SIAM J. NUMER. ANAL.
Vol. 7, No. 1, March 1970

ESTIMATION OF LINEAR FUNCTIONALS ON SOBOLEV SPACES WITH APPLICATION TO FOURIER TRANSFORMS AND SPLINE INTERPOLATION*

J. H. BRAMBLE† AND S. R. HILBERT‡

1. Introduction. In this paper some general theorems on estimation for classes of linear functionals on Sobolev spaces are given. These are applied to the study of convergence properties of discrete Fourier transforms in N-dimensional Euclidean space, E^N. In addition, a class of spline functions on uniform meshes in E^N is considered.

Specifically, in § 2, definitions and notation are introduced.

Section 3 is devoted to the estimation of bounded linear functionals on Sobolev spaces. The particular functionals of interest are those which annihilate polynomials of a certain degree (or less). Such functionals are of central importance in the study of errors in approximation and interpolation of functions. Our estimates can often be used to replace standard Taylor series approaches to the estimation of local errors such as in the comparison of difference quotients with derivatives (in E^N) or the estimation of the remainder term in the Taylor series itself. In addition, our results can frequently serve as a substitute for estimates based on an ad hoc use of Peano kernel theorems (c.f. Sard [5, p. 25]). In such estimates, the particular form of the kernel must be utilized whereas for our theorems only properties which are easily verified are required. For example, the use of kernel theorems by Birkhoff, Schultz and Varga [1] in the study of errors in Hermite interpolation could now be avoided by applying our theorems. This would seem to be of particular importance in more than one dimension where the kernel representations are a bit cumbersome.

Section 4 is devoted to the study of the behavior of the difference between the discrete and continuous Fourier transforms in E^N as the mesh size tends to zero. Our approximation theorems are applied to obtain these estimates via certain lemmas which are also employed in § 5.

The last section deals with a class of spline interpolants of order k on Sobolev spaces. We investigate the error in interpolation by such splines as the mesh size tends to zero. We also obtain a connection between the discrete Fourier transform and the N-dimensional analogue of the so-called *cardinal series*. It is shown finally that this series is obtained as a limiting case of splines of order k as $k \to \infty$. In this connection, Schoenberg [6] has considered this problem in one dimension but for a somewhat more general class of functions and for splines of even order (piecewise polynomials of odd degree). We also want to mention the interesting paper of Golomb [3] in which he uses Fourier methods to study periodic splines on uniform meshes in one dimension.

* Received by the editors May 29, 1969. This research was supported in part by the National Science Foundation under Grant NSF-GP-9467.

† Department of Mathematics, Cornell University, Ithaca, New York 14850.

‡ Department of Mathematics, Ithaca College, Ithaca, New York 14850.

2. Notation and preliminaries. Let R with boundary ∂R be a bounded domain in Euclidean N-space, E^N. Let ρ be the diameter of R. We shall assume that R satisfies a strong cone property; that is, there exists a finite open covering $\{O_i\}$, $i = 1, \cdots, n$, of ∂R and corresponding cones $\{C_i\}$ with vertices at the origin such that $x + C_i$ is contained in R for any $x \in R \cap O_i$.

We shall consider complex-valued functions defined on R. As usual we denote by $L_p(R)$ the completion of the space of complex-valued functions defined on R such that

$$\left(\frac{1}{\rho^N}\int_R |f(x)|^p\,dx\right)^{1/p} = \|f\|_{p,R}$$

is finite. We shall need the following seminorms:

$$|u|_{p,k,R} = \sum_{|\alpha|=k} \|D^\alpha u\|_{p,R} \tag{2.1}$$

and

$$|u|_{k,R} = \sum_{|\alpha|=k} |D^\alpha u|_R, \tag{2.2}$$

where $|u|_R = \sup_{x\in R} |u(x)|$.

In (2.1), (2.2) and the sequel, α is a multi-index;

$$\alpha = (\alpha_1, \cdots, \alpha_N) \quad \text{and} \quad |\alpha| = \sum_{i=1}^N \alpha_i, \quad D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}.$$

Now for $1 \leqq p < \infty$ and m a nonnegative integer let $H_p^m(R)$ be the set of all functions in L_p with distributional (weak) derivatives of order j for $0 \leqq j \leqq m$ in L_p. In this paper we take the norm on $H_p^m(R)$ to be

$$\|u\|_{p,m,R}^p = \sum_{k=0}^m \rho^{kp}|u|_{p,k,R}^p. \tag{2.3}$$

It is trivial that this is equivalent to the usual norm for $H_p^m(R)$.

We shall also consider the space of functions which have continuous derivatives of order up to and including m in R; this space will be denoted by $C^m(R)$. For the purposes of this paper we take the norm on $C^m(R)$ to be:

$$\|u\|_{m,R} = \sum_{k=0}^m \rho^k|u|_{k,R}. \tag{2.4}$$

Again, the usual norm on $C^m(R)$ is equivalent to (2.4).

We shall denote by P_k the set of polynomials of degree less than or equal to k, restricted to R.

Let h be a (small) positive parameter and define the set of mesh points E_h^N as $E_h^N = \{x | x = (n_1 h, \cdots, n_N h), n_j \text{ an integer}, j = 1, \cdots, N\}$.

Throughout this paper we shall use C to denote a generic constant not necessarily the same in any two places.

We shall also use Sobolev norms on E^N. As usual these are given by $\|u\|_{H_p^m}^p = \sum_{|\alpha|\leqq m} \|D^\alpha u\|_p^p$, where $\|D^\alpha u\|_p = \left(\int_{E^N} |D^\alpha u(x)|^p\,dx\right)^{1/p}$ and we denote by $\|u\|_0$ the L_2-norm, $\|u\|_0 = \left(\int_{E^N} |u(x)|^2\,dx\right)^{1/2}$. By $\|u\|_m$ we shall mean the Sobolev norm of $u \in H_2^m(E^N)$. The notation $|u|_m$ will be used to denote the seminorm $\sum_{|\alpha|=m} \|D^\alpha u\|_0$.

