ESTIMATES OF THE SPECTRUM OF COMPACT PSEUDODIFFERENTIAL OPERATORS IN UNBOUNDED DOMAINS

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Abstract. We study the spectrum of compact selfadjoint pseudodifferential operators with matrix symbol in unbounded sets in Euclidean space. Two-sided bounds for the spectrum are obtained by the variational method. Additional regularity assumptions on the symbol lead to a formula for the spectral asymptotics with a remainder term estimate.

The spectrum of compact selfadjoint pseudodifferential operators (ΨDO) in unbounded domains has been studied relatively little. In [1] the asymptotics of the spectrum of ΨDO with anisotropically homogeneous symbols in $\mathbf{R}^m$ was obtained. In [1] the smoothness assumptions on the symbol were minimal, there was no assumption of positive or negative definiteness, but it was assumed that the order of homogeneity $l < m$. A remainder term estimate was not given in [1]. The approximate spectral projection method of [2] has been used in the study of the spectrum of pencils of linear ΨDO of positive order in unbounded domains. In [2] the asymptotics of the spectrum was established with a remainder term estimate. The ΨDO in [1] and [2] have matrix symbols. In [3] and [4] a classical formula for the spectral asymptotics of scalar compact ΨDO in $\mathbf{R}^m$ was found. In this connection the symbol of the operator belongs to the Beals-Hörmander calculus, and, in general, need not possess any homogeneity properties. The assumptions in [3] and [4] imply essential restrictions of the form constant direction of the gradient of the symbol. It was shown that the remainder term of the spectral asymptotics has power order (in the spectral parameter) relative to the principal term. However, the dependence of this order on the symbol of the ΨDO was not obtained in [3, 4]. The proof is connected in an essential way with a specific problem in $\mathbf{R}^m$.

In the present work we study the spectrum of a matrix ΨDO in an unbounded set in $\mathbf{R}^m$. The smoothness hypotheses on the boundary of the domain coincide locally with the corresponding conditions in [5]. The symbol of the ΨDO is assumed to be a smooth matrix function. We do not make any assumption of definiteness or constancy of the multiplicity of the eigenvalues of the matrix symbol. We do not assume that the ΨDO belongs to the Beals-Hörmander class.

Two-sided bounds for the spectrum of the operator are obtained by the variational method. The proof rests mainly on local results from [5]. Under additional smoothness assumptions on the symbol these bounds yield the spectral asymptotics and the order of the remainder term.

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§1. Statement of the main results

1. Basic notation. $\mathbf{R}^m$ denotes the Euclidean space of points $x = (x_1, \dots, x_m)$; $\mathbf{Z}^m$ is the set of points with integer coordinates in $\mathbf{R}^m$; $\mathbf{S}^{m-1}$ is the unit sphere in $\mathbf{R}^m$; $\mathbf{E}^m$ denotes the dual space of $\mathbf{R}^m$ consisting of points $\xi = (\xi_1, \dots, \xi_m)$; $\mathbf{N}$ is the set of natural numbers. For real t, $[t]$ denotes its integer part, and $t_+ = (t + |t|)/2$, $t_- = (|t| - t)/2$.

If $\mathcal{D}$ is a set in $\mathbf{R}^m$, then $\sum_{\mathcal{D}}^{(k)}$ denotes summation over $k \in \mathcal{D} \cap \mathbf{Z}^m$. In case $\mathcal{D} = \mathbf{R}^m$ we omit $\mathcal{D}$. Integrals without indicated domain of integration are assumed to be extended over the whole space.

For a multi-index α, $|\alpha| = \alpha_1 + \dots + \alpha_m$, $\partial_x^\alpha = \partial^{|\alpha|}/\partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}$, and similarly for ∂_ξ^α.

Let $\mathcal{D}$ be an open set, $\mathcal{D} \subseteq \mathbf{R}^m$. By $L_2(\mathcal{D})$ we denote the space of vector valued functions $u = (u_1, \dots, u_s)$ $(1 \le s < \infty)$ with the standard norm. In the usual way we define the function spaces $C_0^\infty(\mathcal{D})$, $C^\infty(\mathbf{E}^m)$, etc.

If T is a compact selfadjoint operator in a Hilbert space H and $\mathcal{E}_T(\cdot)$ its spectral measure, then $N_+(\mu, T) = \dim \mathcal{E}_T(\mu, +\infty)H$, $N_-(\mu, T) = \dim \mathcal{E}_T(-\infty, -\mu)H$. Everywhere in the following the values of the spectral parameters $\lambda, \mu \in (0, 1)$.

For $s \times s$ matrices $\mathcal{F} = \{\mathcal{F}_{ij}\}$ we let $|\mathcal{F}|_s = \sup_{i,j} |\mathcal{F}_{ij}|$.

By c, C we denote various constants.

2. Let G be an unbounded set in $\mathbf{R}^m$ (the case $G = \mathbf{R}^m$ is not excluded) and let χ_G be the characteristic function of G. We cover G by cubes. To be precise, for $k \in \mathbf{Z}^m$, $Q_k = \{x : k_i < x_i < k_i + 1 \ \forall\, i\}$. We write $G_k = G \cap Q_k$, $K = \{k : G_k \ne \emptyset\}$, $Q = \bigcup_{k \in K} \overline{Q_k}$.

Everywhere in what follows we make the following assumption about the smoothness of the boundary of G:

$$\operatorname{meas}\{x : \operatorname{dist}(x, \partial G_k) < \delta\} \le c(G)\delta \qquad \forall \delta > 0, \forall k \in K. \tag{1.1}$$

3. We now consider functions that are useful in describing the symbols of ΨDO. Let $f(x, \xi) : Q \times \mathbf{E}^m \to \mathbf{R}_+$ such that $\forall x, y \in Q$, $\forall \xi \in \mathbf{E}^m$

$$f(x, \xi)/f(y, \xi) \le c_1 (1 + |x - y|^{l_1}), \tag{1.2}$$

$$f(x, \xi) \le c_2 (1 + |x| + |\xi|)^{-l_2}, \qquad l_2 > 0, \tag{1.2a}$$

where $c_i = c_i(f)$, $l_i = l_i(f)$.

We write $\varphi(x) = \sup f(x, \xi)$, $\xi \in \mathbf{E}^m$; $\mathcal{D}(\lambda) = \{x : x \in Q,\ \varphi(x) > \lambda\}$, $V(\lambda, x) = \operatorname{meas}\{\xi : f(x, \xi) > \lambda\}$.

