

Let $R^*(z,x,y)$ be the kernel of the operator

(G) display

Let the point $y \in R_n$ be fixed. Then the expression $R^*(z,x,y)$ may be treated as an integral of the result of the action of the operator $\Phi^*(w)$ (at the point x) on the function $G_0^*(z-w,\cdot,y)$. Therefore

(H) display

Hence and from the estimate $|\Phi^*(w)| \leq M|\operatorname{Re} z|^{-\kappa}$ follows that

(I) display

Since $G(z) = G(z/2)G^*(\bar{z}/2)$, we have

(J) display

Thus, for the kernel $F(z,x,y)$ of the operator $F(z) = G(z) - G_0(z/2)G_0^*(\bar{z}/2)$ we have the bound

(K) display

From the obtained inequalities we conclude that

(L) display

The kernel of the operator $G_0(z/2)G_0^*(\bar{z}/2)$ is estimated by means of formula (3), which gives explicit expressions for the operators $G_0(z/2)$, $G_0^*(\bar{z}/2)$. Finally, we obtain the following bound:

(M) display

where $G(z,x,y)$ is the kernel of the operator $G(z) = \exp(-zP)$,

(N) display

We remark that

(O) display

Since $e(\lambda,x,x), \theta(\lambda,x,x)$ are non-decreasing functions, if we apply the appropriate Tauberian theorem, we obtain the bound (2) for $x = y$. The proof of inequality (2) for $x \neq y$ is based on applying Tauberian theorems to the Laplace-Stieltjes transforms of the non-decreasing functions

(P) display

where $\theta_*(\lambda,x,y) = (\theta(\lambda,x,y) + \overline{\theta(\lambda,x,y)})/2$. The Laplace-Stieltjes transform of the function $\theta_*(\lambda,x,y) - \theta(\lambda,x,y)$ satisfies a sufficiently good bound in some parabolic region $W_{\sigma,c}$, $\sigma \in (0,1)$ $c > 0$.

3. In the case, when the inequalities $a > 2n$, $8\rho n < 1$ do not hold simultaneously, we consider the operator P^k, where k is an even number such that $ak > 2n$, $8\rho n < k$. Let $P \geq E$. Using convolution theorems we represent the operator P^k in the form $P^k = \mathcal{P} + Q$, where Q is a bounded operator, and $\mathcal{P}$ is a pseudodifferential operator, whose Weyl symbol $\mathcal{P}(x,s)$ can be computed in explicit form. For the symbol $\mathcal{P}(x,s)$ there hold conditions analogous to a)–c), with the parameters $ak, \rho/k$ instead of a, ρ.

The spectral functions $e_k(\lambda,x,y), e(\lambda,x,y)$ of the operators P^k, P are connected by the relation

(Q) display

The symbols $P(x,s)$, $\mathcal{P}(x,s)$ satisfy a bound of the form

(R) display

where M, $\lambda_0 > 0$, $0 < \nu < 1$.

By studying the kernel of the operator function $\exp(-zP^k) - \exp(-z\mathcal{P})$, one obtains a bound for the difference between the spectral functions of the operators $P^k, \mathcal{P}$.

4. Let the conditions a)–c) hold for the symbol $P(x,s)$. We consider the operator $P'_\Omega = \chi_\Omega P_0$, $D(P'_\Omega) = C_0^\infty(\Omega)$, where the operator P_0 is defined by formula (1), $\Omega \subset R_n$ is an open set, $\operatorname{meas}\Omega < +\infty$, and χ_Ω is the characteristic function of the set Ω.

Since P_0 is a symmetric operator, semi-bounded from below, the operator P'_Ω is also symmetric and semi-bounded from below.

We denote by P_Ω the Friedrichs extension of the operator P'_Ω. We put

(S) display

We assume that for any number $\sigma \in (0,1)$ and for all $x \in \Omega$

(T) display

where $\omega_\sigma(\lambda) = o(1)$, $\lambda \to +\infty$. We also assume that there exists a positive non-decreasing function $\psi(\lambda) \in C^1(0,+\infty)$ such that

(5) display

Theorem 2. *Under the above conditions the operator P_Ω has discrete spectrum. For the function $N(\lambda)$, the number of eigenvalues of the operator P_Ω not exceeding λ, we have the following asymptotic formula*

(U) *display*

5. We give a *sketch of the proof.* Applying Berezin's theorem and Theorem 1 as in [4], we establish the estimate

(6) display

For this it is necessary to take $l(\lambda) = \varphi(\lambda, \Omega)$ in Theorem 1.

In order to obtain a bound from below we fix a number $\epsilon > 0$ and consider a compact set $K_\epsilon \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus K_\epsilon) < \epsilon$. Let the function $\zeta_\epsilon(x) \in C_0^\infty(\Omega)$ be non-negative and equal to unity in a neighborhood of the set K_ϵ, whereby $\zeta_\epsilon(x) \le 1$ for all $x \in \Omega$. Let P be an operator as in no. 1. We have the identity

(V) display

Hence follows that

(W) display

where $|\cdot|_2$ denotes the Hilbert-Schmidt norm.

Using (4) as in [1, § 4] we establish the bound

(X) display

with some $\delta \in (0,1)$. The expression $|e^{-tP_\Omega}|_2 = |e^{-2tP_\Omega}|_1^{1/2}$, where $|\cdot|_1$ denotes the kernel norm, is estimated from above by means of (5). The bounds for $|\zeta_\epsilon e^{-tP}\zeta_\epsilon|_2,\ |e^{-\frac{1}{2}\tau P}\zeta_\epsilon|_2$ utilize the asymptotic behavior of the spectral function of the operator P (cf. Theorem 1).

Writing

(Y) display

we finally obtain

(7) display

where $r(t,\zeta_\epsilon)=o(1),\ t\to 0+$, for any $\epsilon>0$. We divide both sides of (7) by $V(t,\zeta_\epsilon)$ and pass to the limit $t\to 0+$. As a result we obtain

(8) display

By condition c) we have

(Z) display

where the numbers M_1, M_2 do not depend on ϵ. From condition (5) follows the bound $V(t)=O(1)V(M_2^{-1}t)$.

Hence

(a) display

where the number M_3 does not depend on ϵ. Hence and from (8), in view of the arbitrariness of $\epsilon>0$, we conclude that

(b) display

Since

(c) display

(cf. (6)) we have $\omega(t)\sim V(t),\ t\to 0+$. It remains only to apply the appropriate Tauberian theorem.

References

1. K. Kh. Boimatov, Memoirs of the Petrovskij seminar **7** (1981), 50–100.
2. M. Sh. Birman and M. Z. Solomjak, Itogi nauki i tekhniki. Ser. matem. analiz **14** (1977), 5–58.
3. S. Z. Levendorskij, Uspekhi Mat. Nauk **43** (1988), 123–157.
4. A. B. Khmel'nitskaja, Uspekhi Mat. Nauk **31** (1977), 279–280.

Mathematical Institute and Computing Center
Academy of Science of the Tadzhik SSR, Dushanbe