1 Introduction

This thesis considers finite element solutions of some partial differential equations. All our results concern a posteriori error estimates in the maximum norm. In order to illustrate these concepts we first discuss two model problems: a stationary one and a time-dependent one. We then summarise the three papers that make up the thesis.

1.1 A stationary problem

Let Ω be a convex, polygonal and bounded domain in $\mathbf{R}^d$. Let u be the solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Delta = \sum_{i=1}^d \partial^2_{x_i x_i}$ and $\partial\Omega$ is the boundary of Ω. This is Dirichlet's problem for Poisson's equation and is a prototype of an elliptic differential equation. Let $(\varphi, \psi) = \int_\Omega \varphi(x)\psi(x)\,dx$. The weak formulation of (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{1.2}$$

We now introduce the finite element method. Let $\mathcal{F} = \{\mathcal{T}\}$ be a nondegenerate family of triangulations, where a triangulation is a partition of Ω into open simplices K which are face to face so that $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$. Let $h = h(x)$ denote the function that measures the pointwise size of the triangulation. More precisely $h|_K = \operatorname{diam}(K)$. We now introduce the space of trial and test functions. This is the subspace $S \subset H_0^1(\Omega)$ consisting of the functions that are piecewise linear with respect to the triangulation $\mathcal{T}$.

The finite element problem reads as follows. Find $U \in S$ such that

$$(\nabla U, \nabla \chi) = (f, \chi), \quad \forall \chi \in S. \tag{1.3}$$

It can be proved that this problem has a unique solution U.

A natural question is whether U is a good approximation of u and in what sense. We consider two ways of approaching this question: a priori error estimates and a posteriori error estimates. In the a priori approach we estimate the error in terms of a priori quantities such as data and derivatives of the exact solution u. An a posteriori error estimate bounds the error in terms of data and computed quantities such as U and its derivatives. Further, the error can be measured in different norms. To make this discussion more precise we introduce the L_p norms. Let

$$\|\varphi\|_{L_p(\Omega)} = \Big(\int_\Omega |\varphi(x)|^p\,dx\Big)^{1/p}, \quad 1 \le p < \infty, \tag{1.4}$$

and, for $p=\infty$,

$$\|\varphi\|_{L_\infty(\Omega)} = \operatorname{ess}\sup_{x\in\Omega} |\varphi(x)|. \tag{1.5}$$

We now state an a priori estimate of the error $u-U$. It can be shown that

$$\begin{aligned} \|u-U\|_{L_2(\Omega)} &\le C\|h^2 D^2 u\|_{L_2(\Omega)} \le C h_{\max}^2 \|D^2 u\|_{L_2(\Omega)} \\ &\le C h_{\max}^2 \|f\|_{L_2(\Omega)}, \end{aligned} \tag{1.6}$$

where $\|D^2u\|_{L_2(\Omega)}$ denotes the norm of all second order partial derivatives of u and $h_{\max}$ is the maximum value of h in Ω. We note that a priori error estimates similar to (1.6) can be proved in other norms, particularly, in the L_∞ norm. This estimate shows that for functions f in $L_2(\Omega)$ the error measured in the L_2 norm can be made arbitrarily small by choosing $h_{\max}$ sufficiently small. This proves that the finite element approximation U is convergent. However, this result does not tell us how to choose the triangulation. If we knew D^2u, we see in the first inequality of (1.6) that we could adapt the triangulation by choosing h small in regions where D^2u is large. Since D^2u is not known in general, adaptivity is often based on a posteriori error estimates instead.

We now turn our attention to an a posteriori error estimate. We will use a residual-based approach, see [5] and [7]. For a review of other approaches see [25]. One can prove that the error $u-U$ satisfies

$$\|u-U\|_{L_p(\Omega)} \le C\|h^2 R(U)\|_{L_p(\Omega)}, \quad 1<p<\infty. \tag{1.7}$$

The function $R(U)$, restricted to triangle K, takes the form

$$R(U)(x) = |f(x)+\Delta U(x)| + h_K^{-1/p'} \max_{\partial K} |[\partial_\nu U]|, \quad x\in K, \tag{1.8}$$

where $[\partial_\nu U]$ denotes the jump across ∂K in the outward normal derivative $\partial_\nu U = \nu\cdot\nabla U$ and $p' = p/(p-1)$. In order to introduce some important features of our work we discuss the proof of this result, see [5], [4] and [10]. The proof is based on a duality argument and Galerkin orthogonality combined with a regularity estimate for the solution of the adjoint problem.