3. Estimation of linear functionals. Consider B a Banach space with norm $\|\cdot\|_B$ and let B_1 be a closed linear subspace of B. We define Q to be the quotient or factor space of B with respect to B_1, denoted by B/B_1. The elements of Q are equivalence classes $[u]$, where $[u]$ is the class containing u. The equivalence relation is given by $\sim$ where for $u, v \in B$, $u \sim v$ if and only if $u - v \in B_1$. The usual norm on Q is given by $\|[u]\|_Q = \inf_{v \in [u]} \|v\|_B$. It is easy to show that $\|[u]\|_Q = \inf_{v \in B_1} \|u + v\|_B$. Under the assumptions we have made for B and B_1, it is well known that Q is a Banach space with norm $\|\cdot\|_Q$.

Now consider the (closed) finite-dimensional subspace of $H_p^k(R)$ given by P_{k-1}. Therefore $p(x) \in P_{k-1}$ if and only if $p(x) = \sum_{|\gamma| \leqq k-1} a_\gamma x^\gamma$ for $x \in R$, where the a_γ are complex numbers and γ is a multi-index.

THEOREM 1. *Let $Q = H_p^k(R)/P_{k-1}$. Then $|u|_{k,p,R}$ is a norm on Q equivalent to $\|[u]\|_Q$. Further, there exists C independent of ρ and u such that for any $u \in H_p^k(R)$*

$$\rho^k |u|_{k,p,R} \leqq \|[u]\|_Q \leqq C\rho^k |u|_{k,p,R}. \tag{3.1}$$

Proof. We shall make use of two lemmas which can be found in Morrey [4, p. 85].

LEMMA 1. *For any $u \in H_p^k(R)$ there is a unique polynomial p of degree less than or equal to $k - 1$ (or 0) such that $\int_R D^\alpha(u + p) = 0$ for all α with $0 \leqq |\alpha| \leqq k - 1$.*

LEMMA 2. *Let R satisfy a strong cone condition. Then (since R is contained in a sphere of radius ρ) $|u|_{j,p,R} \leqq C\rho^{k-j}|u|_{k,p,R}$ for $0 \leqq j \leqq k - 1$ for all $u \in H_p^k(R)$ such that the average over R of each $D^\alpha u$ is 0 for $0 \leqq |\alpha| \leqq k - 1$, where C is a constant independent of ρ and u.*

Note. Morrey assumes that his domain is strongly Lipschitz, but the proof is exactly the same if the domain satisfies a strong cone condition.

We shall now prove the right-hand inequality in Theorem 1. By Lemma 1 we can choose $\bar{p} \in P_{k-1}$ such that $\int_R D^\gamma(u + \bar{p}) = 0$ for $|\gamma| \leqq k - 1$. Hence using Lemma 2 it follows that $\|u + \bar{p}\|_{k,p,R} \leqq C\rho^k|u + \bar{p}|_{k,p,R} = C\rho^k|u|_{k,p,R}$. However, since $\bar{p} \in P_{k-1}$ we have that $\|[u]\|_Q \leqq \|u + \bar{p}\|_{k,p,R}$. Hence $\|[u]\|_Q \leqq C\rho^k|u|_{k,p,R}$ for $u \in H_p^k(R)$.

The other inequality is easily seen from the observation that $\rho^k|u + p|_{k,p,R} = \rho^k|u|_{k,p,R}$ for any $p \in P_{k-1}$ from which we immediately obtain

$$\rho^k|u|_{k,p,R} \leqq \inf_{p \in P_{k-1}} \|u + p\|_{k,p,R} = \|[u]\|_Q.$$

We shall now use this theorem to obtain error estimates for linear functionals. The main result of this section is the following theorem.

THEOREM 2. *Let F be a linear functional on $H_p^k(R)$ which satisfies*

(i) *$|F(u)| \leqq C\|u\|_{k,p,R}$ for all $u \in H_p^k(R)$ with C independent of ρ and u and*

(ii) *$F(p) = 0$ for all $p \in P_{k-1}$.*

Then $|F(u)| \leqq C_1\rho^k|u|_{k,p,R}$ for any $u \in H_p^k(R)$ with C_1 independent of ρ and u.

Proof. Since F is linear and satisfies condition (ii),

(3.2) $$|F(u)| = |F(u + p)| \quad \text{for all } p \in P_{k-1}.$$

By condition (i) and (3.2) we have

(3.3) $$|F(u)| \leqq C\|u + p\|_{k,p,R}.$$

Taking the infimum over P_{k-1} in (3.3) we have

(3.4) $$|F(u)| \leqq C\|[u]\|_Q.$$

The result now follows from Theorem 1.

THEOREM 3. *Let F be a linear functional satisfying*

(i) $|F(u)| \leqq C\|u\|_{j,R}$ *for all* $u \in C^j(R)$, *where C is independent of ρ and u and*

(ii) $F(p) = 0$ *for all* $p \in P_{k-1}$.

Then $|F(u)| \leqq C_1\rho^k|u|_{k,p,R}$ *for* $p > N/(k - j)$, *where C_1 does not depend on ρ or u.*

Proof. Since R satisfies a strong cone condition it follows easily from Sobolev's lemma (c.f. [4, p. 78]) that $\|u\|_{j,R} \leqq C\|u\|_{k,p,R}$ with C independent of ρ and u provided $p > N/(k - j)$. Clearly F satisfies the hypotheses of Theorem 2.

For the final result of this section we define the usual Lipschitz spaces. Let s be any positive real number with $s = S + \sigma$, $0 < \sigma \leqq 1$, S a nonnegative integer. We denote by $C^s(R)$ those elements of $C^s(R)$ such that

$$\sup_{x,y \in R} \sum_{|\alpha| = S} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\sigma}$$

is bounded.

THEOREM 4. *Let $u \in C^s(R)$ and let F be a linear functional on $C^0(R)$ which satisfies*

(i) $|F(u)| \leqq C|u|_{0,R}$ *for all* $u \in C^0(R)$ *with C independent of ρ and u and*

(ii) $F(q) = 0$ *for all* $q \in P_{k-1}$.