Further, $U(\lambda, x)$ is a function defined for $\lambda \in (0, 1)$, $x \in Q$. We assume that $\forall t \in (0, 1)$

$$U(\lambda, x) \le U(t\lambda, x) \le c_3 t^{-\omega} U(\lambda, x) \qquad \forall x \in Q; \tag{1.3}$$

$$1 \le U(\lambda, x) \le c_4 \lambda^{-l_3} \qquad \forall x \in \mathcal{D}(\lambda); \tag{1.3a}$$

$c_i = c_i(U)$, $l_i = l_i(U)$, $\omega = \omega(U) > 0$. The concrete values of the constants c_i, l_i are not essential in the following.

By definition the function $f(x, \xi)$ belongs to the class $\Lambda_\omega(U)$ if the conditions (1.2)–(1.3a) hold and

$$V(\lambda, x) \le U(\lambda, x) \qquad \forall x \in Q. \tag{1.4}$$

4. We shall describe the symbol of the ΨDO. Let the $s \times s$ matrix function $\mathcal{A} = \mathcal{A}(x, y, \xi) \in C^\infty(Q \times Q \times \mathbf{E}^m)$, $\mathcal{A}^*(x, y, \xi) = \mathcal{A}(y, x, \xi)$. The eigenvalues of the matrix $\mathcal{A}(x, x, \xi)$ are denoted by $a_j(x, \xi)$; $a_{j+1}(x, \xi) \geq a_j(x, \xi)$, $j = 1, \ldots, s-1$; and

$$v_\pm(\lambda, x) = \sum_{j=1}^{s} \operatorname{meas}\{\xi : \pm a_j(x, \xi) > \lambda\}.$$

By definition the matrix function $\mathcal{A}$ belongs to the class $\Sigma_\rho(f, U)$, if the following conditions are satisfied:

$$(1.5) \qquad |(\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \mathcal{A})(x, y, \xi)|_s \leq c(\mathcal{A}, \alpha, \beta, \gamma)(f(x, \xi) + f(y, \xi))(1 + |\xi|)^{-\rho|\alpha|}$$

$$\forall x, y \in Q,\ \forall \xi \in \mathbf{E}^m,\ \forall \alpha, \beta, \gamma; \quad f \in \Lambda_\omega(U),\ \rho > 0; \quad \forall t \in (0, \tfrac{1}{2}),\ \forall x \in Q,$$

$$(1.6) \qquad v_\pm((1-t)\lambda, x) \leq v_\pm(\lambda, x) + c_1(\mathcal{A}) t U(\lambda, x) + c_2(\mathcal{A}).$$

Condition (1.5) guarantees pseudo-locality of the ΨDO. The class $\Sigma_\rho(f, U)$ may contain symbols, that do not satisfy Beals-Hörmander type inequalities. As an example, consider for $m = s = 1$ the symbol $\mathcal{A}(x, y, \xi) = (\xi^2 + x^2 + 1)^{-1}(\xi^2 + y^2 + 1)^{-1} \sin x \sin y$. The function $\mathcal{A} \in \Sigma_\rho(f, U)$ for $f(x, \xi) = (\xi^2 + x^2 + 1)^{-2}$, $U(\lambda, x) = \lambda^{-1/4}$, $\rho = 1$.

5. Let the function $\mathcal{A} \in \Sigma_\rho(f, U)$, $f \in \Lambda_\omega(U)$. For a function $u \in C_0^\infty(G)$ we write

$$(\langle \mathcal{A} \rangle u)(x) = (2\pi)^{-m} \int d\xi \int \mathcal{A}(x, y, \xi) u(y) \exp(i(x-y)\xi)\, dy, \qquad x \in \mathbf{R}^m;$$

$$(A^{(0)} u)(x) = (\chi_G \langle \mathcal{A} \rangle u)(x).$$

The operator $A^{(0)}$ is extended by continuity to a compact selfadjoint operator A in the space $L_2(G)$. Spectral bounds for the ΨDO A form the main contents of this work.

We shall write:

$$J_\pm(\mu) = (2\pi)^{-m} \int_G v_\pm(\mu, x)\, dx, \quad I_a(\lambda) = \int_{\mathcal{D}(\lambda)} U^a(\lambda, x)\, dx,$$

$a \geq 0$, (clearly, $I_0(\lambda) = \operatorname{meas} \mathcal{D}(\lambda)$); $\sigma_0 = m(m+\rho)^{-1}$, $\sigma = m(\omega+1)^2(m(\omega+1)^2 + \omega\rho)^{-1}$. The quantity $U(\lambda, x)$ is subject to the following conditions:

$$(1.7) \qquad I_0(\lambda) \leq c\lambda^{-q_0},\ I_1(\lambda) \leq c\lambda^{-q},\ I_{\sigma_0}(\lambda) \leq c\lambda^{-p_0},\ I_\sigma(\lambda) \leq c\lambda^{-p},\ c = c(U).$$

Moreover, we may assume that $q_0 \leq p_0 \leq p \leq q$ (cf. (1.3a)).

Theorem 1. *Let $\mathcal{A} \in \Sigma_\rho(f, U)$, $f \in \Lambda_\omega(U)$, and let conditions (1.1) and (1.7) hold. Then for $d = \max(p, p_0(1+p_0)^{-1}(1+q))$, $\forall \epsilon > 0$*

$$(1.8) \qquad |N_\pm(\mu, A) - J_\pm(\mu)| \leq c(\mathcal{A}, \epsilon)\mu^{-d-\epsilon}.$$

Under the conditions $p < q$, $J_\pm(\mu) \geq c\mu^{-q}$ $(c > 0)$ the estimate (1.8) isolates the principal term of the asymptotics of $N_\pm(\mu, A)$ $(d < q)$. The smallest of the possible exponents d in (1.8) is obtained by the optimal choice of the function $f(x, \xi)$ and the quantity $U(\lambda, x)$ for a given symbol $\mathcal{A}(x, y, \xi)$.

§2. Proof of Theorem 1

1. We shall write χ_k for the characteristic function of the set G_k, and $\|\cdot\|_k$ for the norm of the space $L_2(Q_k)$, $k \in K$. Further, $K(\lambda) = \{k : k \in K,\ Q_k \cap \mathcal{D}(\lambda) \neq \emptyset\}$. If $k \in K,\ x \in Q_k$, then in view of (1.2) $\varphi(x) \geq c\lambda,\ c = c(f) > 0$. Therefore

$$Q_k \subseteq \mathcal{D}(c\lambda), \quad \operatorname{card} K(\lambda) \leq \operatorname{meas} \mathcal{D}(c\lambda). \tag{2.1}$$

We denote by Γ the set of points $r \in \mathbf{Z}^m$, such that the cubes Q_r touch the cube Q_0, but do not coincide with it, and by Ω the set of points $r \in \mathbf{Z}^m$, such that $\operatorname{dist}(Q_r, Q_0) \geq 1$.