We first define the residual $r(U) = -\Delta U - f \in H^{-1}(\Omega)$, more precisely,

$$\langle r(U), v\rangle = (f,v) - (\nabla U, \nabla v), \quad \forall v\in H_0^1(\Omega). \tag{1.9}$$

By combining (1.9) and (1.2) we see that $e=u-U$ satisfies

$$e\in H_0^1(\Omega); \quad (\nabla e, \nabla v) = \langle r(U), v\rangle, \quad \forall v\in H_0^1(\Omega), \tag{1.10}$$

which may be regarded as an equation for the error. We next note that by (1.3) we have

$$\langle r(U), \chi\rangle = 0, \quad \forall \chi\in S. \tag{1.11}$$

This property of $r(U)$ is referred to as Galerkin orthogonality. We now introduce the adjoint problem to (1.10). Let g be given and let G be the solution of:

$$\begin{aligned} -\Delta G &= g \quad \text{in } \Omega, \\ G &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.12}$$

or in weak form

$$G \in H_0^1(\Omega); \quad (\nabla w, \nabla G) = (w, g), \quad \forall w \in H_0^1(\Omega). \tag{1.13}$$

Let now $w = e$ in (1.13) and $v = G$ in (1.10). Using also (1.11) we get

$$(e, g) = \langle r(U), G \rangle = \langle r(U), G - \Pi G \rangle, \tag{1.14}$$

where $\Pi : C(\overline{\Omega}) \to S$ is an interpolation operator. The interpolation operator has the following approximation property:

$$\|v - \Pi v\|_{L_p(\Omega)} \leq C \|h^2 D^2 v\|_{L_p(\Omega)}, \quad 1 \leq p \leq \infty, \ \forall v \in W_p^2(\Omega) \cap H_0^1(\Omega). \tag{1.15}$$

Using, essentially, this property it can be shown that

$$|\langle r(U), G - \Pi G \rangle| \leq C \|h^2 R(U)\|_{L_{p'}(\Omega)} \|D^2 G\|_{L_p(\Omega)}, \tag{1.16}$$

where $1/p + 1/p' = 1$. We thus have the inequality

$$|(e, g)| \leq C \|h^2 R(U)\|_{L_{p'}(\Omega)} \|D^2 G\|_{L_p(\Omega)}. \tag{1.17}$$

In order to get an error estimate in the $L_{p'}$ norm we note that

$$\|e\|_{L_{p'}(\Omega)} = \sup_{\|g\|_{L_p} = 1} |(e, g)| \tag{1.18}$$

and using (1.17) we get for, $1 < p < \infty$,

$$\|e\|_{L_{p'}(\Omega)} \leq C \|h^2 R(U)\|_{L_{p'}(\Omega)} \sup_{\|g\|_{L_p} = 1} \|D^2 G\|_{L_p(\Omega)}. \tag{1.19}$$

In order to complete the proof it remains to estimate

$$\sup_{\|g\|_{L_p} = 1} \|D^2 G\|_{L_p(\Omega)}. \tag{1.20}$$

This is based on the regularity estimate

$$\|D^2 v\|_{L_p(\Omega)} \leq C(p) \|\Delta v\|_{L_p(\Omega)}, \quad 1 < p < \infty, \quad \forall v \in W_p^2(\Omega) \cap H_0^1(\Omega). \tag{1.21}$$

This inequality combined with the equation (1.12) gives

$$\|D^2 G\|_{L_p(\Omega)} \leq C(p) \|\Delta G\|_{L_p(\Omega)} \leq C(p) \|g\|_{L_p(\Omega)}, \tag{1.22}$$

and the proof of (1.7) is complete. When one considers the maximum norm the argument differs somewhat. Let $x_0 \in \Omega$ be such that $|e(x_0)| = \|e\|_{L_\infty(\Omega)}$ and let $g = \delta_{x_0}$ be the Dirac measure so that G is the Green function of (1.1). By (1.17) we would have

$$\|e\|_{L_\infty(\Omega)} = |(e, \delta_{x_0})| \leq C\|h^2 R(U)\|_{L_\infty(\Omega)}\|D^2 G\|_{L_1(\Omega)}, \tag{1.23}$$

but $D^2 G$ is not in $L_1(\Omega)$. Therefore the estimate has to be achieved by other methods. One approach, used in [3], is to do a more careful analysis of (1.16) for $p = 1$, using precise information about the singularities in G. In [3] this was carried out for non-convex polygonal domains in $\mathbf{R}^2$.

Our analysis is based on another technique introduced in [20]. This approach uses an approximate Green function which makes it possible to use (1.22). We now introduce an approximate δ function. Let $x_0 \in \Omega$ be as before and let $g = g_{x_0}$ be such that

$$\int_{\mathbf{R}^d} g(x)\,dx = 1; \quad \operatorname{supp} g \subset \mathcal{B}(x_0;\rho); \quad 0 \leq g(x) \leq C\rho^{-d}. \tag{1.24}$$

Here $\mathcal{B}(x_0;\rho)$ denotes the closed ball with center at x_0 and with small radius ρ to be chosen.