Then

$$|F(u)| \leqq C_1\rho^s \sup_{x,y \in R} \sum_{|\alpha| = S} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\alpha}, \qquad 0 \leqq s < k,$$

where C_1 does not depend on ρ or u.

Proof. Since R is bounded, $|u|_{k,p,R} \leqq |u|_{k,R}$ for all $p \geqq 1$. Hence it follows directly from Theorem 3 that

(3.5) $$|F(u)| \leqq C\rho^k|u|_k \quad \text{for any } u \in C^k(R)$$

with C independent of ρ and u. Interpolating between the spaces $C^0(R)$ and $C^k(R)$ we obtain, for $s < k$,

(3.6) $$|F(u)| \leqq C\rho^s\left(|u|_0 + \sup_{x,y \in R} \sum_{|\alpha| = S} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\sigma}\right),$$

where C is independent of ρ and u (c.f. [2] Bramble, Hubbard, Thomée, Lemma 4.1 and 4.2).

Now $F(u) = F(u + q)$ for any $q \in P_S$ since $S \leqq k - 1$. Choosing $q_0 \in P_S$ such that $D^\alpha(u + q_0)(x_0) = 0$ for some $x_0 \in R$ and all $|\alpha| \leqq S$, we may easily obtain

$$|u + q_0|_{0,R} \leqq C \sup_{x,y \in R} \sum_{|\alpha| = S} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\sigma},$$

where C is independent of ρ and u. Hence

$$|F(u)| = |F(u + q_0)| \leqq C\rho^s\left(\sup_{x,y\in R} \sum_{|\alpha|=S} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\sigma}\right),$$

where C is independent of ρ and u.

4. Discrete and continuous Fourier transforms. In this section we shall use the results of the last section to compare the continuous and discrete Fourier transforms.

Let $\mathscr{S}$ be the space of complex-valued infinitely differentiable rapidly decreasing functions on E^N. We remark that $C_0^\infty(R) \subset \mathscr{S}$ for any domain $R \subset E^N$. Now for any function f in $\mathscr{S}$ we define the Fourier transform of f which will be denoted by $\hat{f}$ as $\hat{f}(\xi) = \int_{E^N} f(x)e^{-i\langle\xi,x\rangle}\,dx$, where $\langle\xi, x\rangle = \sum_{i=1}^N \xi_i x_i$. The Fourier transform is defined for a function in $L_2(E^N)$ or $L_1(E^N)$ by using the density of $\mathscr{S}$ in $L_2(E^N)$ or $L_1(E^N)$. It is well known that the Fourier transform is a one-to-one map of $L_2(E^N)$ onto $L_2(E^N)$ and that the Parseval–Plancheral formulas $\|f\|_0 = (2\pi)^{-N/2}\|\hat{f}\|_0$ and $\int_{E^N} f(x)\overline{g(x)}\,dx = (2\pi)^{-N}\int_{E^N} \hat{f}(x)\overline{\hat{g}(x)}\,dx$ hold for any $g, f \in L_2(E^N)$. We define an inverse Fourier transform denoted by $\check{f}$ as $\check{f}(x) = (2\pi)^{-N}\int_{E^N} f(\xi)e^{i\langle x,\xi\rangle}\,d\xi$. For any function in $\mathscr{S}$, we know that $\widehat{D^\alpha f(\xi)} = (i\xi)^\alpha \hat{f}(\xi)$ for any multi-index α, so we can express $\|f\|_k$ as $\int_{E^N} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2\,d\xi$. Finally if $f \in L_2(E^N)$ and $g \in L_1(E^N)$ then $(f * g)(x) = \int_{E^N} f(x - y)g(y)\,dy \in L_2(E^N)$ and $\widehat{f * g}(\xi) = \hat{f}(\xi)\cdot\hat{g}(\xi)$ in $L_2(E^N)$.

We can define a discrete Fourier transform for any function which has bounded support and is defined on all the mesh points E_h^N by $\tilde{u}(\theta) = h^N \sum_{x\in E_h^N} u(x)e^{-i\langle x,\theta\rangle}$. We remark that $\tilde{u}$ is a periodic function of period $2\pi/h$. We shall later show that we can define $\tilde{u}$ for any $u \in H_2^m$ for $m > N/2$.

Let χ_h be the characteristic function of the cube S_h where $S_h = \{\xi | \xi \in E^N, |\xi_j| \leqq \pi/h \text{ for } j = 1, \cdots, N\}$.

Our main aim in this section is to study $\chi_h\tilde{u} - \hat{u}$, as $h \to 0$. We shall first prove the following lemma.

LEMMA 3. *Let $u \in C_0^\infty(E^N)$. Then there exists a constant C independent of h and u such that for $m > N/2$,*

$$\|(\chi_h\tilde{u})^\vee - u\|_j \leqq Ch^{m-j}|u|_m \tag{4.1}$$

for any integer j with $0 \leqq j \leqq m$.

Remark. Our main theorem in this section is the same as this lemma but with $u \in H_2^m$. However, we shall use this lemma to define $\tilde{u}$ for $u \in H_2^m$.

In order to prove Lemma 3 we shall prove two other lemmas which will also be needed in § 5. We need to introduce first some notation.

Let $I_h(y)$ for any $y \in E_h^N$ be the cube given by $I_h(y) = \{x | x \in E^N, y_j - h/2 < x_j \leqq y_j + h/2 \text{ for } j = 1, \cdots, N\}$. Define an extension operator by $P_h u(x) = u(y)$

for $x \in I_h(y)$. Let

$$\psi(x) = \begin{cases} 1/h^N & \text{for } x \in I_h(0), \\ 0 & \text{for } x \notin I_h(0) \end{cases}$$

and

$$\psi_j(x_j) = \begin{cases} 1/h & \text{for } -h/2 < x_j \leqq h/2, \\ 0 & \text{otherwise} \end{cases}$$

so that $\psi(x) = \prod_{j=1}^N \psi_j(x_j)$ and $\hat{\psi}(\xi) = \prod_{j=1}^N (\sin \xi_j h/2)/(\xi_j h/2)$.