2. Let $k \in K,\ k+r \in K$. In the cubes $Q_k,\ Q_{k+r}$ we select points $x_k,\ y_{k+r}$ in an arbitrary way. In view of conditions (1.5) and (1.2) $\forall x \in Q_k,\ \forall y \in Q_{k+r}$

(2.2)
$$|(\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \mathcal{A})(x, y, \xi)|_s \leq c(\mathcal{A}, \alpha, \beta, \gamma)(f(x_k, \xi) + f(y_{k+r}, \xi))(1 + |\xi|)^{-\rho|\alpha|} \quad \forall \alpha, \beta, \gamma.$$

We write $\varphi_k = \varphi(x_k)$.

Consider the following operators in the space $L_2(G)$: $A_k = \chi_k \langle \mathcal{A} \rangle \chi_k$, $T_{kr} = \operatorname{Re}(\chi_k \langle \mathcal{A} \rangle \chi_{k+r})$ (cf. no. 5 of §1).

Lemma 1. *For all $\epsilon > 0$ we have*

$$\left| N_\pm(\lambda, A_k) - (2\pi)^{-m} \int_{G_k} v_\pm(\lambda, x)\, dx \right| \leq c(\mathcal{A}, \epsilon) \lambda^{-\epsilon} U(\lambda, x_k); \tag{2.3}$$

(2.3a)
$$N_\pm(\lambda, A_k) \leq [C\,(V(w\lambda, x_k) + (\varphi_k^{-1}\lambda)^{-\epsilon})],\ C = C(\mathcal{A}, \epsilon),\ w = w(\mathcal{A}, \epsilon) > 0.$$

If $r \in \Gamma,\ \epsilon > 0$, then

$$N_\pm(\lambda, T_{kr}) \leq c(\mathcal{A}, \epsilon) \lambda^{-\epsilon} U^{\sigma_0}(\lambda, x_k); \tag{2.4}$$

$$N_\pm(\lambda, T_{kr}) \leq [C\,(V(w\lambda, x_k) + (\varphi_k^{-1}\lambda)^{-\epsilon})]. \tag{2.4a}$$

Proof. If $r \in \Gamma,\ k \in K,\ k+r \in K$, then in view of (2.2), (1.2) $\forall x \in Q_k,\ \forall y \in Q_{k+r}$

$$|(\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \mathcal{A})(x, y, \xi)|_s \leq c(\mathcal{A}, \alpha, \beta, \gamma) f(x_k, \xi)(1 + |\xi|)^{-\rho|\alpha|} \quad \forall \alpha, \beta, \gamma. \tag{2.5}$$

Let $U(\lambda, x) = V(\lambda, x)$. Then in view of (1.3), (1.6)

(2.6)
$$V(t\lambda, x_k) \leq c(f) t^{-\omega} V(\lambda, x_k) \qquad \forall t \in (0, 1);$$
$$v_\pm((1-t)\lambda, x) \leq v_\pm(\lambda, x) + c_1(\mathcal{A}) t V(\lambda, x_k) + c_2(\mathcal{A}) \quad \forall x \in Q_k,\ \forall t \in (0, \tfrac{1}{2}).$$

In view of (2.5), (2.6) and (1.1) inequalities (2.3)–(2.4a) follow from results in [5] (Lemmas 3,4,6, Theorem 1).

Let $U(\lambda, x) \geq V(\lambda, x)$. In this case we repeat the proofs of [5] replacing $V(\lambda, x_k)$ by $U(\lambda, x_k)$ in all remainder terms and using (1.3), (1.6). This leads to the bounds (2.3), (2.4). The proof of (2.3a), (2.4a) can be repeated in its entirety. $\square$

Corollary. *Let $r \in \Gamma$, $k \in K$, $k+r \in K$. Then there is a constant $c^* = c^*(\mathcal{A}) > 0$ such that, if $x_k \notin \mathcal{D}(c^*\lambda)$, then $N_\pm(\lambda, A_k) = N_\pm(\lambda, T_{kr}) = 0$.*

This statement follows from (2.3a), (2.4a) for any $\epsilon > 0$.

Lemma 2. *Let $k \in K$, $k+r \in K$, $r \in \Omega$. Then $\forall N, M \in \mathbf{N}$, $M < 2\rho N - m$,*

$$N_\pm(\lambda, T_{kr}) \leq [c(\mathcal{A}, N, M)(|r|^{2N}(\varphi_k + \varphi_{k+r})^{-1}\lambda)^{mM^{-1}}]. \tag{2.7}$$

Proof. Let $u \in L_2(G_k)$. For $x \in Q_{k+r}$, $|\alpha| \leq M$

$$(\partial_x^\alpha \langle \mathcal{A} \rangle \chi_k u)(x) = \int_{G_k} \mathcal{A}_\alpha(x,y) u(y)\, dy,$$

$$\mathcal{A}_\alpha(x,y) = c_\alpha \int \exp(i(x-y)\xi)|x-y|^{-2N}\xi^\alpha \, d\xi \, ((-\Delta_\xi)^N \mathcal{A})(x,y,\xi).$$

From estimate (2.2) follows that for $y \in Q_k$, $x \in Q_{k+r}$

$$|\mathcal{A}_\alpha(x,y)|_s \leq c(\mathcal{A}, N, M)(\varphi_k + \varphi_{k+r})|r|^{-2N}.$$

Therefore

$$|(\partial_x^\alpha \langle \mathcal{A} \rangle \chi_k u)(x)| \leq c(\mathcal{A}, N, M)(\varphi_k + \varphi_{k+r})|r|^{-2N}\|u\|_k \quad \forall x \in Q_{k+r}, \ |\alpha| \leq M. \tag{2.8}$$

This inequality is true also with k and $k+r$ interchanged.

We denote by $W_2^M(Q_k)$ the Sobolev space with standard norm $\|\cdot\|_{(M,k)}$ and scalar product $(\cdot,\cdot)_{(M,k)}$, and $D_{(M)}$ is the operator corresponding to the quadratic form $\|\cdot\|_k^2$ in the space $W_2^M(Q_k)$. It is well known that (cf., for example, [6])

$$N_\pm(\mu, D_{(M)}) \leq c(m, M)\mu^{-m(2M)^{-1}}. \tag{2.9}$$

Let $v = T_{kr}u$, $u \in L_2(Q_k \cup Q_{k+r})$. In view of (2.8)

$$\frac{\|v\|_k^2 + \|v\|_{k+r}^2}{\|u\|_k^2 + \|u\|_{k+r}^2} \leq c(\mathcal{A}, N, M)\frac{\varphi_k^2 + \varphi_{k+r}^2}{|r|^{4N}} \frac{(D_{(M)}v, v)_{(M,k)} + (D_{(M)}v, v)_{(M,k+r)}}{\|v\|_{(M,k)}^2 + \|v\|_{(M,k+r)}^2}. \tag{2.10}$$

The statement of the lemma follows from (2.10) and (2.9). $\square$

3. We represent the operator A in the following form:

$$\begin{gathered} A = \tilde{A} + T_\Gamma + T_\Omega, \\ \tilde{A} = \textstyle\sum_K^{(k)} \oplus A_k, \ T_\Gamma = \sum_\Gamma^{(r)} \sum_K^{(k)} \oplus T_{kr}, \ T_\Omega = \sum_\Omega^{(r)} \sum_K^{(k)} \oplus T_{kr}. \end{gathered} \tag{2.11}$$

(If $k + r \notin K$, then the corresponding operator $T_{kr} = 0$.)