In this approach one uses the following observation. By the mean value theorem there is an $x_1 \in \mathcal{B}(x_0,\rho)$ such that $e(x_1) = (e, g)$. It can be proved that the quantity $|e(x_0) - e(x_1)|$ can be made arbitrarily small, using an a priori estimate of u, by choosing ρ small. Therefore it is enough to prove an a posteriori estimate of $|(e, g)|$ based on (1.17). The required estimate of $\|D^2 G\|_{L_1(\Omega)}$ follows by using Hölder's inequality, (1.21) together with the fact that $C(p) = C/(p-1)$ for $1 < p \leq 2$ and (1.24):

$$\begin{aligned} \|D^2 G\|_{L_1(\Omega)} &\leq |\Omega|^{1/p'}\|D^2 G\|_{L_p(\Omega)} \leq \frac{C}{p-1}\|\Delta G\|_{L_p(\Omega)} \\ &\leq \frac{C}{p-1}\|g\|_{L_p(\Omega)} = \frac{C}{p-1}\rho^{-2/p'} \leq CL, \end{aligned} \tag{1.25}$$

where $L = |\log\rho| = 1/(p-1)$. Note that $p \approx 1$ since ρ is small, we will choose $\rho \approx h_{\min}^\sigma$. This leads to an estimate of, essentially, the form

$$\|u - U\|_{L_\infty(\Omega)} \leq CL\|h^2 R(U)\|_{L_\infty(\Omega)}, \tag{1.26}$$

where now $L = |\log h_{\min}|$. We note that this argument is based on $C(p) = C/(p-1)$ in (1.21). This is only known for convex polyhedral domains and smooth domains, but it is plausible that $C(p) = C/(p-1)$ in (1.21) with $p \approx 1$ also for non-convex polyhedral domains, see [10]. The case of non-convex polyhedral domains was treated by replacing (1.25) by a more complicated argument in [20] (for $\Omega \subset \mathbf{R}^2$) and [2] (for $\Omega \subset \mathbf{R}^3$).

We now discuss the value of an a posteriori error estimate of the type (1.7) or (1.26). A drawback is that it is difficult to determine the constant C, both the contribution from the interpolation error estimate (1.15) and the contribution from the regularity estimate of G. Further, numerical examples show that these a posteriori error estimates are not sharp.

In a more general situation, for example a nonlinear differential equation, it may be impossible to estimate the regularity of the adjoint problem by analytic means. In order to circumvent this difficulty several numerical methods have been developed. Some are based on a more or less direct evaluation of the right hand side of (1.14) instead of estimating it, see [21], [17].

We illustrate such a method using our model example above. By combining (1.9) and (1.14) we have, with $g = g_{x_0}$ as in (1.24),

$$|e(x_0)| \approx (e, g) = (f, G - \Pi G) - (\nabla U, \nabla(G - \Pi G)). \tag{1.27}$$

We note that Π is the interpolation operator associated with the mesh that we used in (1.3). We now solve the dual problem (1.12) numerically, which is possible since we know $g = g_{x_0}$ from (1.24), with another triangulation, for instance, a finer mesh. In this way we get an approximation of G, which we can insert into (1.27) to get an approximation of (e, g). This is then an approximation of the error, but only at the specified point x_0.

An advantage of (1.26) compared to (1.27) is that it gives an estimate of the global error. Further, (1.27) may be combined with an a priori estimate of the type (1.6) to show that $\|h^2 R(U)\|_{L_\infty(\Omega)} \to 0$ as $h_{\max} \to 0$, which indicates that the a posteriori estimate is reliable.

1.2 A time-dependent problem

We now consider a time-dependent problem. Let Ω be a convex, polyhedral and bounded domain in $\mathbf{R}^d$. Let u be the solution of

$$\begin{aligned} u_t - \Delta u = f \quad &\text{in} \quad \Omega \times (0, T), \\ u = 0 \quad &\text{in} \quad \partial\Omega \times (0, T), \\ u = u_0 \quad &\text{in} \quad \Omega \times \{0\}, \end{aligned} \tag{1.28}$$

where $f(x, t)$ and $u_0(x)$ are given functions. In addition to the spatial discretisation we now also have to discretise in time. For this we use a finite element method, namely the Discontinuous Galerkin method, see [8], [5], and [7]. These works establish both a priori and a posteriori error estimates in L_2 and L_∞. As a general reference see [24].

