LECTURE 1.2

In this lecture we present properties of the integral. This covers AMBS Ch 28.

1. Properties of the integral

The Fundamental Theorem of Calculus says that the initial-value problem

$$\begin{cases} u'(x) = f(x), & x \in [a,b], \\ u(a) = u_a. \end{cases}$$

has a unique solution. This function can be written

$$u(x) = u_a + \int_a^x f(y)\,dy, \quad x \in [a,b].$$

If we only want the integral $\int_a^b f(y)\,dy$, then we take $u_a = 0$ and $x = b$. The integral is constructed as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{N} f(x_{j-1}^n) h_n, \quad h_n = (b-a)2^{-n},\ x_j^n = a + jh_n,\ N = 2^n.$$

1.1. **The backward integral.**

If $b < a$ then we define the backward integral

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example.

$$\int_2^1 x^2\,dx = -\int_1^2 x^2\,dx = -\frac{7}{3}.$$

1.2. **Division of the interval.**

If $a < b < c$ and f is Lipschitz continuous on $[a,c]$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof. Consider the initial-value problems

$$\begin{cases} u'(x) = f(x), & x \in [a,c], \\ u(a) = 0, \end{cases}$$

and

$$\begin{cases} v'(x) = f(x), & x \in [b,c], \\ v(a) = u(b). \end{cases} \tag{1}$$

They have unique solutions $u(x) = \int_a^x f(y)\,dy$ and $v(x) = u(b) + \int_b^x f(y)\,dy$. But clearly u satisfies (1) also. Therefore, by uniqueness, $u(x) = v(x)$ for $x \in [b,c]$. In particular, with $x = c$ we get $u(c) = v(c)$ which means

$$\int_a^c f(y)\,dy = u(c) = v(c) = u(b) + \int_b^c f(y)\,dy = \int_a^b f(y)\,dy + \int_b^c f(y)\,dy.$$

□

Date: October 30, 2005, Stig Larsson, Mathematical Sciences, Chalmers University of Technology.

1.3. **Piecewise integral.** If f is only piecewise ("styckvis") Lipschitz continuous, then we define the integral of f as a piecewise integral. For example, if $a < b < c$ and f is Lipschitz continuous on $[a, b)$ and $(b, c]$ but discontinuous at b, then we define

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Example. Let $f(x) = 1$ if $x < 0$ and $f(x) = x$ if $x > 0$. Then f is Lipschitz continuous on $[-1, 0)$ and on $(0, 1]$ and we get

$$\int_{-1}^1 f(x)\,dx = \int_{-1}^0 1\,dx + \int_0^1 x\,dx = 1 + \frac{1}{2} = \frac{3}{2}.$$

1.4. **Linearity.** The integral is a linear operator: if $\alpha, \beta \in \mathbf{R}$ and f, g are Lipschitz continuous on $[a, b]$ then

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Proof. We know that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^N f(x_{j-1}^n) h_n, \quad h_n = (b-a)2^{-n},\ x_j^n = a + jh_n,\ N = 2^n.$$

Therefore

$$\begin{aligned}
\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx &= \lim_{n\to\infty} \sum_{j=1}^N \big(\alpha f(x_{j-1}^n) + \beta g(x_{j-1}^n)\big) h_n \\
&= \lim_{n\to\infty} \alpha \sum_{j=1}^N f(x_{j-1}^n) h_n + \lim_{n\to\infty} \beta \sum_{j=1}^N g(x_{j-1}^n) h_n \\
&= \alpha \lim_{n\to\infty} \sum_{j=1}^N f(x_{j-1}^n) h_n + \beta \lim_{n\to\infty} \sum_{j=1}^N g(x_{j-1}^n) h_n \\
&= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.
\end{aligned}$$

□

This means that the integration operator preserves linear combinations of functions: if we integrate a linear combination of f and g then the integral is the same linear combination of the integrals of f and g. Such an operator is called a *linear operator.*

Here we think of the integral as an operator:

- input: f
- output: $\int_a^b f(x)\,dx$

Example.