Lemma 3. *For all $\epsilon > 0$ we have*

$$|N_\pm(\lambda, \tilde{A}) - J_\pm(\lambda)| \leq c(\mathcal{A}, \epsilon)\lambda^{-p-\epsilon}. \tag{2.12}$$

Proof. We bound the spectrum of $\tilde{A}$ from above. We note that in (2.3)–(2.4a) the choice of the points $x_k \in Q_k$ is arbitrary. In particular, if $k \notin K(c^*\lambda)$ (see the corollary of Lemma 1), then we may assume that in (2.3a) $x_k \notin \mathcal{D}(c^*\lambda)$, so that $N_\pm(\lambda, A_k) = 0$. Therefore,

$$N_\pm(\lambda, \tilde{A}) = \textstyle\sum_K^{(k)} N_\pm(\lambda, A_k) = \sum_{K(c^*\lambda)}^{(k)} N_\pm(\lambda, A_k).$$

By virtue of (2.3) $\forall \epsilon > 0$

$$N_\pm(\lambda, \tilde{A}) \le (2\pi)^{-m} \textstyle\sum_{K(c^*\lambda)}^{(k)} \displaystyle\int_{G_k} v_\pm(\lambda, x)\, dx + c(\mathcal{A}, \epsilon)\lambda^{-\epsilon} \textstyle\sum_{K(c^*\lambda)}^{(k)} U^\sigma(\lambda, x_k).$$

Using (2.1) and with appropriately chosen points $x_k \in Q_k$, we conclude that

$$\textstyle\sum_{K(c^*\lambda)}^{(k)} U^\sigma(\lambda, x_k) \le I_\sigma(c\lambda), \quad c = c(\mathcal{A}, f) > 0. \tag{2.13}$$

Hence, taking condition (1.7) into account, $\forall \epsilon > 0$

$$N_\pm(\lambda, \tilde{A}) \le J_\pm(\lambda) + c_1(\mathcal{A}, \epsilon)\lambda^{-p-\epsilon}. \tag{2.14}$$

We now bound the quantity $N_\pm(\lambda, \tilde{A})$ from below. From (1.5) follows that $\forall x \in \mathcal{D}(C\lambda)$ $v_\pm(\lambda, x) = 0$, $C = C(\mathcal{A}) > 0$. Therefore

$$(2\pi)^{-m} \textstyle\sum_{K(C\lambda)}^{(k)} \displaystyle\int_{G_k} v_\pm(\lambda, x)\, dx = J_\pm(\lambda).$$

Moreover, $N_\pm(\lambda, \tilde{A}) \ge \sum_{K(C\lambda)}^{(k)} N_\pm(\lambda, A_k)$. Using (2.3) we obtain that $\forall \epsilon > 0$

$$N_\pm(\lambda, \tilde{A}) \ge J_\pm(\lambda) - c(\mathcal{A}, \epsilon) \textstyle\sum_{K(C\lambda)}^{(k)} U^\sigma(\lambda, x_k)\lambda^{-\epsilon}.$$

Analogously to (2.13), (2.14) we arrive at the estimate

$$N_\pm(\lambda, \tilde{A}) \ge J_\pm(\lambda) - c_2(\mathcal{A}, \epsilon)\lambda^{-p-\epsilon} \qquad \forall \epsilon > 0. \tag{2.15}$$

The statement of the lemma follows from (2.14) and (2.15). $\square$

Lemma 4. *For all $\epsilon > 0$ we have*

$$N_\pm(\lambda, T_\Gamma) \le c(\mathcal{A}, \epsilon)\lambda^{-p_0-\epsilon}. \tag{2.16}$$

Proof. By Weyl's inequality

$$N_\pm(\lambda, T_\Gamma) \le \textstyle\sum_\Gamma^{(r)} \sum_K^{(k)} N_\pm(c_m\lambda, T_{kr}), \quad c_m = (\operatorname{card}\Gamma)^{-1}.$$

The estimate of $\sum_K^{(k)} N_\pm(c_m\lambda, T_{kr})$, $r \in \Gamma$, reiterates the estimate from above of $N_\pm(\lambda, \tilde{A})$ (cf. the proof of Lemma 3). Instead of inequality (2.3) one uses (2.4). This proves the lemma. $\square$

Lemma 5. *For all $\epsilon > 0$ we have*

$$N_\pm(\lambda, T_\Omega) \le c(\mathcal{A}, \epsilon)\lambda^{-q_0-\epsilon}. \tag{2.17}$$

Proof. Using Weyl's inequality we obtain that

$$N_\pm(\lambda, T_\Omega) \le {\sum}_\Omega^{(r)} {\sum}_K^{(k)} N_\pm(c_m^*|r|^{-m-1}\lambda, T_{kr}), \quad c_m^* = \left({\sum}_\Omega^{(r)} |r|^{-m-1}\right)^{-1}. \tag{2.18}$$

In view of estimate (2.7) and condition (1.2)

$$N_\pm(c_m^*|r|^{-m-1}\lambda, T_{kr}) \le [c(\mathcal{A}, N, M)|r|^{-mM^{-1}(2N-l_1-m-1)}(\varphi_k\lambda)^{-mM^{-1}}], \tag{2.19}$$

$M < 2\rho N - m$. Here we put $N = [2^{-1}((M+m)(1+\rho^{-1}) + l_1 + 3)]$. The right side of (2.19) equals zero $\forall k \notin \mathcal{D}(c\lambda)$, $c = c(\mathcal{A}, M) > 0$. Hence from (2.18), (2.19) follows that

$$N_\pm(\lambda, T_\Omega) \le c(\mathcal{A}, M)\left({\sum}_\Omega^{(r)} |r|^{-m-\kappa}\right) {\sum}_{\mathcal{D}(c\lambda)}^{(k)} (\varphi_k\lambda)^{-mM^{-1}}, \quad \kappa > 0.$$

Applying this bound for "large" M we arrive at

$$N_\pm(\lambda, T_\Omega) \le C(\mathcal{A}, \epsilon)\,\mathrm{card}\, K(c\lambda) \qquad \forall \epsilon > 0,\ c = c(\mathcal{A}, \epsilon).$$