We now describe the Discontinuous Galerkin method. Let $0 = t_0 < \cdots < t_N = T$ be a partition of $[0, T]$ into subintervals $I_n = (t_{n-1}, t_n)$ of lengths $k_n = t_n - t_{n-1}$, and associate with each such time interval a triangulation $\mathcal{T}_n \in \mathcal{F}$ with corresponding mesh function h_n and function space S_n. Let $q \geq 0$ be an integer. We define the following function spaces:

$$V_n = \{v : v(x, t) = \sum_{j=0}^{q} t^j \varphi_j(x),\ \varphi_j \in S_n\}, \tag{1.29}$$

$$V = \{\varphi : \varphi|_{\overline{\Omega}\times I_n} \in V_n,\ n = 1, ..., N\}, \tag{1.30}$$

and the mesh functions $h = h(x,t)$ and $k = k(t)$ defined by

$$\begin{aligned} h|_{I_n} &= h_n, \\ k|_{I_n} &= k_n. \end{aligned} \tag{1.31}$$

We discretise (1.28) as follows: Find $U \in V$ such that, for $n = 1, 2, ..., N$,

$$\int_{I_n} \{(U_t, v) + (\nabla U, \nabla v)\}\, dt + ([U]_{n-1}, v_{n-1}^+) = \int_{I_n} (f, v)\, dt, \quad \forall v \in V_n, \tag{1.32}$$

where $U_0^- = u_0$ and

$$[\varphi]_n = \varphi_n^+ - \varphi_n^-, \quad \varphi_n^\pm = \lim_{s\to 0^+} \varphi(t_n \pm s). \tag{1.33}$$

We now state an a posteriori error estimate for this problem in the maximum norm. In order to do so we need some notation. Let $[\partial_\nu U]$ denote the jump across ∂K in the outward normal derivative. We define the following computable residuals R_e^t, R_i^t, R_e^x, and R_i^x by

$$\begin{aligned} R_e^t|_{I_n} &= k_n^{-1}[U]_{n-1}, \\ R_i^t|_{I_n} &= |U_t - \Delta U - f|, \\ R_e^x|_{K\times I_n} &= \left\| h_K^{-1} k_n^{-1} \int_{I_n} |[\partial_\nu U]|\, dt \right\|_{L_\infty(\partial K\setminus\partial\Omega)}, \\ R_i^x|_{I_n} &= k_n^{-1} \int_{I_n} |U_t - \Delta U - f|\, dt, \end{aligned} \tag{1.34}$$

where the indices e and i refer to the edge and the interior parts of the residual, x and t refer to space and time discretisation. The a posteriori error estimate, essentially, takes the form:

$$\begin{aligned} &\max_{1\le n\le N} \|u(t_n) - U_n^-\|_{L_\infty(\Omega)} \\ &\quad \le CL \max_{1\le n\le N} \Big(\|kR_e^t\|_{L_\infty(I_n;L_\infty)} + \|kR_i^t\|_{L_\infty(I_n;L_\infty)} \\ &\qquad\qquad + \|h^2 R_e^x\|_{L_\infty(I_n;L_\infty)} + \|h^2 R_i^x\|_{L_\infty(I_n;L_\infty)} \Big), \end{aligned} \tag{1.35}$$

where L is a logarithmic factor. Note that the last two terms are essentially as in (1.26), see (1.8) with $p = \infty$.

The proof follows the same lines as in the stationary case. It is based on an $L_1 - L_\infty$ duality argument involving a residual and estimates of the regularity of the solution G of a linear adjoint problem. Let $B(\cdot,\cdot)$ be the following bilinear form:

(1.36)

$$\begin{aligned}B(v,w) &= \sum_{n=1}^{N}\int_{I_n}\{(v_t,w)+(\nabla v,\nabla w)\}\,dt+\sum_{n=1}^{N-1}([v]_n,w_n^+)+(v_0^+,w_0^+)\\ &= \sum_{n=1}^{N}\int_{I_n}\{(v,-w_t)+(\nabla v,\nabla w)\}\,dt-\sum_{n=1}^{N-1}(v_n^-,[w]_n)+(v_N^-,w_N^-),\end{aligned}$$

where the second variant was obtained by integration by parts in t. By summing (1.32) we see that U satisfies

$$U\in V;\quad B(U,v)=(u_0,v_0^+)+\int_0^{t_N}(f,v)\,dt,\quad \forall v\in V. \tag{1.37}$$

Let $\mathcal{V}=\{w : w|_{I_n}$ is smooth in time, $w(t)\in H_0^1(\Omega)$ and $w_N^\pm\in H_0^1(\Omega)\}$. The exact solution u satisfies

$$u\in \mathcal{V};\quad B(u,v)=(u_0,v_0^+)+\int_0^{t_N}(f,v)\,dt,\quad \forall v\in \mathcal{V}. \tag{1.38}$$

