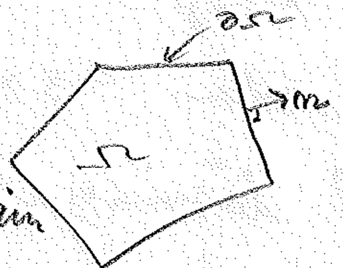
Lecture 1 Jan 17 - 2012

Introduction to FEM (Larsson + Thomee Chapt 5)

Linear elliptic BVP: (heat eq)

$$\begin{cases} -\nabla\cdot(a\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



$a(x) \geq a_0 > 0 \quad \forall x \in \Omega$

$\Omega \subset \mathbb{R}^d$, polygonal domain

$j = -a\nabla u$ heat flux density

f heat source density

$$\underbrace{\int_\Omega f\varphi\,dx}_{=(f,\varphi)} = -\int_\Omega \nabla\cdot(a\nabla u)\varphi\,dx = -\int_{\partial\Omega} n\cdot a\nabla u \underbrace{\varphi}_{=0}\,dS + \underbrace{\int_\Omega a\nabla u\cdot\nabla\varphi\,dx}_{=a(u,\varphi)} \quad \forall\varphi\in C_0^\infty(\Omega)$$

Weak formulation

$$\begin{cases} u \in H_0^1 \\ a(u,\varphi) = (f,\varphi) \quad \forall\varphi\in H_0^1 \end{cases}$$

$L_2 = L_2(\Omega)$, $(f,g) = \int_\Omega fg\,dx$, $\|f\| = \sqrt{\int_\Omega f^2\,dx}$

$H^1 = H^1(\Omega)$ $\quad \|f\|_{H^1} = \left(\|f\|^2 + \|\nabla f\|^2\right)^{1/2} = \left(\int_\Omega f^2 + |\nabla f|^2\,dx\right)^{\frac{1}{2}}$

$H_0^1 = H_0^1(\Omega) = \{v \in H^1 : v|_{\partial\Omega} = 0\}$

$|v|_{H^1} = \|\nabla v\|$ seminorm on H^1 ($v = \text{const} \Rightarrow |v|_1 = 0$)

Poincaré ineq: $\|v\| \leq C|v|_{H^1} \quad \forall v \in H_0^1$

$|v|_{H^1}^2 \leq \|v\|_{H^1}^2 = \|v\|^2 + |v|_{H^1}^2 \leq C|v|_{H^1}^2$ so $|v|_{H^1}$ and $\|v\|_{H^1}$ eq norms on H_0^1

energy norm:

$$\|v\|_E = \sqrt{a(v,v)} = \left(\int_\Omega a |\nabla v|^2 dx\right)^{1/2}$$

$$a_0 |v|_{H^1}^2 = a_0 \int_\Omega |\nabla v|^2 \le \|v\|_E^2 = \int_\Omega a |\nabla v|^2 dx \le \|a\|_{L_\infty} \int a |\nabla v|^2 dx$$

$$a_0 \le a \le \max a = \|a\|_{L_\infty}$$

So $\|v\|_E$ eq. to $|v|_{H^1}$ on H_0^1.

FEM

$h_K = \mathrm{diam}(K)$
$h = \max h_K$
$T_h = \{K\}$

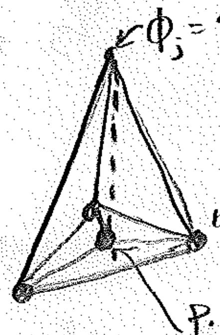
Family of triangulation $\{T_h\}$

Family of function spaces:

$$S_h = \{v_h \in C(\bar\Omega) : v_h|_K \in \Pi_1,\ v_h|_{\partial\Omega} = 0\}$$

$$S_h \subset H_0^1$$

Galerkin method
$$\begin{cases} u_h \in S_h \\ a(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h \end{cases}$$



(pyramid functions)

Basis for S_h: $\{\phi_j\}_{j=1}^{N_h}$ $\quad \phi_j(P_i) = \delta_{ij}$

$$u_h(x) = \sum_{j=1}^{N_h} U_j \phi_j(x), \quad U_j = u_h(P_j)$$

$$\chi = \phi_i$$

$$\sum_{j=1}^{N_h} U_j\, a(\phi_j, \phi_i) = (f, \phi_i) \quad \forall i$$

$$K\mathbf{U} = F$$

Interpolator $I_h : C(\bar\Omega) \to S_h$

$$(I_h v)(x) = \sum_{j=1}^{N_h} v(P_j)\phi_j(x)$$

Interpolation error:

$$|I_h v - v|_{H^1} \le C h |v|_{H^2}$$

$$|v|_{H^2}^2 = \left\|\sum_{i,j} \frac{\partial^2 v}{\partial x_i \partial x_j}\right\|^2$$

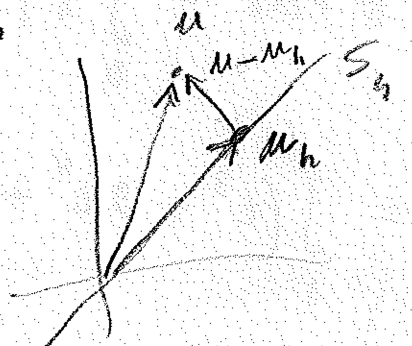
$a(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H^1_0$

$a(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h$

Take $\varphi = \chi \in S_h \subset H^1_0$, subtract:

$a(u - u_h, \chi) = 0 \quad \forall \chi \in S_h$

(Galerkin orthogonality)