The statement of the lemma now follows from (2.1) and (1.7). $\square$

4. We complete the proof of Theorem 1. In view of (2.11) $\forall t \in (0, \frac{1}{2})$

$$N_\pm(\mu, A) \le N_\pm((1-t)\mu, \tilde{A}) + N_\pm(t\mu, A_\Gamma + A_\Omega). \tag{2.20}$$

Using (2.12) we conclude that $\forall \epsilon > 0$

$$N_\pm((1-t)\mu, \tilde{A}) \le J_\pm((1-t)\mu) + c(\mathcal{A}, \epsilon)\mu^{-p-\epsilon}.$$

Note that $\forall x \notin \mathcal{D}(c\mu)$ $v_\pm((1-t)\mu, x) = 0$, $c = c(\mathcal{A}) > 0$. Hence

$$J_\pm((1-t)\mu) = (2\pi)^{-m} \int_{\mathcal{D}(c\mu)} \chi_G(x) v_\pm((1-t)\mu, x)\, dx.$$

By (1.6) and (1.7)

$$\begin{aligned} &\int_{\mathcal{D}(c\mu)} \chi_G(x) v_\pm((1-t)\mu, x)\, dx \\ &\le (2\pi)^m J_\pm(\mu) + c_1(\mathcal{A}) t \int_{\mathcal{D}(c\mu)} U(\mu, x)\, dx + c_2(\mathcal{A})\,\mathrm{meas}\,\mathcal{D}(c\mu) \\ &\le (2\pi)^m J_\pm(\mu) + c(\mathcal{A})(t\mu^{-q} + \mu^{-q_0}). \end{aligned}$$

Finally, recalling that $q_0 \le p$, we obtain the following bound: $\forall \epsilon > 0$

$$N_\pm((1-t)\mu, \tilde{A}) \le J_\pm(\mu) + c(\mathcal{A}, \epsilon)(t\mu^{-q} + \mu^{-p}).$$

The quantity $N_\pm(t\mu, A_\Gamma + A_\Omega)$ is estimated using (2.16) and (2.17). Since $q_0 \le p_0$, we have

$$N_\pm(t\mu, A_\Gamma + A_\Omega) \le c(\mathcal{A}, \epsilon) t^{-p_0-\epsilon} \qquad \forall \epsilon > 0.$$

Hence, from (2.20) follows that

$$N_\pm(\mu, A) \le J_\pm(\mu) + c(\mathcal{A}, \epsilon)(t\mu^{-q} + t^{-p_0}\mu^{-p_0} + \mu^{-p})\mu^{-\epsilon} \qquad \forall \epsilon > 0.$$

Minimizing the right-hand side of this inequality with respect to t, we obtain

$$N_\pm(\mu, A) \le J_\pm(\mu) + c(\mathcal{A}, \epsilon)\mu^{-d-\epsilon} \qquad \forall \epsilon > 0. \tag{2.21}$$

Further, we note that $N_\pm(\mu, A) \ge N_\pm((1+t)\mu, \tilde{A}) - N_\mp(t\mu, A_\Gamma + A_\Omega)$, $t \in (0, \frac{1}{2})$. Repeating the above argument we establish the following estimate:

$$N_\pm(\mu, A) \ge J_\pm(\mu) - c(\mathcal{A}, \epsilon)\mu^{-d-\epsilon} \qquad \forall \epsilon > 0. \tag{2.22}$$

The statement of the theorem follows from (2.21) and (2.22).

§3. Quasiregular spectral asymptotics of ΨDO

1. In this paragraph the function $f(x,\xi)$, occurring in the definition of the symbol of the ΨDO (see no. 3 of §1), will be represented as a product of two factors depending on x and ξ, respectively.

More precisely, $f(x,\xi)=\varphi(x)g(\xi)$, where $\varphi(x): Q\to \mathbf{R}_+$ and $g(\xi):\mathbf{E}^m\to\mathbf{R}_+$. We shall write: $\mathcal{D}(\lambda)=\{x\in Q:\varphi(x)>\lambda\}$, $I_0(\lambda)=\operatorname{meas}\mathcal{D}(\lambda)$, $v(\lambda)=\operatorname{meas}\{\xi: g(\xi)>\lambda\}$. The functions $\varphi(x)$, $g(\xi)$ are subject to the following conditions:

$$\varphi(x)\le c_1\,(1+|x|)^{-l_1};\quad \varphi(x)/\varphi(y)\le c_2\,(1+|x-y|^{l_2}), \tag{3.1}$$
$$\forall x,y\in Q,\ c_i=c_i(\varphi),\ l_i=l_i(\varphi),\ l_1>0;$$

$$I_0(\lambda)\le c_3\lambda^{-q_0},\quad c_3=c_3(\varphi); \tag{3.2}$$

$$g(\xi)\le(1+|\xi|)^{-l_3},\quad l_3=l_3(g)>0; \tag{3.3}$$

$$c_4\lambda^{-\omega}\le v(\lambda)\le c_5\lambda^{-\omega},\quad \omega=\omega(g)>0, \tag{3.4}$$

$c_i=c_i(g)>0$. The concrete values of the constants c_i, l_i are not essential in the following.

Clearly $V(\lambda,x)=v(\lambda/\varphi(x))$. Put $U(\lambda,x)=V(\lambda,x)+1$. From (3.1)–(3.4) follows that conditions (1.2)–(1.3a) are satisfied. The function $f\in\Lambda_\omega(U)$.

2. Consider the ΨDO A with symbol $\mathcal{A}\in\Sigma_\rho(f,U)$ in the space $L_2(G)$ (cf. nos. 4, 5 of §1).

Theorem 2. *Let the conditions (1.1) and (3.1)–(3.4) hold with $\omega>q_0$. Then for $d_0=\max(q_0(1+\omega)(1+q_0)^{-1},\sigma_0\omega(1+\omega)(1+\sigma_0\omega)^{-1},\sigma\omega)$, $\forall\epsilon>0$*

$$|N_\pm(\mu,A)-J_\pm(\mu)|\le c(\mathcal{A},\epsilon)\mu^{-d_0-\epsilon}. \tag{3.5}$$

Proof. We will show that $\forall a\ge 0$

$$I_a(\lambda)\le c(f,a)(\lambda^{-a\omega}+\lambda^{-q_0})(1+|\ln\lambda|). \tag{3.6}$$

For $n=0,1,\dots$ we write: $\mathcal{D}_n(\lambda)=\mathcal{D}(2^n\lambda)\backslash\mathcal{D}(2^{n+1}\lambda)$. Hence $\mathcal{D}(\lambda)=\bigcup_{n=0}^{N(\lambda)}\mathcal{D}_n(\lambda)$, $N(\lambda)=c(\varphi)(1+|\ln\lambda|)$. From (3.4) follows that $U(\lambda,x)\le c(f)(\varphi^\omega(x)\lambda^{-\omega}+1)$. Therefore

$$I_a(\lambda)\le c_1(a,f)\left(\lambda^{-a\omega}\sum_{n=0}^{N(\lambda)}\int_{\mathcal{D}_n(\lambda)}\varphi^{a\omega}(x)\,dx+I_0(\lambda)\right). \tag{3.7}$$

Further, in view of (3.2),

$$\int_{\mathcal{D}_n(\lambda)}\varphi^{a\omega}(x)\,dx\le(2^{n+1}\lambda)^{a\omega}I_0(2^n\lambda)\le c(f,a,\omega)(2^{n(a\omega-q_0)}\lambda^{a\omega-q_0}+2^{na\omega}\lambda^{a\omega}). \tag{3.8}$$

Inequality (3.6) follows from (3.7), (3.8).