$$\int_1^2 (5x^2 + 4x)\,dx = 5 \int_1^2 x^2\,dx + 4 \int_1^2 x\,dx = 5\frac{7}{3} + 4\frac{3}{2} = \frac{35}{3} + 6.$$

1.5. **Monotonicity.** The integral is a monotone operator: if f, g are Lipschitz continuous on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{j=1}^{N}\underbrace{f(x_{j-1}^n)}_{\le g(x_{j-1}^n)}\underbrace{h_n}_{>0} \le \lim_{n\to\infty}\sum_{j=1}^{N} g(x_{j-1}^n)h_n = \int_a^b g(x)\,dx.$$

$\square$

This means that the integral preserves inequalities.

Example. We know that x^{-1} is Lipschitz continuous on $[1,2]$ and $x^{-1}\ge 0$. Therefore

$$\int_1^2 x^{-1}\,dx \ge \int_1^2 0\,dx = 0,$$

so $\int_1^2 x^{-1}\,dx \ge 0$, i.e., the integral is non-negative. We shall soon learn what this integral is.

1.6. **The triangle inequality for integrals.** If f is Lipschitz continuous on $[a,b]$ then

$$\Big|\int_a^b f(x)\,dx\Big| \le \int_a^b \big|f(x)\big|\,dx.$$

Proof. Remember the triangle inequality for sums: $|x_1+x_2|\le|x_1|+|x_2|$ and more generally

$$\Big|\sum_{j=1}^{N} x_j\Big| \le \sum_{j=1}^{N}|x_j|.$$

We use it together with the Lipschitz continuity of the absolute value:

$$\begin{aligned}\Big|\int_a^b f(x)\,dx\Big| &= \Big|\lim_{n\to\infty}\sum_{j=1}^{N} f(x_{j-1}^n)h_n\Big| = \lim_{n\to\infty}\Big|\sum_{j=1}^{N} f(x_{j-1}^n)h_n\Big|\\ &\le \lim_{n\to\infty}\sum_{j=1}^{N}\big|f(x_{j-1}^n)\big|h_n = \int_a^b \big|f(x)\big|\,dx.\end{aligned}$$

$\square$

Example. By using the monotonicity and the triangle inequality we can estimate the value of a complicated integral by a simpler one:

$$\Big|\int_0^1 x\cos(x^5)\,dx\Big| \le \int_0^1 |x\cos(x^5)|\,dx = \int_0^1 \underbrace{x}_{\ge 0}\underbrace{|\cos(x^5)|}_{\le 1}\,dx \le \int_0^1 x\,dx = \frac{1}{2}.$$

This is very important because it allows us to calculate with things that cannot be calculated exactly. Of course, we do not get an exact value of the integral but this is not always needed. In this case, we conclude that the integral is between $-\frac{1}{2}$ and $\frac{1}{2}$.

Example. Assume that the derivative of f is bounded:

$$|f'(x)| \le M_f, \quad \forall x\in[a,b].$$

Then if $x\le y$ we have

$$|f(y)-f(x)| = \Big|\int_x^y f'(x)\,dx\Big| \le \int_x^y \big|f'(x)\big|\,dx \le \int_x^y M_f\,dx = M_f(y-x) = M_f|y-x|.$$

If $y\le x$ then we have in the same way

$$\begin{aligned}|f(y)-f(x)| = |f(x)-f(y)| &= \Big|\int_y^x f'(x)\,dx\Big|\\ &\le \int_y^x \big|f'(x)\big|\,dx \le \int_y^x M_f\,dx = M_f(x-y) = M_f|y-x|.\end{aligned}$$

(Note that we must have a forward integral in the triangle inequality.) We conclude that $L_f\le M_f$. This was proved in a more complicated way in AMBS 23.15.