Further, we define the residual $r(U)$ by

$$\langle r(U),v\rangle=(u_0,v_0^+)+\int_0^{T}(f,v)\,dt-B(U,v),\quad \forall v\in V. \tag{1.39}$$

By combining (1.38) and (1.39) we get

$$e\in\mathcal{V};\quad B(e,v)=\langle r(U),v\rangle,\quad \forall v\in\mathcal{V}. \tag{1.40}$$

By subtracting (1.38) and (1.37) we get

$$B(e,v)=0,\quad \forall v\in V, \tag{1.41}$$

which is the Galerkin orthogonality. We now introduce the adjoint problem to (1.40). Given g, find $G=G(x,t)$ such that

$$\begin{aligned}-G_t-\Delta G&=0,\quad \text{in } \Omega\times(0,t_N),\\ G&=0,\quad \text{in } \partial\Omega\times(0,t_N),\\ G&=g,\quad \text{in } \Omega\times\{t_N\}.\end{aligned} \tag{1.42}$$

By the second variant of (1.36) we see that G satisfies

$$G\in\mathcal{V};\quad B(w,G)=(w_N^-,g),\quad \forall w\in\mathcal{V}. \tag{1.43}$$

Set now $v=G$ in (1.40) and $w=e$ in (1.43), and use (1.41) to get the error representation

$$(e_N^-,g)=\langle r(U),G\rangle=\langle r(U),G-\tilde{G}\rangle, \tag{1.44}$$

where $\tilde{G} \in V$ is arbitrary. In order to estimate $|(e_N^-, g)|$ it remains to estimate $\langle r(U), G - \tilde{G} \rangle$. As in the stationary case this can be done in terms of the computable residuals (1.34) and certain regularity estimates of G. In order to get an estimate of the error in L_2 one chooses $g = e$. In order to get an error estimate in the maximum norm we choose g as in (1.24). In this case we, essentially, finish the proof by showing the estimate

$$\int_0^{t_{N-1}} \|G_t\|_{L_1(\Omega)}\, dt + \int_0^{t_{N-1}} \|D^2 G\|_{L_1(\Omega)}\, dt \leq CL, \tag{1.45}$$

which corresponds to (1.25).

2 Summary of the appended papers

[I] A posteriori error analysis in the maximum norm for a penalty finite element method for the time-dependent obstacle problem

In this paper we consider finite element approximations of the time- dependent obstacle problem. Let $\Omega \subset \mathbf{R}^d$ be a convex, polyhedral and bounded domain and $Q_T = \Omega \times (0, T)$. Let $\psi(x,t)$ be a smooth function with $\psi|_{\partial\Omega} \leq 0$. Let

$$\mathcal{K} = \{v \in H^1(Q_T), v|_{\partial\Omega} = 0, v(\cdot, 0) = v_0, v \geq \psi\}, \tag{2.1}$$

The time-dependent obstacle problem is the following parabolic variational inequality. Find u such that

$$u \in \mathcal{K}; \quad (u_t, u - v) + (\nabla u, \nabla(u - v)) \geq (f, u - v), \quad \forall v \in \mathcal{K}, \quad t > 0. \tag{2.2}$$

One can show, see [11] and [12], that (2.2) possesses a unique solution. Further, the solution u has the following properties

$$\begin{aligned} u_t - \Delta u &\geq f, && \text{in } Q_T, \\ u &\geq \psi, && \text{in } Q_T, \\ (u_t - \Delta u - f)(u - \psi) &= 0 && \text{in } Q_T, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{2.3}$$

These relations can be expressed as the following differential inclusion: Find $u = u(x,t)$ such that

$$\begin{aligned} u_t - \Delta u + \omega(u - \psi) &\ni f, && \text{in } \Omega \times (0, T), \\ u &= 0, && \text{in } \partial\Omega \times (0, T), \\ u &= v_0, && \text{in } \Omega \times \{0\}, \end{aligned} \tag{2.4}$$

where ω is the maximal monotone graph defined by

$$\omega(s) = \begin{cases} \{0\}, & s > 0, \\ (-\infty, 0], & s = 0, \\ \emptyset, & s < 0. \end{cases}$$

One way of finding a numerical approximation is to dicretise the variational inequality directly, see [14], [9] and [1]. Another approach is based on discretising the penalised problem: Find $u_\epsilon = u_\epsilon(x,t)$ such that

$$
\begin{aligned}
u_{\epsilon,t} - \Delta u_\epsilon + \epsilon^{-1}\beta(u_\epsilon) &= f, && \text{in } \Omega \times (0,T),\\
u_\epsilon &= 0, && \text{in } \partial\Omega \times (0,T),\\
u_\epsilon &= v_0, && \text{in } \Omega \times \{0\},
\end{aligned} \tag{2.5}
$$

where $\epsilon = \epsilon(x,t)$ is a positive function and

$$\beta(w(x,t)) = (w(x,t) - \psi(x,t))^- = \min(w(x,t) - \psi(x,t), 0).$$

This approach has been analysed in [22], [23], [19], [15] and [10], and this is the approach we take in Paper I.