It is easy to see that for any u such that $\tilde{u}$ exists,

$$\widehat{P_h u(\xi)} = \hat{\psi}(\xi)\tilde{u}(\xi). \tag{4.2}$$

For fixed $x \in E^N$ we define a linear functional on C_0^∞ by

$$\begin{aligned} F_m(x;u) &= (\psi^{(m-1)} * P_h u)(x) - (\psi^{(m)} * u)(x) \\ &= h^N \sum_{z \in E_h^N} \psi^{(m)}(x-z)u(z) - \int_{E^N} \psi^{(m)}(x-z)u(z)\,dz, \end{aligned}$$

where $\psi^{(k)} = \psi * \cdots * \psi$, k times.

We have the following lemma.

LEMMA 4. *There exist constants C, C_1 and K independent of x, h and u such that*

$$|F_m(x;u)| \leqq C|u|_{I_{Kh}(x)} \leqq C_1 \|u\|_{m,2,I_{Kh}(x)}$$

for $m > N/2$.

Proof. Choose K such that $I_{Kh}(0) = \operatorname{supp} \psi^{(m)}$. Then the first inequality is obvious and the second an immediate application of Sobolev's lemma.

The next lemma is more difficult to prove.

LEMMA 5. *For any polynomial $p \in P_{m-1}$,*

$$F_m(x, p) \equiv 0.$$

Proof. Let J be any set of mesh points which are translates of E_h^N, i.e., $J = \{x | x + a = y$, where $y \in E_h^N$ for fixed $a \in E^N\}$. Now define $F_{m,J}(x;u)$ for u continuous by $h^N \sum_{z \in J} \psi^{(m)}(x-z)u(z) - \int_{E^N} \psi^{(m)}(x-z)u(z)\,dz$ and we shall prove the following proposition which contains our lemma.

PROPOSITION. *For any J and any $x \in E^N$, $F_{m,J}(x,p) = 0$ for all $p \in P_{m-1}$.*

We prove this by induction. It is easy to see that the result is true for $m = 1$ and 2 and if m is an integer greater than two, then $\psi^{(m)}$ belongs to C^{m-2} in E^N. Now assume the lemma is true for m; we want to show that $F_{m+1,J}(x,p) = 0$ for all $p \in P_m$ and any J or x. Consider $\partial/\partial x_j(F_{m+1,J}(x,p))$ for $j = 1, \cdots, N$. (For $m \geqq 3$ we know that $F_{m+1,J}$ is at least a continuously differentiable function of x.) Now

$$\begin{aligned} \frac{\partial}{\partial x_j}\psi^{(m+1)}(x) &= \frac{\partial}{\partial x_j}\int_E \psi_j(x_j - y_j)\psi_j^{(m)}(y_j)\,dy_j \prod_{l \neq j} \psi_l^{(m+1)}(x_l) \\ &= \frac{1}{h}\left[\psi_j^{(m)}\left(x_j + \frac{h}{2}\right) - \psi_j^{(m)}\left(x_j - \frac{h}{2}\right)\right] \prod_{l \neq j} \psi_l^{(m+1)}(x_l). \end{aligned}$$

We define $\partial_j f(x) = [f(x_1, \cdots, x_j + h/2, x_{j+1}, \cdots, x_N) - f(x_1, \cdots, x_j - h/2, x_{j+1}, \cdots, x_N)]/n$ and easily obtain

$$\frac{\partial}{\partial x_j} F_{m+1,J}(x, p) = h^N \sum_{y \in J} \prod_{l \neq j} \psi_l^{(m+1)}(x_l - y_l) \partial_j \psi_j^{(m)}(x_j - y_j) p(y)$$
$$- \int_{E^N} \prod_{l \neq j} \psi_l^{(m+1)}(x_l - y_l) \partial_j \psi_j^{(m)}(x_j - y_j) p(y)\, dy.$$

Now defining a new set of mesh points $\bar{J} = \{z | z_l = y_l, l \neq j, z_j = y_j - h/2, y \in J\}$ we have

$$\frac{\partial}{\partial x_j} F_{m+1,J}(x, p) = h^N \sum_{z \in \bar{J}} \prod_{l \neq j} \psi_l^{(m+1)}(x_l - z_l) \psi_j^{(m)}(x_j - y_j) \partial_j p(z)$$
$$- \int_{E^N} \prod_{l \neq j} \psi_l^{(m+1)}(x_l - z_l) \psi_j^{(m)}(x_j - z_j) \partial_j p(z)\, dz$$
$$= \left[\left(\delta_j \prod_{l \neq j} \psi_l \right) * F_{m,\bar{J}}(\cdot, \partial_j p) \right](x),$$

where δ_j is the one-dimensional Dirac measure with respect to x_j. However, this is zero since $\partial_j p$ is in P_{m-1} if p is in P_m. Since $\partial/(\partial x_j) F_{m+1,J}(x, p) = 0$ for any x, J and $p \in P_m$ and $j = 1, \cdots, N$, then $F_{m+1,J}(x, p) = C$, where C does not depend on x. Hence

$$C = h^N \sum_{y \in J} \psi^{(m+1)}(x - y) p(y) - \int_{E^N} \psi^{(m+1)}(x - y) p(y)\, dy.$$

Using the fact that $F_{m+1,J}$ annihilates polynomials in P_{m-1} we can replace $p(y)$ by $p(y - x)$, and noting that $\psi^{(m+1)}$ is even, by a change of variable we obtain

$$C = h^N \sum_{y \in J} \psi^{(m+1)}(x + y) p(x + y) - \int_{E^N} \psi^{(m+1)}(t) p(t)\, dt. \tag{4.3}$$

Averaging both sides of (4.3) over $I_h(0)$ we find that

$$C = \frac{1}{h^N} \int_{I_h(0)} \Big(h^N \sum_{y \in J} \psi^{(m+1)}(x + y) p(x + y)\Big)\, dx - \int_{E^N} \psi^{(m+1)}(x) p(x)\, dx$$
$$= \sum_{y \in J} \int_{I_h(y)} \psi^{(m+1)}(x) p(x)\, dx - \int_{E^N} \psi^{(m+1)}(x) p(x)\, dx = 0.$$

This proves the proposition.