1.7. **The derivative of the integral.** The fundamental theorem says that

$$u(x) = u_a + \int_a^x f(y)\,dy$$

is uniformly differentiable in $[a,b]$ and that $u'(\bar{x}) = f(\bar{x})$. This means that

$$u(x) = u(\bar{x}) + f(\bar{x})(x-\bar{x}) + E_u(x,\bar{x}), \quad x,\bar{x} \in [a,b], \tag{2}$$

with

$$|E_u(x,\bar{x})| \le K_u(x-\bar{x})^2, \quad x,\bar{x} \in [a,b]. \tag{3}$$

This is the missing Step 3 of the proof of the fundamental theorem. We prove it now. More precisely, we must prove (3).

Proof. Assume that $x \ge \bar{x}$. Then starting from (2) we get

$$\begin{aligned}
|E_u(x,\bar{x})| &= \big|\underbrace{u(x)-u(\bar{x})}_{=\int_{\bar{x}}^x f(y)\,dy} - f(\bar{x})\underbrace{(x-\bar{x})}_{=\int_{\bar{x}}^x \,dy}\big| \\
&= \Big|\int_{\bar{x}}^x f(y)\,dy - \int_{\bar{x}}^x f(\bar{x})\,dy\Big| \\
&= \Big|\int_{\bar{x}}^x \big(f(y)-f(\bar{x})\big)\,dy\Big| \quad \Big\{\text{by the triangle inequality}\Big\} \\
&\le \int_{\bar{x}}^x \big|f(y)-f(\bar{x})\big|\,dy \quad \Big\{\text{by the Lipschitz condition}\Big\} \\
&\le \int_{\bar{x}}^x L_f\big|y-\bar{x}\big|\,dy \quad \Big\{\text{because } y \ge \bar{x}\Big\} \\
&= \int_{\bar{x}}^x L_f\big(y-\bar{x}\big)\,dy \\
&= \frac{1}{2}L_f\big(x-\bar{x}\big)^2.
\end{aligned}$$

If $x \le \bar{x}$ then we get in a similar way:

$$|E_u(x,\bar{x})| \le \int_x^{\bar{x}} \big|f(y)-f(\bar{x})\big|\,dy \le \int_x^{\bar{x}} L_f\big|y-\bar{x}\big|\,dy = \int_x^{\bar{x}} L_f\big(\bar{x}-y\big)\,dy = \frac{1}{2}L_f\big(x-\bar{x}\big)^2.$$

This is (3) with $K_u = \frac{1}{2}L_f$. □

This can also be expressed as

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x), \quad x \ge a. \tag{4}$$

We shall soon see that this holds also for $x < a$, see (6). Together with the chain rule we have

$$\frac{d}{dx}\int_a^{g(x)} f(y)\,dy = \frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x), \tag{5}$$

where $F(x) = \int_a^x f(y)\,dy$.

Example.

$$\frac{d}{dx}\int_1^x y^2\,dy = x^2, \quad \frac{d}{dx}\int_x^1 y^2\,dy = -\frac{d}{dx}\int_1^x y^2\,dy = -x^2,$$

$$\frac{d}{dy}\int_1^y \frac{x^{12}-565x^4+x^{-35}}{1+x^{18}}\,dx = \frac{y^{12}-565y^4+y^{-35}}{1+y^{18}},$$

$$\frac{d}{dx}\int_1^{x^3} \frac{1+y^2}{1+y^6}\,dy = \frac{1+(x^3)^2}{1+(x^3)^6}\,3x^2 = \frac{1+x^6}{1+x^{18}}\,3x^2.$$

1.8. **Substitution of variables.**

$$\int_a^b f(g(y))g'(y)\,dy = \left\{\begin{array}{l} z = g(y) \\ \dfrac{dz}{dy} = g'(y) \\ dz = g'(y)dy \\ y = a \implies z = g(a) \\ y = b \implies z = g(b) \end{array}\right\} = \int_{g(a)}^{g(b)} f(z)\,dz.$$

The stuff within the braces is not a proof of the formula. It is only a convenient way to remember it.

Proof. We define two functions: $v(x) = \int_a^x f(g(y))g'(y)\,dy$ and $w(x) = \int_{g(a)}^{g(x)} f(z)\,dz$. We shall show that they are equal: $v(x) = w(x)$ for all $x \in [a, b]$.