We first prove that the penalised solution u_ϵ converges to u in the maximum norm. More precisely, we prove

$$\|u - u_\epsilon\|_{L_\infty(0,T;L_\infty(\Omega))} \leq \|\epsilon(f + \Delta\psi - \psi_t)\chi_{\hat{\Omega}}\|_{L_\infty(0,T;L_\infty(\Omega))}, \tag{2.6}$$

where $\hat{\Omega} = \{(x,t) \in \Omega \times [0,T] : u(x,t) - \psi(x,t) = 0,\ u_\epsilon(x,t) - \psi(x,t) \leq 0\}$ is the "contact set", and $\chi_{\hat{\Omega}}$ is the characteristic function of $\hat{\Omega}$.

In order to estimate the discretisation error $u_\epsilon - U_\epsilon$ we argue as described in Section 1 using $L_1 - L_\infty$ duality with the solution G of the adjoint problem acting as an approximate Green function. Instead of (1.40) the equation for the error now takes the form

$$e \in \mathcal{V}; \quad B(e,v) + \int_0^{t_N} (be,v)\,dt = \langle r(U), v\rangle, \quad \forall v \in \mathcal{V}, \tag{2.7}$$

where $b = b(x,t)$ is defined by

$$b = \frac{\beta_\epsilon(u_\epsilon - \psi) - \beta_\epsilon(U_\epsilon - \psi)}{u_\epsilon - U_\epsilon}.$$

We note that

$$0 \leq b(x,t) \leq \frac{1}{\epsilon(x,t)} \leq \frac{1}{\epsilon_{\min}}. \tag{2.8}$$

The adjoint problem to (2.7) takes the form: Let g be as in (1.24) find $G = G(x,t)$ such that

$$
\begin{aligned}
-G_t - \Delta G + bG &= 0, && \text{in } \Omega \times [0,t_N],\\
G &= 0, && \text{in } \partial\Omega \times [0,t_N],\\
G &= g, && \text{in } \Omega \times \{t_N\}.
\end{aligned} \tag{2.9}
$$

The term bG introduces an extra difficulty because of (2.8). It turns out that in L_2 this term degrades the regularity estimates. More precisely, we are only able to prove

$$\int_0^{t_{N-1}} \|G_t\|_{L_2(\Omega)}\,dt + \int_0^{t_{N-1}} \|D^2G\|_{L_2(\Omega)}\,dt \leq C{\epsilon_{\min}}^{-1/2}\|g\|_{L_2(\Omega)}, \tag{2.10}$$

where the factor ${\epsilon_{\min}}^{-1/2}$ is unacceptable, since $\epsilon \approx h^2$ is suggested by a priori analysis. However in L_1 it is possible to get almost ϵ independent regularity estimates of the same form as (1.45). Our result is an estimate of $\|u_\epsilon(t_N) - U^-_{\epsilon,N}\|_{L_\infty(\Omega)}$ of the same form as (1.35). Our regularity analysis is based on a combination of a maximal regularity theorem from [13] and results from the theory of analytic semigroups.

[II] Global and localised a posteriori error analysis in the maximum norm for finite element approximations of a convection-diffusion problem

In this paper we study finite element solutions of the following problem. Let $\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$, be a bounded convex polyhedral domain. Find u such that

$$
\begin{aligned}
\alpha u + \beta \cdot \nabla u - \epsilon \Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \partial\Omega,
\end{aligned}
\tag{2.11}
$$

where α, β, and f are given functions and ϵ is a positive number. For $\epsilon \approx 1$ we discretise using the standard Galerkin method, see (1.3), but for small ϵ we use the Streamline Diffusion method: Find $U \in S$ such that

$$
\begin{aligned}
(\alpha U, \chi + \delta\beta \cdot \nabla\chi) + (\beta \cdot \nabla U, \chi + \delta\beta \cdot \nabla\chi) + \epsilon(\nabla U, \nabla\chi) \\
= (f, \chi + \delta\beta \cdot \nabla\chi), \qquad \forall \chi \in S.
\end{aligned}
\tag{2.12}
$$

The streamline diffusion coefficient $\delta = \delta(x)$ is defined by

$$
\delta = c_1 \max(0, h - \epsilon),
\tag{2.13}
$$

where the parameter $c_1 \geq 0$ is a constant. Note that $c_1 = 0$ gives the standard Galerkin method.