We can now complete the proof of Lemma 3. Consider $j = 0$. Now by Parseval's identity $\|(\chi_h \tilde{u})^\vee - u\|_0 = (2\pi)^{-N/2} \|\chi_h \tilde{u} - \hat{u}\|_0$, and

$$\|\chi_h \tilde{u} - \hat{u}\|_0^2 = \int_{S_h} |\tilde{u} - \hat{u}|^2\, d\xi + \int_{|\xi_j| > \pi/h} |\hat{u}(\xi)|^2\, d\xi.$$

The second integral is easily estimated since for $\xi \notin S_h$ there is a constant C_m such that $C_m h^{2m}|\xi|^{2m} \geqq 1$ for all $\xi \notin S_h$. Hence

(4.4)
$$\int_{|\xi|>\pi/h} |\hat{u}(\xi)|^2\, d\xi \leqq C_m h^{2m} \int_{|\xi_j|>\pi/h} |\xi|^{2m}|\hat{u}(\xi)|^2\, d\xi \leqq C_m h^{2m}|u|_m^2.$$

Now consider the first integral. Since $\hat{\psi}(\xi) = \prod_{j=1}^N (\sin \xi_j h/2)/(\xi_j h/2)$, there are positive constants C_1 and C_2 independent of h such that $0 < C_1 \leqq \hat{\psi}(\xi) \leqq C_2$ for any $\xi \in S_h$. Thus in S_h, $\tilde{u}(\xi) = \hat{\psi}^{-1}(\xi)\widehat{P_h u(\xi)}$. We have

$$\int_{S_h} |\tilde{u} - \hat{u}|^2\, d\xi = \int_{S_h} |\hat{\psi}^{-1}\widehat{P_h u} - \hat{u}|^2\, d\xi = \int_{S_h} |\hat{\psi}^{-m}(\hat{\psi}^{(m-1)}\widehat{P_h u} - \widehat{\psi^{(m)}}\hat{u}|^2\, d\xi$$
$$\leqq C \int_{S_h} |\hat{\psi}^{(m-1)}\widehat{P_h u} - \widehat{\psi^{(m)}}\hat{u}|^2\, d\xi$$
$$\leqq C\|\hat{\psi}^{(m-1)}\widehat{P_h u} - \widehat{\psi^{(m)}}\hat{u}\|_0^2,$$

and so, by Parseval's formula we obtain

(4.5)
$$\int_{S_h} |\tilde{u} - \hat{u}|^2\, d\xi \leqq C\|\psi^{(m-1)} * P_h u - \psi^{(m)} * u\|_0^2 = C\|F_m(\cdot, u)\|_0^2.$$

Clearly by Lemma 4 we can extend $F_m(x; u)$ to $H_2^m(I_{Kh}(x))$ by continuity. Now by Lemmas 4 and 5, F_m satisfies the hypotheses of Theorem 2 with $R = I_{Kh}(x)$. Hence $|F_m(x; u)| \leqq Ch^m|u|_{m,2,I_{Kh}}(x)$, where C is independent of x, h and u. Explicitly

$$|F_m(x; u)|^2 \leqq Ch^{2m}\left(1/\text{meas } I_{Kh}(x) \int_{I_{Kh}(x)} \sum_{|\alpha|=m} |D^\alpha u(z)|^2\, dz\right).$$

Let

$$\varphi_m(y) = \begin{cases} 1/\text{meas } I_{Kh}(0) & \text{if } y \in I_{Kh}(0), \\ 0 & \text{if } y \notin I_{Kh}(0) \end{cases}$$

so that $\|F_m(\cdot\,; u)\|_0 \leqq Ch^m\left(\int_{E^N} (\varphi_m * \sum_{|\alpha|=m} |D^\alpha u|^2)(x)\, dx\right)^{1/2}$. However, since $\int_{E^N} \varphi_m(x)\, dx = 1$, by interchanging the integration in the convolution with the integration with respect to x we have

(4.6)
$$\|F_m(\cdot, u)\|_0 \leqq Ch^m|u|_m.$$

Thus (4.4) and (4.6) prove Lemma 3 for $j = 0$. Now, for $0 < j \leqq m$, following the same steps as before we are easily led to

$$\|(\chi_h\tilde{u})^\vee - u\|_j \leqq C(\|F_m(\cdot, u)\|_j + h^{m-j}|u|_m).$$

The estimate for the first term on the right is obtained by applying Theorem 2 to the functional $G_{m,\alpha}(x; u) = h^{|\alpha|}D^\alpha F_m(x; u)$ for each α with $|\alpha| \leqq j$, which clearly satisfies the hypotheses. The proof is completed as before. Thus we have proved Lemma 3.

We now wish to extend the definition of the discrete Fourier transform to any function in H_2^m where $m > N/2$. Define l_2 as the set of all square summable functions defined on E_h^N, with norm $\|\varphi\|_{l_2}^2 = h^N \sum_{x \in E_h^N} |\varphi(x)|^2$. It is easy to see that for any function $\varphi \in C_0^\infty$ we have

$$h^N \sum_{x \in E_h^N} |\varphi(x)|^2 = \frac{1}{(2\pi)^N} \int_{S_h} |\varphi(\xi)|^2 \, d\xi. \tag{4.7}$$

For any $u \in H_2^m$ there exists a sequence $\{\varphi_j\} \in C_0^\infty$ such that $\varphi_j \to u$ in H_2^m. Now since

$$\int_{S_h} |\tilde{\varphi}_j - \tilde{\varphi}_k|^2 \, d\xi \leqq \int_{S_h} |\tilde{\varphi}_j - \tilde{\varphi}_k - \hat{\varphi}_j + \hat{\varphi}_k|^2 \, d\xi + \int_{E^N} |\hat{\varphi}_j - \hat{\varphi}_k|^2 \, d\xi,$$

using Lemma 1 and the fact that $\{\varphi_j\}$ is a Cauchy sequence in H_2^m, we have $\{\tilde{\varphi}_j\}$ is a Cauchy sequence in $L_2(S_h)$. We define $\tilde{u}$ as the limit in $L_2(S_h)$ of $\tilde{\varphi}_j$ and extend it periodically to all of E^N. It is easy to see that $\tilde{u}$ is independent of the choice of the sequence $\varphi_j \to u$ in H_2^m, so $\tilde{u}$ is well-defined.

