$a = 1 \Rightarrow u_S = u_I$

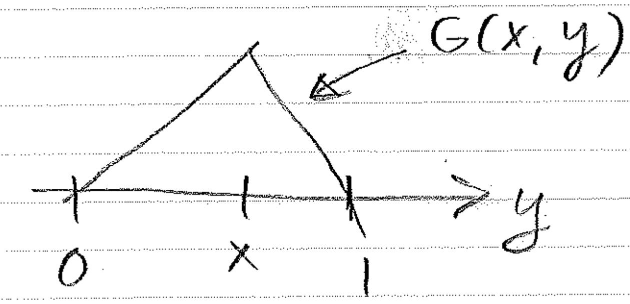
$a=1 \quad \begin{cases} -u'' = f & \text{in } I \\ u(0) = u(1) = 0 \end{cases}$

(Larsson-Thomée Problem 2.4 Problem 5.4(a))

Green's function

(1) $u(x) = \int_0^1 G(x,y) f(y)\, dy$

$= (G(x,\cdot), f)$



$\begin{cases} u \in H_0^1 \\ (u', \varphi') = (f, \varphi) \quad \forall \varphi \in H_0^1 \end{cases}$

Note: $G(x_j, \cdot) \in S$ when x_j is a mesh point

take $\varphi = G(x, \cdot)$

$(u', G_y'(x,\cdot)) = (f, G(x,\cdot)) \overset{(1)}{=} u(x) \quad \forall u \in H_0^1$

Now forget f and conclude

(2) $\begin{cases} G(x,\cdot) \in H_0^1 \\ (v', G_y'(x,\cdot)) = v(x) \quad \forall v \in H_0^1 \end{cases}$

i.e. $-G_{yy}''(x,y) = \delta(x-y)$

(3) $\begin{cases} u_S \in S \\ (u_S', \chi') = (f, \chi) \quad \forall \chi \in S \end{cases}$

Take $\chi = G(x_j, \cdot) \in S$ in (3) and $v = u_S \in S \subset H_0^1$ in (2):

$u_S(x_j) \overset{(2)}{=} (u_S', G_y'(x_j, \cdot)) \overset{(3)}{=} (f, G(x_j, \cdot)) \overset{(1)}{=} u(x_j)$

Thus: $\quad u_S = u_I \quad$ because they $\in S$ and agree in the nodes.

Therefore the FEM is exact in the nodes when $a=1$. But it is still necessary to compute u_S from (3) because we do not know u and hence not u_I.