From (3.6) follows that conditions (1.7) are satisfied for $q=\omega+\delta$, $p_0=\max(\sigma_0\omega,q_0)+\delta$, $p=\max(\sigma\omega,q_0)+\delta$ $\forall\delta>0$. Hence the hypotheses of Theorem 1 are satisfied and (3.5) is a special case of (1.8). □

3. Let for some point $x_0 \in G$, and some j

$$\pm\lambda_j(x_0, x_0, \xi) \geq cg(\xi), \qquad c > 0. \tag{3.9}$$

We construct the ball $T_\delta = \{x : |x - x_0| < \delta\}$, $\delta < \operatorname{dist}(x_0, \partial G)$. Using (1.5) it is easy to show that, for sufficiently small δ, $\pm\lambda_j(x, x, \xi) \geq cg(\xi)/2$ $\forall x \in T_\delta$. Hence $v_\pm(\mu, x) \geq v(2c^{-1}\mu)$ $\forall x \in T_\delta$. By (3.4)

$$\int_{T_\delta} v_\pm(\mu, x)\, dx \geq c(x_0, \mathcal{A})v(\mu), \qquad c(x_0, \mathcal{A}) > 0. \tag{3.10}$$

If condition (3.9) holds and $\omega > q_0$, then the quantities $N_\pm(\mu)$ and $v(\mu)$ agree in order. In fact, we note that the function $\varphi^\omega(x)$ is integrable in Q (the proof of this differs from the proof of (3.6) only by simplifications). Hence, in view of (1.5), (3.4),

$$J_\pm(\mu) \leq \int V(c_1\mu, x)\, dx \leq c_2 \left(\int_Q \varphi^\omega(x)\, dx\right) v(\mu) \leq c_3 v(\mu), \qquad c_i = c_i(\mathcal{A}),\ c_1 > 0.$$

Simultaneously, by (3.10), $J_\pm(\mu) \geq c_4 v(\mu)$, $c_4 > 0$. Since $d_0 < \omega$, estimate (3.5) isolates the principal term of the spectral asymptotics of the operator A.

Consider the ΨDO A_δ with symbol $\mathcal{A}(x, y, \xi)$ in the space $L_2(T_\delta)$. In view of the results of [5]

$$N_\pm(\mu, A_\delta) \sim (2\pi)^{-m} \int_{T_\delta} v_\pm(\mu, x)\, dx, \qquad \mu \to +0.$$

Thus, the spectral asymptotics of the operator A and the operator A_δ of the regular problem agree in order. We remark that condition (3.9) is guaranteed by a sufficiently precise choice of the functions $\varphi(x)$, $g(\xi)$ for the symbol $\mathcal{A}(x, y, \xi)$. The inequality $\omega > q_0$ plays the role of a regularity condition for the spectral asymptotics of the ΨDO A.

4. We shall now apply the result of Theorem 2 to ΨDO with symbols that are anisotropically homogeneous with respect to the variable ξ.

We write: $\Xi_\delta = \{\xi : |\xi| > \delta\}$, $\delta > 0$, and we let $h(\xi)$ be a cut-off function: $h \in C^\infty(\mathbf{E}^m)$, $0 \leq h(\xi) \leq 1$, $h(\xi) = 0$, if $|\xi| < 1$, and $h(\xi) = 1$, if $|\xi| > 2$. Further, $\tau = (\tau_1, \ldots, \tau_m)$, $\tau_j > 0$, $\tau_1 + \cdots + \tau_m = m$, $W_\tau(t) = \{t^{\tau_j}\delta_{ij}\}$, $i, j = 1, \ldots, m$, $t > 0$.

Let the $s \times s$ matrix function $\mathcal{A}_0(x, y, \xi)$ be defined for $x, y \in Q$, $\xi \in \mathbf{E}^m \setminus \{0\}$, with $\mathcal{A}_0^*(x, y, \xi) = \mathcal{A}_0(y, x, \xi)$. The function $\mathcal{A}_0$ is assumed to be τ-homogeneous, i.e., $\forall t > 0$ $\mathcal{A}_0(x, y, W_\tau(t)\xi) = t^{-l}\mathcal{A}_0(x, y, \xi)$, $l > 0$. Moreover,

$$\begin{gathered} |(\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \mathcal{A}_0)(x, y, \xi)|_s \leq c(\mathcal{A}_0, \alpha, \beta, \gamma, \delta)(\varphi(x) + \varphi(y)) \\ \forall x, y \in Q,\ \forall \xi \in \Xi_\delta,\ \forall \delta > 0,\ \forall \alpha, \beta, \gamma. \end{gathered} \tag{A}$$

The function $\varphi(x)$ satisfies conditions (3.1) and (3.2).

The eigenvalues of the matrix $\mathcal{A}_0(x, x, \xi)$ are denoted by $a_j^{(0)}(x, \xi)$. Moreover,

$$v_\pm^{(0)}(x) = \sum_{j=1}^{s} \operatorname{meas}\{\xi : \pm a_j^{(0)}(x, \xi) > 1\},\ C_\pm^{(0)} = (2\pi)^{-m} \int_G v_\pm^{(0)}(x)\, dx.$$

We shall write $\omega = ml^{-1}$, $\rho_0 = \min_{i,j}(\tau_i \tau_j^{-1})$, $\sigma_0 = m(m + \rho_0)^{-1}$, $\sigma = (m + l)^2((m + l)^2 + l\rho_0)^{-1}$, $d_0 = \max(q_0(\omega + 1)(q_0 + 1)^{-1}, \sigma_0\omega(m + l)(l + m\sigma_0)^{-1}, \omega\sigma)$.

We consider in the space $L_2(G)$ a ΨDO A with symbol $\mathcal{A}(x, y, \xi) = h(\xi)\mathcal{A}_0(x, y, \xi)$. In the case $G = \mathbf{R}^m$, $\omega > 1$, the principal term of the spectral asymptotics of this operator was established in [1].