We now have the main result of this section.

THEOREM 5. *Let $u \in H_2^m$. Then there exists a constant C independent of h and u such that for $m > N/2$,*

$$\|(\chi_h \tilde{u})^\vee - u\|_j \leqq C h^{m-j} |u|_m$$

for any integer j with $0 \leqq j \leqq m$.

Proof. Let $\{\varphi_n\}$ be a sequence such that $\varphi_n \in C_0^\infty$, $n = 1, 2, \cdots$, and $\varphi_n \to u$ in H_2^m as $n \to \infty$. Now

$$\|(\chi_h \tilde{u})^\vee - u\|_j \leqq \|(\chi_h(\tilde{u} - \tilde{\varphi}_n))^\vee\|_j + \|(\chi_h \tilde{\varphi}_n)^\vee - \varphi_n\|_j + \|\varphi_n - u\|_j.$$

By Lemma 3 and the definition of χ_h and $\|\cdot\|_j$ it follows that

$$\|(\chi_h \tilde{u})^\vee - u\|_j \leqq C h^{-j} \|\chi_h(\tilde{u} - \tilde{\varphi}_n)\|_0 + C h^{m-j} |\varphi_n|_j + \|\varphi_n - u\|_j.$$

By letting $n \to \infty$ the theorem follows.

We shall also give a version of the Poisson summation formula relating the discrete and continuous Fourier transforms. This will be needed in the next section.

THEOREM 6. *Let $u \in H_2^m$ with $m > N/2$; then*

$$\tilde{u}(\xi) = \sum_{\beta \in E_1^N} \hat{u}(\xi + 2\pi\beta/h) \quad a.e.$$

We shall not give a proof of this formula here since it is essentially a well-known result.

Finally we wish to remark that a proof of Theorem 5 can be based on Theorem 6, but since Lemmas 4 and 5 are required in the next section it seemed preferable to present a self-contained proof based on these lemmas.

5. Splines in E^N. In this section we shall apply the results of § 3 and § 4 to certain types of splines. If v is defined on E_h^N, then a function of the form $h^N \sum_{y \in E_h^N} \psi^{(k)}(x - y) v(y)$ is called a *spline of order k*. It is easy to see that regarded as a function of x, a spline of order k has continuous partial derivatives of order $k-2$ and is a piecewise polynomial of degree $k-1$. Now let u be defined on E_h^N. Then we call $S_k(x;u)$ a *spline interpolant of order* k for u, provided $S_k(x;u) = h^N \sum_{y\in E_h^N} \psi^{(k)}(x-y)v(y)$ and $S_k(x;u) = u(x)$ for all $x \in E_h^N$. We remark that the functions $\psi^{(k)}(x-z)$, $h = 1$, $N = 1$, are all the so-called B-splines of Schoenberg [6].

We have the following existence and uniqueness theorem.

THEOREM 7. *Let $u \in l_2$. Then there exists a unique $v \in l_2$ such that $S_k(x,u) = h^N \sum_{y\in E_h^N} \psi^{(k)}(x-y)v(y)$ is a spline interpolant of order k for u, $k = 1, 2, \cdots$.*

Proof. In order to prove this theorem we need the following lemma whose proof will be deferred.

LEMMA 6. *For each $k = 1, 2, \cdots$ there exists a constant C_k such that*

$$\tilde{\psi}^{(k)}(\xi) \geqq C_k > 0 \qquad \text{for all } \xi \in E^N.$$

Let L be an operator defined on l_2 by

$$(Lv)(x) = h^N \sum_{y\in E_h^N} \psi^{(k)}(x-y)v(y), \qquad x \in E_h^N.$$

For any $\varphi \in l_2$ let $\{\varphi_n\}$ be a sequence such that $\varphi_n \in l_2$ has bounded support for each n and $\varphi_n \to \varphi$ as $n \to \infty$ in l_2. Now clearly $\widetilde{L\varphi_n} = \widetilde{\psi^{(k)}}\tilde{\varphi}_n$, and by the Parseval identity (4.7) and Lemma 6 it follows that

$$\|L\varphi_n\|_{l_2}^2 = \frac{1}{(2\pi)^N}\int_{S_h} |\widetilde{L\varphi_n}|^2\, d\theta = \frac{1}{(2\pi)^N}\int_{S_h} |\widetilde{\psi^{(k)}}|^2 |\tilde{\varphi}_n|^2\, d\theta$$
$$\geqq \frac{C_k^2}{(2\pi)^N}\|\varphi_n\|_{l_2}^2.$$

The operator $L: l_2 \to l_2$ is obviously continuous. Hence we obtain

$$\|\varphi\|_{l_2} \leqq \frac{C_k^{-1}}{(2\pi)^{-N/2}}\|L\varphi\|_{l_2}$$

for all $\varphi \in l_2$. Denote by $R(L)$ the range of L, regarded as a subspace of l_2.

Define a functional F on the range of L by $F(L\varphi) = (u, \varphi)$ for all $\varphi \in l_2$. Now $|F(L\varphi)| \leqq \|\varphi\|_{l_2}\|u\|_{l_2} \leqq C\|L\varphi\|_{l_2}\|u\|_{l_2}$ and hence F is well-defined and bounded on $R(L)$. By the Hahn-Banach theorem there is an extension $\bar{F}$ of F to all of l_2. Now by the Riesz-Fréchet theorem, there exists a unique $v \in l_2$ such that $\bar{F}(w) = (v, w)$ for all $w \in l_2$. Thus we have, for all $\varphi \in l_2$,

$$(v, L\varphi) = \bar{F}(L\varphi) = F(L\varphi) = (u, \varphi).$$

Clearly L is symmetric so that

$$(Lv, \varphi) = (u, \varphi) \quad \text{for all } \varphi \in l_2,$$

which proves the theorem.