Best approximation property:

$$\|u - u_h\|_E = \inf_{v_h \in S_h} \|u - v_h\|_E \quad \text{(optimal)}$$

Norm equivalence:

$$|u - u_h|_{H^1} \le C \|u - u_h\|_E \le C \inf \|u - v_h\|_E$$

$$\le C \inf |u - v_h|_{H^1}$$

(quasi-optimal)

$$\le C |u - I_h u|_{H^1}$$

$$\le C h |u|_{H^2} \quad \left(\text{if } \Omega \text{ is convex so that } u \in H^2\right)$$

<u>Duality argument</u> (Aubin-Nitsche trick):

dual problem with data $e = u - u_h$

$$\begin{cases} -\nabla \cdot (a \nabla w) = e & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} w \in H^1_0 \\ a(\varphi, w) = (\varphi, e) \quad \forall \varphi \in H^1_0 \end{cases}$$

Take $\varphi = e = u - u_h \in H^1_0$

Cauchy-Schwarz

$$\|e\|^2 = a(e, w) = a(e, w - I_h w) \le$$

$$\le \|e\|_E \, \|w - I_h w\|_E \le$$

$$\le C\|e\|_E \, C h \underbrace{|w|_{H^2}}_{\le C\|e\| \text{ if } \Omega \text{ is convex.}} \le C h \|e\|_E$$

Thus

$$\|u-u_h\| \le C h \|u-u_h\|_E$$

$$\le C h^2 |u|_{H^2}$$

<u>Summary</u> $|I_h v - v|_{H^1} \le C h |v|_{H^2}$

implies (if Ω convex)

$$\|u-u_h\| + h|u-u_h|_{H^1} \le C h^2 |u|_{H^2}$$

<u>Outline</u> (core of the course)

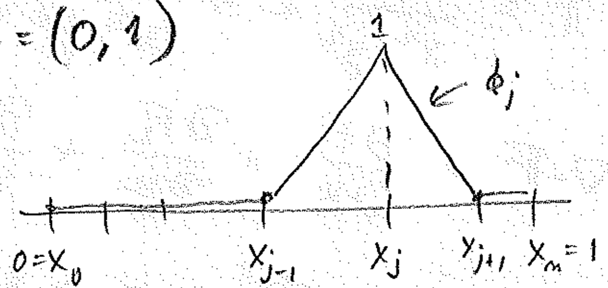
<u>Chapt 3</u> Construction of finite element spaces

<u>Chapt 4</u> Approximation theory (interpolation error estimates)

This is much easier in 1-D.

<u>Chapt 0</u>: $d=1$ $\Omega=(0,1)$

mesh



$h_j = x_j - x_{j-1}$, $h = \max h_j$

$I_j = (x_{j-1}, x_j)$ $\quad$ $\dim S_h = n-1 = \#$ interior nodes

$S_h =$ as above $= \{ v \in C[0,1] : v|_{I_j} \in \Pi_1,\ v(0)=v(1)=0 \}$

<u>(0.4.1) Lemma</u> $\{\phi_i\}_{i=1}^{n-1}$ is a basis for S_h

<u>Proof</u> (i) $\sum_{i=1}^{n-1} c_i \phi_i(x) \equiv 0 \overset{x=x_j}{\Rightarrow} c_j = 0$ <u>linearly indep</u>

(ii) $\{\phi_i\}$ span S_h because

Assume $v \in S_h$. Let $v_I = I_h v = \sum v(x_j)\phi_j$

Then $v - v_I$ is $=0$ at nodes and linear

so $v - v_I = 0$.

in Brenner-Scott only $v(0)=0$
so $\dim(S_h) = n$ instead of $n-1$

0.4.5 Theorem

$$|u-u_I|_{H^1} \le c\,h\,\|u''\|$$

Proof. Let $e = u-u_I$. Sufficient to show

$u_I''=0$ on I_j

$$\int_{I_j} |e(x)|^2\,dx \le c\,h_j^2 \int_{I_j} |u''(x)|^2\,dx = c\,h_j^2 \int |e''|^2\,dx$$

or with

$$\begin{cases} x = x_{j-1} + h_j \tilde{x} \\ \tilde{e}(\tilde{x}) = e(x_{j-1}+h_j\tilde{x}) \\ \dfrac{de}{dx} = \dfrac{d\tilde{e}}{d\tilde{x}}\dfrac{d\tilde{x}}{dx} = h_j^{-1}\tilde{e}' \end{cases}$$

$$\int_0^1 h_j^{-2}|\tilde{e}'|^2\,h_j\,d\tilde{x} \le c\,h_j^2 \int h_j^{-4}|\tilde{e}''|^2\,h_j\,dx$$

Thus, with $w=\tilde{e}$, we must show

$$\int_0^1 |w'(x)|^2\,dx \le c\int |w''(x)|^2\,dx$$

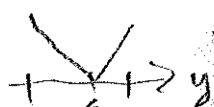
(Note: no h_j here after scaling!)

$w(0)=w(1)=0 \Rightarrow w'(\xi)=0$ for some ξ (Rolle's)

Then
$$w'(y) = \underbrace{w'(\xi)}_{=0} + \int_\xi^y w''(x)\,dx$$

$$|w'(y)| = \left|\int_\xi^y 1\cdot w''(x)\,dx\right| \le \left(\left|\int_\xi^y 1^2\,dx\right|\right)^{\frac{1}{2}}\left(\left|\int_\xi^y |w''|^2\,dx\right|\right)^{1/2}$$

$$\le |y-\xi|^{1/2}\left(\int_0^1 |w''|^2\,dx\right)^{1/2}$$



$$\int_0^1 |w'|^2\,dy \le \int_0^1 |y-\xi|\,dy \int_0^1 |w''|^2\,dx$$

$$\le C\int_0^1 |w''|^2\,dx,\quad c = \sup_\xi \int_0^1 |y-\xi|\,dy = \tfrac{1}{2}$$