Theorem 3. *Let conditions (1.1) and (A) hold with $q_0 < \omega$. Then $\forall \epsilon > 0$*

$$|N_\pm(\mu, A) - C_\pm^{(0)}\mu^{-\omega}| \le c(\mathcal{A}, \epsilon)\mu^{-d_0-\epsilon}. \tag{3.11}$$

Proof. We write

$$g_n(\xi) = \left(\sum_{j=1}^{m} |m\xi_j|^{n\tau_j^{-1}}\right)^{-n^{-1}l}, \qquad n > 0.$$

Clearly, $\forall \xi \in \Xi_\delta,\ \forall \alpha,\ |\alpha| < nm^{-1}$

$$|(\partial_\xi^\alpha g_n)(\xi)| \le c(n, \tau, \delta) g_n(\xi)(1 + |\xi|)^{\rho_0 \alpha}, \quad g_n(\xi) \le c(n, \tau) g_1(\xi). \tag{3.12}$$

We represent the function $\mathcal{A}_0(x, y, \xi)$ as $\mathcal{A}_0(x, y, \xi) = \mathcal{A}_0(x, y, W_\tau(t)\xi) g_n(\xi),\ t = g_n^{l^{-1}}(\xi)$. Here $W_\tau(t)\xi \in \Xi_\delta,\ \delta = \delta(\tau, n) > 0$. Using this representation for sufficiently large n, and inequalities (A), (3.12), we obtain

$$|(\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \mathcal{A})(x, y, \xi)|_s \le c(\mathcal{A}, \alpha, \beta, \gamma)(\varphi(x) + \varphi(y)) g_1(\xi)(1 + |\xi|)^{-\rho_0|\alpha|} \tag{3.13}$$
$$\forall x, y \in Q,\ \forall \xi \in \mathbf{E}^m,\ \forall \alpha, \beta, \gamma.$$

Hence, it follows that

$$v \pm^{(0)} (x) \le c(\mathcal{A}) \varphi^\omega(x), \qquad x \in Q. \tag{3.14}$$

Moreover,

$$v_\pm(\lambda, x) = v_\pm^{(0)}(x)\lambda^{-\omega} + O(1),\ v_\pm(\lambda, x) = 0\ \forall x \notin \mathcal{D}(w\lambda),\ w = w(\mathcal{A}) > 0. \tag{3.15}$$

We put $c_\tau = \max g_1(\xi),\ \xi \in \Xi_1$ and $g(\xi) = (1 + c_\tau)^{-1}(h(\xi) g_1(\xi) + 1 - h(\xi))$. The function $g(\xi)$ clearly satisfies (3.3) and (3.4). Let $f_\tau(x, \xi) = \varphi(x) g(\xi)$, $U_\tau(\lambda, x) = c^*(\mathcal{A})\varphi^\omega(x)\lambda^{-\omega} + 1$. In view of (3.13)–(3.15) the symbol $\mathcal{A} \in \Sigma_{\rho_0}(f_\tau, U_\tau)$ for the appropriate value of the constant $c^*(\mathcal{A})$. From (3.14) and (3.15) follows that

$$J_\pm(\mu) = C_\pm^{(0)}\mu^{-\omega} + R_\pm(\mu),$$
$$|R_\pm(\mu)| \le c(\mathcal{A}) \left(I_0(w\mu) + \mu^{-\omega} \int_{Q \setminus \mathcal{D}(w\mu)} \varphi^\omega(x)\, dx \right).$$

It is easy to see (cf. the proof of (3.6)) that

$$\int_{Q \setminus \mathcal{D}(w\mu)} \varphi^\omega(x)\, dx \le c(\varphi, w)\mu^{\omega - q_0},$$

whence $|R_\pm(\mu)| \le c(\mathcal{A})\mu^{-q_0}$. The estimate (3.11) now follows as a consequence of Theorem 2. □

5. We now consider the case, where the symbol $\mathcal{A}(x, y, \xi)$ is isotropically homogeneous with respect to the variable ξ, while the function $\varphi(x)$ has power order: $\mathcal{A}(x, y, t\xi) = t^{-l}\mathcal{A}(x, y, \xi)\ \forall \xi \in \Xi_2,\ \forall t > 0$, and $\varphi(x) = (1 + |x|)^{-l_0},\ l > 0,\ l_0 > 0$. Clearly here $\tau_j = 1\ \forall j,\ \rho_0 = 1$. The exponent d_0 in (3.11) is defined for $\sigma_0 = m(m + 1)^{-1},\ \sigma = (m + l)^2((m + l)^2 + l)^{-1}$.

If $G = \mathbf{R}^m$, then $q_0 = ml_0^{-1}$ and the regularity condition of the spectral asymptotics takes the form $l_0 > l$.

Let G be a parabolic domain in $\mathbf{R}^m$: $G = \{x : x_m > \sum_{j=1}^{m-1} x_j^2\}$. Then $q_0 = (2l_0)^{-1}(m + 1)$. The regularity condition is improved: $l_0 > (2m)^{-1}(m + l)l$. The form of the asymptotics remains the same as before with the corresponding coefficient $C_\pm^{(0)} = C_\pm^{(0)}(\mathcal{A}, G)$.

§4. On a class of ΨDO with nonregular spectral asymptotics

1. We now consider some compact ΨDO occurring in the study of the negative spectrum of the Schrödinger equation.

Let the functions $P(x)$, $F(x)$, $T(x)$ be defined $\forall x \in Q$, and $P(x) \geq 1$, $F(x) \in \mathbf{R}$, $T(x) > 0$. It is assumed that $F, P \in C^\infty(Q)$, with

$$
\begin{gathered}
|(\partial_x^\alpha P)(x)| \leq c(P,\alpha)P(x), \quad |(\partial_x^\alpha F)(x)| \leq c(F,\alpha)T(x), \\
P(x) \leq c(P)(1+|x|)^{l_1}, \quad T(x) \leq c(T)(1+|x|)^{l_2}, \quad \forall x \in Q, \ \forall \alpha, \ l_i = l_i(P,T).
\end{gathered} \tag{4.1}
$$

We write: $\varphi(x) = T(x)P^{-b}(x)$, $b > 0$. As before, $\mathcal{D}(\lambda) = \{x : x \in Q, \ \varphi(x) > \lambda\}$, $I_0(\lambda) = \operatorname{meas} \mathcal{D}(\lambda)$, and

$$
\begin{gathered}
\varphi(x)/\varphi(y) \leq c(\varphi)(1+|x-y|^{l_3}), \quad \varphi(x) \leq c(\varphi)(1+|x|)^{-l_4}, \\
l_i = l_i(\varphi), \ l_4 > 0, \ x, y \in Q; \quad I_0(\lambda) \leq c(\varphi)\lambda^{-q_0}.
\end{gathered} \tag{4.2}
$$

The concrete values of the exponents l_i in (4.1) and (4.2) are not essential.