Proof of Lemma 6. It is sufficient to prove the result in one dimension since $\psi^{(k)}(\xi) = \prod_{j=1}^N \psi_j^{(k)}(\xi_j)$. Now by the Poisson summation formula $\widetilde{\psi_j^{(k)}}(\xi_j)$

$= \sum_{l=-\infty}^{\infty} \widehat{\psi_j^{(k)}}(\xi_j + 2\pi l/h)$ a.e. Hence, we have

$$\widetilde{\psi_j^{(k)}}(\xi_j) = \sum_{l=-\infty}^{\infty} \left(\frac{\sin(\xi_j h/2 + \pi l)}{\xi_j h/2 + \pi l}\right)^k$$

$$= \left(\frac{\sin \xi_j h/2}{\xi_j h/2}\right)^k + (\sin \xi_j h/2)^k \sum_{l=1}^{\infty} \left\{(-1)^{lk}\left[\left(\frac{1}{\xi_j h/2 + l\pi}\right)^k + \left(\frac{-1}{l\pi - \xi_j h/2}\right)^k\right]\right\} \quad \text{a.e.}$$

Now since $\widetilde{\psi_j^{(k)}}$ has period $2\pi/h$ and is even, we have for any positive integer k, $\widetilde{\psi_j^{(k)}}(\xi_j) \geqq [(\sin \xi_j h/2)/(\xi_j h/2)]^k$ a.e. for $0 \leqq \xi_j \leqq \pi/h$ and therefore we have $\widetilde{\psi_j^{(k)}}(\xi_j) \geqq (\cos \xi_0)^k$ a.e. for $0 \leqq \xi_j \leqq \pi/h$, when $\xi_0 = \tan \xi_0$, $0 < \xi_0 < \pi/2$. Then by the continuity, the periodicity and the evenness of ψ_j^k we obtain

$$\psi^{(k)}(\xi) \geqq \prod_{j=1}^{N} (\cos \xi_0)^k = (\cos \xi_0)^{kN} > 0 \quad \text{for all } \xi \in E^N.$$

We shall define $S_k(x; u)$, $k = 1, 2, \cdots$, for $u \in H_2^m$ with $m > N/2$. By Sobolev's lemma for any $u \in H_2^m$ there exists $u_1 \in H_2^m$ such that $u - u_1 = 0$ a.e. and $u_1 \in C^0$. Again by Sobolev's lemma, the restriction of u_1 to E_h^N belongs to l_2. Hence $S_k(x; u_1)$ exists and is unique. We define $S_k(x; u)$ to be equal to $S_k(x; u_1)$. We remark that it follows from the definition of $\tilde{u}$ that $\tilde{u} = \tilde{u}_1$.

We shall now obtain a representation for $S_k(x;u)$ in terms of u.

THEOREM 8. *Let $u \in H_2^m$ with $m > N/2$. Then*

$$S_k(x; u) = ((\widehat{\psi^{(k)}}/\widetilde{\psi^{(k)}})\tilde{u})^{\vee}. \tag{5.1}$$

Proof. From the definition of $S_k(x; u)$ for $u \in H_2^m$ and Theorem 7 there exists $v \in l_2$ such that

$$S_k(x; u) = h^N \sum_{y \in E_h^N} \psi^{(k)}(x - y) v(y).$$

Hence

$$\widehat{S_k} = \widehat{\psi^{(k)}}\tilde{v}$$

and

$$\tilde{u} = \tilde{u}_1 = \tilde{S}_k = \widetilde{\psi^{(k)}}\tilde{v}.$$

By Lemma 6 it follows that

$$\widehat{S}_k = \widehat{\psi^{(k)}}/\widetilde{\psi^{(k)}}\tilde{u}.$$

Taking the inverse transform we obtain (5.1). Using this representation we obtain the following error estimate.

THEOREM 9. *Let $u \in H_2^k$, $k > N/2$. Then there exists a constant C independent of h and u such that*

$$\|S_k - u\|_j \leqq Ch^{k-j}\|u\|_k, \qquad 0 \leqq j \leqq k.$$

Proof. Now

$$\|S_k - u\|_j^2 \leqq C \int_{E^N} (1 + |\theta|^2)^j |\widehat{S_k} - \hat{u}|^2 \, d\theta.$$

Since we can express $\widehat{S_k} - \hat{u}$ as

$$(\widetilde{\psi^{(k)}})^{-1}[\widetilde{\psi^{(k)}\tilde{u}} - \widehat{\psi^{(k)}\hat{u}}] + (\widetilde{\psi^{(k)}})^{-1}[\widehat{\psi^{(k)}\hat{u}} - \widetilde{\psi^{(k)}\hat{u}}]$$

we have by Lemma 6

$$\|S_k - u\|_j \leqq C_k^{-1}\left[\|F_k(\cdot, u)\|_j + \left(\int_{E^N} (1 + |\theta|^2)^j|\widehat{\psi^{(k)}} - \widetilde{\psi^{(k)}}|^2|\hat{u}|^2\, d\theta\right)^{1/2}\right]. \tag{5.2}$$

However, we have

$$\begin{aligned}|\widehat{\psi^{(k)}}(\theta) - \widetilde{\psi^{k}}(\theta)| &= |h^N \sum_{x\in E_h^N} \psi^{(k)}(x)e^{-i\langle x,\theta\rangle} - \int_{E^N} \psi^{(k)}(x)e^{-i\langle x,\theta\rangle}\, dx| \\ &= |F_k(0, e^{-i\langle\cdot,\theta\rangle})| \leqq Ch^l|\theta|^l \quad \text{for any } 0 \leqq l \leqq k,\end{aligned}$$

by Theorem 4. Take $l = k - j$. Then it is clear that the second term on the right of (5.2) is bounded by $Ch^{k-j}\|u\|_k$. We have already seen in the proof of Theorem 5 that $\|F_k(\cdot, u)\|_j \leqq Ch^{k-j}\|u\|_k$. Thus the proof is complete.