We compute their derivatives:

$$v'(x) = \frac{d}{dx}\int_a^x f(g(y))g'(y)\,dy = f(g(x))g'(x),$$

$$w'(x) = \frac{d}{dx}\int_{g(a)}^{g(x)} f(z)\,dz = f(g(x))g'(x),$$

where in the last one we used the chain rule as in (5). We also compute the initial values:

$$v(a) = 0, \quad w(a) = 0.$$

Therefore they both satisfy the initial-value problem:

$$\begin{cases} u'(x) = f(g(x))g'(x), & x \in [a, b], \\ u(a) = 0. \end{cases}$$

The fundamental theorem says that the solution is unique. Therefore: $v = w$. □

Example.

$$\int_0^2 (1 + x^8)x\,dx = \left\{\begin{array}{l} z = x^2 \\ \dfrac{dz}{dx} = 2x \\ dz = 2x\,dx \\ x = 0 \implies z = 0 \\ x = 2 \implies z = 4 \end{array}\right\} = \frac{1}{2}\int_0^4 (1 + z^4)\,dz = \frac{1}{2}\Big[z + \tfrac{1}{5}z^5\Big]_0^4.$$

Example. Assume that $x < a$. We want to differentiate the backward integral:

$$\int_a^x f(y)\,dy = -\int_x^a f(y)\,dy.$$

Substitution of variables gives:

$$\int_a^x f(y)\,dy = \left\{\begin{array}{l} z = a - y \\ \dfrac{dz}{dy} = -1 \\ dz = -dy \\ y = a \implies z = 0 \\ y = x \implies z = a - x > 0 \end{array}\right\} = -\int_0^{a-x} f(a - z)\,dz,$$

which is a forward integral. Hence, by the chain rule,

$$\frac{d}{dx}\int_a^x f(y)\,dy = -\frac{d}{dx}\int_0^{a-x} f(a - z)\,dz = -f(a - (a - x))\frac{d}{dx}(a - x) = f(x).$$

We conclude that (4) holds also for $x < a$:

$$\frac{d}{dx}\int_a^x f(y)\,dy = f(x). \tag{6}$$

1.9. **Partial integration.**

$$\int_a^b u'(x)v(x)\,dx = \Big[u(x)v(x)\Big]_{x=a}^b - \int_a^b u(x)v'(x)\,dx.$$

Proof. This is an integrated version of the product differentiation rule:

$$(uv)' = u'v + uv',$$

$$\Big[u(x)v(x)\Big]_{x=a}^b = \int_a^b (uv)'\,dx = \int_a^b (u'v + uv')\,dx = \int_a^b u'v\,dx + \int_a^b uv'\,dx.$$

□

1.10. **The mean value theorem.**

Theorem. (The mean value theorem for functions.) *If u is uniformly differentiable on $[a,b]$, then there is an $\bar{x} \in [a,b]$ such that*

$$u'(\bar{x}) = \frac{u(b) - u(a)}{b - a}.$$

If $u(x)$ is the distance travelled at time x, then this means that there is a time $\bar{x}$ where the velocity is equal to the average velocity.

We skip the proof. It can be found in AMBS 28.11. It is based on the Intermediate Value Theorem.

If we apply this to the function $u(x) = \int_a^x f(y)\,dy$ we get the following:

Theorem. (The mean value theorem for integrals.) *If f is Lipschitz continuous on $[a,b]$, then there is an $\bar{x} \in [a,b]$ such that*

$$f(\bar{x}) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. With $u(x) = \int_a^x f(y)\,dy$ we get

$$f(\bar{x}) = u'(\bar{x}) = \frac{u(b) - u(a)}{b - a} = \frac{\int_a^b u'(x)\,dx}{b-a} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

□

The quantity $\frac{1}{b-a}\int_a^b f(x)\,dx$ is the *average* of f over $[a,b]$. The theorem means that there is (an unknown) point $\bar{x}$ such that $f(\bar{x})$ is equal to the average of f over $[a,b]$.