Let

$$
\mathcal{A}(x,y,\xi) = 2^{-1}(|\xi|^{2a} + P(x))^{-b/2}(F(x)+F(y))(|\xi|^{2a} + P(y))^{-b/2}, \ a \in \mathbf{N}.
$$

We consider a scalar ΨDO with symbol $\mathcal{A}(x,y,\xi)$ in the space $L_2(G)$. This operator is compact in view of (4.2).

2. We shall write: $\omega = (2ab)^{-1}m$, $\sigma_0 = m(m+1)^{-1}$, $\sigma = m(\omega+1)^2(m(\omega+1)^2 + \omega)^{-1}$. In addition, the function $T(x)$ is subject to the following conditions:

$$
\begin{gathered}
\int_{\mathcal{D}(\lambda)} T^\omega(x)\,dx \leq c\lambda^{-w}, \ \int_{\mathcal{D}(\lambda)} T^{\sigma_0\omega}(x)\,dx \leq c\lambda^{-r_0}, \\
\int_{\mathcal{D}(\lambda)} T^{\sigma\omega}(x)\,dx \leq c\lambda^{-r}, \qquad c = c(T).
\end{gathered} \tag{4.3}
$$

We write: $q = \max(\omega + w, q_0)$, $p_0 = \max(r_0 + \omega\sigma_0, q_0)$, $p = \max(r + \omega\sigma, q_0)$, $d = \max(p, p_0(1+p_0)^{-1}(1+q))$.

Theorem 4. *Let conditions (1.1) and (4.1)–(4.3) hold, and let $m \geq 2a$. Then $\forall \epsilon > 0$*

$$
|N_\pm(\mu, A) - J_\pm(\mu)| \leq c(\mathcal{A}, \epsilon)\mu^{-d-\epsilon}. \tag{4.4}
$$

Proof. Put $f(x,\xi) = (|\xi|^{2a} + P(x))^{-b}T(x)$, $U(\lambda,x) = c_m\lambda^{-\omega}T^\omega(x) + 1$, $c_m = m^{-1}\operatorname{meas}\mathbf{S}^{m-1}$. Clearly, $V(\lambda,x) = c_m\lambda^{-\omega}(T^{b^{-1}}(x) - \lambda^{b^{-1}}P(x))_+^{\omega b}$. Condition (1.2) holds for $f(x,\xi)$ in view of (4.2). Since (1.3)–(1.4) are also satisfied, we have $f \in \Lambda_\omega(U)$. From (4.1) follows that inequality (1.5) is true for $\rho = 1$.

Moreover, $v_\pm(\lambda,x) = c_m\lambda^{-\omega}(F_\pm^{b^{-1}}(x) - \lambda^{b^{-1}}P(x))_+^{\omega b}$, $\omega b = (2a)^{-1}m \geq 1$. It is easy to see that

$$
\left(\frac{\partial v_\pm}{\partial \lambda}\right)(\lambda,x) \leq c(\mathcal{A})\lambda^{-\omega-1}F_\pm^\omega(x) \leq c(\mathcal{A})\lambda^{-\omega-1}T^\omega(x),
$$

so that (1.6) holds. Hence, the symbol $\mathcal{A} \in \Sigma_1(f,U)$. Inequality (1.7) is satisfied in view of (4.3). The bound (4.4) is a consequence of (1.8). □

3. The quantity $J_\pm(\mu) = c_m \mu^{-\omega} \Psi_\pm(\mu)$, where

$$\Psi_\pm(\mu) = \int_G (F_\pm^{b^{-1}}(x) - \mu^{b^{-1}} P(x))_+^{\omega b}\, dx.$$

If the function $F_\pm^\omega(x)$ is integrable in G, then $\Psi_\pm(\mu) = c_\pm(\mathcal{A}, G) + o(1),\ \mu \to +0$. Under the condition $q_0 < \omega$ the spectral asymptotics of the operator A becomes regular: $N_\pm(\mu, A) \sim c_\pm c_m \mu^{-\omega}$.

Let the function $F_\pm^\omega(x)$ be nonintegrable in G, and $\Psi_\pm(\mu) \geq c\mu^{-\omega_0},\ \omega_0 > 0,\ c > 0$. If $d < \omega + \omega_0$, then the estimate (4.4) isolates the principal term of the spectral asymptotics. These asymptotics are different in order than the regular ones.

4. We consider the case of homogeneous "potentials" $P(x)$ and $F(x)$. More precisely, $G = \mathbf{R}^m,\ z = |x|^{-1}x \in \mathbf{S}^{m-1},\ P(x) = h(x)\mathcal{P}(z)|x|^l + 1 - h(x/2),\ F(x) = h(x)\mathcal{F}(z)|x|^n$. Here $\mathcal{P}, \mathcal{F} \in C^\infty(\mathbf{S}^{m-1})$, with $\mathcal{P}(z) \geq 1,\ \mathcal{F}(z) \in \mathbf{R},\ z \in \mathbf{S}^{m-1}$, and $h(x)$ is a cut-off function, analogous to the one introduced in no. 4 of §3. Concerning the orders of homogeneity $l,\ n$, we assume that $l \geq 0,\ n < bl,\ \omega n > -m$. As before, $m \geq 2a$.

Put $T(x) = h(x)|x|^n + 1 - h(x)$. Clearly, condition (4.1) is then satisfied, and (4.2) holds for $q_0 = m(bl-n)^{-1}$. Inequality (4.3) is true for $w = (m+\omega n)(bl-n)^{-1},\ r_0 = (m+\sigma_0\omega n)(bl-n)^{-1},\ r = (m+\sigma\omega n)(bl-n)^{-1}$. Calculation of the quantity $\Psi_\pm(\mu)$ leads to the following result: $\Psi_\pm(\mu) = c_\pm(\mathcal{A})\mu^{-\omega_0} + O(1)$, $\omega_0 = (m + l\omega b)(bl-n)^{-1}$ and

$$c_\pm(\mathcal{A}) = b(bl-n)^{-1} B((\omega n + m)(bl-n)^{-1}, \omega b + 1) \int_{\mathbf{S}^{m-1}} (\mathcal{F}_\pm(z)\mathcal{P}^{-b}(z))^{\omega_0}\, ds(z),$$

where $B(\cdot,\cdot)$ is the beta function.

Since in the case under consideration $d < \omega + \omega_0$, the bound (4.4) isolates the spectral asymptotics of the operator A:

$$N_\pm(\mu, A) \sim c_\pm(\mathcal{A}) c_m \mu^{-\omega-\omega_0} + O(\mu^{-d-\epsilon}) \qquad \forall \epsilon > 0.$$

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