Now in § 4, we showed that $\|(\chi_h\tilde{u})^\vee - u\|_j \leqq Ch^{m-j}\|u\|_m$. We shall now show that we can regard $(\chi_h\tilde{u})^\vee$ as a limiting case of $S_k(x; u)$. Consider $(\chi_h\tilde{u})^\vee$ for u in C_0^∞; then

$$\chi_h\tilde{u} = \begin{cases} h^N \sum_{x\in E_h^N} u(x)e^{-i\langle x,\theta\rangle}, & \theta\in S_h, \\ 0 & \text{for } |\theta > \pi/h. \end{cases}$$

Hence

$$\begin{aligned}(\chi_h\tilde{u})^\vee(\xi) &= \frac{h^N}{(2\pi)^N}\int_{S_h} \sum u(x)e^{-i\langle x,\theta\rangle}e^{i\langle\theta,\xi\rangle}\, d\theta \\ &= \frac{h^N}{(2\pi)^N}\sum_{x\in E_h^N} u(x)\int_{S_h} e^{i\langle\theta,\xi - x\rangle}\, d\theta \\ &= \sum_{x\in E_h^N} u(x)\prod_{j=1}^N \left(\frac{\sin(\xi_j - x_j)\pi/h}{(\xi_j - x_j)\pi/h}\right).\end{aligned} \tag{5.3}$$

Note that $(\chi_h\tilde{u})^\vee$ is just the N-dimensional cardinal series of Whittaker [7]. We shall denote $(\chi_h\tilde{u})^\vee$ by $S_\infty(x; u)$.

The behavior of S_k as $k \to \infty$ is studied in the following theorem.

THEOREM 10. *Let $u \in H_2^m$ with $m > N/2$. Then S_k converges uniformly to S_∞ on E^N.*

Proof. By Theorem 8 we have

$$S_k(x; u) - S_\infty(x; u) = \left(\frac{1}{2\pi}\right)^N \int_{E^N}\left(\frac{\psi^{(k)}}{\widetilde{\psi^{(k)}}} - \chi_h\right)\tilde{u}e^{i\langle x,\theta\rangle}\, d\theta,$$

and hence

$$|S_k(x; u) - S_\infty(x; u)| \leqq \left(\frac{1}{2\pi}\right)^N\left[\int_{S_h} |\widehat{\psi^{(k)}}/\widetilde{\psi^{(k)}} - 1||\tilde{u}|\, d\theta + \int_{|\theta_j| > \pi/h} |\widehat{\psi^{(k)}}/\widetilde{\psi^{(k)}}||\tilde{u}|\, d\theta\right].$$

By an application of the monotone convergence theorem and the Poisson summation formula we can write the first integral as

$$\sum_{\beta \neq 0} \int_{S_h} \frac{\widehat{\psi^{(k)}}(\theta + 2\pi\beta/h)}{\widetilde{\psi^{(k)}}\theta} |\tilde{u}(\theta)| \, d\theta.$$

Using the periodicity of $\tilde{u}$ and $\tilde{\psi}$ and regarding the integral over $|\theta_j| > \pi/h$ as the sum of the integrals over $S_h + 2\beta\pi/h$ for all $\beta \neq 0$, after making a change of variable for each β, we can express the second integral as

$$\sum_{\beta \neq 0} \int_{S_h} \frac{\widehat{\psi^{(k)}}(\theta + 2\beta\pi/h)}{\widetilde{\psi^{(k)}}\theta} |\tilde{u}(\theta)| \, d\theta.$$

Now since $\widetilde{\psi^{(k)}} \geqq \widehat{\psi^{(k)}}$ in S_h, and since $|\sin (x + \pi)| = |\sin x|$, we obtain

$$|S_k(x; u) - S_\infty(x; u)| \leqq \frac{2}{(2\pi)^N} \sum_{\beta \neq 0} \int_{S_h} \prod_{j=1}^{N} \left| \frac{\theta_j h}{\theta_j h + 2\pi\beta_j} \right|^k |\tilde{u}(\theta)| \, d\theta.$$

Using the Cauchy–Schwarz inequality we have

$$|S_k(x; u) - S_\infty(x; u)| \leqq \frac{2}{(2\pi)^N} \|\tilde{u}\|_{0,S_h} \sum_{\beta \neq 0} \left(\int_{S_h} \prod_{j=1}^{N} \left| \frac{\theta_j h}{\theta_j h + 2\pi\beta_j} \right|^{2k} d\theta \right)^{1/2}.$$

By an elementary estimate it follows easily that there is a constant C independent of k such that

$$2 \sum_{\beta \neq 0} \left(\int_{S_h} \prod_{j=1}^{N} \left| \frac{\theta_j h}{\theta_j h + 2\pi\beta_j} \right|^{2k} d\theta \right)^{1/2} \leqq C(2k - 1)^{-N/2}.$$

Thus we have $|S_k(x; u) - S_\infty(x; u)| \leqq C(2k - 1)^{-N/2}$, where C is independent of χ; by letting $k \to \infty$ the theorem follows.

REFERENCES

[1] G. BIRKHOFF, M. SCHULTZ AND R. VARGA, *Piecewise Hermite interpolation in one and two variables with applications to partial differential equations*, Numer Math., 11 (1968), pp. 232–256.

[2] J. BRAMBLE, B. HUBBARD AND V. THOMÉE, *Convergence estimates for essentially positive type discrete Dirichlet problems*, Math. Comp., to appear.

[3] M. GOLOMB, *Approximation by periodic spline interpolants on uniform meshes*, J. Approx. Theor., 1 (1968), pp. 26–65.

[4] C. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York, 1966.

[5] A. SARD, *Linear Approximation*, Math. Surveys, No. 9, American Mathematical Society, Providence, 1963.

[6] I. J. SCHOENBERG, *On cardinal spline interpolation and a summability method for the cardinal series*, Lecture Notes, Symposium on Approximation, University of Cincinnati, Cincinnati, Ohio, 1969.

[7] J. M. WHITTAKER, *Interpolatory Function Theory*, Cambridge University Press, Cambridge, 1935.