There is an extensive literature about finite element methods for (2.11) both in the elliptic case, $\epsilon \approx 1$, and the case $\epsilon << 1$. Our work was inspired by the works [6] and [16] on the convection dominated case.

We analyse (2.11) concerning a posteriori error estimates in the maximum norm for different values of ϵ. As in Section 1.1 we establish an equation for the error. Let the residual be defined by $r(U) = \alpha U + \beta \cdot \nabla U - \epsilon \Delta U$, and let $e = u - U$. Then, cf. (1.10),

$$
e \in H^1_0(\Omega); \quad (\alpha e + \beta \cdot \nabla e, v) + \epsilon(\nabla e, \nabla v) = \langle r(U), v \rangle, \quad \forall v \in H^1_0(\Omega).
\tag{2.14}
$$

If we are using the Streamline Diffusion method the orthogonality equation takes a different form. More precisely, instead of (1.11) we have

$$
\langle r(U), \chi \rangle = (\alpha U + \beta \cdot \nabla U - f, \delta\beta \cdot \nabla\chi), \quad \forall \chi \in S,
\tag{2.15}
$$

but using the definition of δ we can still prove a result similar to (1.16). In our analysis of (2.14) we use the following adjoint problem. Let g be given by (1.24) find G such that

$$\begin{aligned} \alpha G - \nabla \cdot (\beta G) - \epsilon \Delta G &= g \quad \text{in } \Omega, \\ G &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{2.16}$$

We are able to prove the following regularity estimates of G, cf. (1.25),

$$\begin{aligned} \|G\|_{L_1(\Omega)} &\leq C, \\ \|DG\|_{L_1(\Omega)} &\leq CL\epsilon^{-1/2}, \\ \|D^2G\|_{L_1(\Omega)} &\leq CL^{3/2}\epsilon^{-3/2}. \end{aligned} \tag{2.17}$$

The first two estimates in (2.17) seem optimal with respect to ϵ, but we believe that one should have the estimate

$$\|D^2G\|_{L_1(\Omega)} \leq CL\epsilon^{-1}, \tag{2.18}$$

which we get in the case $d = 1$. The key step in the proof is an estimate of $\|DG\|_{L_p(\Omega)}$, $1 \leq p \leq 2$, which we obtain by an energy argument. A standard argument based on the interpolation inequality

$$\|DG\|_{L_p(\Omega)} \leq \eta\|D^2G\|_{L_p(\Omega)} + \eta^{-1}\|G\|_{L_p(\Omega)}, \quad \eta > 0, \tag{2.19}$$

would lead to

$$\begin{aligned} \|DG\|_{L_1(\Omega)} &\leq CL\epsilon^{-1}, \\ \|D^2G\|_{L_1(\Omega)} &\leq CL^2\epsilon^{-2}. \end{aligned} \tag{2.20}$$

Our error estimate is essentially of the form:

$$\|u - U\|_{L_\infty(\Omega)} \leq L^{3/2}\Big\|\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)R(U)\Big\|_{L_\infty(\Omega)}. \tag{2.21}$$

Our result in the elliptic case $\epsilon \approx 1$ is essentially included in the results of [3], [20] and [2].

It is known that if ϵ is small compared to β, then the solution u is non-smooth in certain regions, typically near characteristic layers and boundary layers. Adaptivity based on a global estimate, such as (2.21), would lead to heavy refinement near such singularities even if we are only interested in the behaviour in regions where the solution is smooth. In order to see that it may be possible to avoid this we prove, in a special case, that the above result may be localised to special regions Ω_0 oriented along the streamlines.

More precisely, we consider the following equation. Let $\Omega \subset \mathbf{R}^2$, find u such that

$$\begin{aligned} u + u_x - \epsilon\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

Our localised result, essentially, takes the form

$$\|\varphi(u-U)\|_{L_\infty(\Omega)} \le CL^{3/2}\Big\|\varphi \min\Big(1, \frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big)R(U)\Big\|_{L_\infty(\Omega)}, \tag{2.22}$$

where the cut-off function φ is such that $\varphi \approx 1$ in Ω_0 and φ decays exponentially with $s/\sqrt{\epsilon}$ where s is the distance to Ω_0.

For this result to be useful it is necessary that the residual $R(U)$ can be made small in Ω_0 without resolving the singular layers. The standard Galerkin finite element method does not have this property.

However, a priori analysis indicates that this is possible for the Streamline Diffusion method. In [16] it is shown that in regions Ω_0, of the same type as above, and where u is assumed to be smooth, we have $\|u-U\|_{L_\infty(\Omega_0)} \le Ch_{\max}^{5/4}$. This result is improved in [18] to $O(h_{\max}^{11/8})$. Further, under special conditions on the triangulation, [29] proves that the error behaves as $O(h_{\max}^2)$ in such regions. We also refer to the discussion in [28]. An important feature of these localised results is that they are valid for triangulations that do not resolve the singular layers.