Example. The average of $f(x) = x^3$ over $[-1,1]$ is

$$\frac{1}{2}\int_{-1}^1 x^3\,dx = 0.$$

It is clear that $\bar{x} = 0$ is as in the mean value theorem. Note, by the way, that the reason why this integral is zero is that the integrand is an odd function: $f(-x) = -f(x)$, and the interval of integration is symmetric: $[-a,a]$. In this situation we get:

$$\begin{aligned}\int_{-a}^a f(x)\,dx &= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = \Big\{z = -x\Big\} = -\int_a^0 f(-z)\,dz + \int_0^a f(x)\,dx \\ &= \int_0^a f(-z)\,dz + \int_0^a f(x)\,dx = \Big\{\text{odd}\Big\} = -\int_0^a f(z)\,dz + \int_0^a f(x)\,dx = 0.\end{aligned}$$

Example. The average of $f(x) = x^3$ over $[0, 1]$ is

$$\int_0^1 x^3\,dx = \frac{1}{4}.$$

It is clear that $\bar{x} = 4^{-1/3}$ is as in the mean value theorem.

Example. Let $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x > 0$. The average of f over $[-1, 1]$ is

$$\frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{0} 0\,dx + \frac{1}{2}\int_{0}^{1} 1\,dx = \frac{1}{2}.$$

But there is no $\bar{x}$ such that $f(\bar{x}) = \frac{1}{2}$. This shows that a discontinuous function may skip over its average value.

In these examples we can easily find an $\bar{x}$. The importance of the theorem is that it guarantees that (at least) one such point exists even if we cannot compute it exactly. So we conclude that there is an unknown point $\bar{x}$ where f is equal to its average. Often we do not need to know what the point is.

Example. Assume that the derivative of f is bounded in $[a, b]$:

$$|f'(x)| \le M_f, \quad \forall x \in [a, b].$$

Then we use the mean value theorem for functions on the interval $[x, y]$ with $x \le y$ (or on the interval $[y, x]$ if $y \ge x$):

$$|f(x) - f(y)| = |(x - y)f'(\bar{x})| = |x - y|\,|f'(\bar{x})| \le M_f|x - y|.$$

We conclude that $L_f \le M_f$. This was proved in a more complicated way in AMBS 23.15. Note that we need not know what $\bar{x}$ is. This is another example where we calculate something important without computing exactly.

Example. Assume that the derivative is non-negative in $[a, b]$:

$$f'(x) \ge 0, \quad \forall x \in [a, b].$$

Then the mean value theorem for functions applied on the interval $[x, y]$ with $x \le y$ says

$$f(y) - f(x) = \underbrace{(y - x)}_{\ge 0}\underbrace{f'(\bar{x})}_{\ge 0} \ge 0.$$

We conclude that $x \le y$ implies $f(x) \le f(y)$. This means that f is an *increasing function.* Note that we need not know what $\bar{x}$ is.

In the same way: if the derivative is positive, $f'(x) > 0$ for all $x \in [a, b]$, and $x < y$, then

$$f(y) - f(x) = \underbrace{(y - x)}_{>0}\underbrace{f'(\bar{x})}_{>0} > 0.$$

We conclude that $x < y$ implies $f(x) < f(y)$. This means that f is a *strictly increasing function.*

Similar calculations can be made if the derivative is non-positive, negative, and zero. We summarize this:

Theorem. *Assume that f is uniformly differentiable on $[a, b]$. Then*

$$\begin{aligned}
f'(x) \ge 0, \quad \forall x \in [a, b] &\implies f \text{ is increasing in } [a, b],\\
f'(x) > 0, \quad \forall x \in [a, b] &\implies f \text{ is strictly increasing in } [a, b],\\
f'(x) \le 0, \quad \forall x \in [a, b] &\implies f \text{ is decreasing in } [a, b],\\
f'(x) < 0, \quad \forall x \in [a, b] &\implies f \text{ is strictly decreasing in } [a, b],\\
f'(x) = 0, \quad \forall x \in [a, b] &\implies f \text{ is constant in } [a, b].
\end{aligned}$$

1.11. **Taylor's theorem.** We will do this later (in week 6).

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