[III] A posteriori error analysis in the maximum norm for finite element approximations of a time-dependent convection-diffusion problem

In this paper we consider the following problem. Let Ω be a convex, polyhedral, bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$. Let u be the solution of

$$\begin{aligned} u_t + \alpha u + \beta \cdot \nabla u - \epsilon \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u &= u_0 \quad \text{in } \Omega \times \{0\}, \end{aligned} \tag{2.23}$$

where α, β, and f are functions of x and t. The initial value u_0 is a function of x and ϵ is a positive number.

We analyse this problem for different values of ϵ. As for the corresponding stationary problem the standard Galerkin method works well for $\epsilon \approx 1$, but not for small ϵ. We therefore use the Streamline Diffusion method when ϵ is small.

In order to discretise this equation in time we use a Discontinuous Galerkin method, see the notation in Section 1. Find $U \in V$ such that, for $n = 1, 2, ..., N$,

$$\begin{aligned} \int_{I_n} \big\{(U_t + \alpha U + \beta \cdot \nabla U, w + \delta(w_t + \beta \cdot \nabla w)) + \epsilon(\nabla U, \nabla w)\big\}\, dt \\ + ([U]_{n-1}, w_{n-1}^+) \\ = \int_{I_n} (f, w + \delta(w_t + \beta \cdot \nabla w))\, dt, \quad \forall w \in V_n, \end{aligned} \tag{2.24}$$

where $U_0^- = u_0$ and

$$[\varphi]_n = \varphi_n^+ - \varphi_n^-, \quad \varphi_n^\pm = \lim_{s\to 0^+} \varphi(t_n \pm s). \tag{2.25}$$

The streamline diffusion coefficient $\delta = \delta(x,t)$ is defined by

$$\delta = c_1 \min(0, h - \epsilon), \tag{2.26}$$

where $c_1 \geq 0$. Note that $c_1 = 0$ means that streamline diffusion is not used.

Our a posteriori error estimate takes the form:

$$\begin{aligned} \max_{1 \leq n \leq N} \|u(t_n) - U_n^-\|_{L_\infty(\Omega)} \leq CL\Big(& \Big\| \frac{k}{\epsilon^{1/2}} R^t \Big\|_{L_\infty(I_n; L_\infty(\Omega))} \\ & + \Big\| \min\Big(\frac{h}{\epsilon^{1/2}}, \frac{h^2}{\epsilon^{3/2}}\Big) R^x \Big\|_{L_\infty(I_n; L_\infty(\Omega))} \Big), \end{aligned} \tag{2.27}$$

where L is a logarithmic factor. R^t, R^x are residuals arising from the time and space discretisation, respectively.

The proof follows the outline of Section 1. We use the following adjoint problem. Let g be defined by (1.24). Find G such that

$$\begin{aligned} -G_t + \alpha G - \nabla \cdot (\beta G) - \epsilon \Delta G = 0, \quad & \text{in } \Omega \times (0, t_N), \\ G = 0, \quad & \text{in } \partial\Omega \times (0, t_N), \\ G = g, \quad & \text{in } \Omega \times \{t_N\}. \end{aligned} \tag{2.28}$$

As in the stationary case we believe that our analysis is not optimal for small ϵ. In fact, our regularity estimates are, instead of (1.45),

$$\int_0^{t_{N-1}} \|G_t\|_{L_1(\Omega)}\, dt + \epsilon \int_0^{t_{N-1}} \|D^2 G\|_{L_1(\Omega)}\, dt \leq CL\epsilon^{-1/2}, \tag{2.29}$$

which we believe should be

$$\int_0^{t_{N-1}} \|G_t\|_{L_1(\Omega)}\, dt + \epsilon \int_0^{t_{N-1}} \|D^2 G\|_{L_1(\Omega)} \leq CL, \tag{2.30}$$

The proof of (2.29) combines techniques from Paper I and Paper II.

It is known that the solution u of (1.1) can be nonsmooth in certain regions such as boundary and characteristic layers when ϵ is small, cf. the summary of Paper II. Since its proof is similar to the stationary case, we believe that it is possible to localise (2.27) in a similar way as (2.22). This is also supported by the localised a priori analysis in [27] and [26]. In [27] it is shown that for certain regions Q_0 oriented along the streamlines and where $u \in C^2(Q_0)$ we have $\|u - U\|_{L_\infty(Q_0)} \leq Ch_{\max}^{(12-d)/8}$, where d is the dimension of Ω.

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